Hidden Teleportation Power for Entangled Quantum States

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Introduction

- Quantum Teleportation [1]:

[1] Bennett *et. al.*, Phys. Rev. Lett, **70** (1993)

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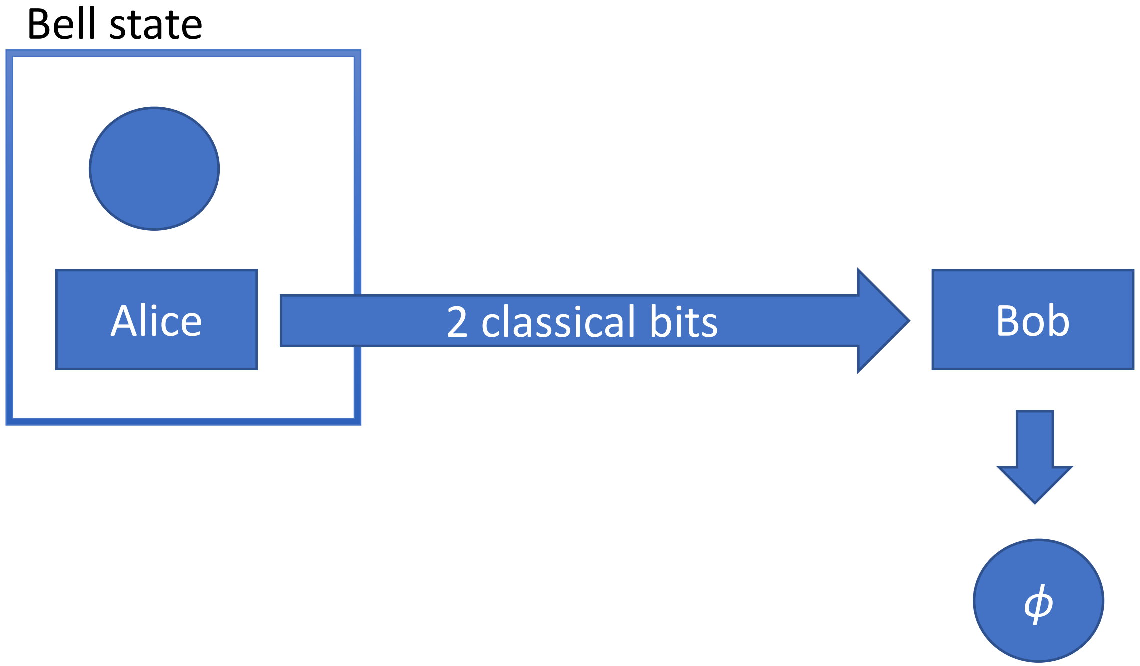
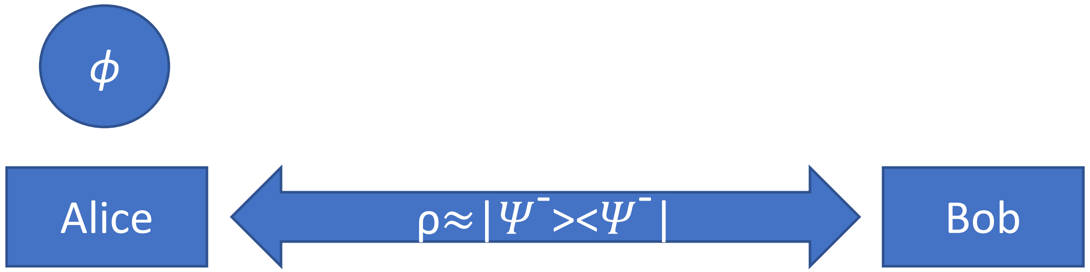


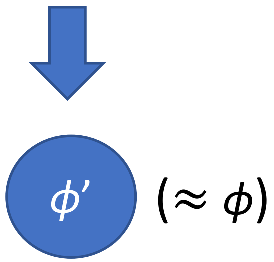
[1] Bennett *et. al.*, Phys. Rev. Lett, **70** (1993)

Introduction

- What if they do not share a maximally entangled state?



- The fidelity $F(|\phi\rangle, |\phi'\rangle) = |\langle\phi|\phi'\rangle|^2 < 1$



Introduction

- For qubit case:
  If Alice and Bob can only have classical resource:

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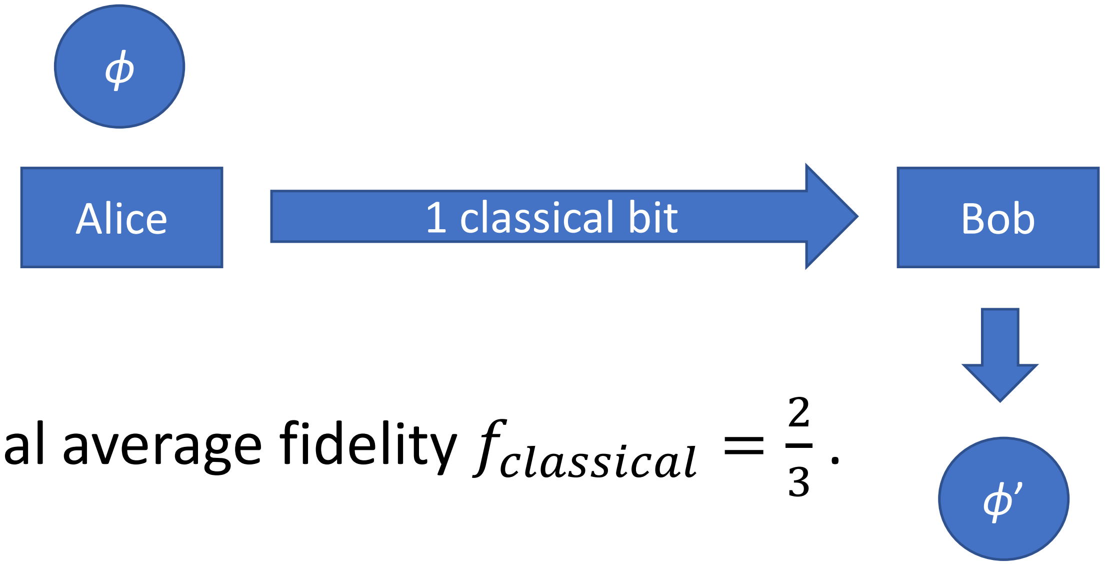

- The maximal average fidelity $f_{classical} = \frac{2}{3}$.

