



Bounding the set of finite dimensional correlations

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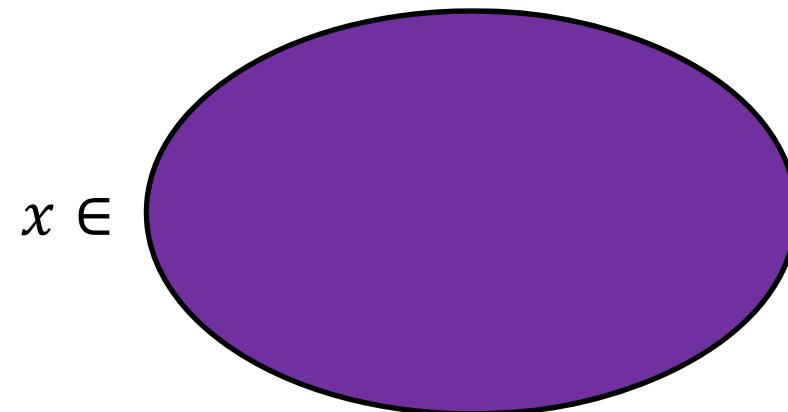


MN and T. Vértesi, Phys. Rev. Lett. 115, 020501 (2015).

MN, A. Feix, M. Araújo and T. Vértesi, Phys. Rev. A 92, 042117 (2015).

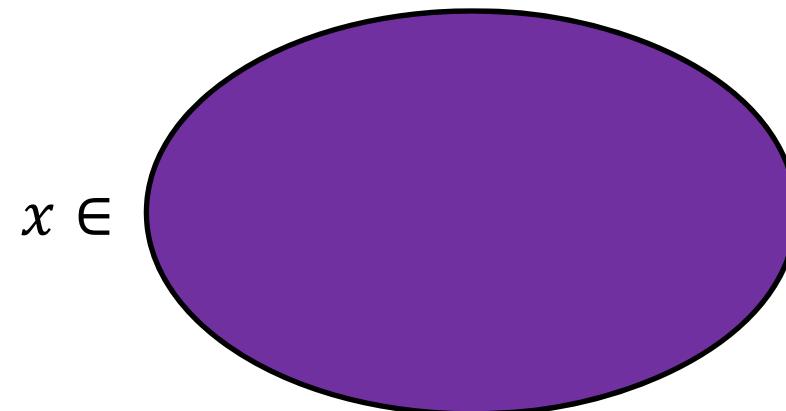
Optimization theory

$$\max f(x)$$



Variational methods

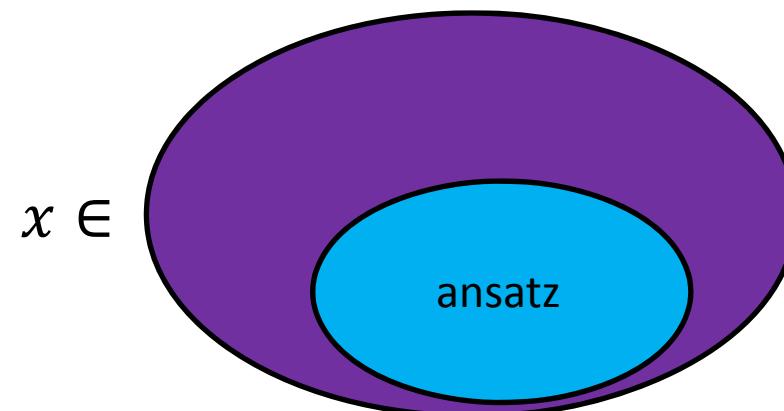
$$\max f(x)$$



$x \in$

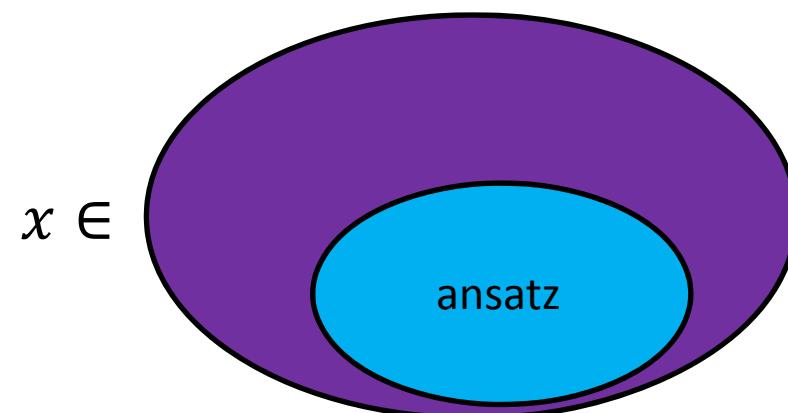
Variational methods

$$\max f(x)$$



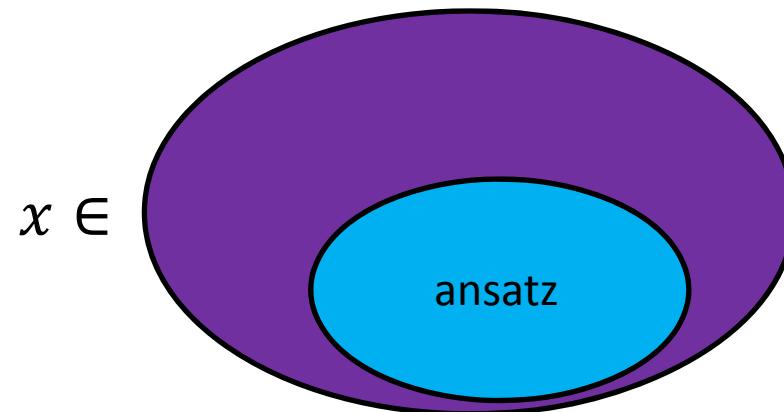
Variational methods

$$\max f(x) \geq \mu$$



Variational methods

$$\max f(x) \geq \mu$$

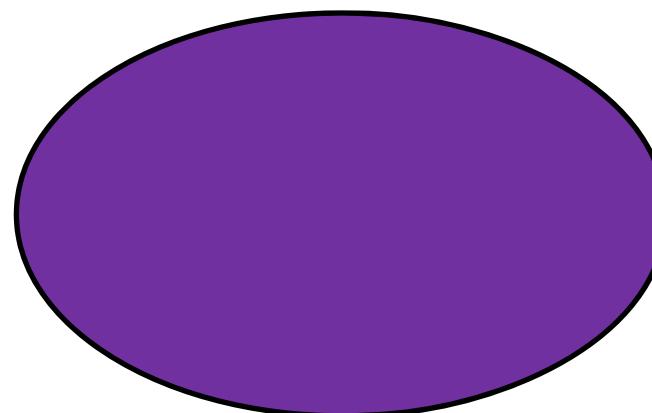


- usually, pretty straightforward
 - usually, universal
 - sometimes arise as intuitions gathered via numerical experiments. E.g.: MPS.
- } E.g.: Newton's method

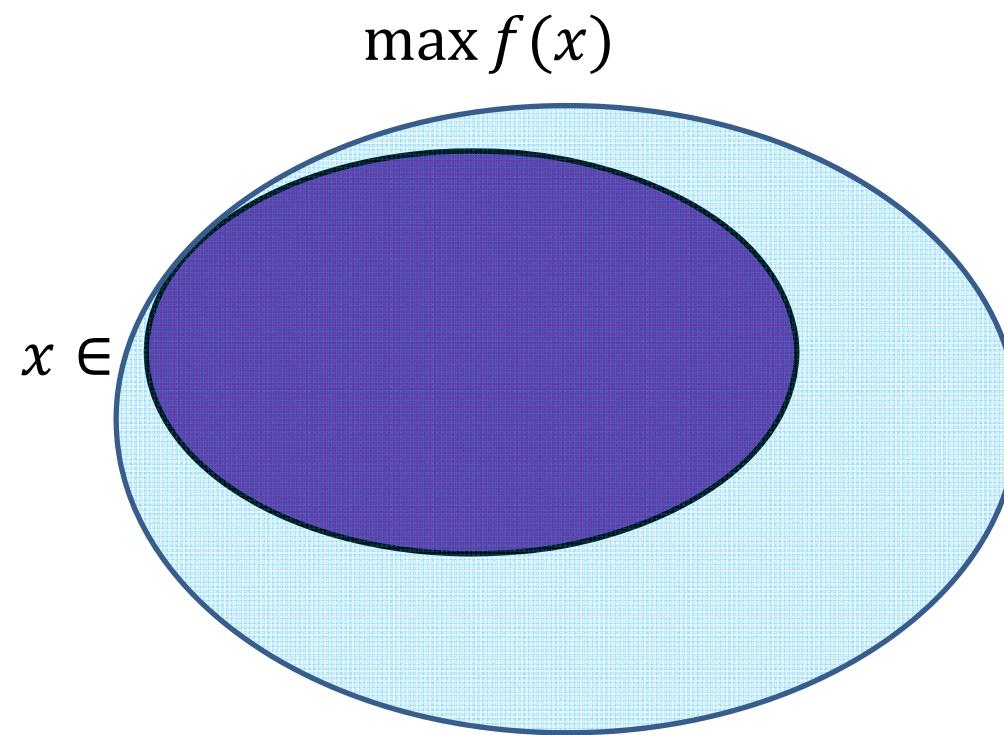
Relaxations

$$\max f(x)$$

$x \in$

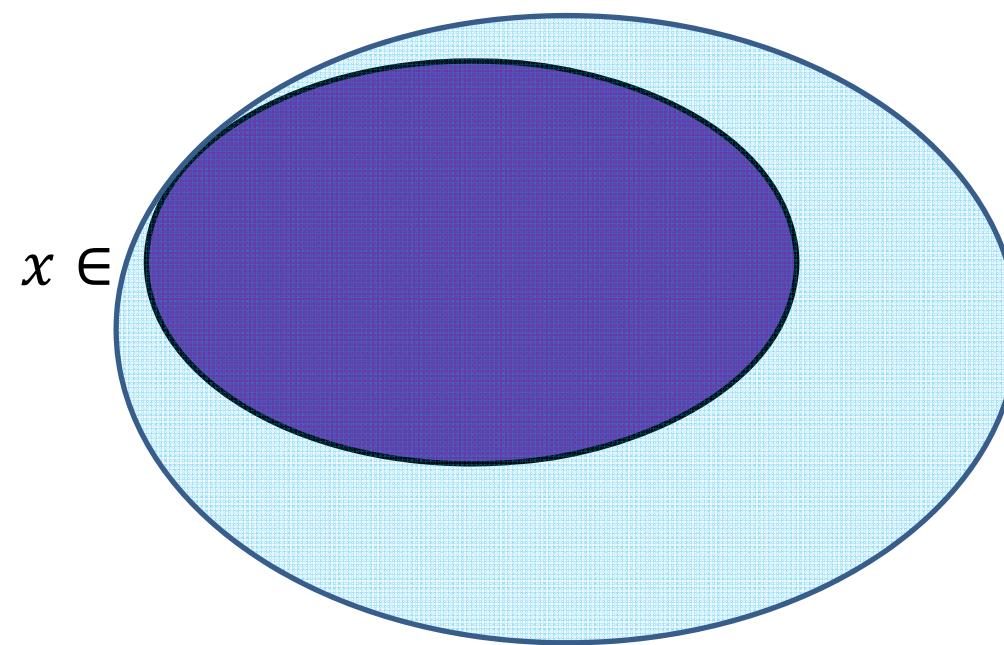


Relaxations



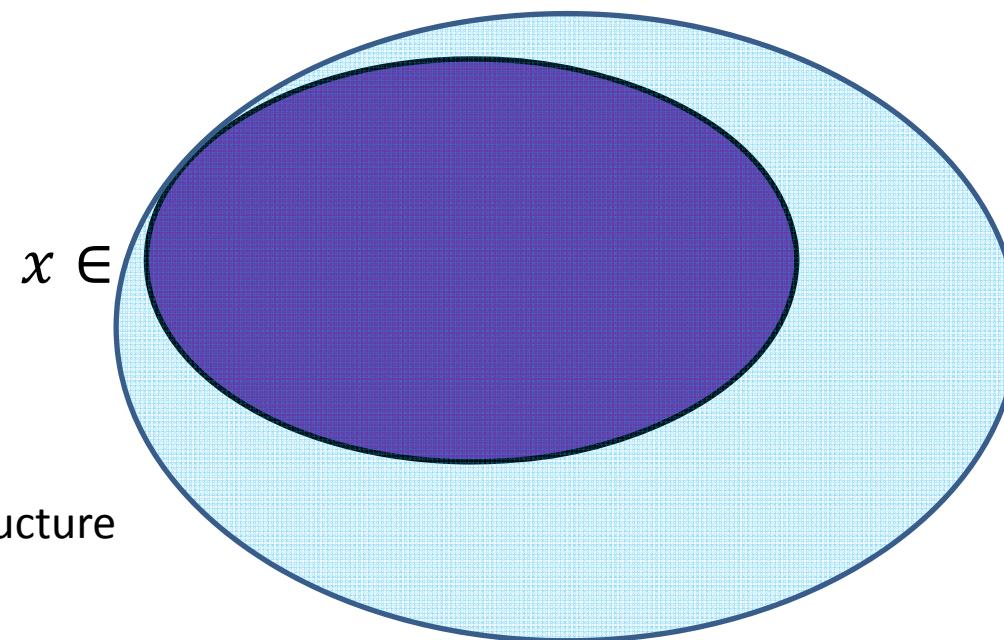
Relaxations

$$\lambda \geq \max f(x)$$



Relaxations

$$\lambda \geq \max f(x)$$



- Depend on the structure of the problem

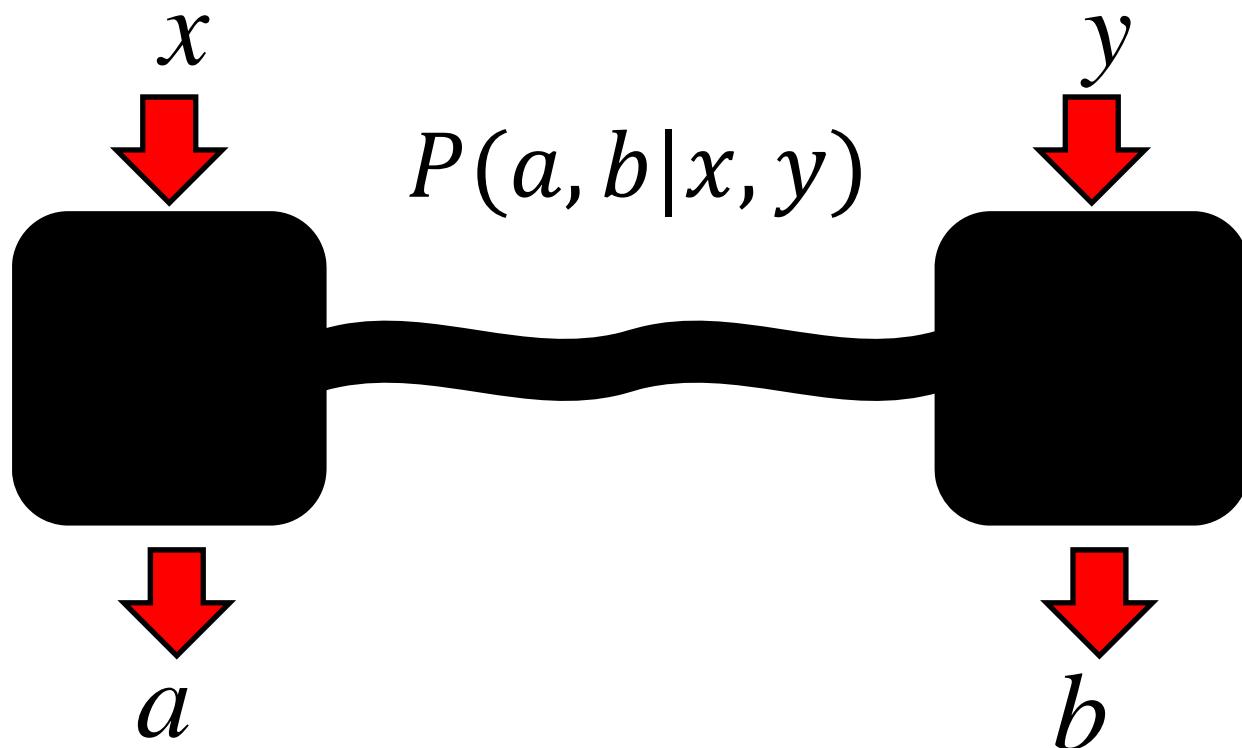
- "Aha!" idea

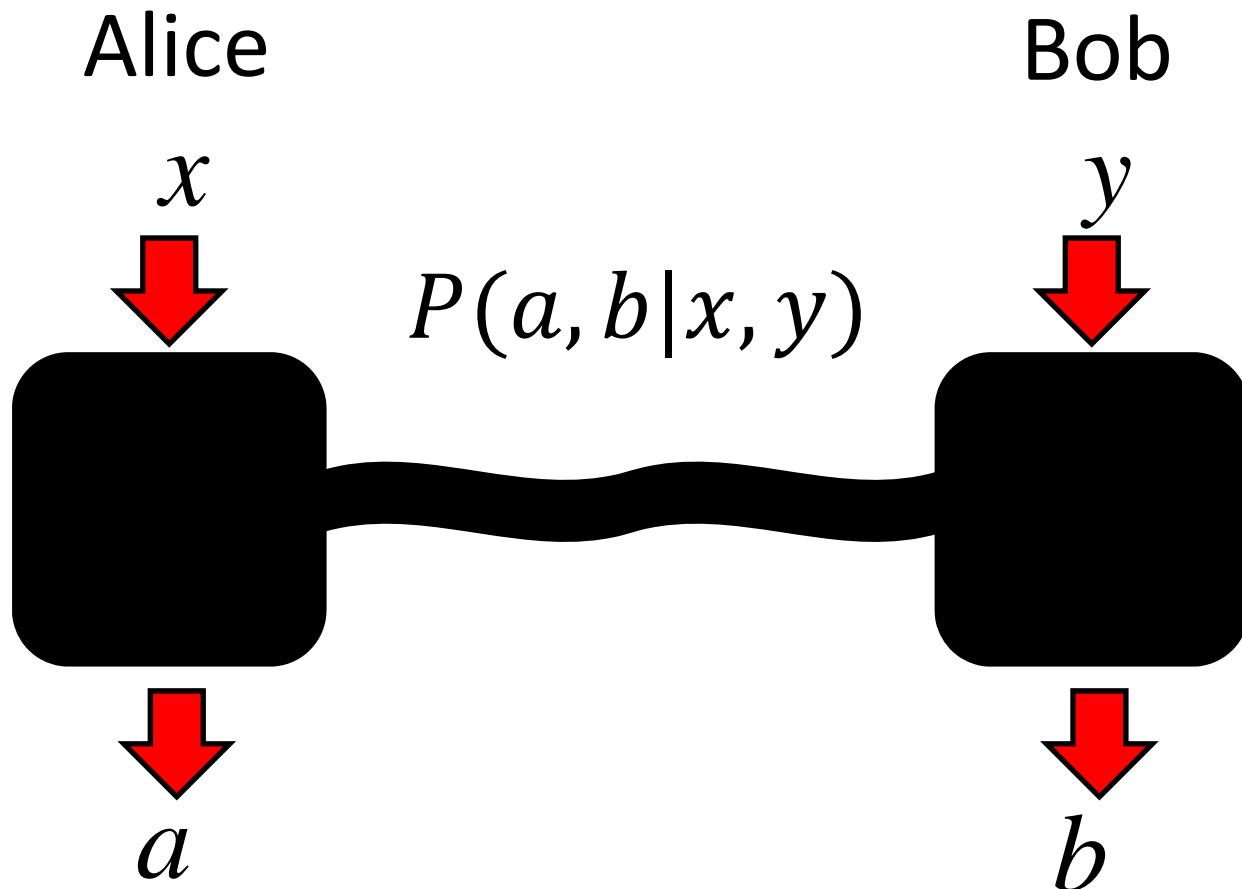
E.g.: the NPA hierarchy

MN, S. Pironio and A. Acín, Phys. Rev. Lett. 98, 010401 (2007).
MN, S. Pironio and A. Acín, New J. Phys. 10, 073013 (2008).

Alice

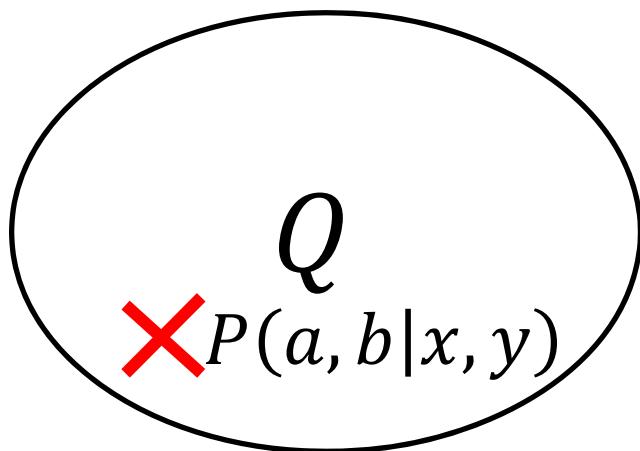
Bob





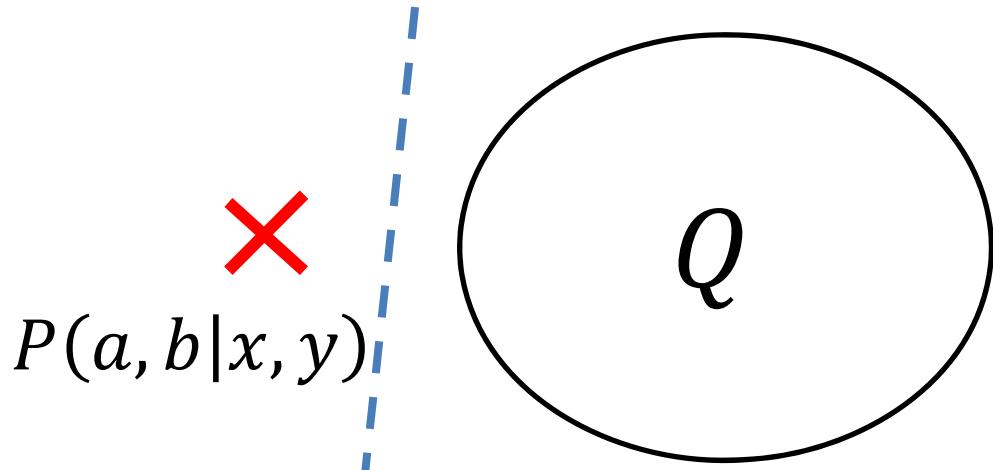
Are the ouputs of this experiment compatible with
quantum mechanics?

