Universally optimal verification of entangled states with non-demolition measurements

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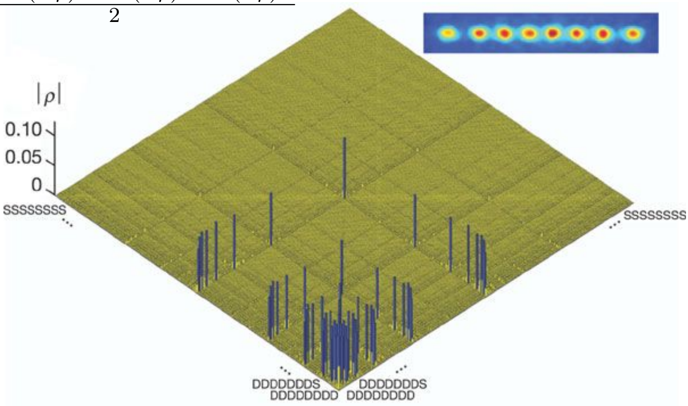
Outline

Quantum State Tomography

an orthogonal set of matrices: $I/\sqrt{2}, X/\sqrt{2}, Y/\sqrt{2}, Z/\sqrt{2}$

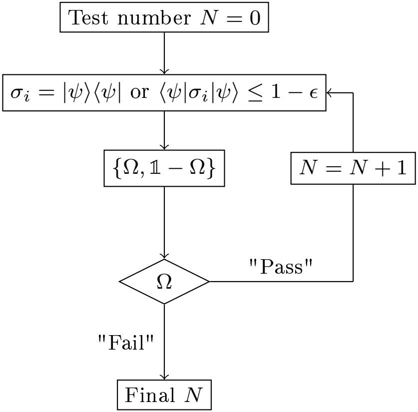
$$\rho = \frac{\mathrm{tr}(\rho)I + \mathrm{tr}(X\rho)X + \mathrm{tr}(Y\rho)Y + \mathrm{tr}(Z\rho)Z}{2}$$



$$\rho = \sum_{\vec{v}} \frac{\mathrm{tr}(\sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n} \rho)\sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n}}{2^n}$$

Häffner *et al.*, Nature **438**, 643–646(2005)

Quantum State Verification



Verification Protocol

$$\Omega = \sum_{j=1}^{m} \mu_j \Omega_j \,,$$

Error probability

$$\max_{\langle\psi|\sigma|\psi\rangle \leq 1-\epsilon} \operatorname{tr}(\Omega\sigma) = 1 - [1 - \lambda_2(\Omega)]\epsilon$$

$$= 1 - \nu(\Omega)\epsilon \,,$$

To achieve confidence level $1 - \delta$, i.e., $[1 - \nu(\Omega)\epsilon]^N \leq \delta$,

$$N \geq \frac{\ln \delta^{-1}}{\ln\left\{[1 - \nu(\Omega)\epsilon]^{-1}\right\}} \approx \frac{1}{\nu(\Omega)} \epsilon^{-1} \ln \delta^{-1} \,.$$

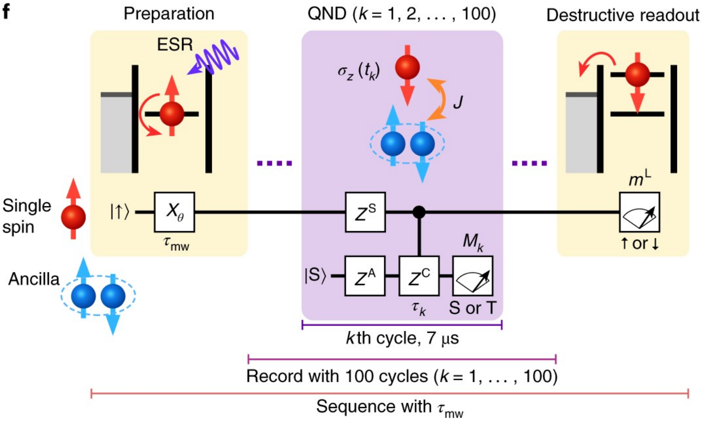
Advantages and Problems

- Advantages:
  - Ω_is are implementable with local projectors only.
  - The overhand for high accuracy is $O(\epsilon^{-1})$.
  - Many quantum states can be verified efficiently or even optimally by QSV.
- Problems:
  - An optimal strategy can rarely be devised.
  - The probabilistic measurements can be very difficult to handle in experiments, especially in the adversarial scenario.
  - The unknown quantum states to be characterized are destroyed after each measurement as the system collapses at the detector, thus, cannot be reused in any subsequent tasks.

Zhang *et al.*, PRL **125**, 030506 (2020).
Jiang *et al.*, arXiv:2002.00640.