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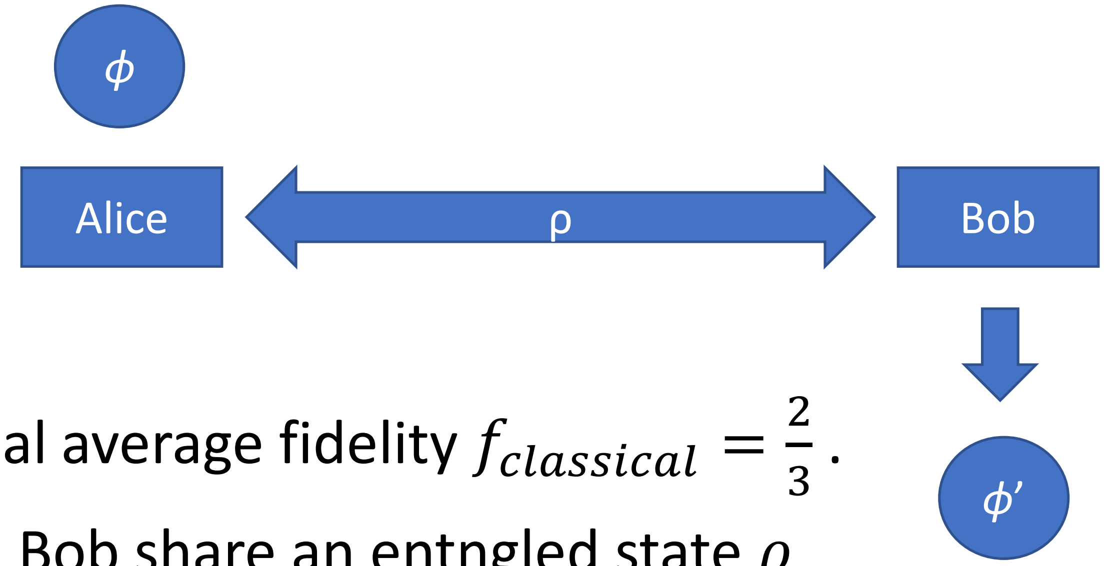
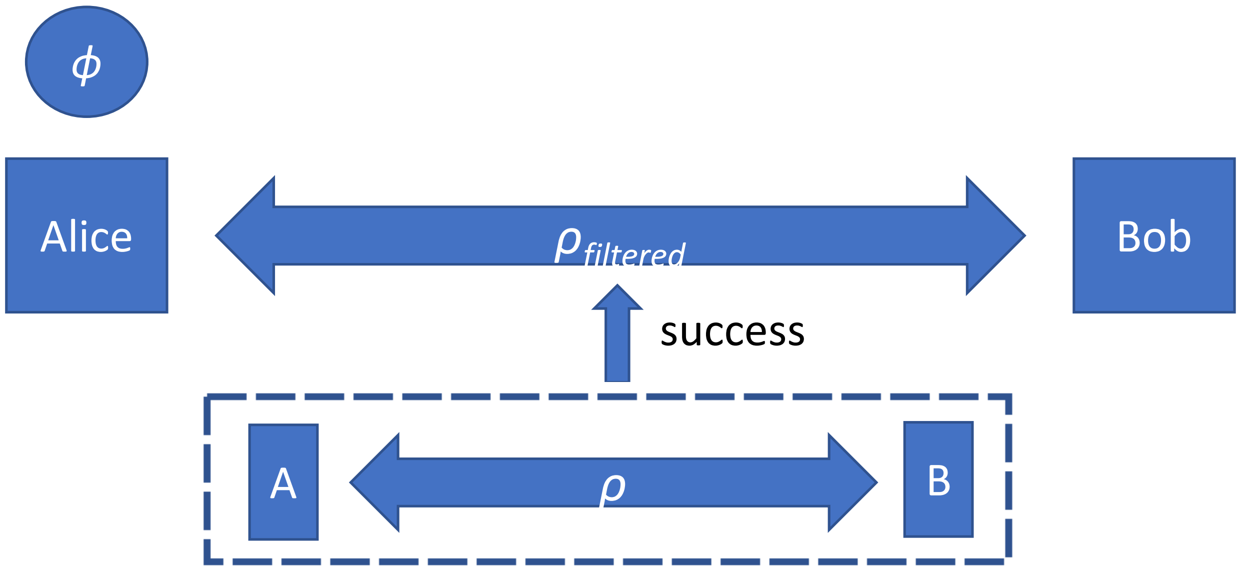
- For qubit case:
  If Alice and Bob can only have classical resource:



- The maximal average fidelity $f_{classical} = \frac{2}{3}$.
- If Alice and Bob share an entngled state ρ
  and $f_{ave}^{max}(\rho) > \frac{2}{3}$, we say this entangled state ρ is useful for teleportation.

Hidden teleportation power

- Local filtering operation:



- For qubit case, if an entangled state is useless $f_{ave}^{max}(\rho) < \frac{2}{3}$ initially, but become useful $f_{ave}^{max}(\rho_{filtered}) > \frac{2}{3}$ after successful local filtering, then we say this entangled state ρ has hidden teleportation power.

Hidden teleportation power for qudit states

- The hidden teleportation power for qubit case has been studied. [2], [3]
- Relating fidelity and singlet fraction [4]:
  Suppose that Alice and Bob share an entangled state ρ acting on $H_A \otimes H_B = \mathbb{C}^d \otimes \mathbb{C}^d$, then

$$f_{ave}^{max}(\rho) = \frac{SF(\rho)d+1}{d+1}$$

  where $SF(\rho) = \max_{\Psi} \langle\Psi|\rho|\Psi\rangle$ is the maximal singlet fraction of the state ρ.

- $f_{classical} = \frac{2}{d+1}$
  An entangled state is useful if $f_{ave}^{max}(\rho) > \frac{2}{d+1}$. (or $SF(\rho) > \frac{1}{d}$)

[4]Horodecki *et. al.*, Phys. Rev. A, **60** (1999)
[2]Horodecki *et. al.*, Phys. Rev. Lett, **78** (1997)   [3] F. Verstraete, H. Verschelde, Phys. Rev. Lett, **90** (2003)

Hidden teleportation power for qudit states

- Consider following entangled state share between Alice and Bob:

$$\rho(q) = q|\Phi^+\rangle\langle\Phi^+| + (1-q)|01\rangle\langle 01|,\ 0 < q \leq 1$$

$$|\Phi^+\rangle = (|00\rangle + |11\rangle + |22\rangle + \cdots + |(\mathrm{d}-1)(\mathrm{d}-1)\rangle)/\sqrt{d}$$

- Goal: find an optimal filter A s.t. the overall teleportation fidelity is maximal:

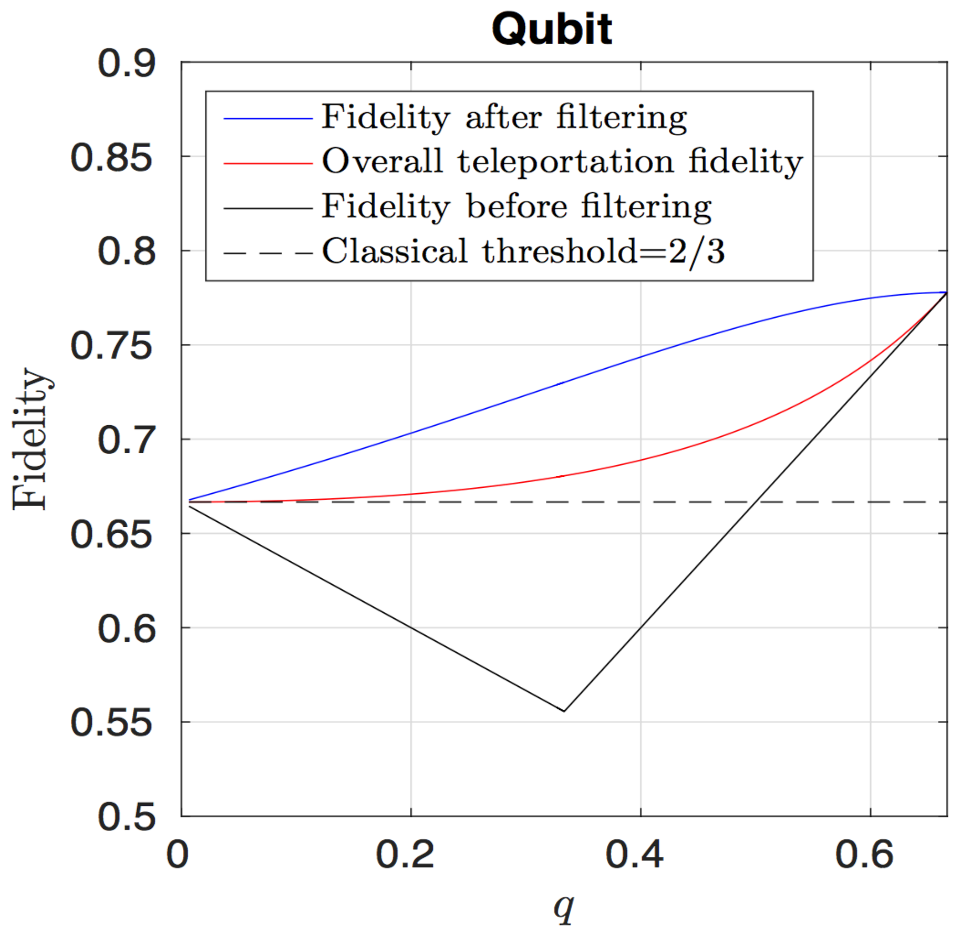
$$f_{overall} = p f_{ave}^{max}(\rho_{filtered}) + (1-p)\frac{2}{d+1}$$