"Easy"



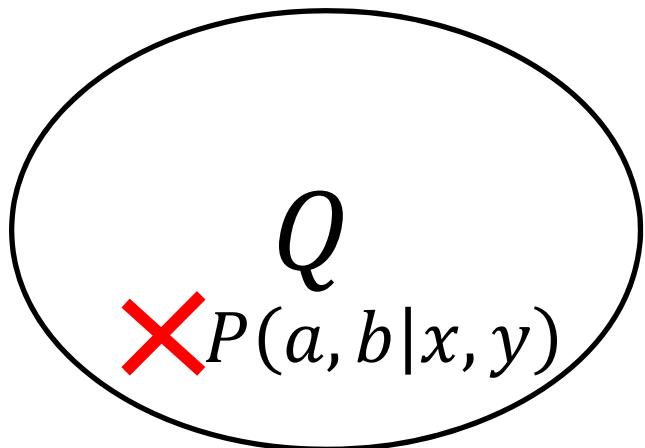
Variational methods

"Difficult"



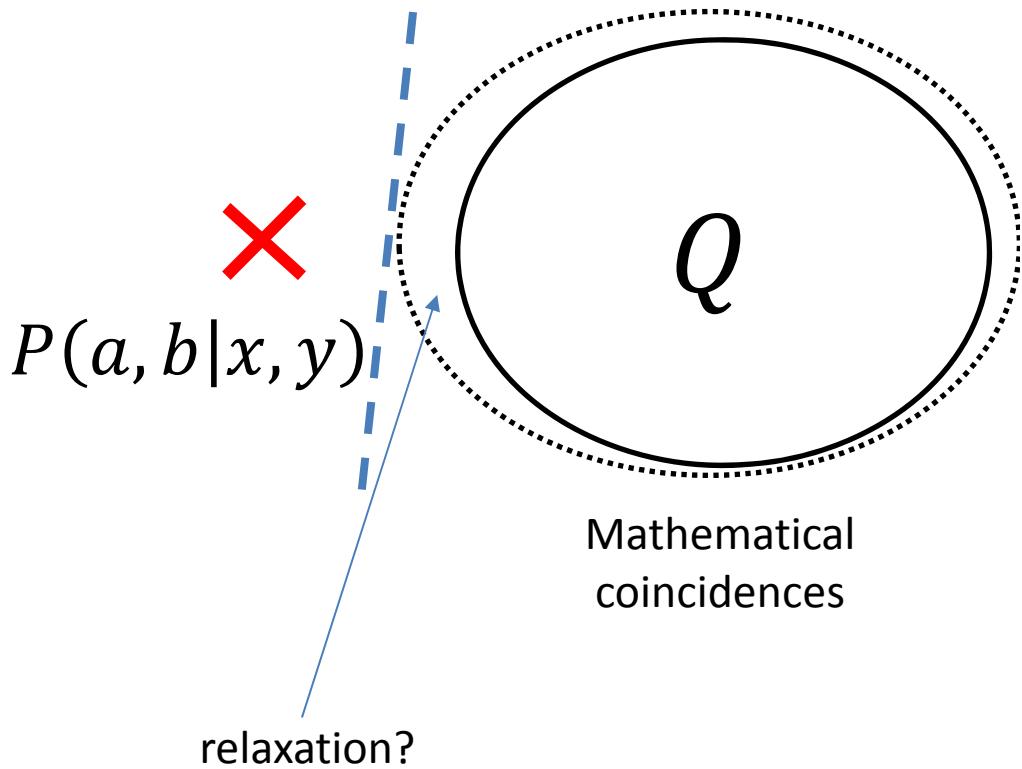
Mathematical
coincidences

"Easy"



Variational methods

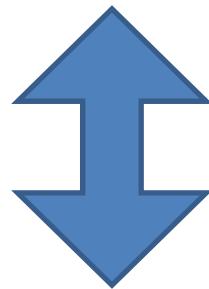
"Difficult"



Mathematical coincidences

relaxation?

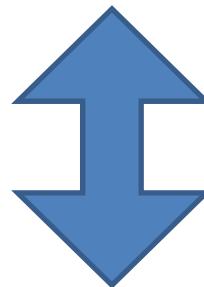
$P(a, b|x, y)$ is not quantum



There do not exist a Hilbert space \mathcal{H} , a quantum state $|\psi\rangle \in \mathcal{H}$ and projector operators $\{E_a^x, F_b^y\} \subset B(\mathcal{H})$, with $\sum_a E_a^x = \sum_b F_b^y = \mathbb{I}$, $[E_a^x, F_b^y] = 0$, such that $P(a, b|x, y) = \langle \psi | E_a^x F_b^y | \psi \rangle$.

The problem resists brute force approach!!

$P(a, b|x, y)$ is not quantum



There do not exist a Hilbert space \mathcal{H} , a quantum state $|\psi\rangle \in \mathcal{H}$ and projector operators $\{E_a^x, F_b^y\} \subset B(\mathcal{H})$, with $\sum_a E_a^x = \sum_b F_b^y = \mathbb{I}$, $[E_a^x, F_b^y] = 0$, such that $P(a, b|x, y) = \langle \psi | E_a^x F_b^y | \psi \rangle$.

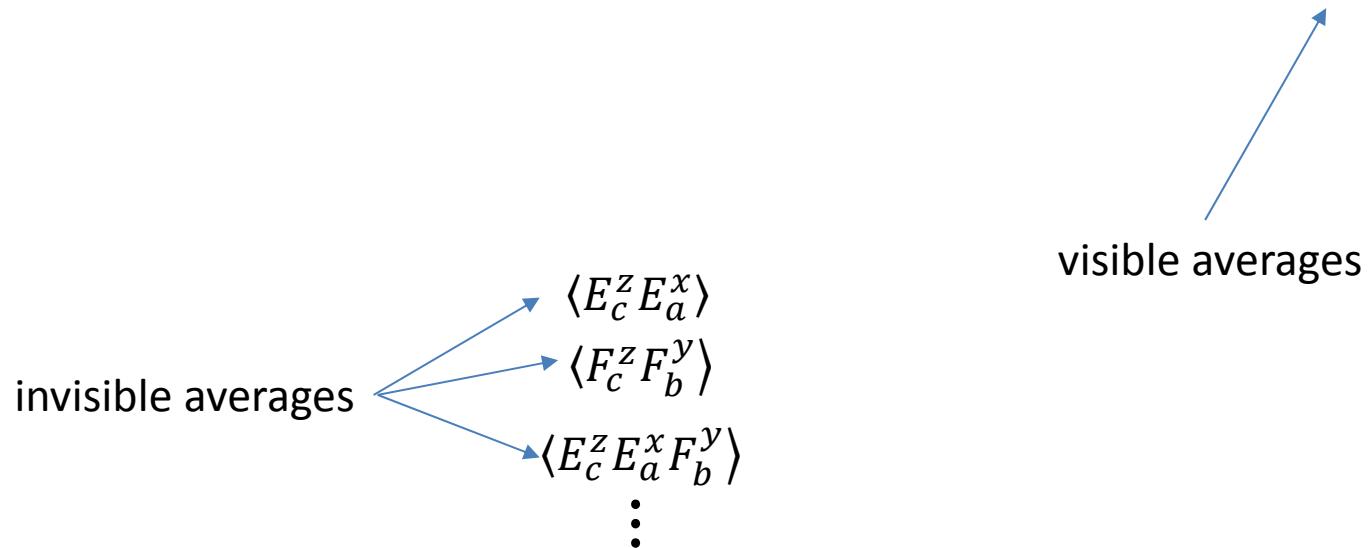
The problem resists brute force approach!!

What is a Hilbert space?

$$\psi = (\psi_1,\psi_2,\psi_3\dots)$$

$$\sum_i |\psi_i|^2 < \infty$$

$$P(a, b | x, y) = \langle \phi, E_a^x F_b^y \phi \rangle = \langle E_a^x F_b^y \rangle$$



Moment matrix

	\mathbb{I}	E_a^x	$E_{a'}^{x'}$	F_b^y	\dots
\mathbb{I}^\dagger	1				
$E_a^{x\dagger}$			$\langle E_a^x E_{a'}^{x'} \rangle$	$P(a, b x, y)$	
$E_{a'}^{x'\dagger}$		$P(a' x')$		$P(a', b x', y)$	
$F_b^{y\dagger}$					
\vdots					

$$\Gamma_{u,v} = \langle u^\dagger v \rangle$$

Moment matrix

$$\Gamma = \begin{array}{c|ccccc} & \mathbb{I} & E_a^x & E_{a'}^{x'} & F_b^y & \dots \\ \hline \mathbb{I}^\dagger & 1 & & & & \\ E_a^{x\dagger} & \vdash & \cdots & \langle E_a^x E_{a'}^{x'} \rangle & P(a, b|x, y) & \\ E_{a'}^{x'\dagger} & P(a'|x') & \cdots & & P(a', b|x', y) & \\ F_b^{y\dagger} & & & & & \\ \vdots & & & & & \end{array} \geq 0$$



Hilbert space



Moment matrix

	\mathbb{I}	E_a^x	$E_{a'}^{x'}$	F_b^y	\dots
\mathbb{I}^\dagger	1				
$E_a^{x\dagger}$			$\langle E_a^x E_{a'}^{x'} \rangle$	$P(a, b x, y)$	≥ 0
$E_{a'}^{x'\dagger}$		$P(a' x')$		$P(a', b x', y)$	
$F_b^{y\dagger}$					
:					

$P(a, b|x, y)$ admits a positive semidefinite moment matrix for the operators \mathbb{I}, E_a^x, F_b^y

1st-order moment matrix

Q^1

$P(a, b|x, y)$ admits a positive semidefinite moment matrix for the operators

$\mathbb{I}, E_a^x, F_b^y, E_a^x F_b^y,$
 $E_a^x E_{a'}^x, F_{b'}^y F_b^y$

2nd-order moment matrix

Q^2

Q^3

...

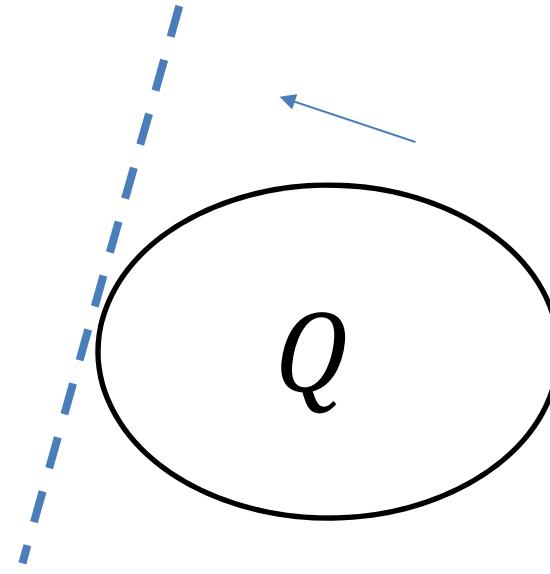
Q

3rd-order moment matrix

$P(a, b|x, y)$ admits a positive semidefinite moment matrix for the operators
 $\mathbb{I}, E_a^x, F_b^y, E_a^x F_b^y,$
 $E_a^x E_{a'}^x, F_{b'}^y F_b^y,$
 $E_a^x E_{a'}^x E_{a''}^x, E_a^x E_{a'}^x F_b^y,$
etc.

MN, S. Pironio and A. Acín, Phys. Rev. Lett. 98, 010401 (2007).

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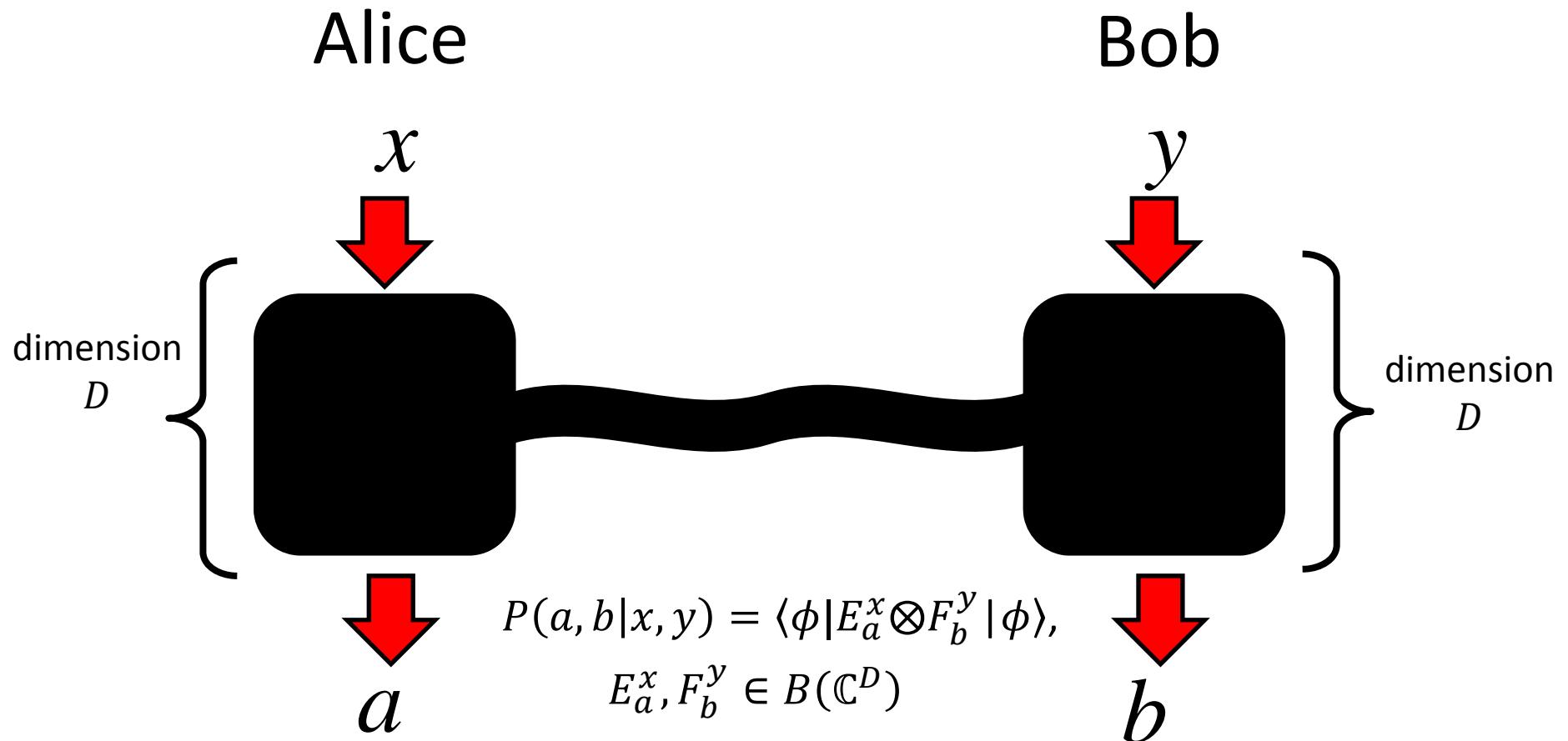
Variational methods

$$L \leq \max \sum c_{abxy} P(a, b | x, y) \leq U$$

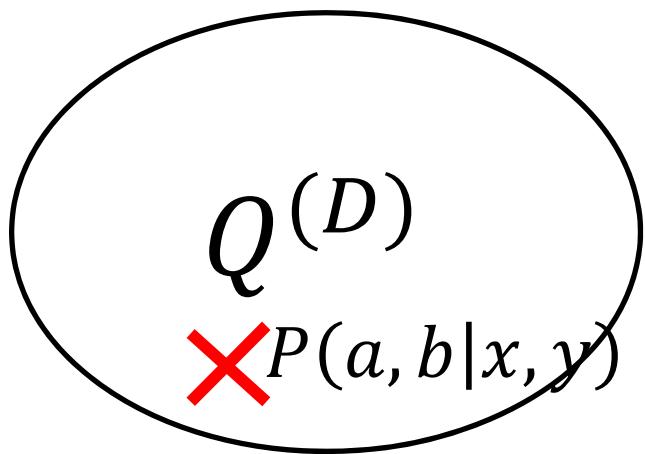
NPA

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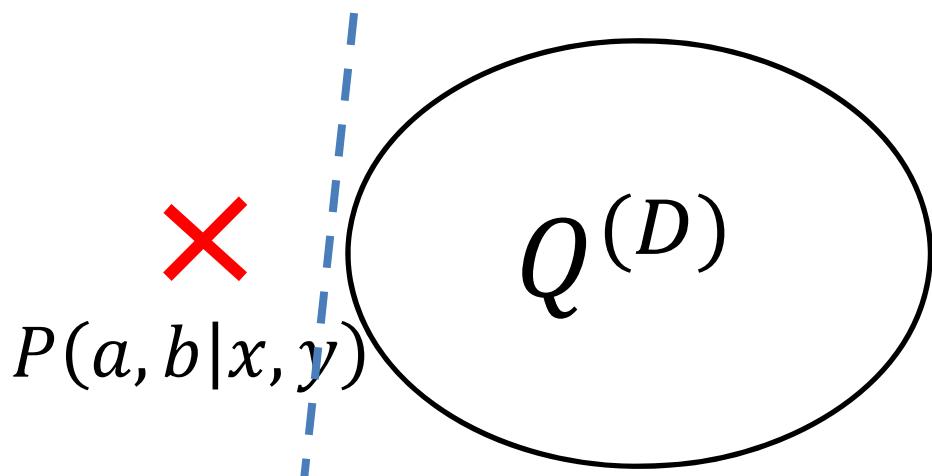


"Easy"

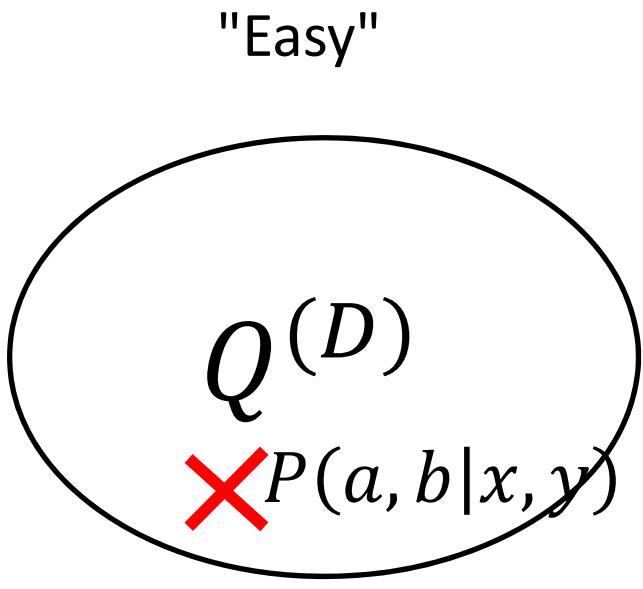


Variational methods

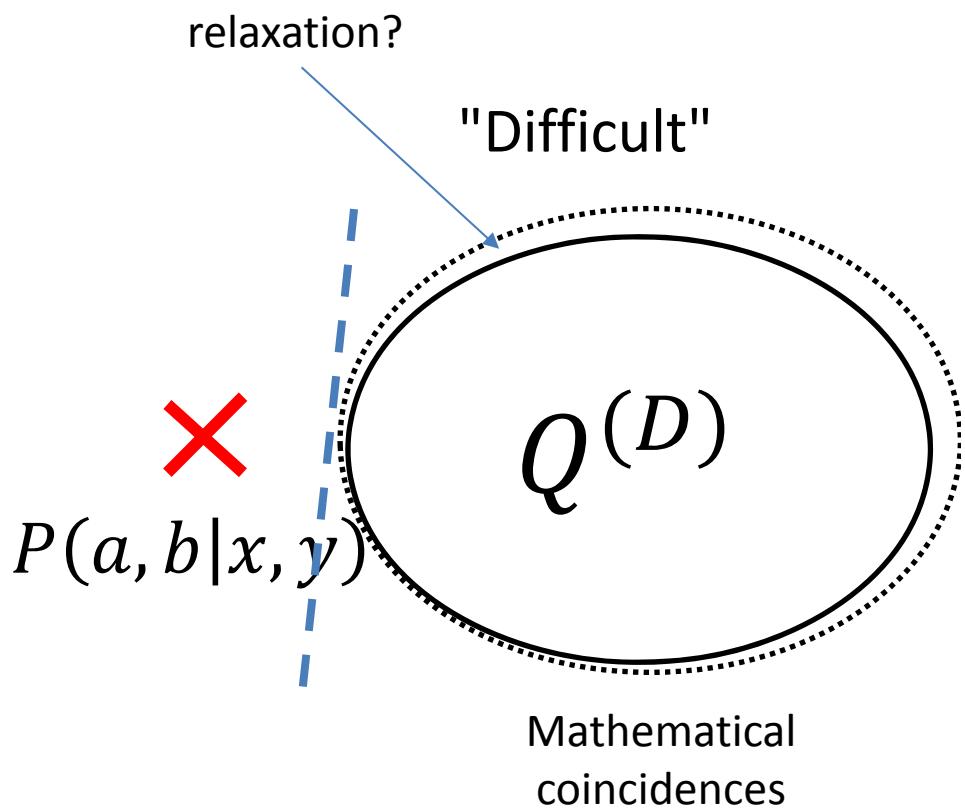
"Difficult"



Mathematical
coincidences



Variational methods



Quantum communication complexity

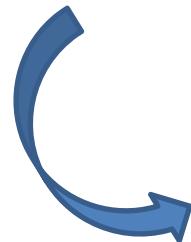
Alice

Bob

H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).