Quantum Nondemolition Measurement

$$(\alpha|0\rangle + \beta|1\rangle)_S|0\rangle_A \xrightarrow{\mathcal{C}_X} \alpha|0\rangle_S \otimes |0\rangle_A + \beta|1\rangle_S \otimes |1\rangle_A$$



Takashi *et al.*, Nat. Nanotechnol. **14**, 555–560(2019)

Verification with QND measurement

$$|\Psi\rangle \otimes |0\rangle \xrightarrow{U_i} \Omega_i|\Psi\rangle \otimes |0\rangle + (\mathbb{1} - \Omega_i)|\Psi\rangle \otimes |1\rangle$$

coupling:

$$U_i = \Omega_i \otimes \mathbb{1} + (\mathbb{1} - \Omega_i) \otimes X$$

QND measurement:

$$\mathcal{M}_i = (\mathbb{1} \otimes |0\rangle\langle 0|)\, U_i$$

Remarks:

- $\mathcal{M}_i(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |0\rangle$
- The ancilla is one two-dimensional qubit initialized as $|0\rangle$.
- U_i can be realized with standard quantum gates.

Theorem 1: Sequential NDQV

Theorem

If a target state $|\psi\rangle$ can be verified by the protocol $\Omega = \sum_i \mu_i \Omega_i$, where Ω_is are local projectors, then it can be verified optimally by

$$\mathcal{M} = \prod_i \mathcal{M}_i$$

The spectral gap of $\mathcal{M}$ is given by

$$\nu(\mathcal{M}) = 1\,,$$

indicating that the verification efficiency of $\mathcal{M}$ is the same as that of the optimal global strategy.

Remarks

- $\mathcal{M}\left[\sigma \otimes \left(|0\rangle\langle 0|\right)^{\otimes l}\right] = \Omega_s \sigma \otimes \left(|0\rangle\langle 0|\right)^{\otimes l} \quad \longrightarrow \quad \mathcal{M} \widehat{=} \Omega_s \otimes \mathbb{1}$
  where $\Omega_s = \prod \Omega_i = |\psi\rangle\langle\psi|$.
- The order of the measurements $\mathcal{M}_i$s in the sequential NDQV protocol can be made arbitrary.

Corollary (No more measurements)

The verification efficiency of the sequential NDQV protocol will not be improved by adding more measurement settings.

Corollary (Fidelity Estimation and State Preparation)

The average fidelity between the output state σ and the target state $|\psi\rangle$, i.e., $\mathcal{F} = \langle F \rangle$, can be directly estimated by the sequential NDQV protocol,

$$\mathcal{M}\left[\sigma \otimes \left(|0\rangle\langle 0|\right)^{\otimes l}\right] = F|\psi\rangle\langle\psi| \otimes \left(|0\rangle\langle 0|\right)^{\otimes l}.$$

Verification of the Bell state

Target state:

$$|\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$

One noise decomposition:

$$\{(|00\rangle - |11\rangle)/\sqrt{2}, (|01\rangle + |10\rangle)/\sqrt{2}, (|01\rangle - |10\rangle)/\sqrt{2}\}$$

Local measurement:

$$\begin{aligned}\Omega_1 &= P_{ZZ}^{+} = |00\rangle\langle 00| + |11\rangle\langle 11|\,,\\ \Omega_2 &= P_{XX}^{+} = |++\rangle\langle ++| + |--\rangle\langle --|\,.\end{aligned}$$

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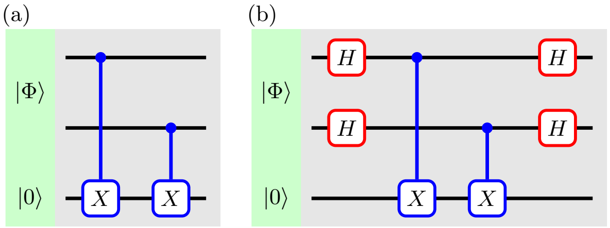
One noise decomposition:

$$\{(|00\rangle - |11\rangle)/\sqrt{2}, (|01\rangle + |10\rangle)/\sqrt{2}, (|01\rangle - |10\rangle)/\sqrt{2}\}$$

QND measurement: $\mathcal{M}_i = (\mathbb{1} \otimes |0\rangle\langle 0|)\, U_i$

$$\begin{aligned} U_1 &= \mathcal{C}_{X13}\mathcal{C}_{X23}\,, \\ U_2 &= (H \otimes H \otimes \mathbb{1})\,\mathcal{C}_{X13}\mathcal{C}_{X23}\,(H \otimes H \otimes \mathbb{1})\,. \end{aligned}$$



Verification of arbitrary two-qubit states

Target state:

$$|\Psi\rangle = \sin\theta|00\rangle + \cos\theta|11\rangle$$

Local measurement:

$$\begin{aligned}
\Omega_1 &= P_{ZZ}^{+} = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\,, \\
\Omega_2 &= \mathbb{1} - |+\rangle\langle +| \otimes |\varphi_+\rangle\langle \varphi_+|\,, \\
\Omega_3 &= \mathbb{1} - |-\rangle\langle -| \otimes |\varphi_-\rangle\langle \varphi_-|\,,
\end{aligned}$$

where $|\varphi_\pm\rangle = \cos\theta|0\rangle \mp \sin\theta|1\rangle$.