$p = tr[\,(A \otimes I)\rho(A \otimes I)^{\dagger}\,]$ is the success probability of local filtering,

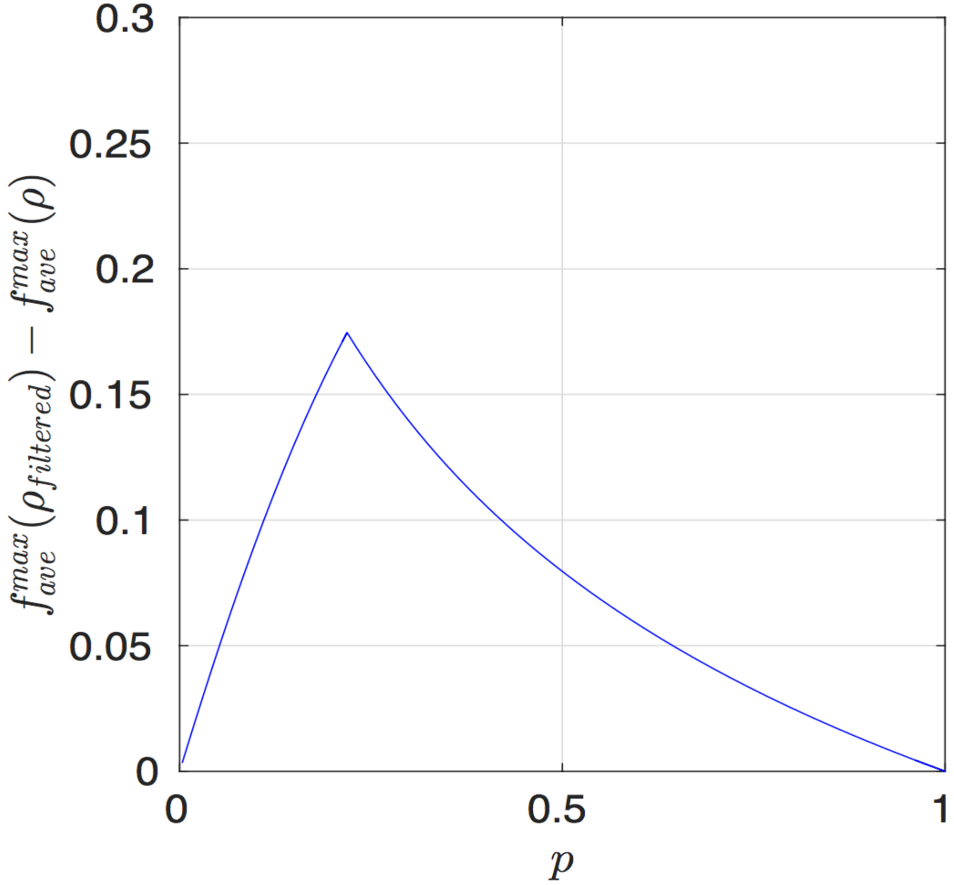
$\rho_{filtered} = \dfrac{(A \otimes I)\rho(A \otimes I)^{\dagger}}{p}$ is the state after successful local filtering.

- The optimal filter $A = \begin{bmatrix} \frac{(d-1)q}{d(q-1)} & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix},\ 0 < q \leq \frac{d}{2d-1}$