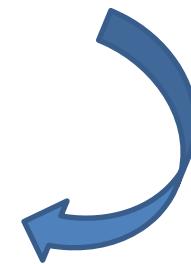
Quantum communication complexity

$$\bar{x} \in \{0,1\}^n$$



Alice

$$\bar{y} \in \{0,1\}^n$$

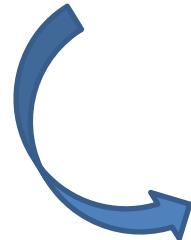


Bob

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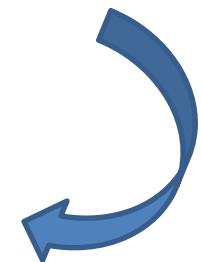
Quantum communication complexity

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Alice

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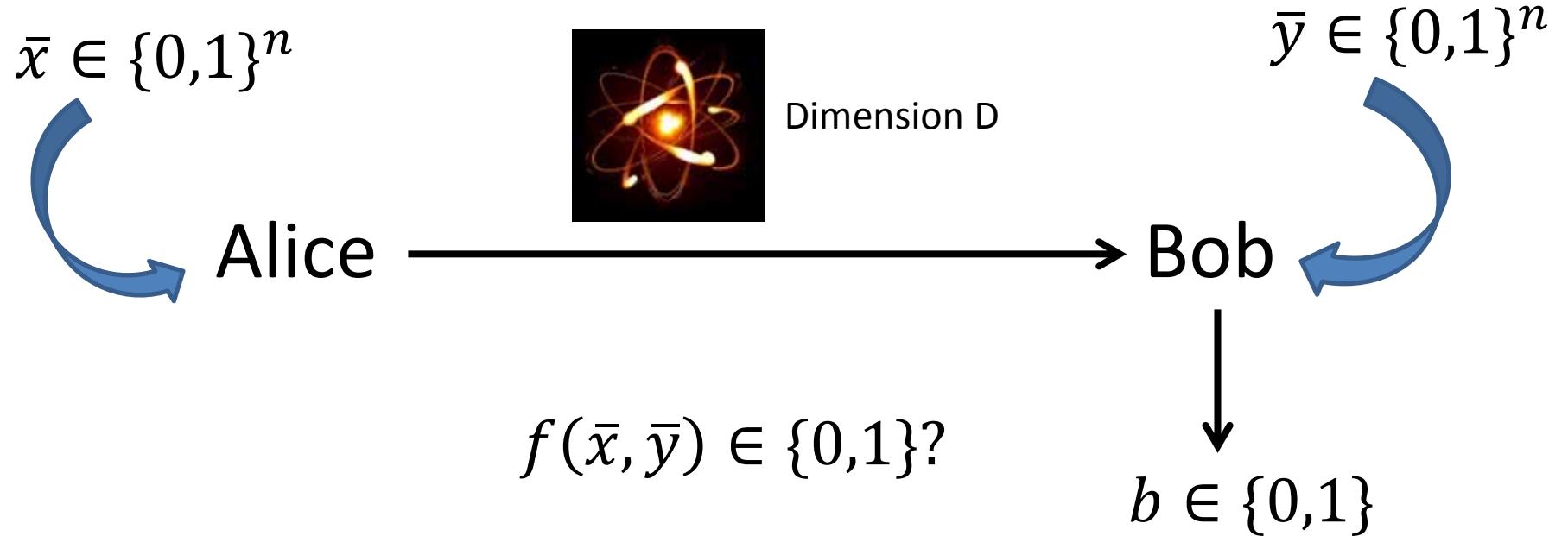


Bob

$$f(\bar{x}, \bar{y}) \in \{0,1\}?$$

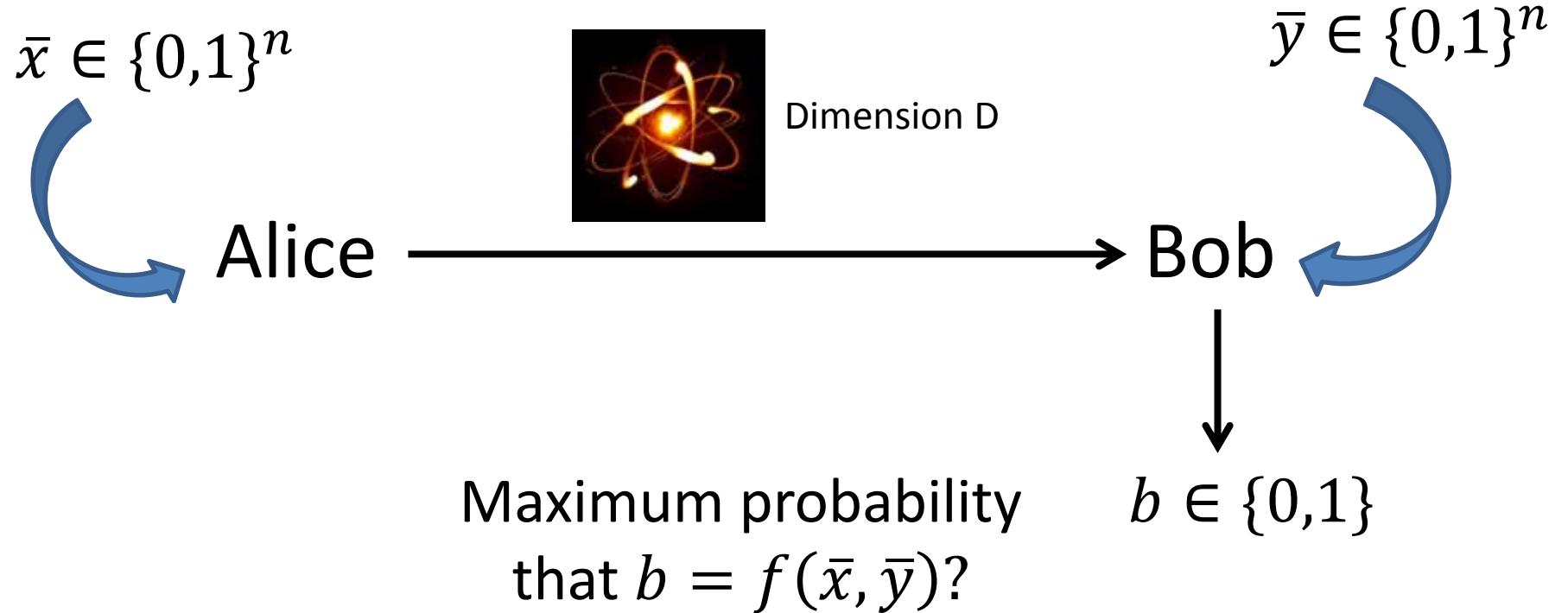
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Quantum communication complexity



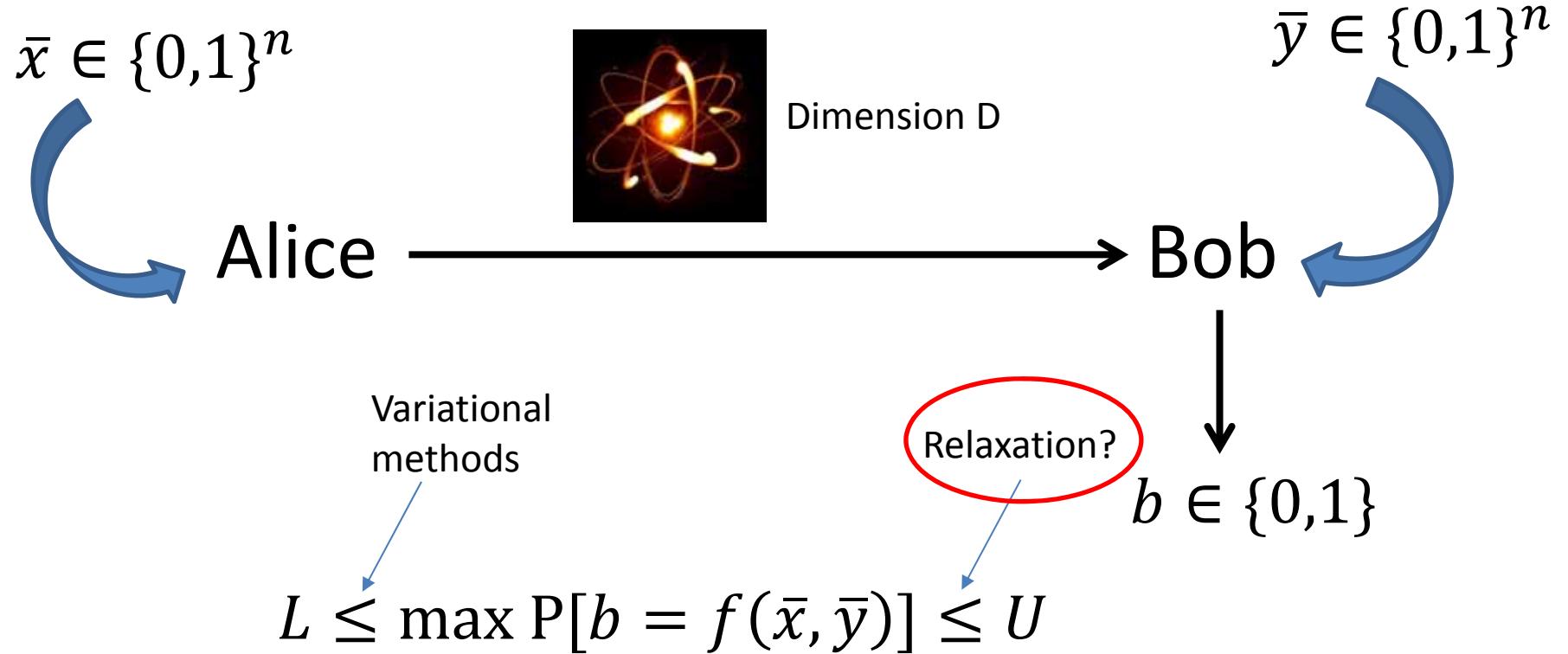
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Quantum communication complexity



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$$\begin{aligned}
p^* &= \frac{1}{2^{2n}} \max \sum_{x,y} \text{tr}(F_{f(x,y)}^y \rho_x) & \max \left\langle \phi, \sum_{a,b,x,y} C_{a,b}^{x,y} E_a^x \otimes F_b^y \phi \right\rangle \\
\text{s.t.} & \quad F_b^y, \rho_x \in B(\mathbb{C}^D), & \text{s.t.} \quad \phi \in \mathcal{H}^{\otimes 2}, \langle \phi, \phi \rangle = 1 \\
& (F_b^y)^2 = F_b^y, & E_a^x, F_b^y \in \text{B}(\mathcal{H}), \text{ projectors}, \\
& \rho_x \geq 0, \text{tr}(\rho_x) = 1 & \sum_a E_a^x = \sum_b F_b^y = 1, \\
& & \dim(\mathcal{H}) \leq D
\end{aligned}$$

Brute force theoretically possible, but impractical

What is a D-dimensional Hilbert space?

MN and T. Vértesi, Phys. Rev. Lett. 115, 020501 (2015) .

$$\psi = \overbrace{(\psi_1, \dots, \psi_D)}^{\text{D}=\#\text{ of entries}}$$

$$X_i = \begin{pmatrix} x_{11}^i & \dots & x_{1D}^i \\ \vdots & \ddots & \vdots \\ x_{D1}^i & \dots & x_{DD}^i \end{pmatrix}^{\text{D}=\#\text{ of columns}}$$

Moment matrix

$$\Gamma = \begin{array}{c|ccccc} & \mathbb{I} & E_a^x & E_{a'}^{x'} & F_b^y & \dots \\ \hline \mathbb{I}^\dagger & 1 & & & & \\ E_a^{x\dagger} & \vdash & \text{---} & \langle E_a^x E_{a'}^{x'} \rangle & P(a, b|x, y) & \geq 0 \\ E_{a'}^{x'\dagger} & P(a'|x') & \text{---} & & P(a', b|x', y) & \\ F_b^{y\dagger} & & & & \Gamma_{u,v} = \langle u^\dagger v \rangle & \\ \vdots & & & & & \end{array}$$

How to incorporate dimension constraints?

Matrix polynomial identities

$$F(X) = 0, \forall X_1, \dots, X_n \in B(\mathbb{C}^d), d \leq D$$

Matrix polynomial identities

$$F(X) = 0, \forall X_1, \dots, X_n \in B(\mathbb{C}^d), d \leq D$$

E.g.:
$$\left\{ \begin{array}{l} D = 1 \longrightarrow [X_1, X_2] \end{array} \right.$$

Matrix polynomial identities

$$F(X) = 0, \forall X_1, \dots, X_n \in B(\mathbb{C}^d), d \leq D$$

E.g.:
$$\begin{cases} D = 1 \rightarrow [X_1, X_2] \\ D = 2 \rightarrow [[X_1, X_2]^2, X_3] \end{cases}$$

Matrix polynomial identities

$$F(X) = 0, \forall X_1, \dots, X_n \in B(\mathbb{C}^d), d \leq D$$

E.g.:
$$\begin{cases} D = 1 \rightarrow [X_1, X_2] \\ D = 2 \rightarrow [[X_1, X_2]^2, X_3] \end{cases}$$

Central polynomial (commutes with everything)

Standard Identity

$X_1, \dots, X_{2D}, D \times D$ matrices



$$F_D(X) = \sum_{\pi \in S_{2D}} sgn(\pi) X_{\pi(1)} \dots X_{\pi(2D)} = 0$$

Moment matrix

$$\Gamma = \begin{array}{c|ccccc} & \mathbb{I} & E_a^x & E_{a'}^{x'} & F_b^y & \dots \\ \hline \mathbb{I}^\dagger & 1 & & & & \\ E_a^{x\dagger} & \vdash & \cdots & \langle E_a^x E_{a'}^{x'} \rangle & P(a, b|x, y) & \geq 0 \\ E_{a'}^{x'\dagger} & P(a'|x') & \cdots & & P(a', b|x', y) & \\ F_b^{y\dagger} & & & & & \\ \vdots & & & & & \end{array}$$

Moment matrix

$$\Gamma = \begin{array}{c|ccccc} & \mathbb{I} & E_a^x & E_{a'}^{x'} & F_b^y & \dots \\ \hline \mathbb{I}^\dagger & 1 & & & & \\ E_a^{x\dagger} & \vdash & \cdots & \langle E_a^x E_{a'}^{x'} \rangle & P(a, b|x, y) & \geq 0 \\ E_{a'}^{x'\dagger} & P(a'|x') & \cdots & & P(a', b|x', y) & \\ F_b^{y\dagger} & & & & & \\ \vdots & & & & & \end{array}$$

$$F(X) = 0, \forall X_1, \dots, X_n \in B(\mathbb{C}^d), d \leq D \rightarrow \langle F(E_a^x, F_b^y) \rangle \geq 0 \rightarrow \sum_{i,j} C_{i,j}^F \Gamma_{i,j} = 0$$

Moment matrix

$$\Gamma = \begin{array}{c|ccccc} & \mathbb{I} & E_a^x & E_{a'}^{x'} & F_b^y & \dots \\ \hline \mathbb{I}^\dagger & 1 & & & & \\ E_a^{x\dagger} & \vdash & \cdots & \langle E_a^x E_{a'}^{x'} \rangle & P(a, b|x, y) & \\ E_{a'}^{x'\dagger} & P(a'|x') & \cdots & & P(a', b|x', y) & \\ F_b^{y\dagger} & & & & & \\ \vdots & & & & & \end{array} \geq 0$$



Moment matrices of D-level quantum systems are subject to non-trivial linear constraints!!!

D-dimensional
Hilbert space



$$\begin{aligned}\Gamma &\geq 0 \\ \Gamma &\in \mathcal{S}^D\end{aligned}$$

D-dimensional
Hilbert space

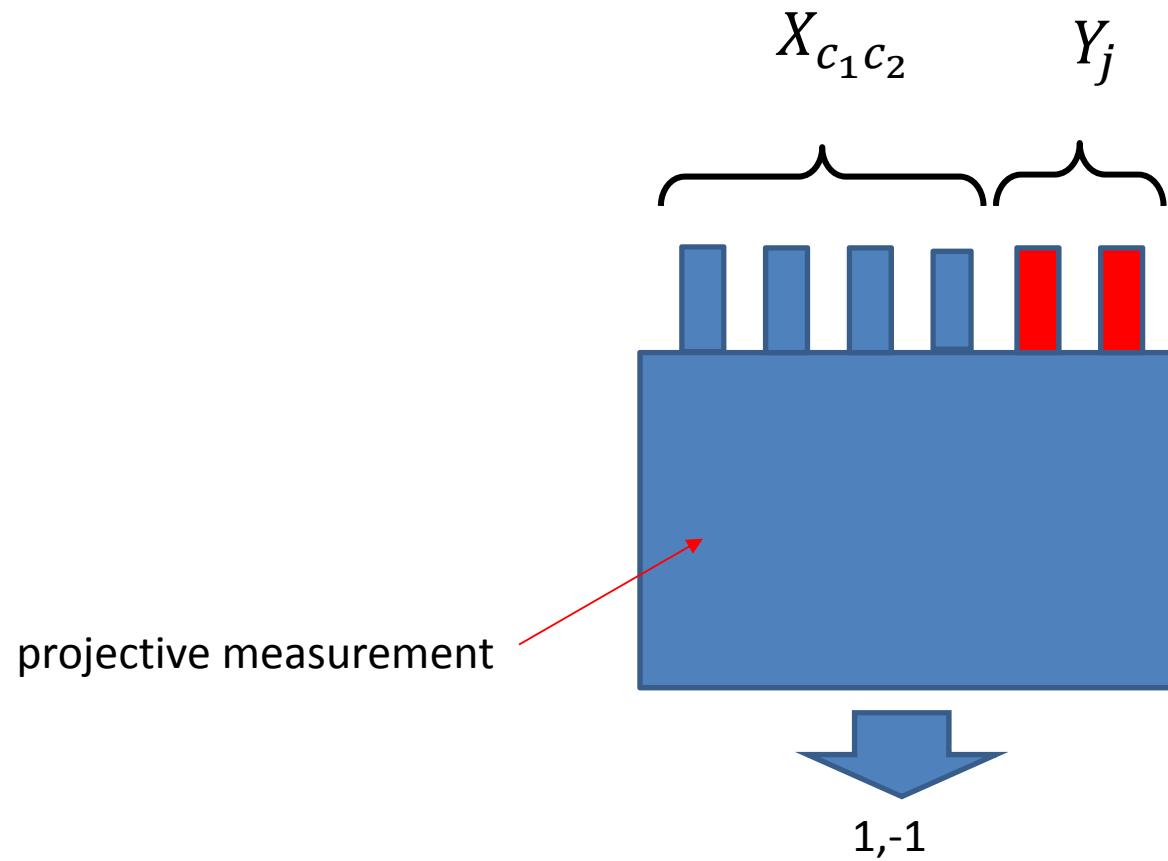


$\Gamma \geq 0$
 $\Gamma \in \mathcal{S}^D$?