Verification of arbitrary two-qubit states

Target state:

$$|\Psi\rangle = \sin\theta|00\rangle + \cos\theta|11\rangle$$

QND measurement: $\mathcal{M}_i = (\mathbb{1} \otimes |0\rangle\langle 0|)\, U_i$

$$\begin{aligned}
\mathcal{M}_1 &= \big(\mathbb{1} \otimes |0\rangle\langle 0|\big)\mathcal{C}_{X1a}\mathcal{C}_{X2a}, \\
\mathcal{M}_i^t &= \big(\mathbb{1} \otimes |0\rangle\langle 0|\big)\big(R_i^\dagger \otimes \mathbb{1}\big)\big(X \otimes X \otimes \mathbb{1}\big) \\
&\quad \mathcal{C}_{X12a}^2\big(X \otimes X \otimes \mathbb{1}\big)\big(R_i \otimes \mathbb{1}\big),
\end{aligned}$$

where $R_{2(3)}$ turns the state $|+\rangle \otimes |\varphi_+\rangle$ $(|-\rangle \otimes |\varphi_-\rangle)$ into $|00\rangle$.

Theorem 2

$$\mathcal{M}_i^t = \left(\mathbb{1} \otimes |0\rangle\langle 0|\right)\left(R_i^\dagger \otimes \mathbb{1}\right)\left(X \otimes X \otimes \mathbb{1}\right) \mathcal{C}_{X12a}^2\left(X \otimes X \otimes \mathbb{1}\right)\left(R_i \otimes \mathbb{1}\right),$$

$$\mathcal{M}_i^b = \mathbb{1} - \left(\mathbb{1} \otimes |00\rangle\langle 00|\right)\left(R_i^\dagger \otimes \mathbb{1}\right)\mathcal{C}_{X1a}\mathcal{C}_{X2a'}\left(R_i \otimes \mathbb{1}\right).$$

Theorem

For the specific setting of NDQV where the ancilla is always prepared in $|0\rangle$ and measured in the Pauli-Z basis, a generalized $(n+1)$-body Toffoli gate can always be replaced by n two-body CNOT *gates with ancilla qubits initially prepared in $|0\rangle^{\otimes n}$.*

Verification of stabilizer states

These QND measurements can be implemented in the same way as the syndrome measurements for stabilizer quantum error correction codes.

Target state:

$$|\psi\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$$

The generator set:

$$\{XXX, ZIZ, ZZI\}$$

QND measurement:

$$\begin{aligned}
\mathcal{M}_1 &= (\mathbb{1} \otimes |0\rangle\langle 0|)\big[(H \otimes H \otimes H \otimes \mathbb{1}) \\
&\quad \mathcal{C}_{X14}\mathcal{C}_{X24}\mathcal{C}_{X34}(H \otimes H \otimes H \otimes \mathbb{1})\big], \\
\mathcal{M}_2 &= (\mathbb{1} \otimes |0\rangle\langle 0|)\big[\mathcal{C}_{X14}\mathcal{C}_{X34}\big], \\
\mathcal{M}_3 &= (\mathbb{1} \otimes |0\rangle\langle 0|)\big[\mathcal{C}_{X14}\mathcal{C}_{X24}\big].
\end{aligned}$$

Conclusion

- Advantages: Maintained
  - ✓ local projectors only
  - ✓ $O(\epsilon^{-1})$ efficiency
  - ✓ many achievable cases
- Problems: Improved
  - optimal strategy $\longrightarrow$ Corollary (No more measurements)
  - probabilistic measurements $\longrightarrow$ sequential NDQV
  - destroyed quantum states $\longrightarrow$ QND measurements
- Overhands:
  - Quantum States: $N \approx \epsilon^{-1} \ln \delta^{-1}$
  - Ancilla: Only one qubit would be enough if it is engineered properly.
  - Couplings: $O(n)$ local rotations and $O(n)$ CNOT gates

THANK YOU

– **Ye-Chao Liu**, Jiangwei Shang, Rui Han, Xiangdong Zhang, arXiv:2005.01106