Results

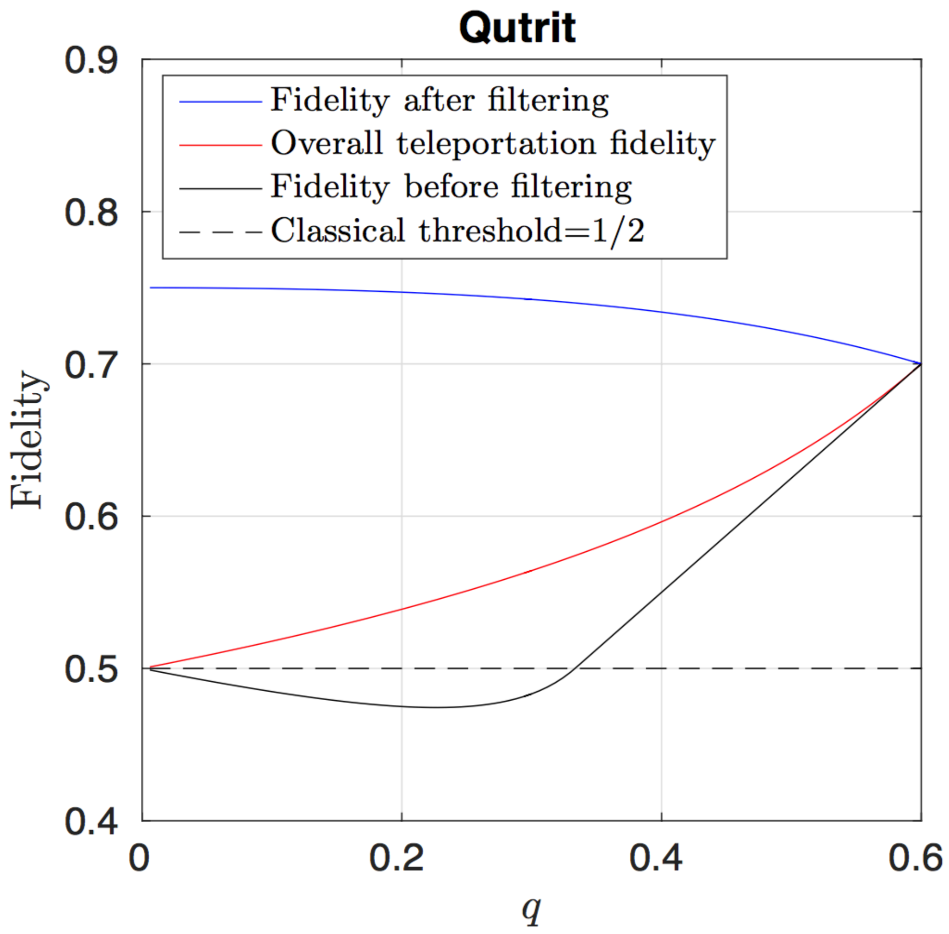


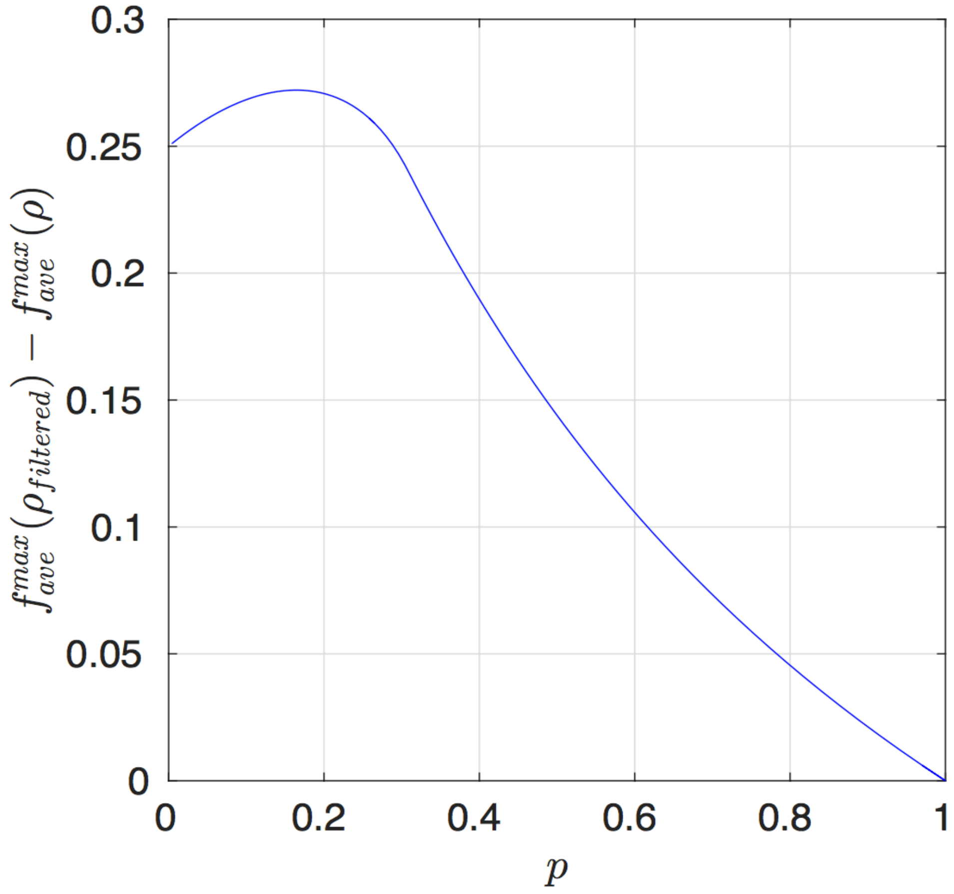
Qubit
Fidelity after filtering
Overall teleportation fidelity
Fidelity before filtering
Classical threshold=2/3
Fidelity
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
0.2
0.4
0.6
q



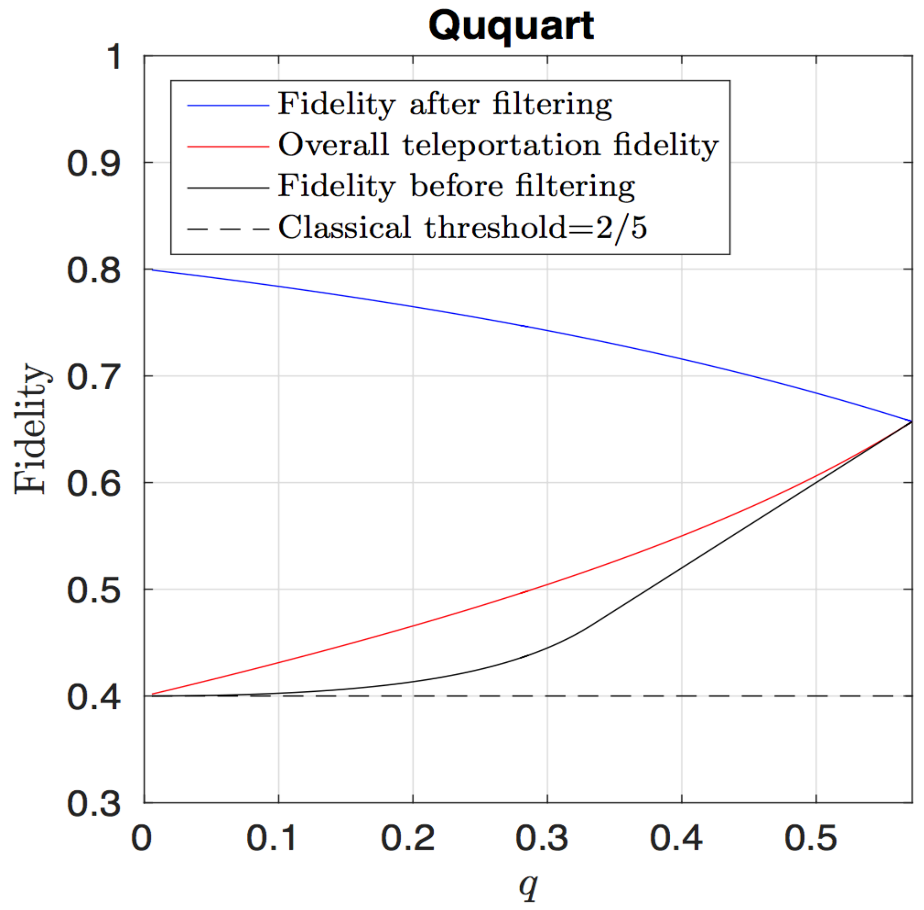
$f_{ave}^{max}(\rho_{filtered}) - f_{ave}^{max}(\rho)$
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
p