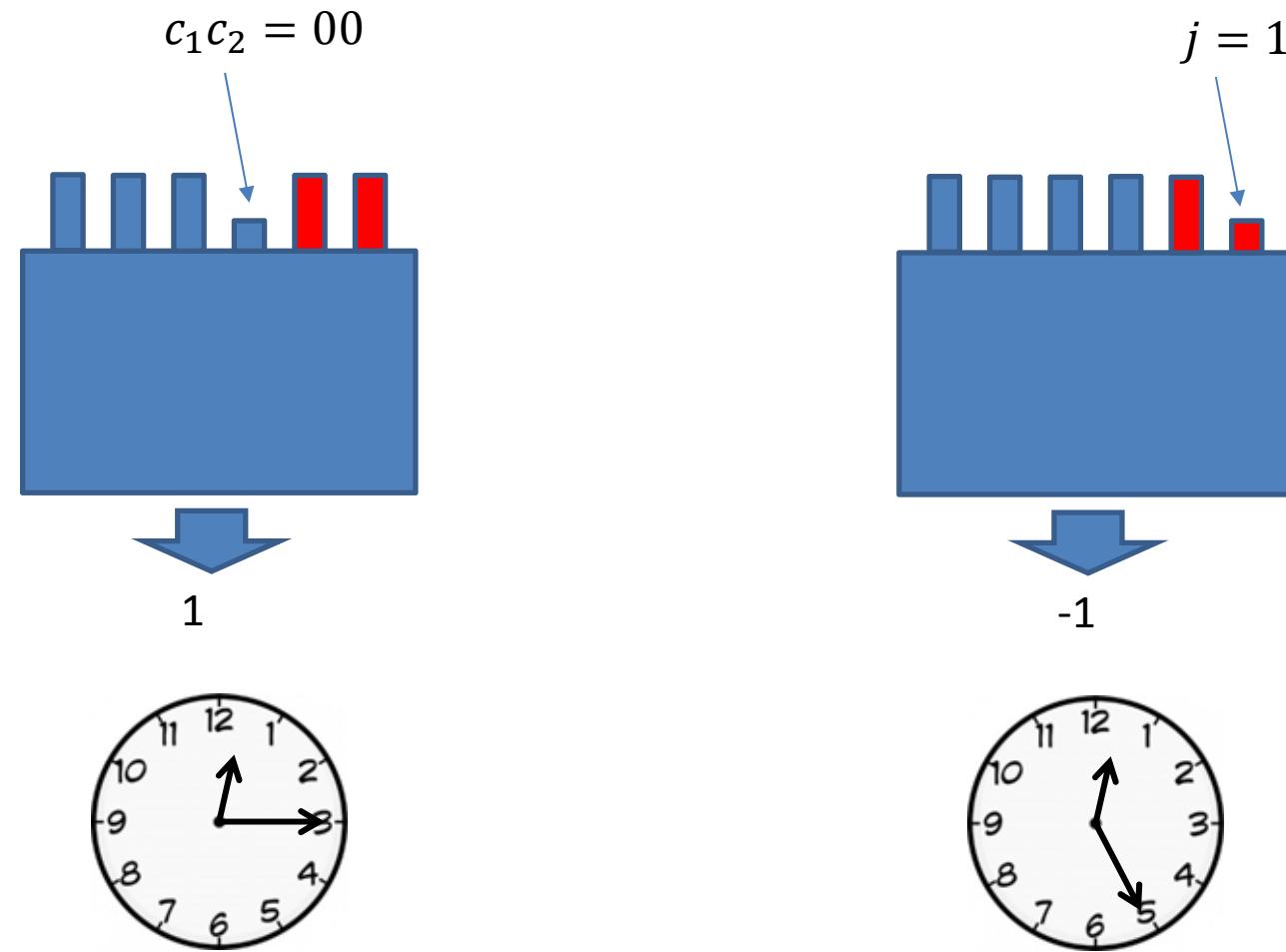
Toy problem

$$\begin{aligned} \max & \sum_{j=1,2} \sum_{c_1, c_2=0,1} (-1)^{c_j} \langle X_{c_1 c_2} Y_j X_{c_1 c_2} \rangle \\ \text{s.t. } & X_{c_1 c_2}^2 = Y_j^2 = 1 \end{aligned}$$

Temporal quantum correlations



Temporal quantum correlations



Toy problem

$$p^* = \max \langle \phi | \sum_{j=1,2} \sum_{c_1, c_2 = 0,1} (-1)^{c_j} X_{c_1 c_2} Y_j X_{c_1 c_2} | \phi \rangle$$

$$\text{s.t. } X_{c_1 c_2}^2 = Y_j^2 = 1,$$

$$\langle \phi | \phi \rangle = 1$$

Toy problem

$$p^* = \max \langle \phi | \sum_{j=1,2} \sum_{c_1, c_2 = 0,1} (-1)^{c_j} X_{c_1 c_2} Y_j X_{c_1 c_2} | \phi \rangle$$

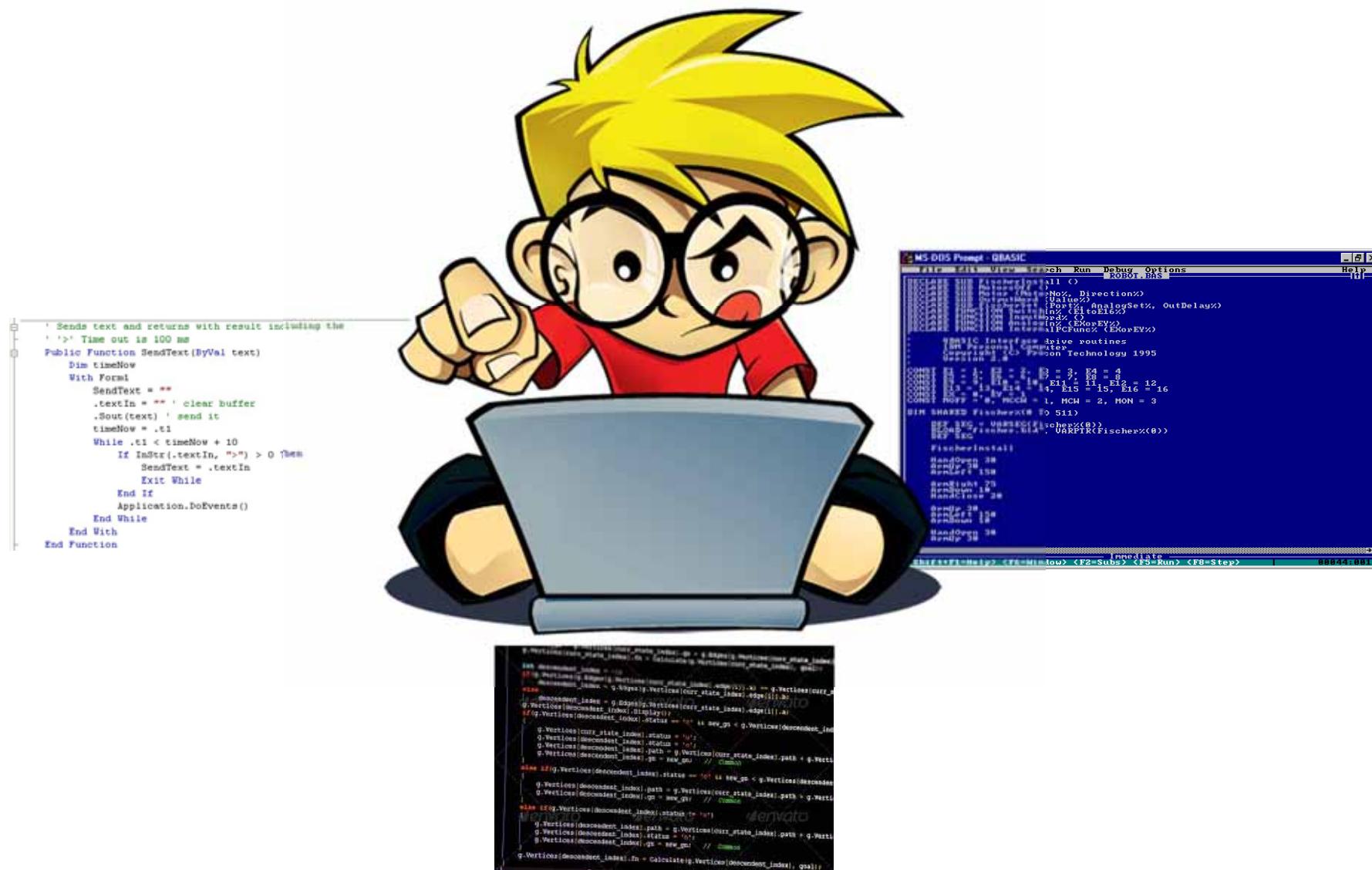
$$\text{s.t. } X_{c_1 c_2}^2 = Y_j^2 = 1,$$

$$\langle \phi | \phi \rangle = 1$$

In the dimension-free case, this kind of problems can be solved with a single SDP

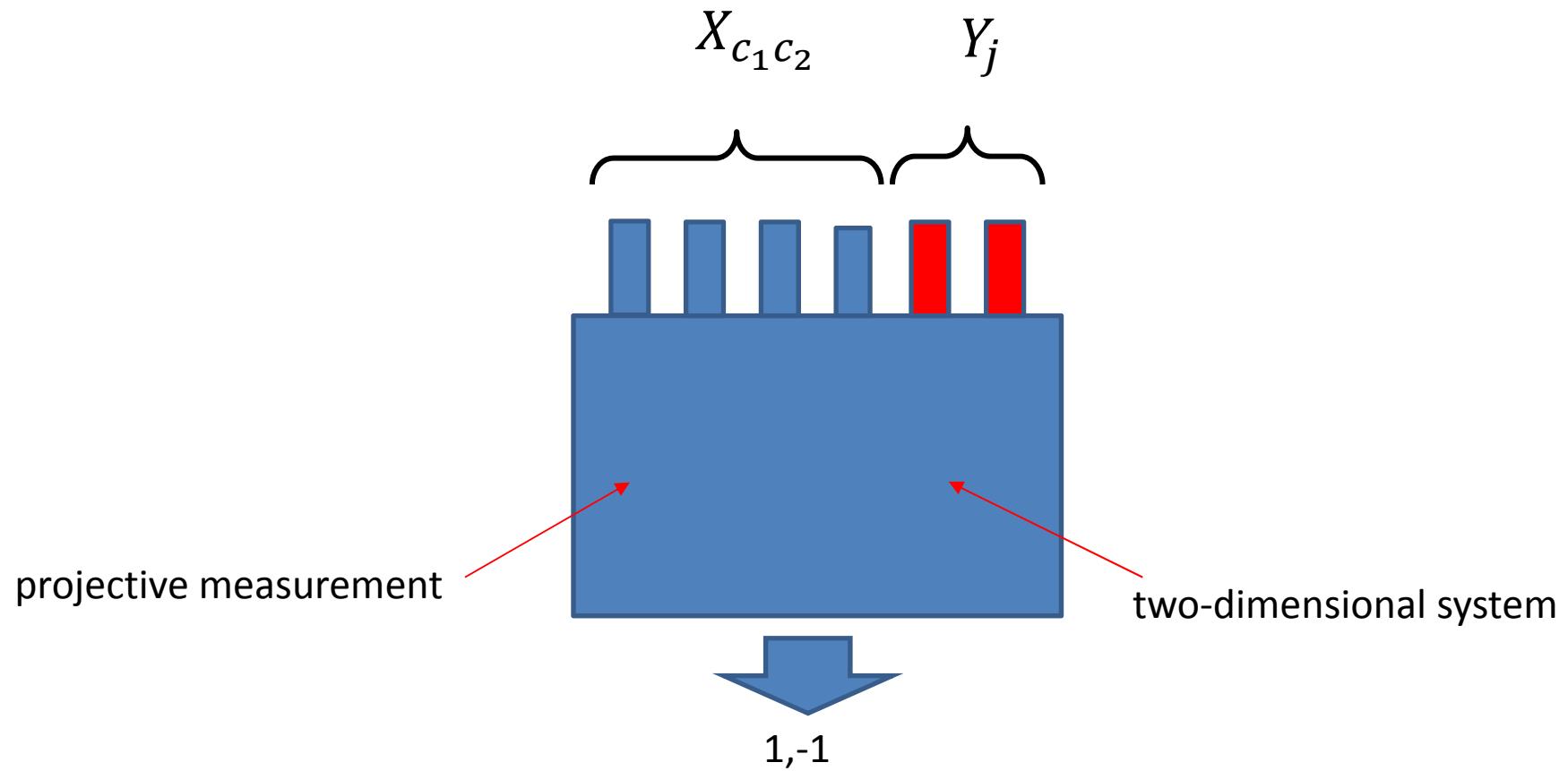
S. Pironio, M. Navascués and A. Acín, SIAM J. Optim. 20, 5, 2157-2180 (2010).

C. Budroni, T. Moroder, M. Kleinmann, O. Gühne, Phys. Rev. Lett. 111, 020403 (2013)



$$p^{\ast}=8$$

Temporal quantum correlations



Toy problem

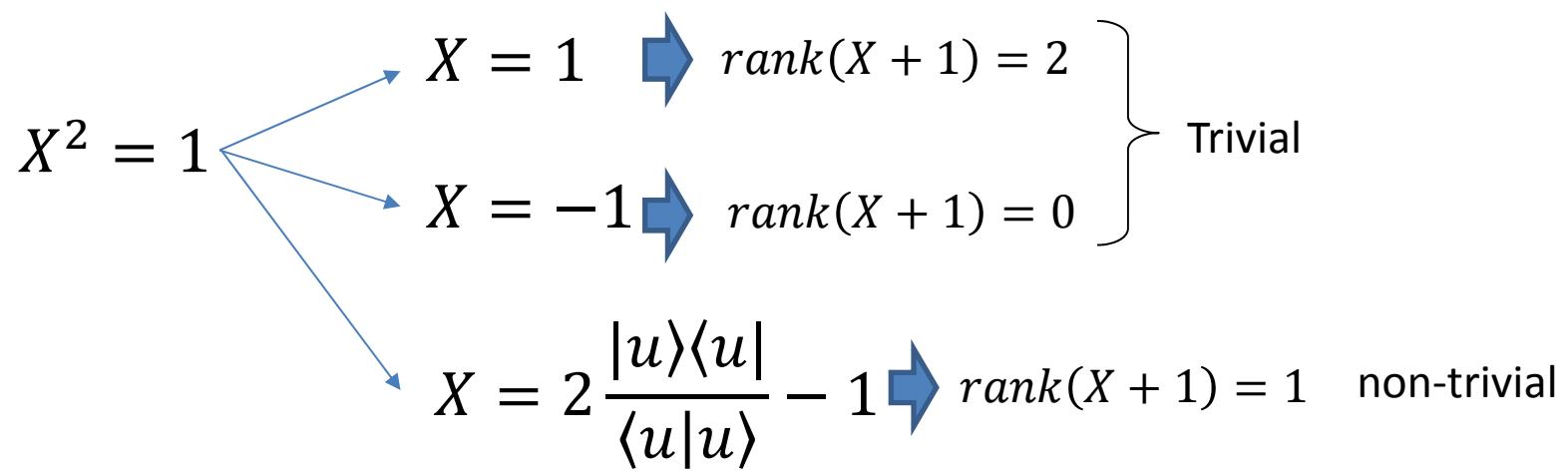
$$p^* = \max \langle \phi | \sum_{j=1,2} \sum_{c_1, c_2=0,1} (-1)^{c_j} X_{c_1 c_2} Y_j X_{c_1 c_2} | \phi \rangle$$

$$\text{s.t. } X_{c_1 c_2}^2 = Y_j^2 = 1,$$

$$\langle \phi | \phi \rangle = 1,$$

$$| \phi \rangle \in \mathbb{C}^2, X_{c_1 c_2}, Y_j \in B(\mathbb{C}^2)$$

Solving the toy problem (I): divide the problem into classes



A random instance can be
generated easily

Solving the toy problem (I): divide the problem into classes

729 classes

Each class labeled by a
vector $\vec{r} \in \{0,1,2\}^6$



$$\text{rank}(X_{c_1 c_2} + 1) = r_{c_1 c_2}$$

$$\text{rank}(Y_j + 1) = r_j$$

Classed toy problem

$$p_{\vec{r}}^* = \max \langle \phi | \sum_{j=1,2} \sum_{c_1, c_2=0,1} (-1)^{c_j} X_{c_1 c_2} Y_j X_{c_1 c_2} | \phi \rangle$$

$$\text{s.t. } X_{c_1 c_2}^2 = Y_j^2 = 1,$$

$$\langle \phi | \phi \rangle = 1,$$

$$| \phi \rangle \in \mathbb{C}^2, X_{c_1 c_2}, Y_j \in B(\mathbb{C}^2),$$

$$\text{rank}(X_{c_1 c_2} + 1) = r_{c_1 c_2},$$

$$\text{rank}(Y_j + 1) = r_j$$

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t.} \quad \Gamma_{1,1}=1$$

$$\Gamma\geq 0,$$

$$\Gamma\in S_{D,\vec r}$$

Solving the toy problem (II): Identify $S_{D,\vec{r}}$

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t. } \Gamma_{1,1} = 1$$

$$\Gamma \geq 0,$$

$$\Gamma \in S_{D,\vec{r}}$$

High level description

$$i = 1$$

High level description

$$i = 1$$

Generate random dichotomic operators $X_{c_1 c_2}^i, Y_j^i \in B(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$

High level description

$$i = 1$$

Generate random dichotomic operators $X_{c_1 c_2}^i, Y_j^i \in B(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$



Build moment matrix Γ^j

$$\Gamma_{\mathbb{I}, \mathbb{I}}^i = \langle \phi^i | \mathbb{I}_d | \phi^i \rangle = 1,$$

$$\Gamma_{X_{c_1 c_2}, Y_j}^i = \langle \phi^i | X_{c_1 c_2}^i, Y_j^i | \phi^i \rangle,$$

⋮

High level description

$i = 1$

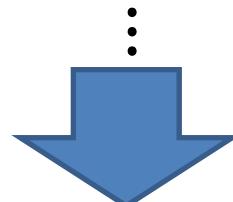
Generate random dichotomic operators $X_{c_1 c_2}^i, Y_j^i \in B(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$



Build moment matrix Γ^j

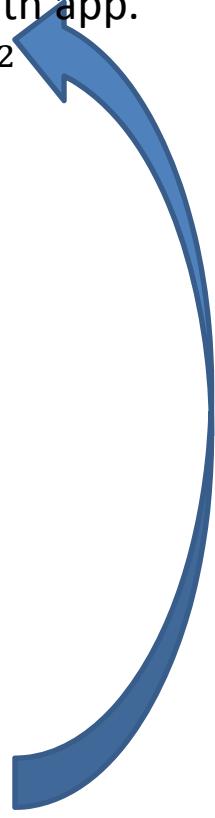
$$\Gamma_{\mathbb{I}, \mathbb{I}}^i = \langle \phi^i | \mathbb{I}_d | \phi^i \rangle = 1,$$

$$\Gamma_{X_{c_1 c_2}, Y_j}^i = \langle \phi^i | X_{c_1 c_2}^i, Y_j^i | \phi^i \rangle,$$



$i = i + 1$

Repeat until the moment matrix $\Gamma^{(N+1)}$ is a linear combination of $\Gamma^{(1)}, \dots, \Gamma^{(N)}$



High level description

$i = 1$

Generate random dichotomic operators $X_{c_1 c_2}^i, Y_j^i \in B(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$



Build moment matrix Γ^j

$$\Gamma_{\mathbb{I}, \mathbb{I}}^i = \langle \phi^i | \mathbb{I}_d | \phi^i \rangle = 1,$$

$$\Gamma_{X_{c_1 c_2}, Y_j}^i = \langle \phi^i | X_{c_1 c_2}^i, Y_j^i | \phi^i \rangle,$$

⋮



$i = i + 1$

At that point, *any* feasible moment matrix Γ must be a linear combination of $\Gamma^{(1)}, \dots, \Gamma^{(N)}$



$$\mathcal{S}_r^D = \text{span}(\Gamma^{(1)}, \dots, \Gamma^{(N)})$$

SDP relaxation

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t. } \Gamma_{1,1} = 1$$

$$\Gamma \geq 0,$$

$$\Gamma = \sum_{s=1}^N c_s \Gamma^s$$

SDP relaxation

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t. } \Gamma_{1,1} = 1$$

$$\Gamma \geq 0,$$

Lack of strict feasibility,
i.e., there are no $\Gamma > 0$

$$\Gamma = \sum_{s=1}^N c_s \Gamma^s \quad |c_s| \gg 1$$

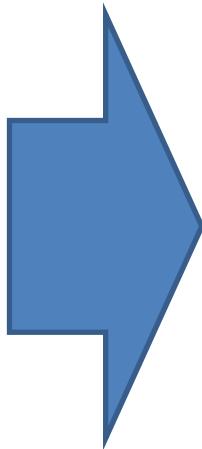
Problems with SDP solvers

Implementation tips



$$\frac{1}{N} \sum_{s=1}^N \Gamma^s = \Gamma_{mix} \geq 0$$

Isometry \bar{V} to the support of Γ_{mix}



$$\Gamma \geq 0 \rightarrow \bar{V}^\dagger \Gamma \bar{V} \geq 0$$

Why?

$$\bar{V}^\dagger \Gamma_{mix} \bar{V} > 0$$

Strict positivity

Implementation tips



Use (modified) Gram-Schmidt as you generate the matrices $\Gamma^1, \Gamma^2 \dots$

Sequence of orthogonal matrices $\widetilde{\Gamma^1}, \widetilde{\Gamma^2} \dots$

at $s=N+1$, $\widetilde{\Gamma^s} = 0$, up to computer precision

$$\Gamma = \sum_{s=1}^N c_s \Gamma^s \quad \xrightarrow{\text{blue arrow}} \quad \Gamma = \sum_{s=1}^N \tilde{c}_s \frac{\widetilde{\Gamma}_s}{\sqrt{\text{tr}(\widetilde{\Gamma}_s^2)}}$$