Results

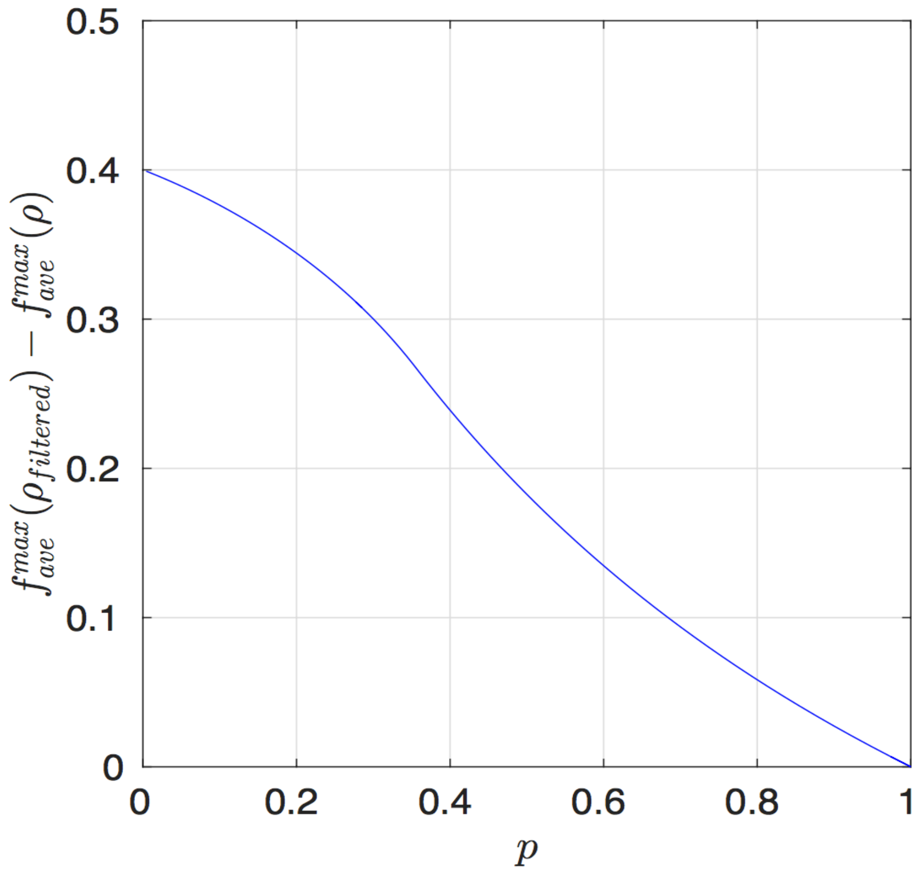

Qutrit
Fidelity after filtering
Overall teleportation fidelity
Fidelity before filtering
Classical threshold=1/2
Fidelity
0.9
0.8
0.7
0.6
0.5
0.4
0
0.2
0.4
0.6
q


$f_{ave}^{max}(\rho_{filtered}) - f_{ave}^{max}(\rho)$
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
p

Results



Ququart
Fidelity after filtering
Overall teleportation fidelity
Fidelity before filtering
Classical threshold=2/5
Fidelity
q
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
0.1
0.2
0.3
0.4
0.5



$f_{ave}^{max}(\rho_{filtered}) - f_{ave}^{max}(\rho)$
p
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1

Hidden teleportation power for Werner state

- Werner State:

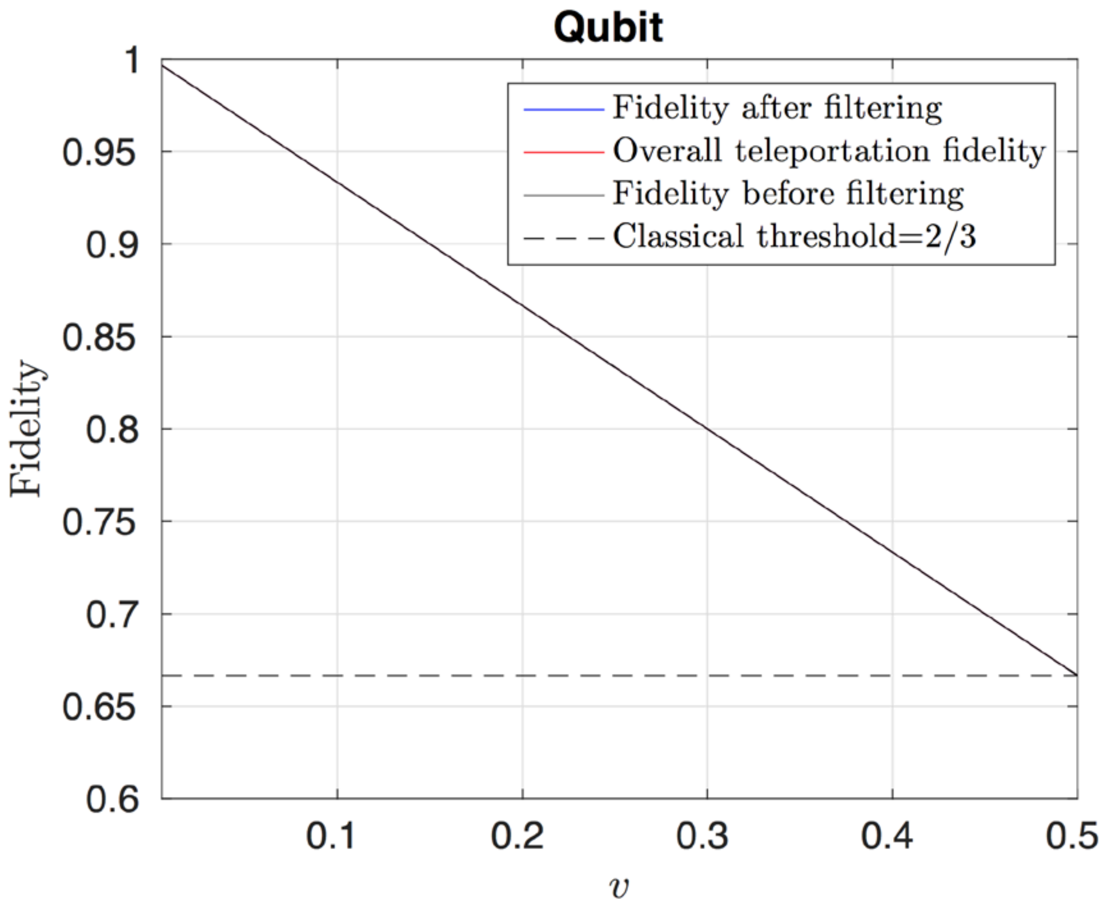
$$\rho(v) = v\frac{2}{d(d+1)}P_{sy} + (1-v)\frac{2}{d(d-1)}P_{asy},\ 0 \le v \le 1$$

where $P = \sum_{i,j} |i\rangle\langle j| \otimes |j\rangle\langle i|$, $P_{sy} = \frac{I+P}{2}, P_{asy} = \frac{I-P}{2}$

- The optimal filters A and B:

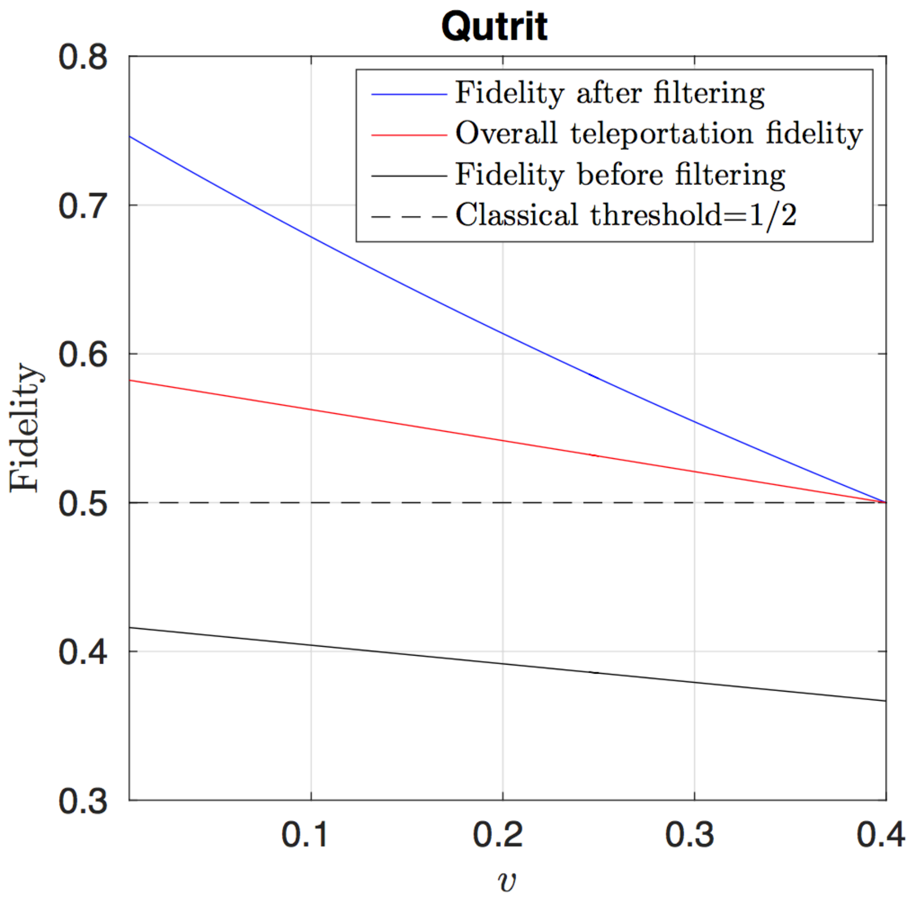
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, 0 \le v < \frac{d+1}{4d-2}$$

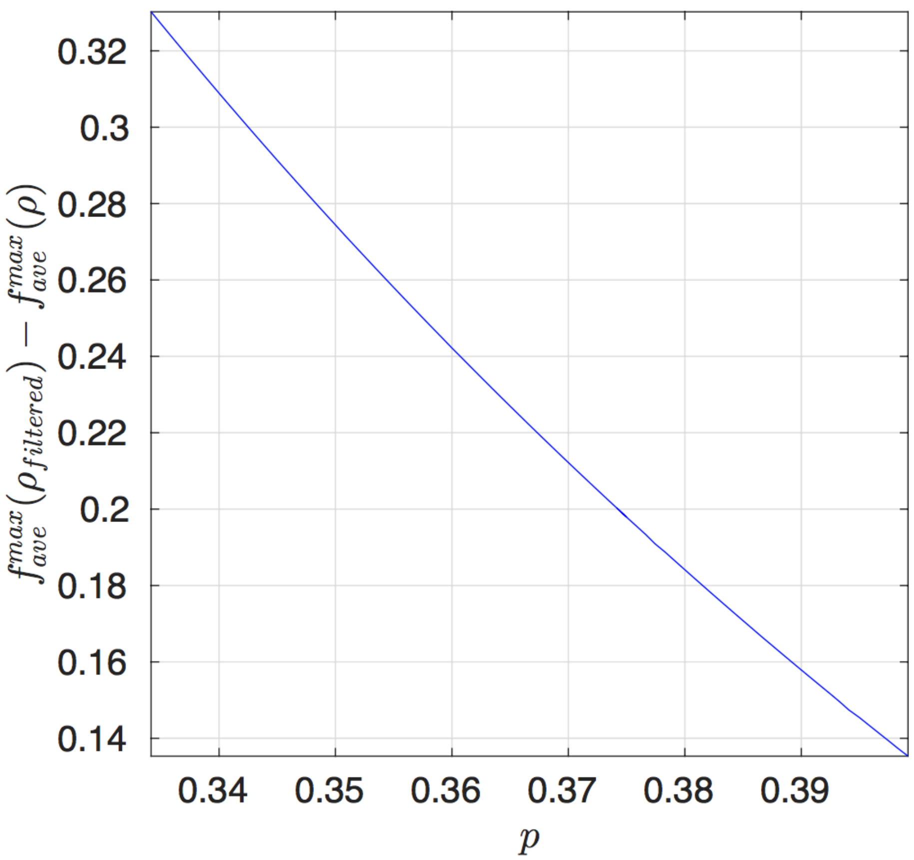
Results



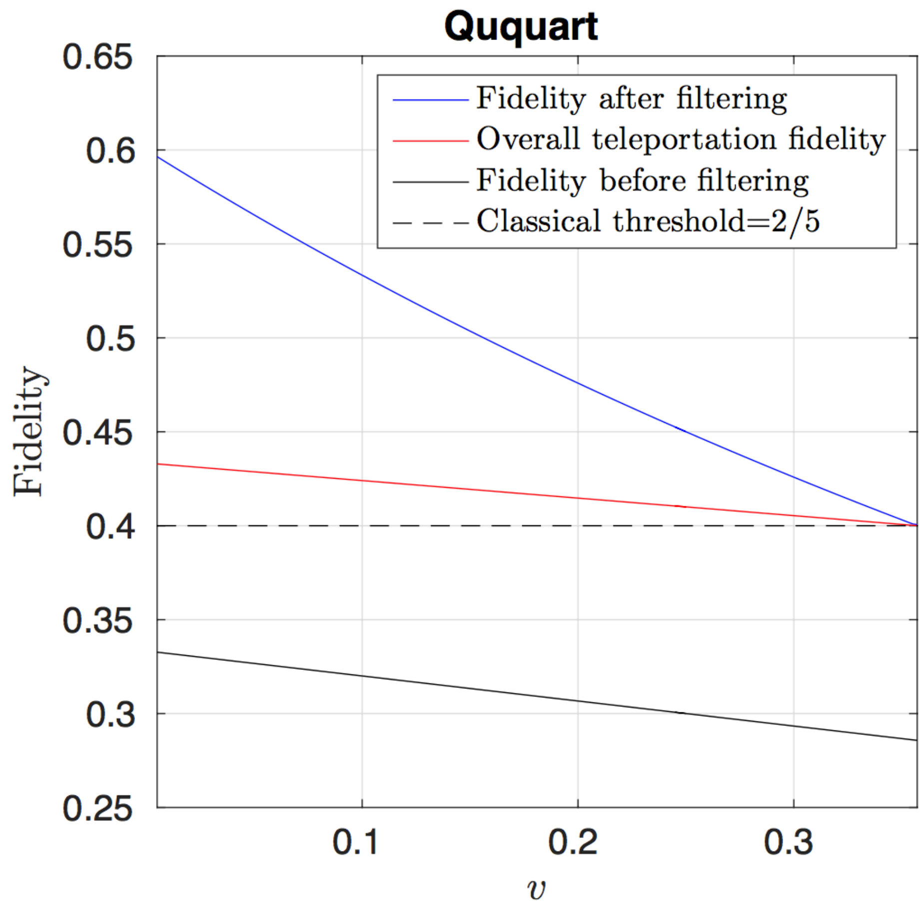
Qubit
Fidelity after filtering
Overall teleportation fidelity
Fidelity before filtering
Classical threshold=2/3
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
Fidelity
0.1
0.2
0.3
0.4
0.5
v