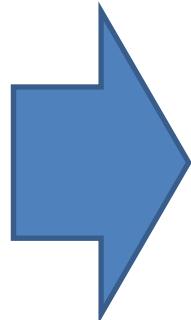
SDP relaxation

$$\min \sum_{a,b,x,y} C_{a,b}^{x,y} \Gamma_{E_a^x, F_b^y}$$

$$\text{s.t. } \Gamma_{1,1} = 1,$$

$$\Gamma \geq 0,$$

$$\Gamma = \sum_{s=1}^N c_s \Gamma^s$$

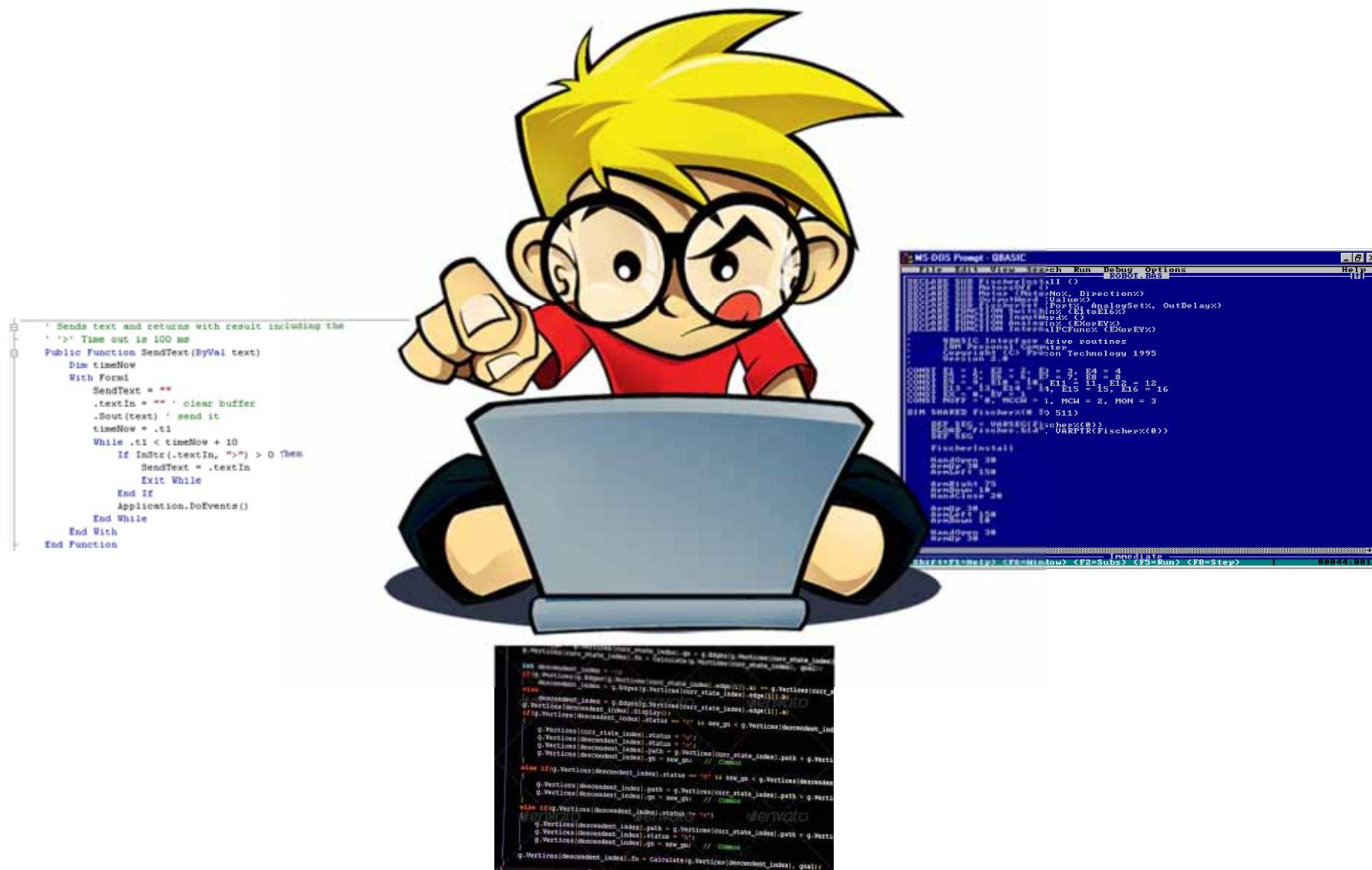


$$\min \sum_{a,b,x,y} C_{a,b}^{x,y} \Gamma_{E_a^x, F_b^y}$$

$$\text{s.t. } \Gamma_{1,1} = 1,$$

$$\bar{V}^\dagger \Gamma \bar{V} \geq 0,$$

$$\Gamma = \sum_{s=1}^N \tilde{c}_s \tilde{\Gamma}^s$$



Free dimensionality

$$p^* = 8$$

Second relaxation, D=2

$$p^2 = 5.656854 \dots$$

Generalization

NPO problem with dimension constraints

$$p^* = \min_{\mathcal{H}, X, \phi} \langle \phi, p(X) \phi \rangle$$

$$\begin{aligned} \text{s.t. } & X_1, \dots, X_n \in B(\mathcal{H}), \\ & q_i(X) \geq 0, i = 1, \dots, s \\ & \phi \in \mathcal{H} \\ & \langle \phi, \phi \rangle = 1 \\ & \dim(\mathcal{H}) \leq D \end{aligned}$$

MN, A. Feix, M. Araújo and T. Vértesi, Phys. Rev. A 92, 042117 (2015).

$$\begin{aligned}
p_k^* &= \min \sum_w p_w y_w \\
\text{s.t. } & y_1 = 1, \\
& M_k(y) \geq 0, \\
& M_{k-[d_i/2]}(q_i y) \geq 0, \\
& y = \sum_{s=1}^{j-1} c_s y^{(s)}, c_s \in \mathbb{C}.
\end{aligned}$$



$$p_k^* \leq p^*$$

$$\begin{aligned}
p_k^* &= \min \sum_w p_w y_w \\
\text{s.t. } & y_1 = 1, \\
& M_k(y) \geq 0, \\
& M_{k-[d_i/2]}(q_i y) \geq 0, \\
& y = \sum_{s=1}^{j-1} c_s y^{(s)}, c_s \in \mathbb{C}.
\end{aligned}$$

Archimedean
condition

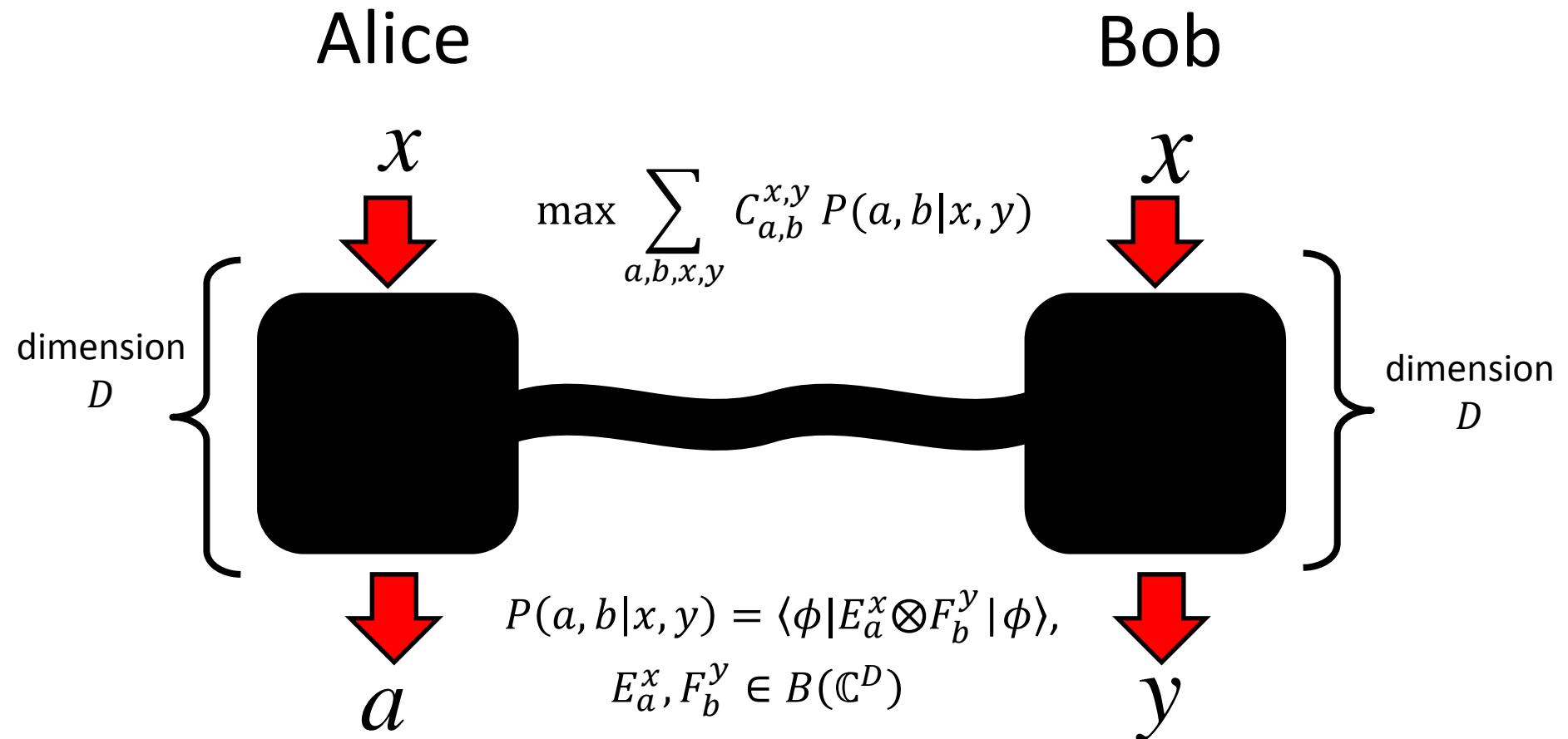
$$C - \sum_r X_r^\dagger X_r + X_r X_r^\dagger = \sum_s f_s f_s^\dagger + \sum_{s,i} g_{s,i} q_i g_{s,i}^\dagger$$



$$\lim_{k \rightarrow \infty} p_k^* = p^*$$

Related hierarchies

SDP hierarchy for quantum non-locality under dimension constraints



$$\max \left\langle \phi, \sum_{a,b,x,y} C_{a,b}^{x,y} E_a^x \otimes F_b^y \phi \right\rangle$$

$$\text{s.t. } \phi \in \mathcal{H}^{\otimes 2}, \langle \phi, \phi \rangle = 1$$

$E_a^x, F_b^y \in \mathcal{B}(\mathcal{H})$, projectors,

$$\sum_a E_a^x = \sum_b F_b^y = 1,$$

$$\dim(\mathcal{H}) \leq D$$

$$\max \left\langle \phi, \sum_{a,b,x,y} C_{a,b}^{x,y} E_a^x \otimes F_b^y \phi \right\rangle$$

$$\text{s.t. } \phi \in \mathcal{H}^{\otimes 2}, \langle \phi, \phi \rangle = 1$$

$E_a^x, F_b^y \in \mathcal{B}(\mathcal{H})$, projectors,

$$\sum_a E_a^x = \sum_b F_b^y = 1,$$

$$\dim(\mathcal{H}) \leq D,$$

$$\text{rank}(E_a^x) = r_a^x, \text{rank}(F_b^y) = t_b^y$$

High level description

$i = 1$

Generate random projectors
 $E_a^{i,x}, F_b^{i,y} \in \mathcal{B}(\mathbb{C}^D)$ (with app. ranks)
and vector $\phi^i \in \mathbb{C}^{D^2}$



Build moment matrix Γ^i

$$\Gamma_{\mathbb{I}, \mathbb{I}}^i = \langle \phi^i | \mathbb{I}_d \otimes \mathbb{I}_d | \phi^i \rangle = 1,$$

$$\Gamma_{E_a^x, F_b^y}^i = \langle \phi^i | E_a^{i,x} \otimes F_b^{i,y} | \phi^j \rangle,$$



$i = i + 1$

Repeat until the
moment matrix $\Gamma^{(N+1)}$ is
a linear combination of
 $\Gamma^{(1)}, \dots, \Gamma^{(N)}$

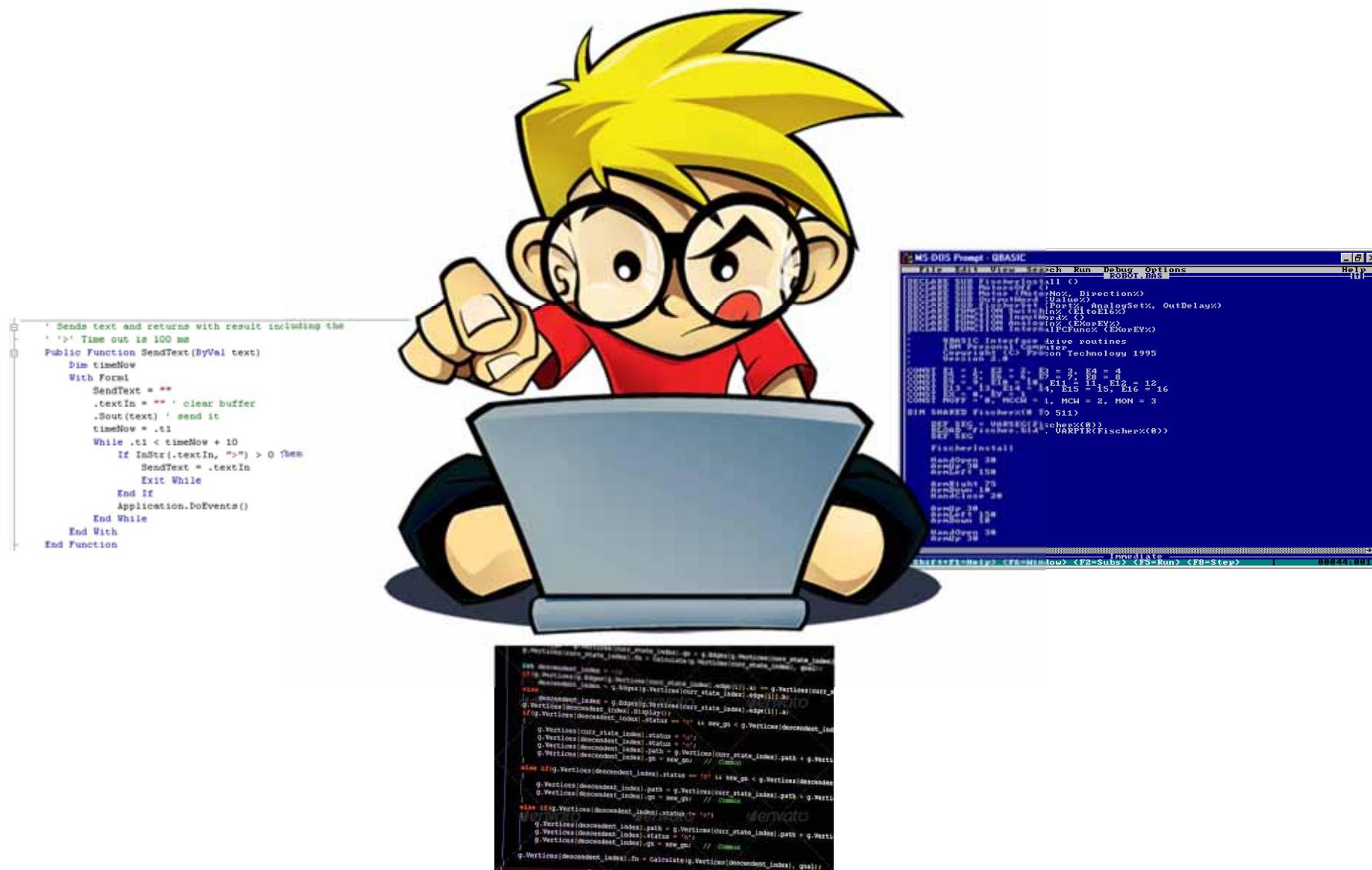
SDP hierarchy

$$\max \sum_{a,b,x,y} C_{a,b}^{x,y} \Gamma_{E_a^x, F_b^y}$$

$$\text{s.t. } \Gamma_{1,1} = 1$$

$$\Gamma \geq 0, \Gamma = \sum_{s=1}^j c_s \Gamma^s$$

Convergence is guaranteed



$$\text{Pironio-Bell} \leq 0.2532$$

NPA

$$\text{Pironio-Bell} \leq 0.2071$$

D=2

$$I_{3322} \leq 0.250875$$

NPA

$$I_{3322} \leq 0.25$$

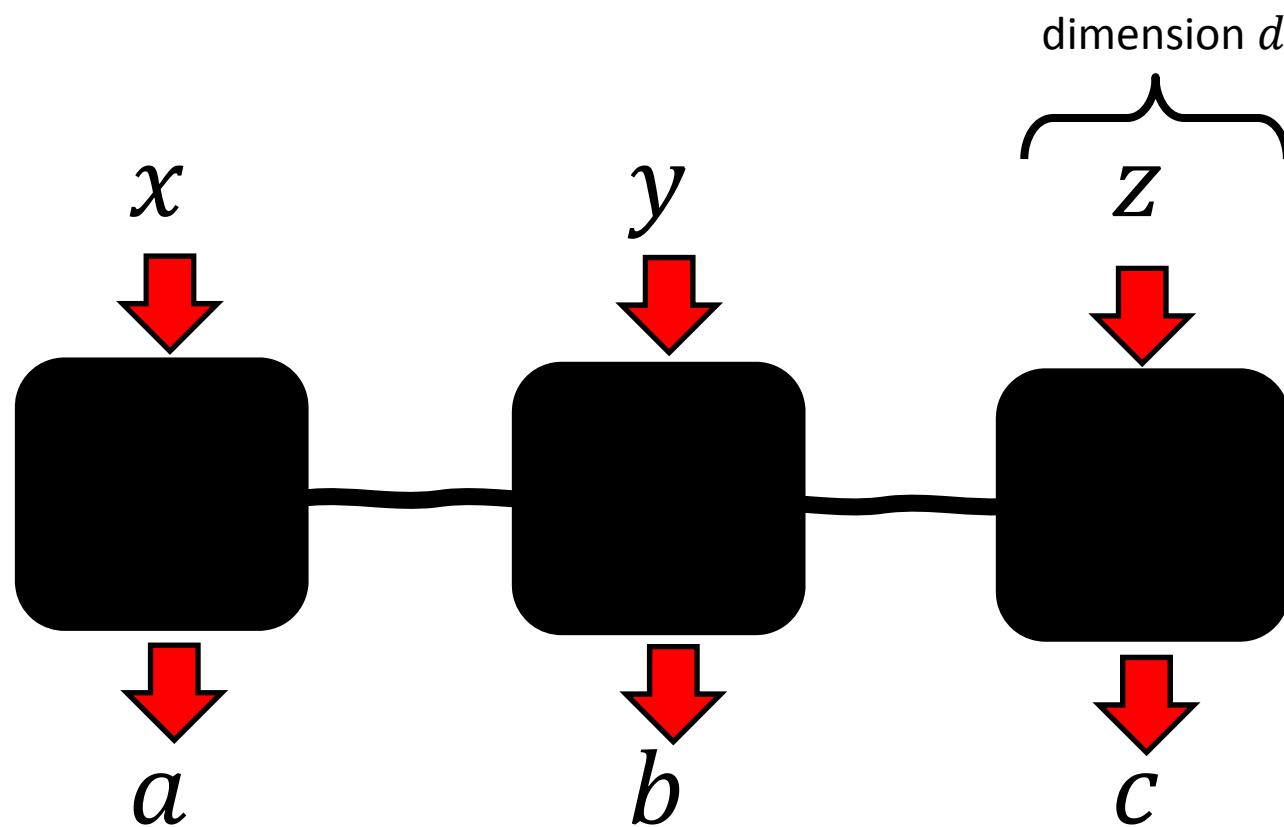
D=2, 3

$$I_{4,4} = E_1^A + E_{1,1} + E_{1,2} + E_{2,1} - E_{2,2} + E_{3,3} + E_{3,4} + E_{4,3} - E_{4,4} \leq 5.9907$$

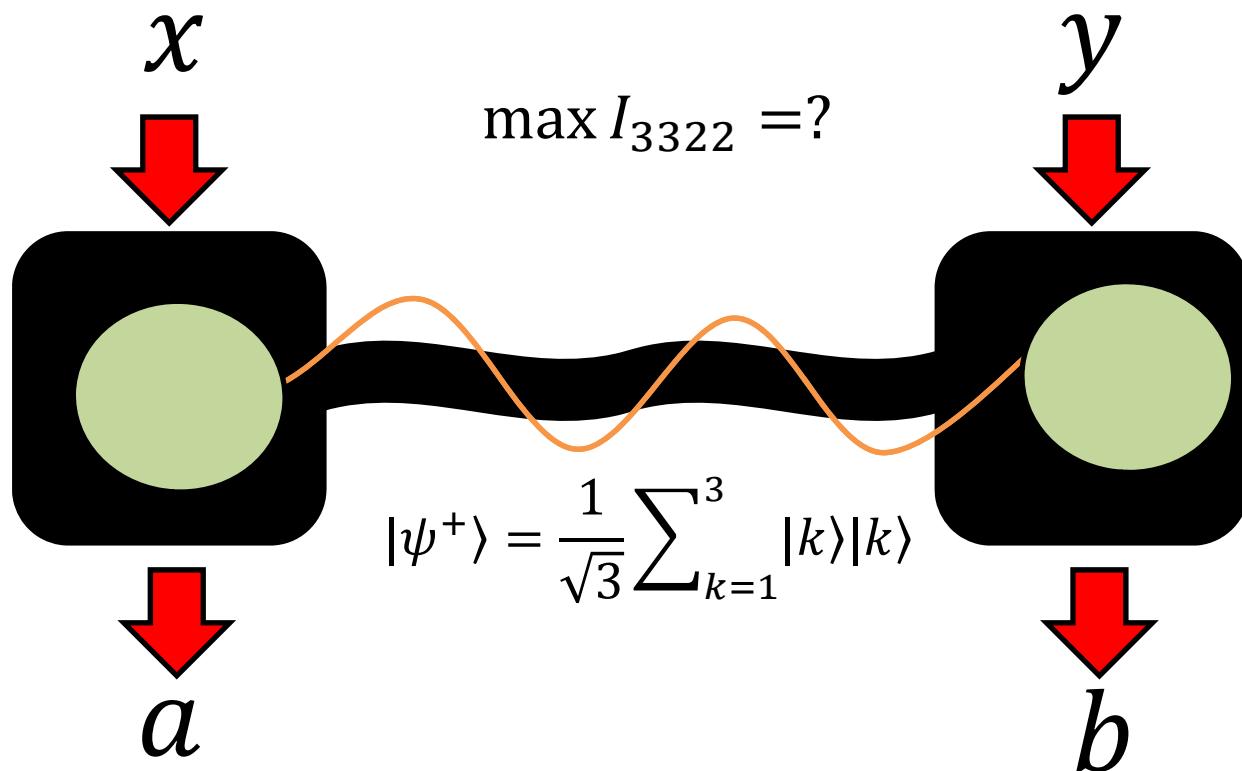
quantum mechanics

$$I_{4,4} = E_1^A + E_{1,1} + E_{1,2} + E_{2,1} - E_{2,2} + E_{3,3} + E_{3,4} + E_{4,3} - E_{4,4} \leq 5.8310$$

D=2, 3



$$P(a, b, c | x, y, z)$$



High level description

$i = 1$

Generate random projectors
 $E_a^{i,x}, F_b^{i,y} \in \mathcal{B}(\mathbb{C}^D)$ (with app. ranks)