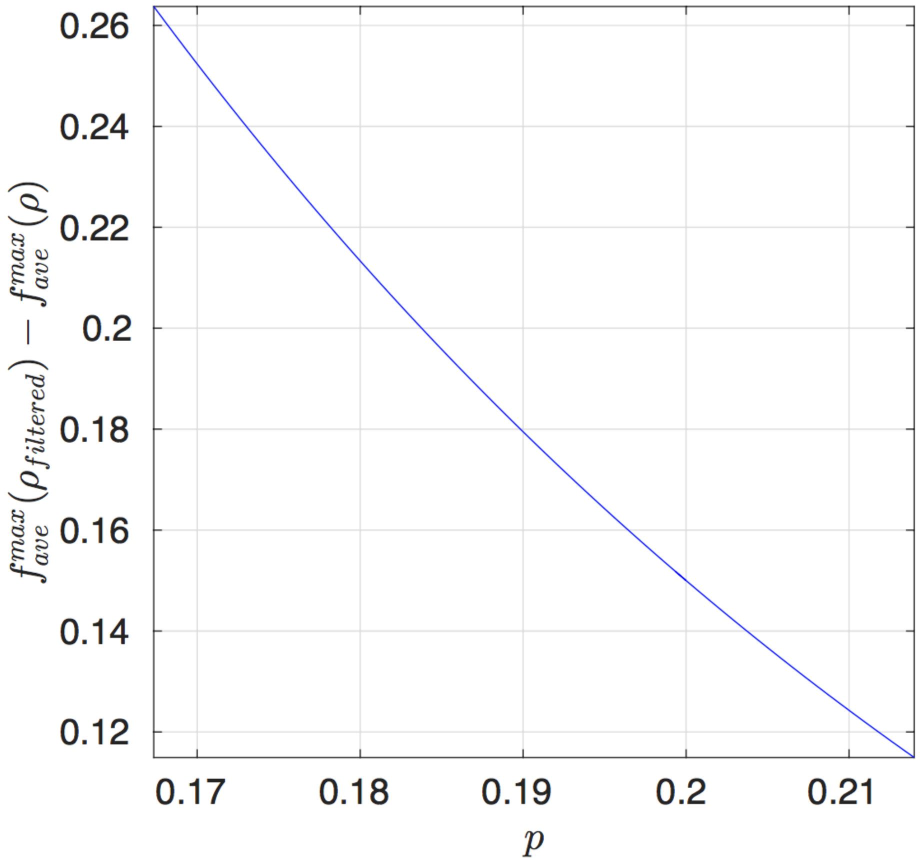
Results


Qutrit
Fidelity after filtering
Overall teleportation fidelity
Fidelity before filtering
Classical threshold=1/2
Fidelity
0.8
0.7
0.6
0.5
0.4
0.3
0.1
0.2
0.3
0.4
v


$f_{ave}^{max}(\rho_{filtered}) - f_{ave}^{max}(\rho)$
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.34
0.35
0.36
0.37
0.38
0.39
p

Results


Ququart
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
Fidelity
Fidelity after filtering
Overall teleportation fidelity
Fidelity before filtering
Classical threshold=2/5
0.1
0.2
0.3
v


0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
$f_{ave}^{max}(\rho_{filtered}) - f_{ave}^{max}(\rho)$
0.17
0.18
0.19
0.2
0.21
p

Summary

- We summarize our results in the following table:

d	Teleportation power	Teleportation power after local filtering	Hidden teleportation power
2	useless ($0 < q \leq \frac{1}{2}$)	always useful	yes
3	useless ($0 < q \leq \frac{1}{3}$)	always useful	yes
4	always useful	always useful	no

d	Teleportation power	Teleportation power after local filtering	Hidden teleportation power
2	always useful	always useful	no
3	always useless	always useful	yes
4	always useless	always useful	yes

- There is a tradeoff between the success probability of local filtering and how much fidelity can be increased by local filtering.
- What about the general entangled state?

Back-up

- For qubit case, the maximal *SF* can be obtained by only consider 4 Bell states; but for higher dimensional case is not:

$$\rho(F) = q|\Psi\rangle\langle\Psi| + (1-q)|01\rangle\langle 01|$$

$$|\Psi\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$$

$$\text{SF} = \max_{\Psi} \langle \Psi|\rho|\Psi \rangle = \max_{U_B} \langle\psi|(I \otimes U)^{\dagger}\rho(I \otimes U)|\psi\rangle$$

- $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = e^{i\theta} \begin{bmatrix} a_{11} & a_{12} + ib_{12} & a_{13} + ib_{13} \\ a_{21} + ib_{21} & a_{22} + ib_{22} & a_{23} + ib_{23} \\ a_{31} + ib_{31} & a_{32} + ib_{32} & a_{33} + ib_{33} \end{bmatrix}$

Back-up

- $f_{overall} = \frac{1}{3}$ [1+ $\frac{q}{3}|A_{11}+A_{22}+A_{33}|^2$+ (1-$q$)$|A_{21}|^2$ - $\frac{q}{3}$ ($|A_{11}|^2+|A_{22}|^2+|A_{33}|^2$) - $\frac{q}{3}$ (all the off-diagonal terms of A) - (1- q) ($|A_{11}|^2+|A_{21}|^2+|A_{31}|^2$)]

  $= \frac{1}{3}$ [1+ $(q\text{-}1)a_{11}^2 + \frac{2q}{3}a_{11}(a_{22}+a_{33}) + \frac{2q}{3}(a_{22}a_{33}+b_{22}b_{33})$]

- Generalize to d-dimensional quantum state:

  $f_{overall} = \frac{1}{d}\left[1+(q\text{-}1)\,a_{11}^2 + \frac{2q}{d}a_{11}\sum_{i=2}^{d} a_{ii} + \frac{2q}{d}\sum_{i=2,j>i}^{i=d-1} a_{ii}a_{jj}\right]$

  $= \frac{1}{d}\{1+(q\text{-}1)\,a_{11}^2 + \frac{q(d-1)}{d}[2a_{22}+(\text{d-2})]\}$

  * $\frac{\partial K_d}{\partial a_{11}} = 0$, we can find $a_{11} = \frac{(d-1)q}{d(q-1)}$

  $f_{overall} = \frac{1}{d}\left[1+ \frac{q^2(d-1)^2}{d^2(1-q)} + \frac{q(d-1)(d-2)}{d}\right].$

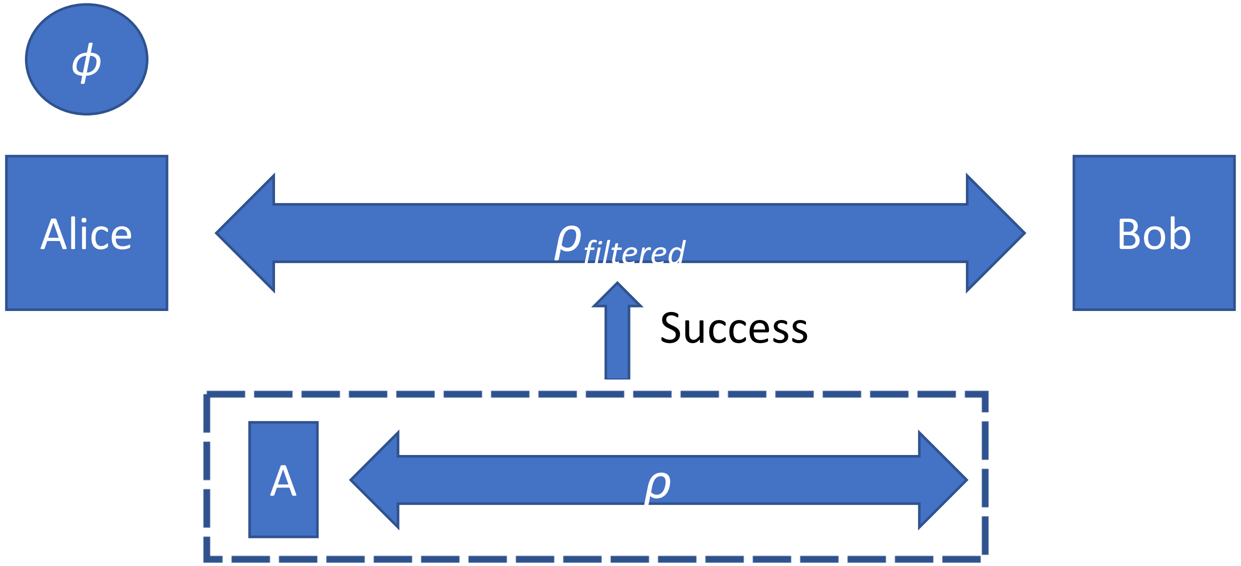
  $$A=\begin{bmatrix} \frac{(d-1)q}{d(q-1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Back-up

- The problem of hidden teleportation for qubit case has been studied:
  Optimal teleportation with a mixed state of two qubit (F. Verstraete, H. Verschelde, 2003)
  Theorem:
  The bipartite local filtering operations probabilistically bring an entangled mixed state of two qubits to a state with highest possible fidelity are given by the filtering operations bringing the state into its unique Bell-diagonal normal form, yielding a fidelity larger than 1/2.