Build moment matrix Γ^i

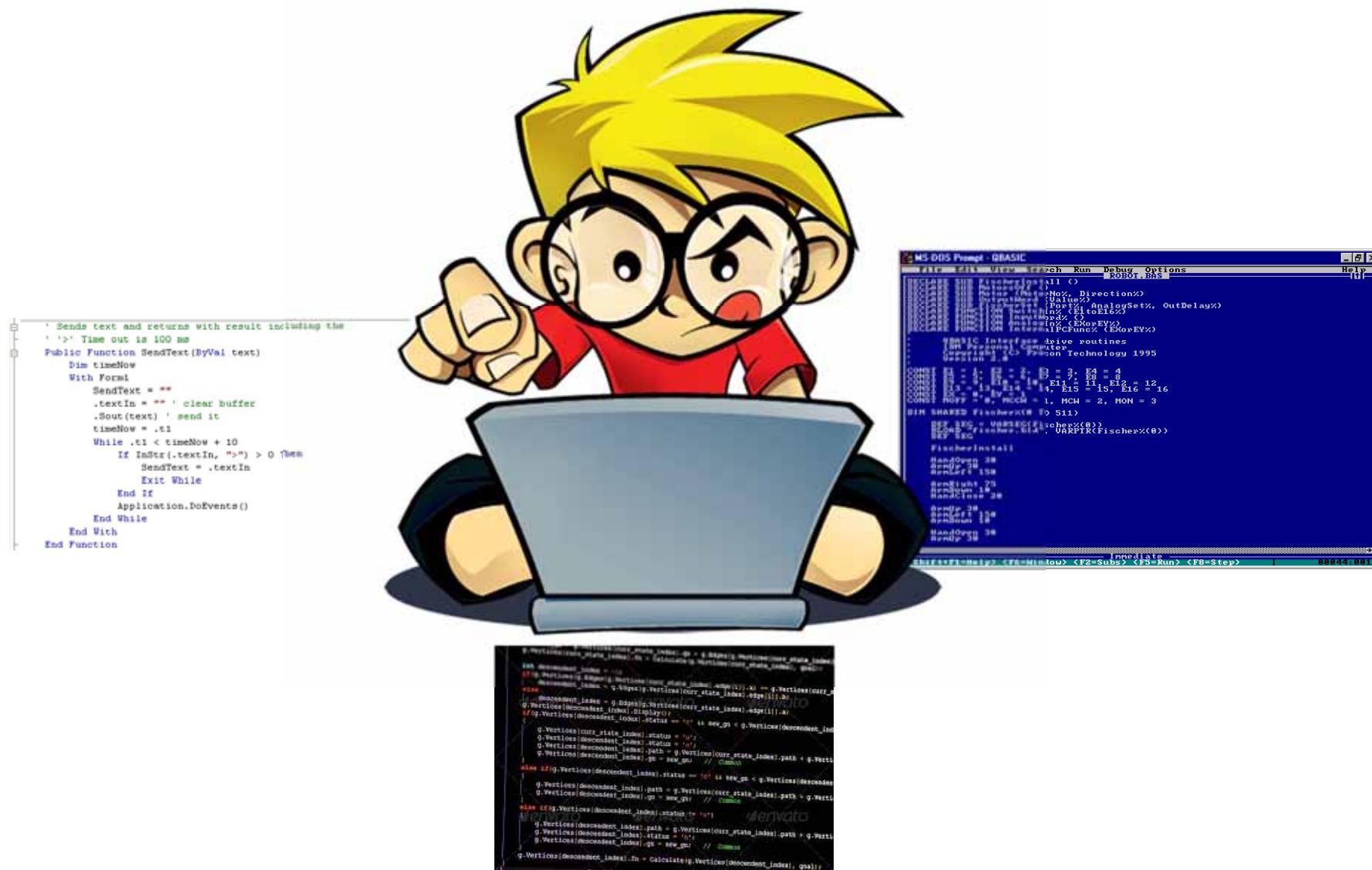
$$\Gamma_{\mathbb{I}, \mathbb{I}}^i = \langle \psi^+ | \mathbb{I}_d \otimes \mathbb{I}_d | \psi^+ \rangle = 1,$$

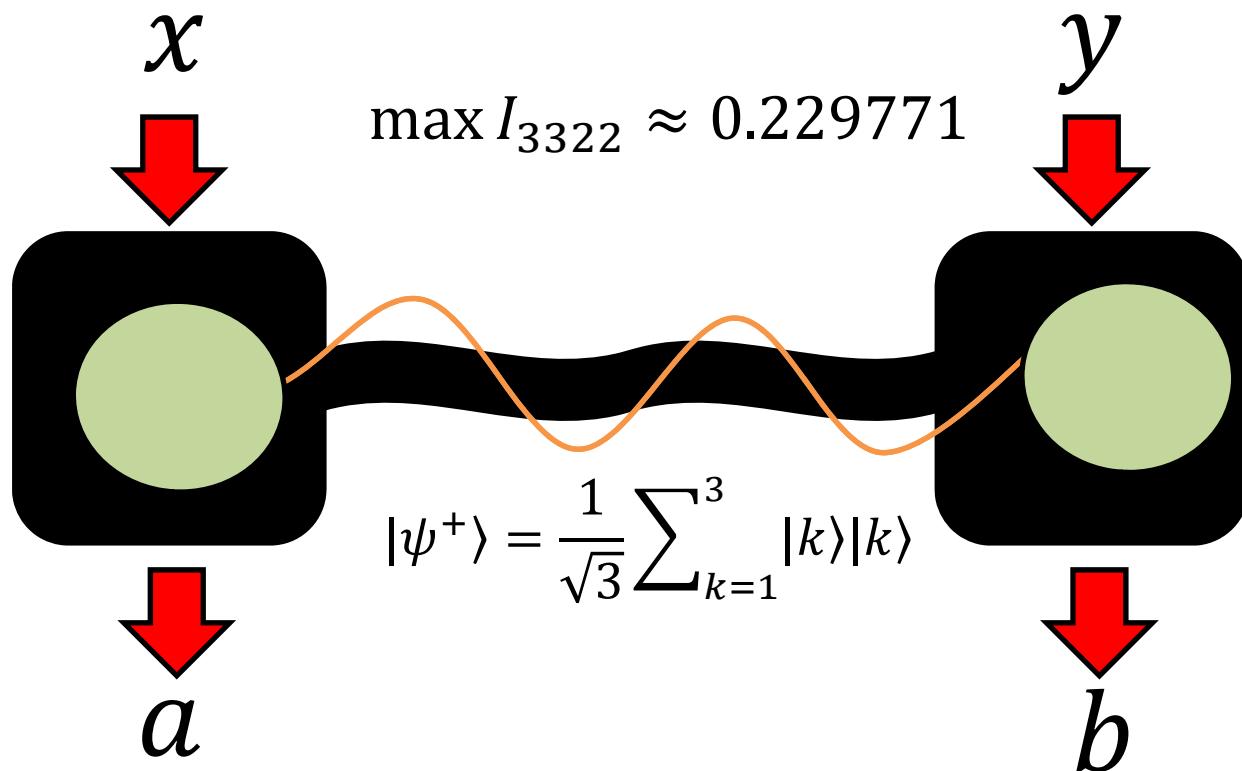
$$\Gamma_{E_a^x, F_b^y}^i = \left\langle \psi^+ \left| E_a^{i,x} \otimes F_b^{i,y} \right| \psi^+ \right\rangle,$$



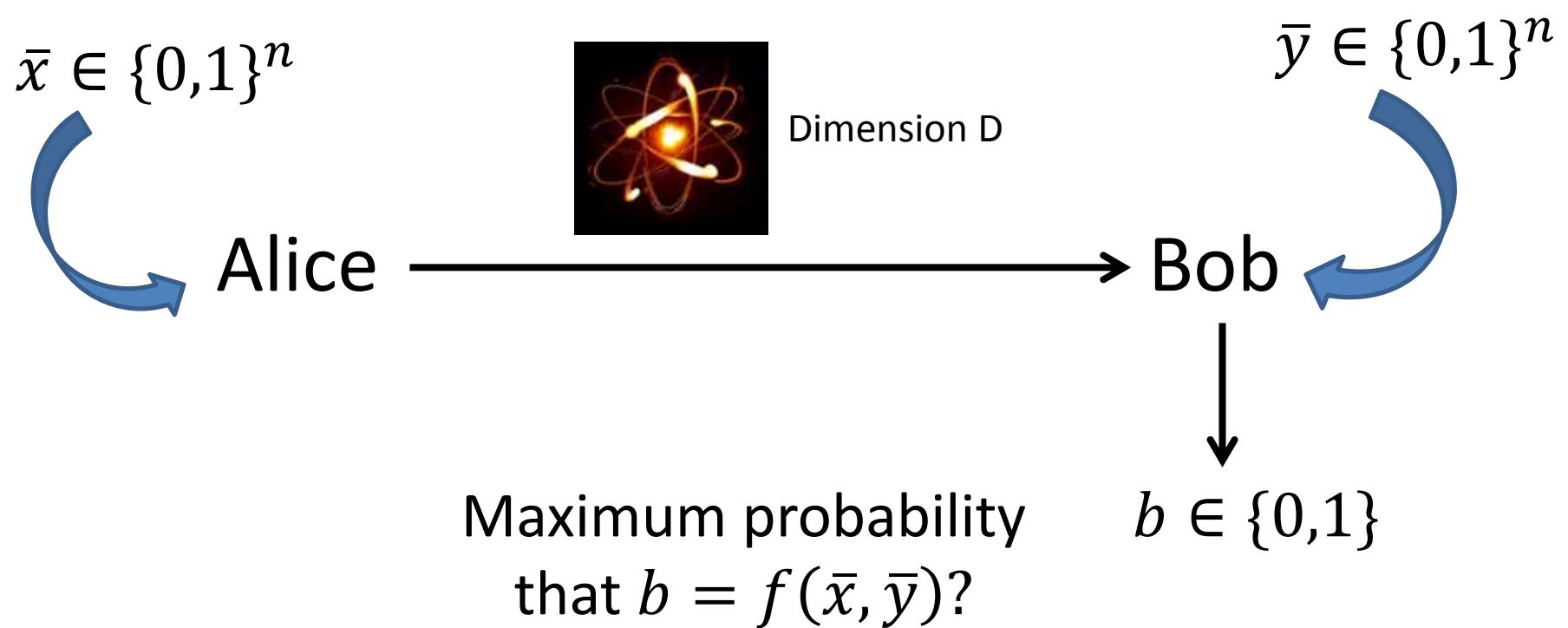
$i = i + 1$

Repeat until the
moment matrix $\Gamma^{(N+1)}$ is
a linear combination of
 $\Gamma^{(1)}, \dots, \Gamma^{(N)}$





SDP hierarchy for quantum communication complexity



$$p^* = \frac{1}{2^{2n}}\max\sum\nolimits_{x,y} tr(F_{f(x,y)}^y\rho_x)$$

$$\text{such that } \quad F_b^y,\rho_x\in B(\mathbb{C}^D),$$

$$\begin{gathered}(F_b^y)^2=F_b^y,\\ \rho_x\geq 0, tr(\rho_x)=1\end{gathered}$$

High level description

$i = 1$

Generate random
projectors $\rho_x^i, F_b^{i,y} \in \mathcal{B}(\mathbb{C}^D)$ (with
app. ranks)



Build moment matrix Γ^i

$$\Gamma_{\mathbb{I}, \mathbb{I}}^i = \text{tr}(\mathbb{I}) = D,$$

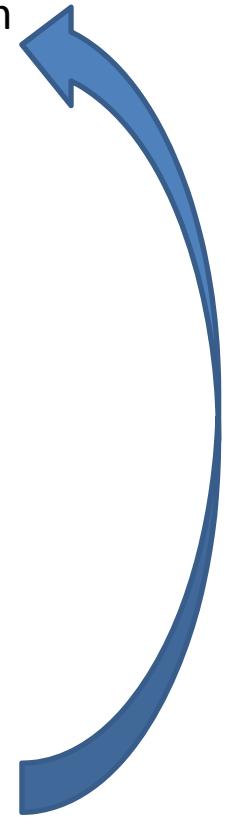
$$\Gamma_{\rho_x, F_b^y}^i = \text{tr}(\rho_x^i F_b^{i,y}),$$

⋮



$i = i + 1$

Repeat until the
moment matrix $\Gamma^{(N+1)}$ is
a linear combination of
 $\Gamma^{(1)}, \dots, \Gamma^{(N)}$



SDP hierarchy

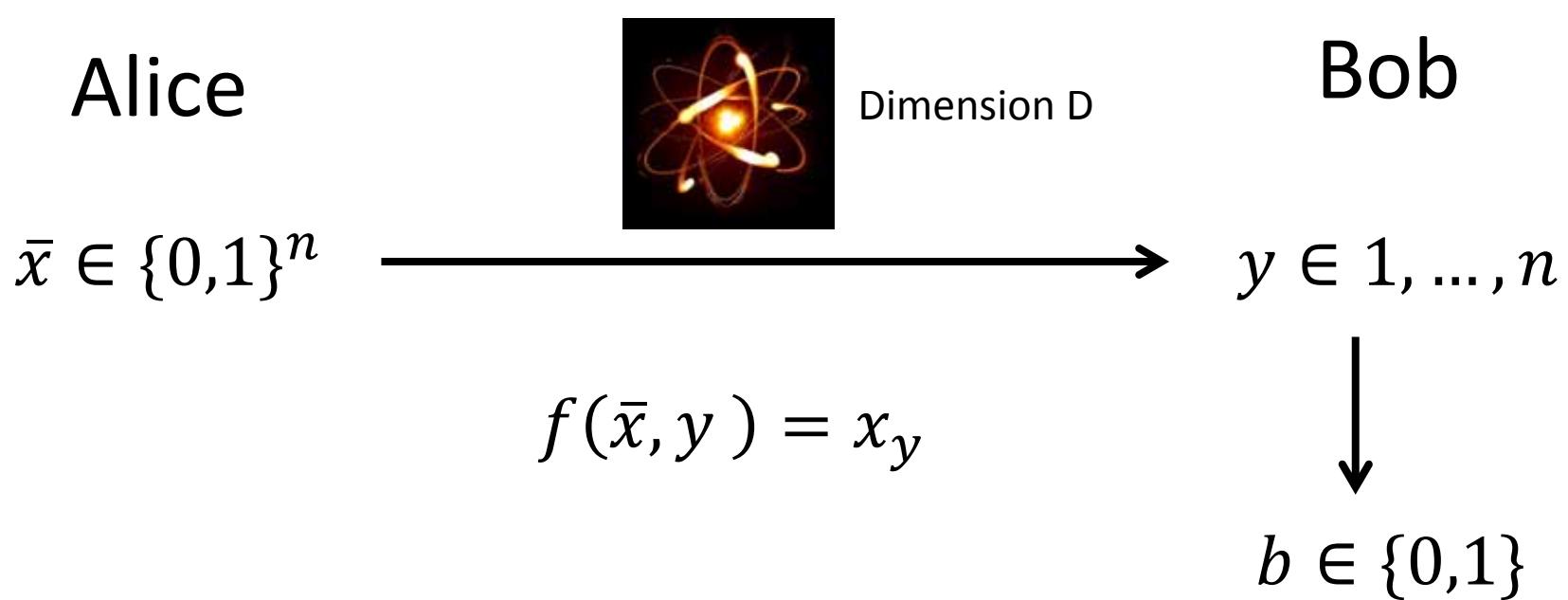
$$\max \frac{1}{2^{2n}} \sum_{b,x,y} \Gamma_{\rho_x, F_{f(x,y)}^y}$$

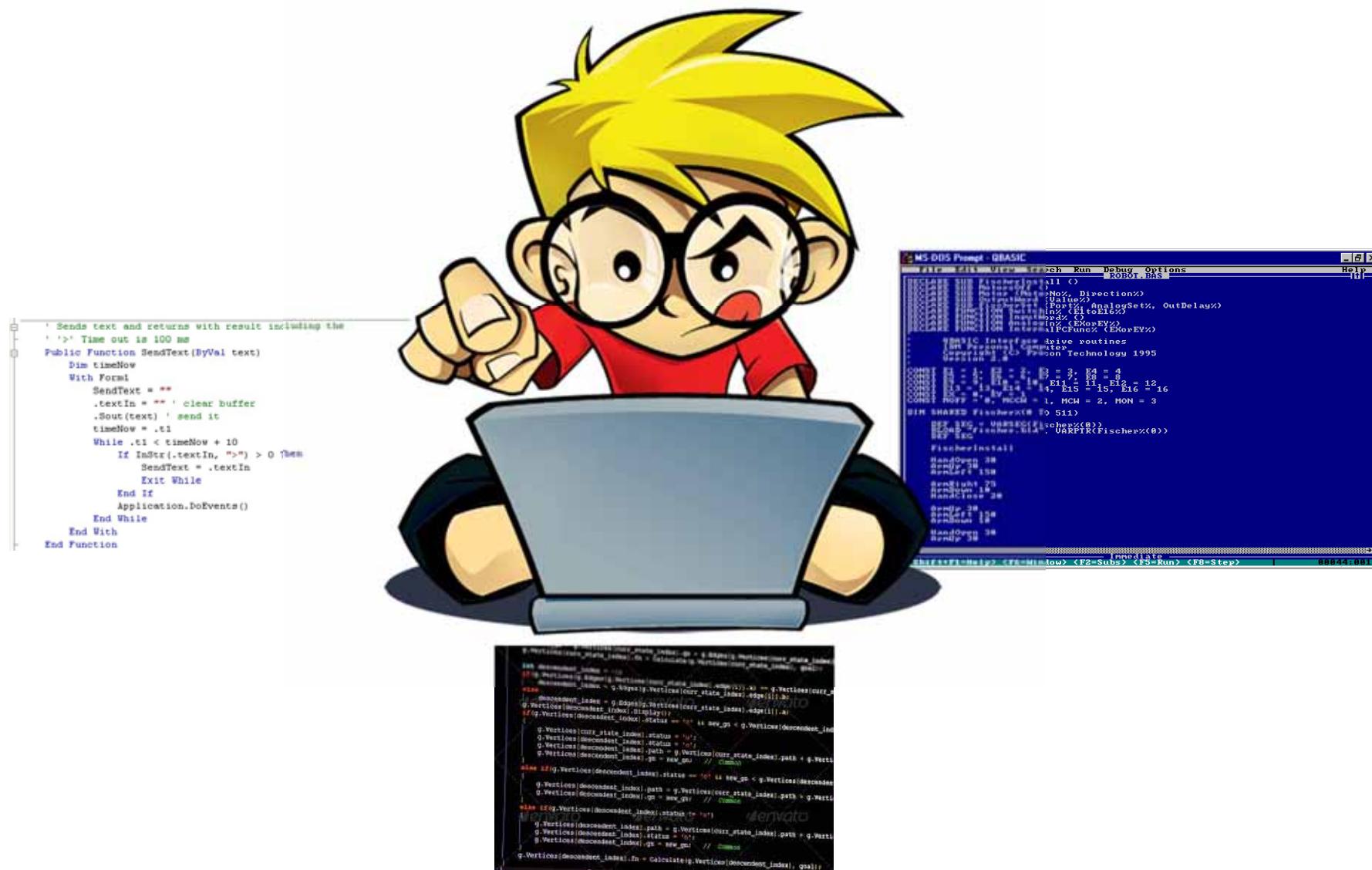
$$\text{s.t. } \Gamma_{1,1} = 1$$

$$\Gamma \geq 0, \Gamma = \sum_{s=1}^j c_s \Gamma^s$$

No proof of convergence!!!

Random Access Codes (RAC)

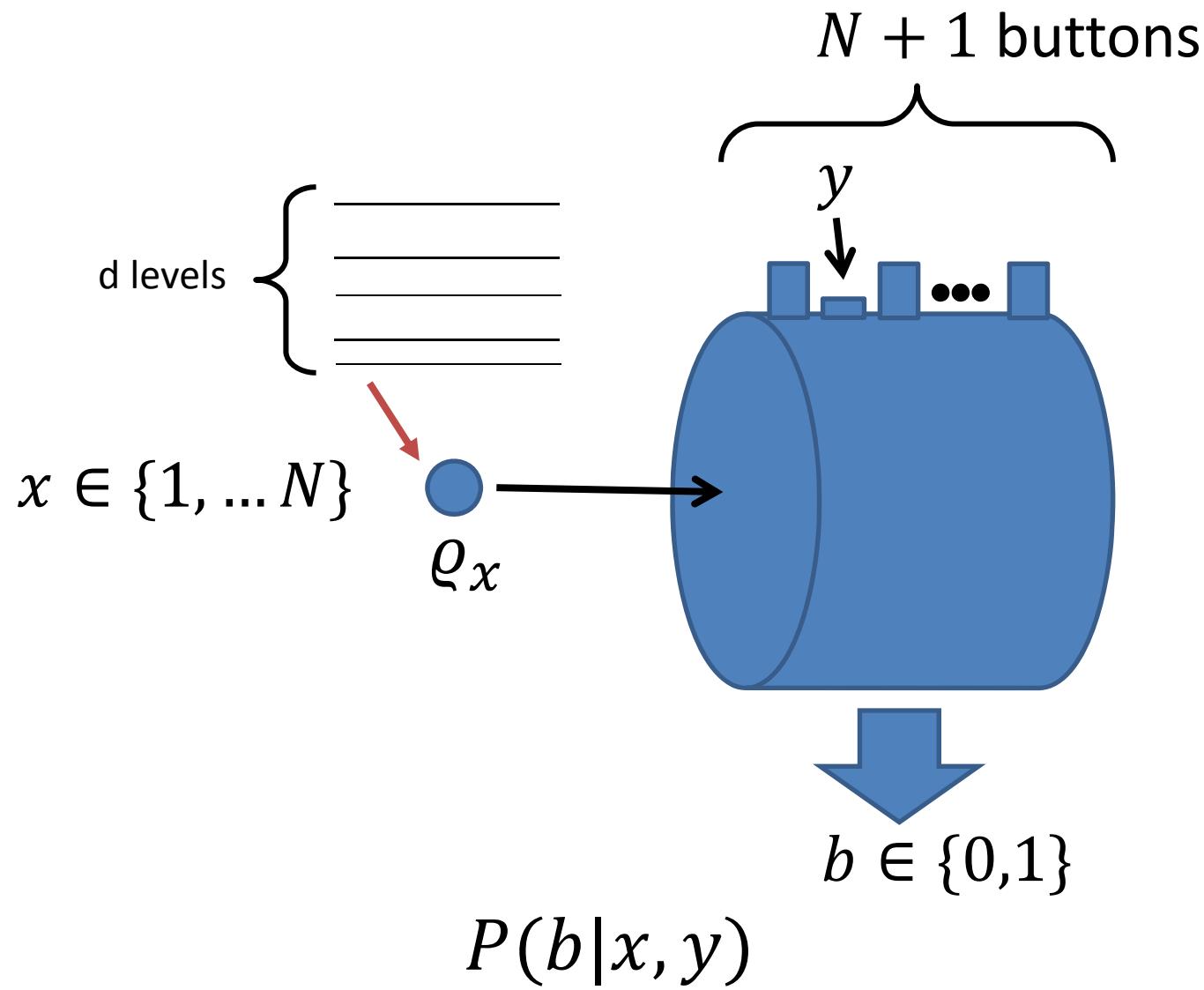




Random Access Codes (RAC)

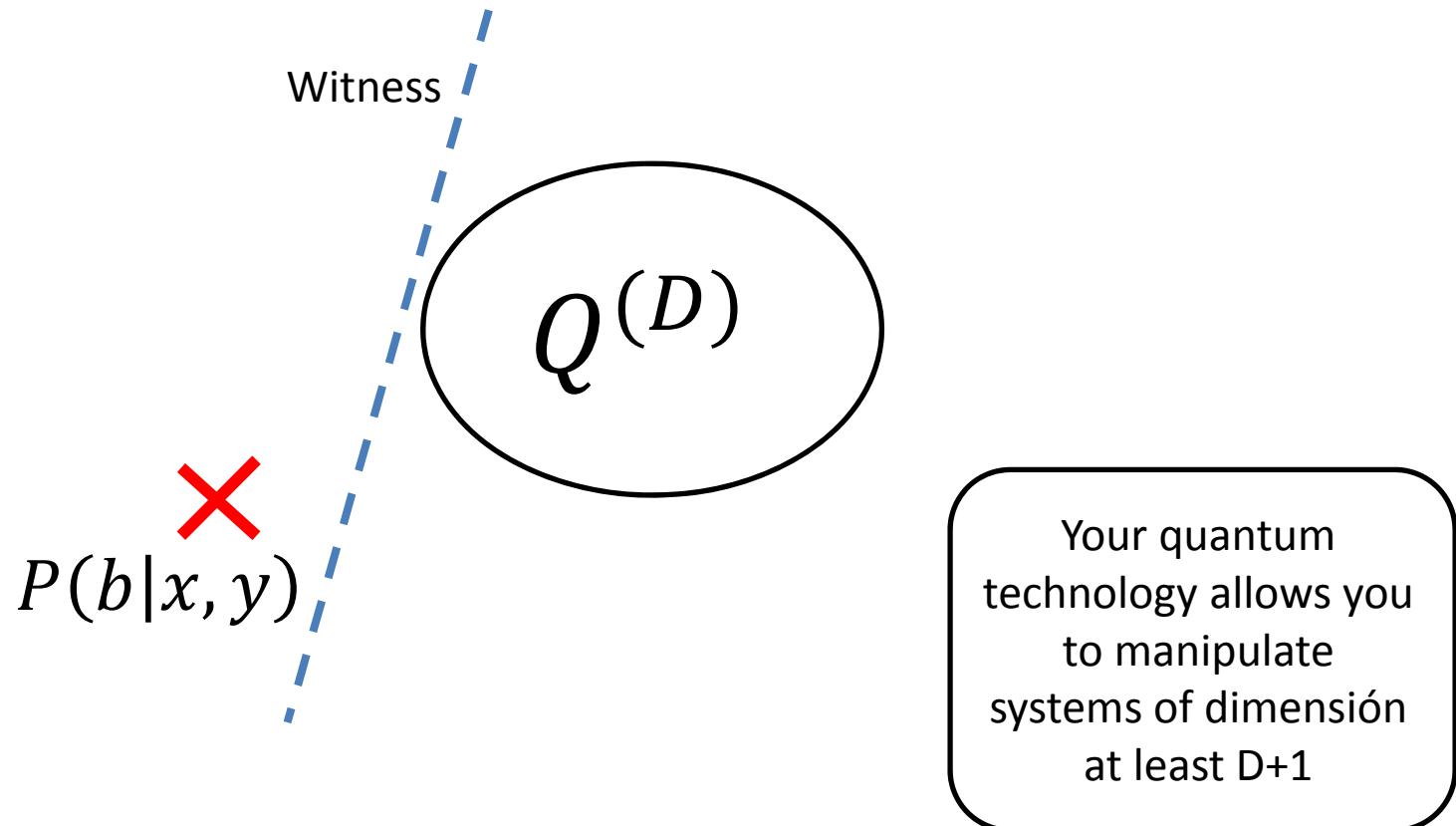
D	2	3	4	5	6	7
LB	0.788675	0.832273	0.908248	0.924431	0.951184	0.969841
UB	0.788675	0.832273	0.908248	0.924445	0.954123	0.969841

TABLE I: Lower and upper bounds on $P_{\max}(3 \rightarrow \log_2(D))$.



R. Gallego, N. Brunner, C. Hadley and A. Acín, Phys. Rev. Lett. 105, 230501 (2010).

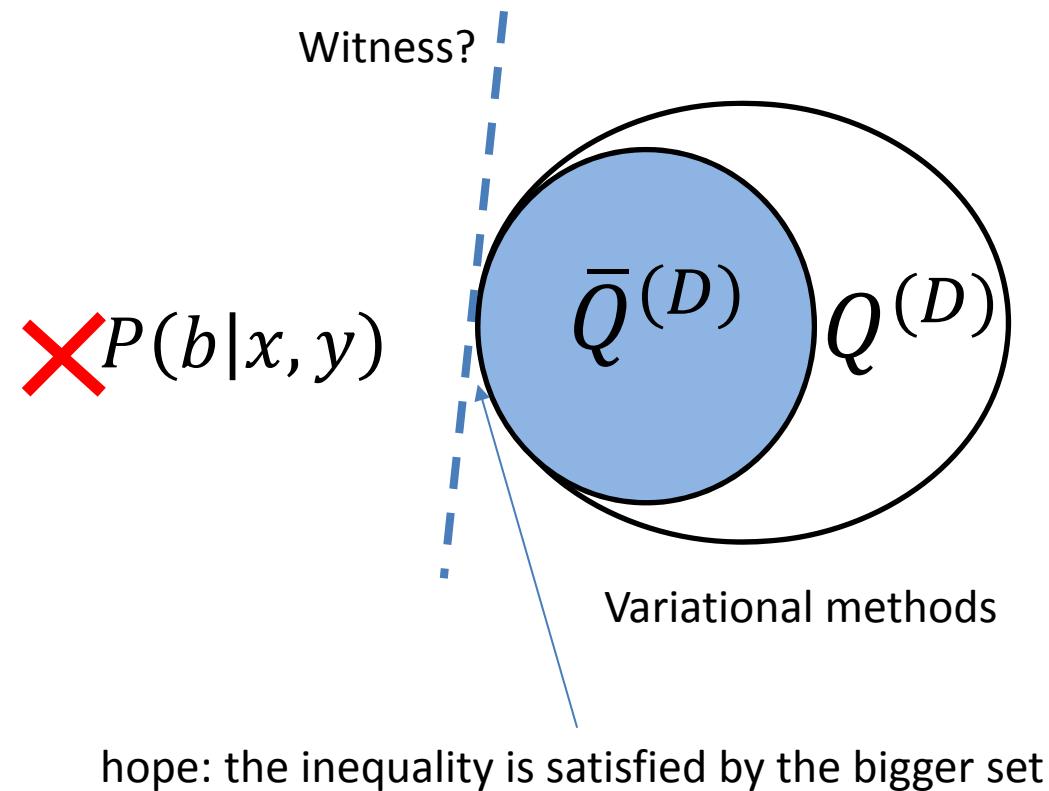
Prepare-and-measure dimension witnesses



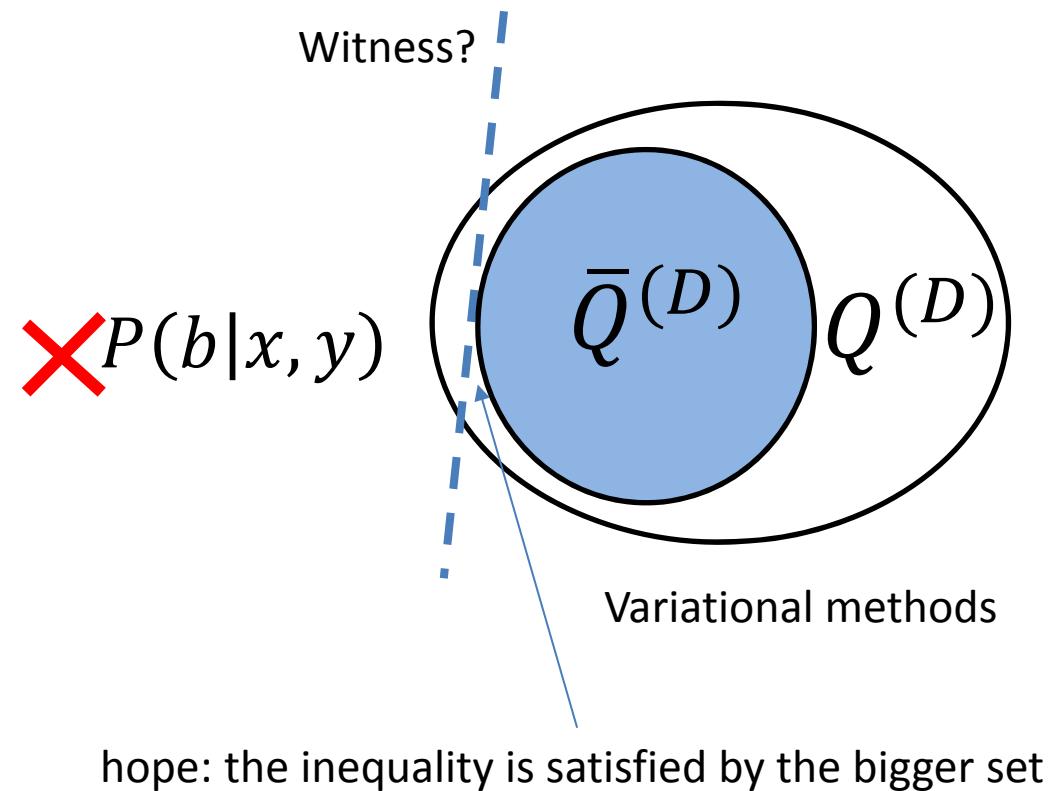
	C_2 (bit)	Q_2 (qubit)	C_3 (trit)	Q_3 (qutrit)	C_4 (quat)
I_3	3	$1 + 2\sqrt{2}$	5	5	5
I_4	5	6	7	7.9689	9

TABLE I: Classical and quantum bounds for the dimension witnesses I_3 and I_4 . Notably, these witnesses can distinguish classical and quantum systems of given dimensions.

In the lab



In the lab



```
' Sends text and returns with result including the
' '>' Time out is 100 ms
Public Function SendText(ByVal text)
    Dim timeNow
    With Form1
        SendText = ""
        .textIn = "" ' clear buffer
        .Sout(text) ' send it
        timeNow = .t1
        While .t1 < timeNow + 10
            If InStr(.textIn, ">") > 0 Then
                SendText = .textIn
                Exit While
            End If
            Application.DoEvents()
        End While
    End With
End Function
```



```
MS-DOS Prompt - QBASIC
File Edit View Search Run Debug Options
ROBOT.BAS
Help [?] ?[?]

FischerInstall()
PortSet(1, Noz, Direction)
PortSet(2, AnalogSetZ, OutDelay)
Inp(1, E1)
Inp(2, E2)
Inp(3, E3)
Inp(4, E4)
Inp(5, E5)
Inp(6, E6)
Inp(7, E7)
Inp(8, E8)
Inp(9, E9)
Inp(10, E10)
Inp(11, E11)
Inp(12, E12)
Inp(13, E13)
Inp(14, E14)
Inp(15, E15)
Inp(16, E16)
COM1 = 1, COM2 = 2, COM3 = 3, E4 = 4
COM4 = 5, COM5 = 6, COM6 = 7, E11 = 8, E8 = 9
COM7 = 10, COM8 = 11, E12 = 12, E13 = 13
COM9 = 14, E14 = 15, E15 = 16, E16 = 17
COM10 = 18, E17 = 19, E18 = 20, E19 = 21
COM11 = 22, E20 = 23, E21 = 24, E22 = 25
COM12 = 26, E23 = 27, E24 = 28, E25 = 29
COM13 = 30, E26 = 31, E27 = 32, E28 = 33
COM14 = 34, E29 = 35, E30 = 36, E31 = 37
COM15 = 38, E32 = 39, E33 = 40, E34 = 41
COM16 = 42, E35 = 43, E36 = 44, E37 = 45
COM17 = 46, E38 = 47, E39 = 48, E40 = 49
COM18 = 50, E41 = 51, E42 = 52, E43 = 53
COM19 = 54, E44 = 55, E45 = 56, E46 = 57
COM20 = 58, E47 = 59, E48 = 60, E49 = 61
COM21 = 62, E50 = 63, E51 = 64, E52 = 65
COM22 = 66, E53 = 67, E54 = 68, E55 = 69
COM23 = 70, E56 = 71, E57 = 72, E58 = 73
COM24 = 74, E59 = 75, E60 = 76, E61 = 77
COM25 = 78, E62 = 79, E63 = 80, E64 = 81
COM26 = 82, E65 = 83, E66 = 84, E67 = 85
COM27 = 86, E68 = 87, E69 = 88, E6A = 89
COM28 = 90, E6B = 91, E6C = 92, E6D = 93
COM29 = 94, E6E = 95, E6F = 96, E6G = 97
COM30 = 98, E6H = 99, E6I = 100, E6J = 101
COM31 = 102, E6K = 103, E6L = 104, E6M = 105
COM32 = 106, E6N = 107, E6O = 108, E6P = 109
COM33 = 110, E6Q = 111, E6R = 112, E6S = 113
COM34 = 114, E6T = 115, E6U = 116, E6V = 117
COM35 = 118, E6W = 119, E6X = 120, E6Y = 121
COM36 = 122, E6Z = 123, E6A1 = 124, E6B1 = 125
COM37 = 126, E6C1 = 127, E6D1 = 128, E6E1 = 129
COM38 = 130, E6F1 = 131, E6G1 = 132, E6H1 = 133
COM39 = 134, E6I1 = 135, E6J1 = 136, E6K1 = 137
COM40 = 138, E6L1 = 139, E6M1 = 140, E6N1 = 141
COM41 = 142, E6O1 = 143, E6P1 = 144, E6Q1 = 145
COM42 = 146, E6R1 = 147, E6S1 = 148, E6T1 = 149
COM43 = 150, E6U1 = 151, E6V1 = 152, E6W1 = 153
COM44 = 154, E6X1 = 155, E6Y1 = 156, E6Z1 = 157
COM45 = 158, E6A2 = 159, E6B2 = 160, E6C2 = 161
COM46 = 162, E6D2 = 163, E6E2 = 164, E6F2 = 165
COM47 = 166, E6G2 = 167, E6H2 = 168, E6I2 = 169
COM48 = 170, E6J2 = 171, E6K2 = 172, E6L2 = 173
COM49 = 174, E6M2 = 175, E6N2 = 176, E6O2 = 177
COM50 = 178, E6P2 = 179, E6Q2 = 180, E6R2 = 181
COM51 = 182, E6S2 = 183, E6T2 = 184, E6U2 = 185
COM52 = 186, E6V2 = 187, E6W2 = 188, E6X2 = 189
COM53 = 190, E6Y2 = 191, E6Z2 = 192, E6A3 = 193
COM54 = 194, E6B3 = 195, E6C3 = 196, E6D3 = 197
COM55 = 198, E6E3 = 199, E6F3 = 200, E6G3 = 201
COM56 = 202, E6H3 = 203, E6I3 = 204, E6J3 = 205
COM57 = 206, E6K3 = 207, E6L3 = 208, E6M3 = 209
COM58 = 210, E6N3 = 211, E6O3 = 212, E6P3 = 213
COM59 = 214, E6Q3 = 215, E6R3 = 216, E6S3 = 217
COM60 = 218, E6T3 = 219, E6U3 = 220, E6V3 = 221
COM61 = 222, E6W3 = 223, E6X3 = 224, E6Y3 = 225
COM62 = 226, E6Z3 = 227, E6A4 = 228, E6B4 = 229
COM63 = 230, E6C4 = 231, E6D4 = 232, E6E4 = 233
COM64 = 234, E6F4 = 235, E6G4 = 236, E6H4 = 237
COM65 = 238, E6I4 = 239, E6J4 = 240, E6K4 = 241
COM66 = 242, E6L4 = 243, E6M4 = 244, E6N4 = 245
COM67 = 246, E6O4 = 247, E6P4 = 248, E6Q4 = 249
COM68 = 250, E6R4 = 251, E6S4 = 252, E6T4 = 253
COM69 = 254, E6U4 = 255, E6V4 = 256, E6W4 = 257
COM70 = 258, E6X4 = 259, E6Y4 = 260, E6Z4 = 261
COM71 = 262, E6A5 = 263, E6B5 = 264, E6C5 = 265
COM72 = 266, E6D5 = 267, E6E5 = 268, E6F5 = 269
COM73 = 270, E6G5 = 271, E6H5 = 272, E6I5 = 273
COM74 = 274, E6J5 = 275, E6K5 = 276, E6L5 = 277
COM75 = 278, E6M5 = 279, E6N5 = 280, E6O5 = 281
COM76 = 282, E6P5 = 283, E6Q5 = 284, E6R5 = 285
COM77 = 286, E6S5 = 287, E6T5 = 288, E6U5 = 289
COM78 = 290, E6V5 = 291, E6W5 = 292, E6X5 = 293
COM79 = 294, E6Y5 = 295, E6Z5 = 296, E6A6 = 297
COM80 = 298, E6B6 = 299, E6C6 = 300, E6D6 = 301
COM81 = 302, E6E6 = 303, E6F6 = 304, E6G6 = 305
COM82 = 306, E6H6 = 307, E6I6 = 308, E6J6 = 309
COM83 = 310, E6K6 = 311, E6L6 = 312, E6M6 = 313
COM84 = 314, E6N6 = 315, E6O6 = 316, E6P6 = 317
COM85 = 318, E6Q6 = 319, E6R6 = 320, E6S6 = 321
COM86 = 322, E6T6 = 323, E6U6 = 324, E6V6 = 325
COM87 = 326, E6W6 = 327, E6X6 = 328, E6Y6 = 329
COM88 = 330, E6Z6 = 331, E6A7 = 332, E6B7 = 333
COM89 = 334, E6C7 = 335, E6D7 = 336, E6E7 = 337
COM90 = 338, E6F7 = 339, E6G7 = 340, E6H7 = 341
COM91 = 342, E6I7 = 343, E6J7 = 344, E6K7 = 345
COM92 = 346, E6L7 = 347, E6M7 = 348, E6N7 = 349
COM93 = 350, E6O7 = 351, E6P7 = 352, E6Q7 = 353
COM94 = 354, E6R7 = 355, E6S7 = 356, E6T7 = 357
COM95 = 358, E6U7 = 359, E6V7 = 360, E6W7 = 361
COM96 = 362, E6X7 = 363, E6Y7 = 364, E6Z7 = 365
COM97 = 366, E6A8 = 367, E6B8 = 368, E6C8 = 369
COM98 = 370, E6D8 = 371, E6E8 = 372, E6F8 = 373
COM99 = 374, E6G8 = 375, E6H8 = 376, E6I8 = 377
COM100 = 378, E6J8 = 379, E6K8 = 380, E6L8 = 381
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COM267 = 1045, E6Q27 = 1046, E6R27 = 1047, E
```

	C_2 (bit)	Q_2 (qubit)	C_3 (trit)	Q_3 (qutrit)	C_4 (quat)
I_3	3	$1 + 2\sqrt{2}$	5	5	5
I_4	5	6	7	7.9689	9

TABLE I: Classical and quantum bounds for the dimension witnesses I_3 and I_4 . Notably, these witnesses can distinguish classical and quantum systems of given dimensions.

We proved that all these witnesses were sound,
hence validating of the conclusions of the
experimental papers

M. Hendrych, R. Gallego, M. Micuda, N. Brunner, A. Acín and J. P. Torres, Nat. Phys. 8, 588 (2012).
J. Ahrens, P. Badziag, A. Cabello, M. Bourennane, Nat. Phys. 8, 592 (2012).