Protocol

- Success



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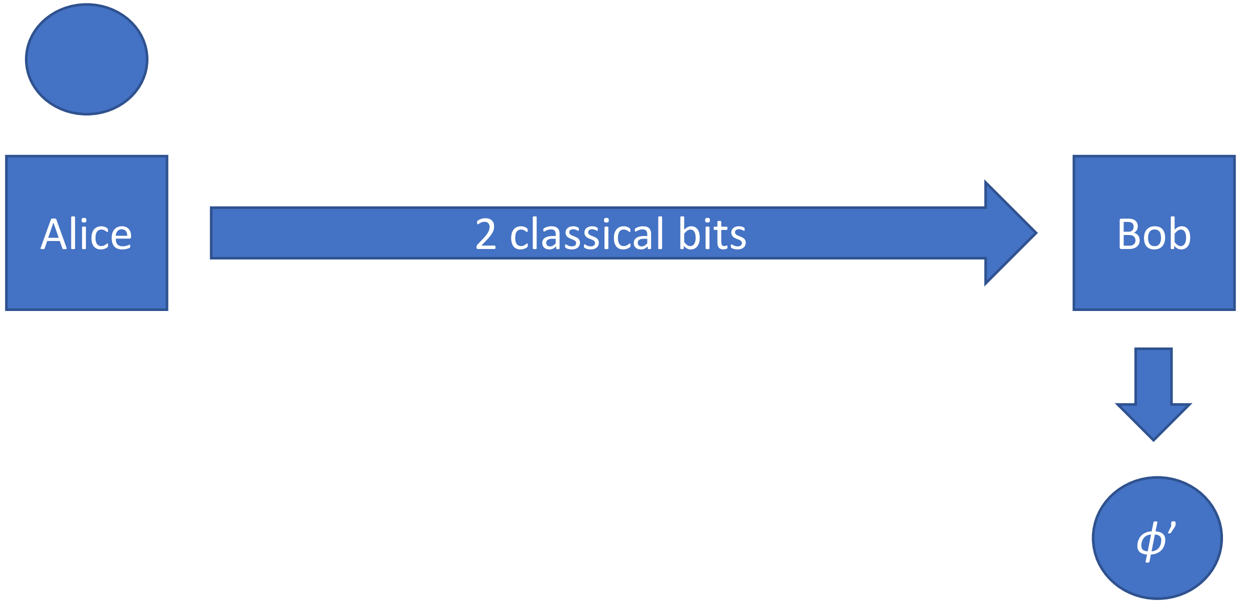


Protocol

- Fail

Protocol

- Fail

Back-up

- Quantum Teleportation [1]
  $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

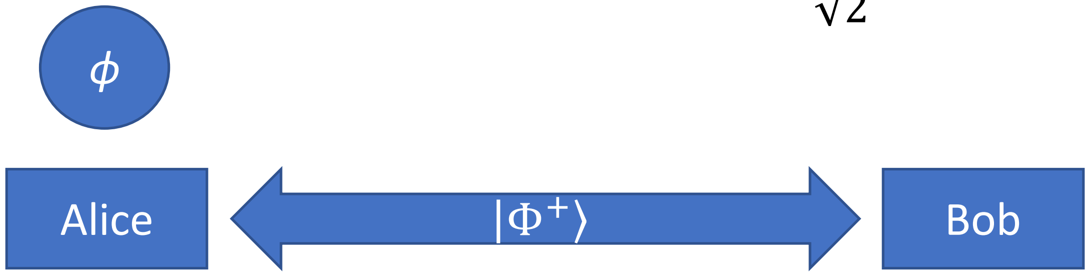
Bell basis :

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad \Rightarrow \quad |01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad |10\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle)$$



- $|\phi\rangle|\Phi^+\rangle = \frac{1}{2}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

$$= \frac{1}{2}[|\Phi^+\rangle(\alpha|0\rangle + \beta|1\rangle) + |\Phi^-\rangle(\alpha|0\rangle - \beta|1\rangle)$$
$$+|\Psi^+\rangle(\alpha|1\rangle + \beta|0\rangle) + |\Psi^-\rangle(\alpha|1\rangle - \beta|0\rangle)$$

[1] Bennett *et. al.*, Phys. Rev. Lett, **70** (1993)

Back-up

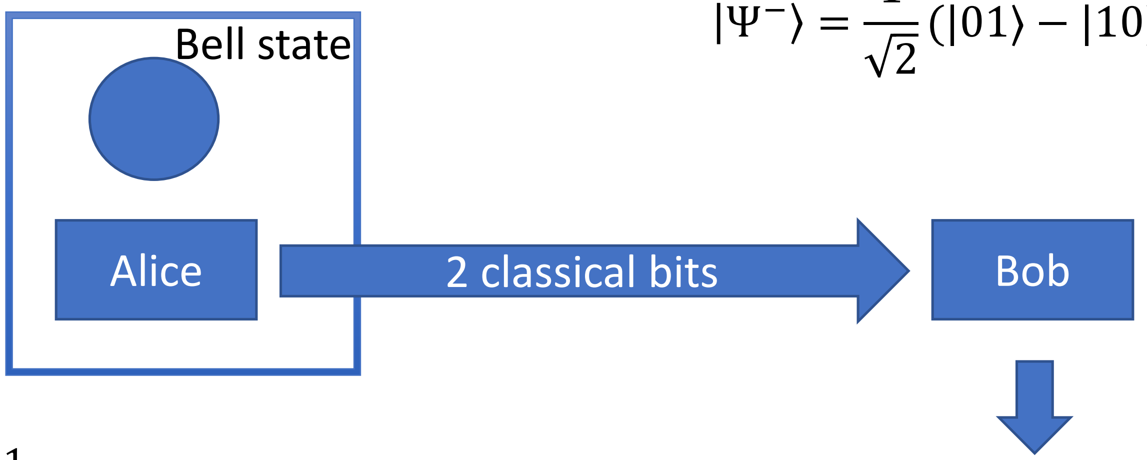
- Quantum Teleportation [1]
  $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

Bell basis :

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$\Rightarrow$$

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$$
$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$$
$$|10\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle)$$



- $|\phi\rangle|\Phi^+\rangle = \frac{1}{2}(\alpha|000\rangle + \alpha|011\rangle + \alpha|100\rangle + \alpha|111\rangle)$
  $= \frac{1}{2}[|\Phi^+\rangle(\alpha|0\rangle + \beta|1\rangle) + |\Phi^-\rangle(\alpha|0\rangle - \beta|1\rangle)$
  $+|\Psi^+\rangle(\alpha|1\rangle + \beta|0\rangle) + |\Psi^-\rangle(\alpha|1\rangle - \beta|0\rangle)$

[1] Bennett *et. al.*, Phys. Rev. Lett, **70** (1993)