Hierarchy for real quantum systems

$i = 1$

Generate *real/random*
projectors $\rho_x^i, F_b^{i,y} \in \mathcal{B}(\mathbb{C}^D)$ (with
app. ranks)



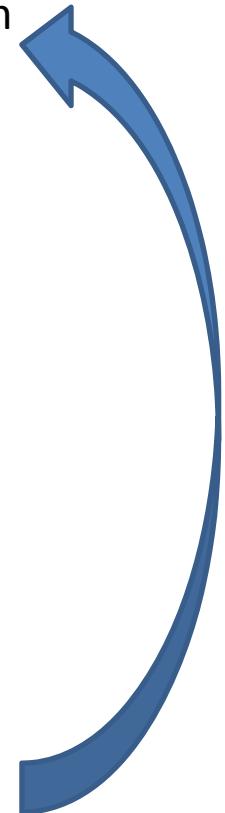
Build moment matrix Γ^i

$$\begin{aligned}\Gamma_{\mathbb{I}, \mathbb{I}}^i &= \text{tr}(\mathbb{I}) = 1, \\ \Gamma_{\rho_x, F_b^y}^i &= \text{tr}(\rho_x^i F_b^{i,y}),\end{aligned}$$

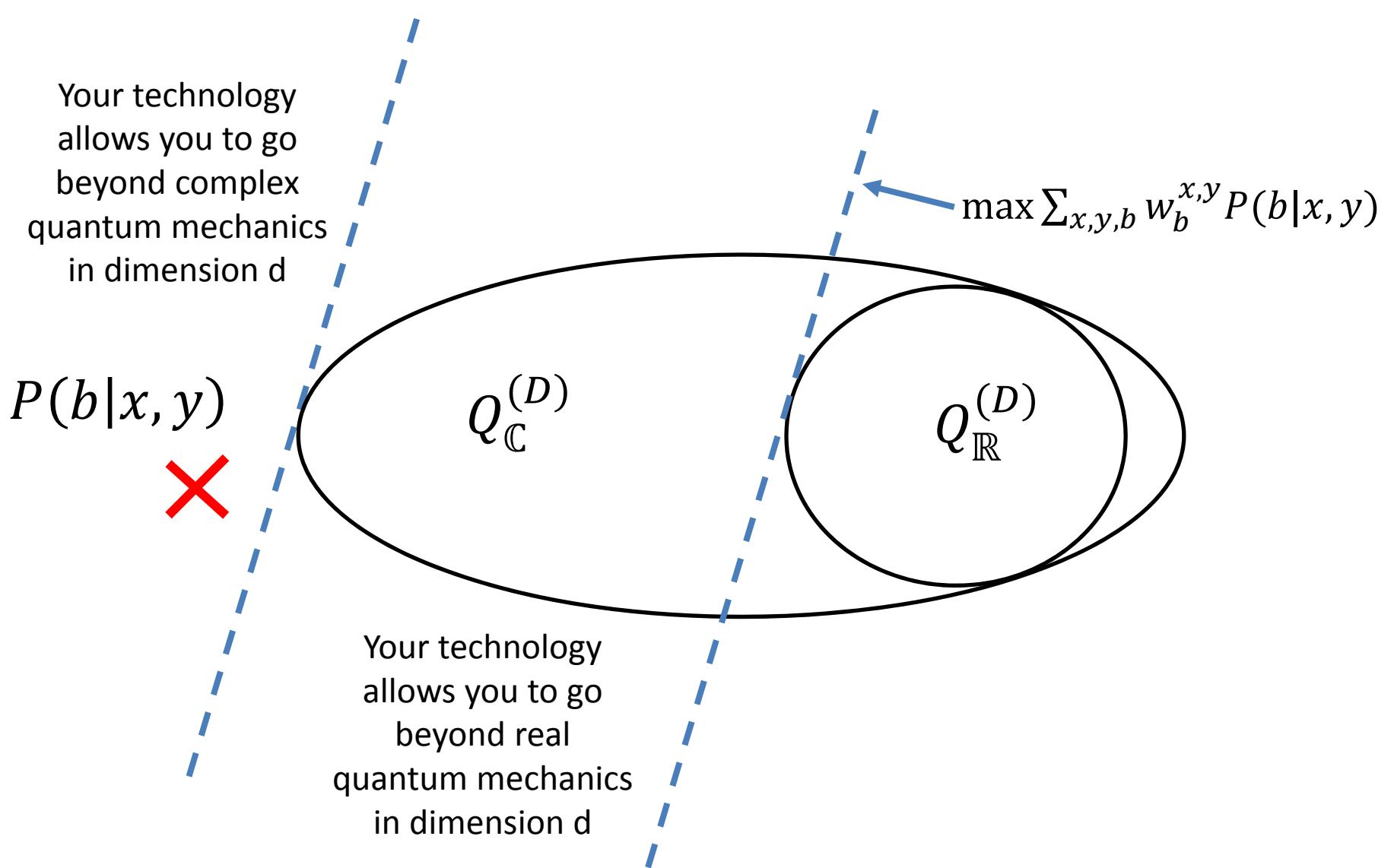


$i = i + 1$

Repeat until the
moment matrix $\Gamma^{(N+1)}$ is
a linear combination of
 $\Gamma^{(1)}, \dots, \Gamma^{(N)}$



Real vs. Complex quantum mechanics



Random Access Codes (RAC)

Alice



Dimension D

Bob

$$\longrightarrow \quad y \in 1, \dots, n$$

$$f(\bar{x}, y) = x_y$$



$$b \in \{0,1\}$$

complex qubits

$$P_{max}(3 \rightarrow 1) \approx 0.788675$$

real qubits

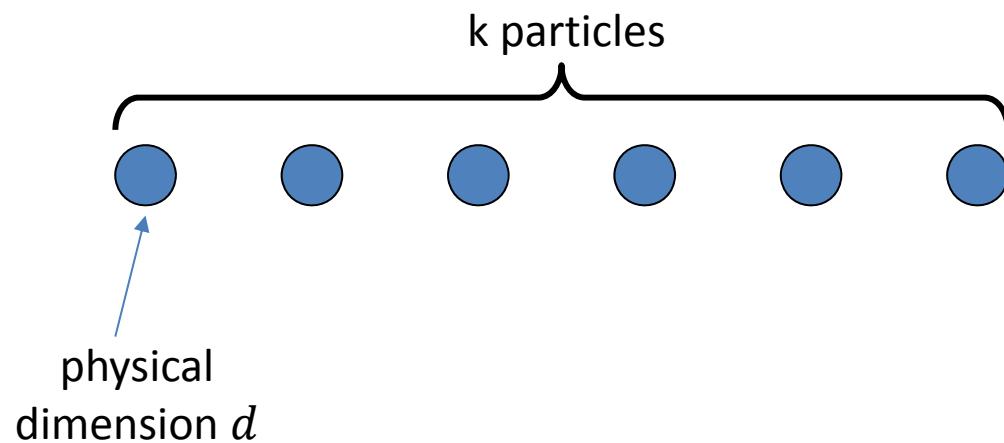
$$P_{max}(3 \rightarrow 1) \approx 0.7696723$$

The structure of Matrix Product States

MN and T. Vértesi, arXiv:1509.04507.

$$H = \sum_{j=1}^{n-s} h_j$$

s-local term
(acts on particles
 $j, \dots, j+s-1$)



ground state $|\varphi\rangle \in \mathbb{C}^{d^k}$ 

Impossible to even
store for high k

Matrix product states

$$A_1 \dots A_d \in B(\mathbb{C}^D)$$

physical dimension d

boundary condition

bond dimension D

$$|\psi\rangle = \sum_{i_1, \dots, i_k} \text{tr}(\sigma A_{i_1} \dots A_{i_k}) |i_1\rangle \otimes \dots \otimes |i_k\rangle$$

D. Perez-Garcia, F. Verstraete, M.M. Wolf and J.I. Cirac, Quantum Inf. Comput. 7, 401 (2007).

Matrix product states



Features:

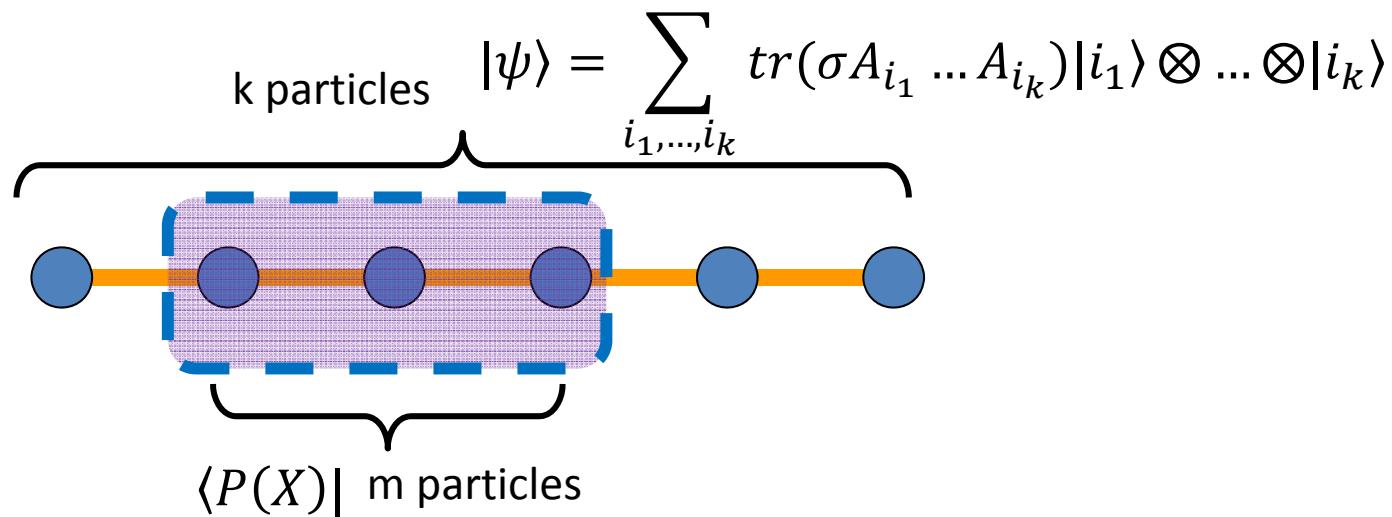
- a) Efficient computation of expectation values
- b) Calculations in the thermodynamical limit $k \rightarrow \infty$

Correspondence between polynomials and states

$P(X) = P(X_1, \dots, X_d)$,
homogeneous polynomial
of degree m

$$P(X) = \sum_{\vec{l}} p_{\vec{l}} X_{i_1} \dots X_{i_m} \quad \rightarrow \quad |P(X)\rangle = \sum_{\vec{l}} p_{\vec{l}}^* |i_1, \dots, i_m\rangle$$

Overlap between a polynomial and a MPS



$$\langle P(X) | \psi \rangle = \sum_{\substack{i_1, \dots, i_s, \\ i_{s+m+1}, \dots, i_k}} \text{tr}(\sigma A_{i_1} \dots A_{i_s} P(A) A_{i_{s+m+1}} \dots A_{i_k}) |i_1, \dots, i_s, i_{s+m+1}, \dots i_k\rangle$$

"defect"

Existence of annihilation operators

$$\langle P(X) | \psi \rangle = \sum_{\substack{i_1, \dots, i_s, \\ i_{s+m+1}, \dots, i_k}} \text{tr}(\sigma A_{i_1} \dots A_{i_s} P(A) A_{i_{s+m+1}} \dots A_{i_k}) |i_1, \dots, i_s, i_{s+m+1}, \dots i_k\rangle$$

$P(X)$, MPI for dimension D



$\langle P(X) | \psi \rangle = 0$, for all MPS with bond dimension D!!

There exist local operators h which annihilate all MPS of bond dimension D or smaller

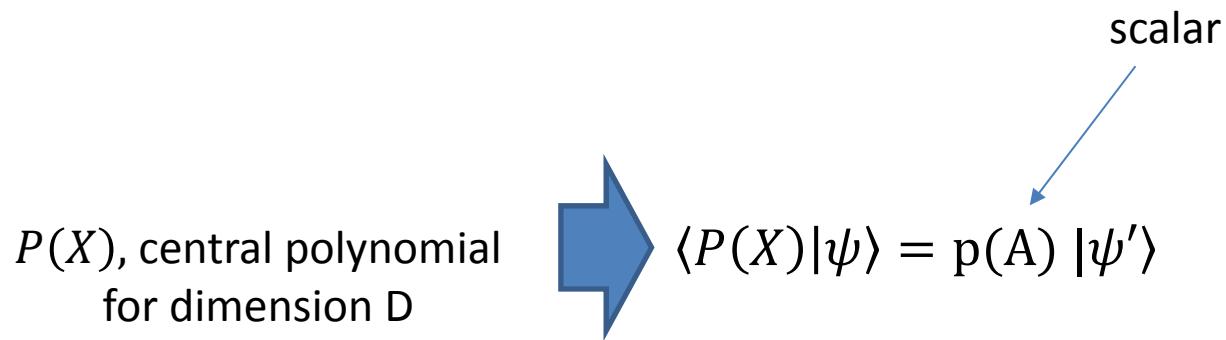
Actually, for high m , almost all m -local operators are annihilation operators, because

$$\text{Local dimension of } m \text{ particles} \quad d^m$$

$$\text{Local dimension of MPS subspace} \quad \text{poly}(m)$$

Existence of cut-and-glue operators

$$\langle P(X)|\psi\rangle = \sum_{\substack{i_1, \dots, i_s, \\ i_{s+m+1}, \dots, i_k}} \text{tr}(\sigma A_{i_1} \dots A_{i_s} P(A) A_{i_{s+m+1}} \dots A_{i_k}) |i_1, \dots, i_s, i_{s+m+1}, \dots i_k\rangle$$

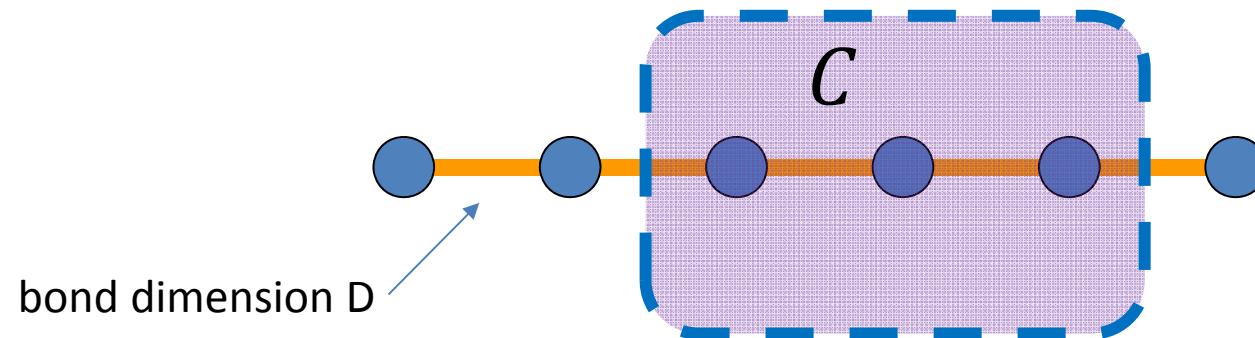


$$|\psi'\rangle = \sum_{\substack{i_1, \dots, i_s, \\ i_{s+m+1}, \dots, i_k}} \text{tr}(\sigma A_{i_1} \dots A_{i_s} A_{i_{s+m+1}} \dots A_{i_k}) |i_1, \dots, i_s, i_{s+m+1}, \dots i_k\rangle$$

$\{P_j(X)\}$, basis of
homogeneous central
polynomials for
dimension D of degree m



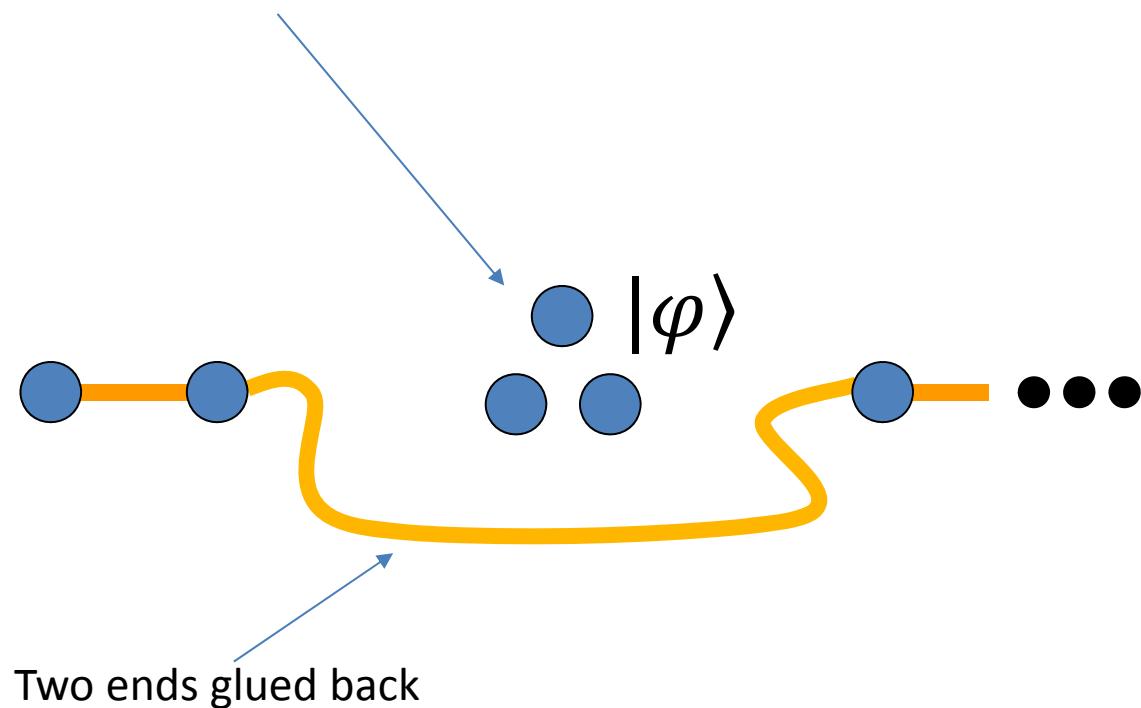
$$C = \sum_j |j\rangle\langle P_j(X)|$$



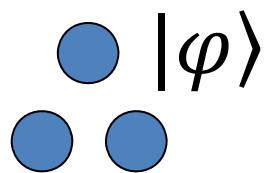
$$|\psi\rangle = \sum_{i_1, \dots, i_k} \text{tr}(\sigma A_{i_1} \dots A_{i_k}) |i_1\rangle \otimes \dots \otimes |i_k\rangle$$

C, cut-and-glue operator

m particles projected onto a pure state (cut from the chain)



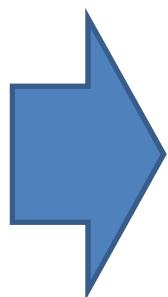
C, cut-and-glue operator



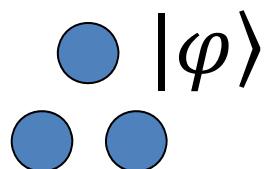
$$|\psi\rangle = \sum_{i_1, \dots, i_k} \text{tr}(\sigma A_{i_1} \dots A_{i_{k-m}}) |i_1\rangle \otimes \dots \otimes |i_{k-m}\rangle$$

C , cut-and-glue operator for
dimension D

ρ_k , k -site MPS of bond
dimension D



$(C \otimes 1)\rho_k(C^\dagger \otimes 1)$, separable



$$|\psi\rangle = \sum_{i_1, \dots, i_k} \text{tr}(\sigma A_{i_1} \dots A_{i_{k-m}}) |i_1\rangle \otimes \dots \otimes |i_{k-m}\rangle$$

Theoretical complexity of the SDP hierarchy and connection with MPS

MN and T. Vértesi, arXiv:1509.04507.

Complexity of implementing the kth relaxation

$$\Gamma_k = \left[\begin{array}{c} \text{SDP constraint} \\ \vdots \end{array} \right] \quad \exp(k)$$

$$|c^k| = \exp(k) \quad \text{free variables}$$

Free-dimensional problems have the same complexity as dimension-constrained ones!?

Connection with MPSs

$$\Gamma_k = \sum_{i_1, \dots, i_k} \sum_{j_1, \dots, j_k} \text{tr}(\tilde{X}_{j_n} \dots \tilde{X}_{j_1} |\psi\rangle\langle\psi| \tilde{X}_{i_1} \dots \tilde{X}_{i_k}) |\vec{i}\rangle\langle\vec{j}|$$

Feasible moment matrices are extendible MPSs of bond dimension D!!!

Complexity of implementing the kth relaxation

SDP constraint

$$\Gamma_k = \left[\begin{array}{c} \\ \\ \\ \end{array} \right] \quad \left. \right\} \exp(k)$$

Supp \in span{poly(k) k-site MPS
of bond dimension D }

$$|c^k| = \exp(k) \quad \text{free variables}$$

$c^k \in$ span{poly(k) 2k-site
MPS of bond dimension D }

Complexity of implementing the k th relaxation

$$\tilde{\Gamma}^k = \left[\quad \right] \} \text{poly}(k)$$

$$|\widetilde{c^k}| = \text{poly}(k)$$

Finite dimensions are exponentially
easier to characterize than infinite
dimensions!!!

Extra PSD conditions to boost speed of convergence

$$\Gamma^k = \sum p_j \rho_{MPS}^j$$

Extra PSD conditions to boost speed of convergence

$$\Gamma^k = \sum p_j \rho_{MPS}^j$$

cut-and-glue-operator



$C\Gamma^k C^\dagger$, separable operator

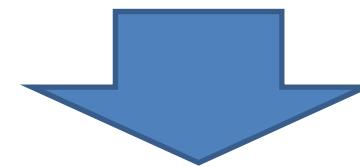
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New PSD conditions

$C\Gamma^k C^\dagger$, PPT

Conclusions

- Simple, easy-to-program technique to enforce dimension constraints in noncommutative polynomial optimization.
- Plenty of numerical evidence suggests that it is effective.
- It can be combined with other hierarchies, such as MLHG or BFS.

T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann and O. Guhne, Phys. Rev. Lett. 111, 030501 (2013).

M. Berta, O. Fawzi and V. B. Scholz, arXiv:1506.08810.

THE END