Bounding the set of finite dimensional correlations

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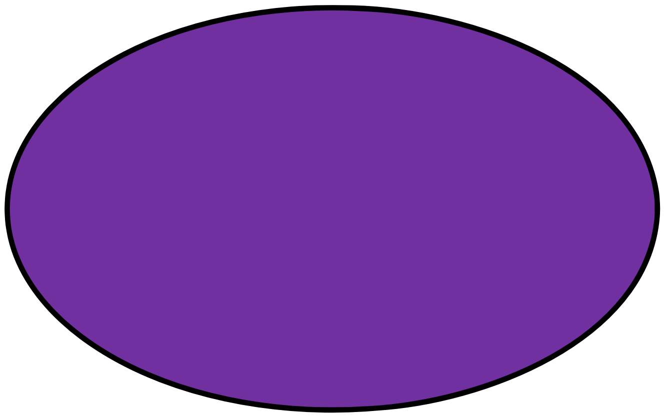
MN and T. Vértesi, Phys. Rev. Lett. 115, 020501 (2015) .

MN, A. Feix, M. Araújo and T. Vértesi, Phys. Rev. A 92, 042117 (2015).

Optimization theory

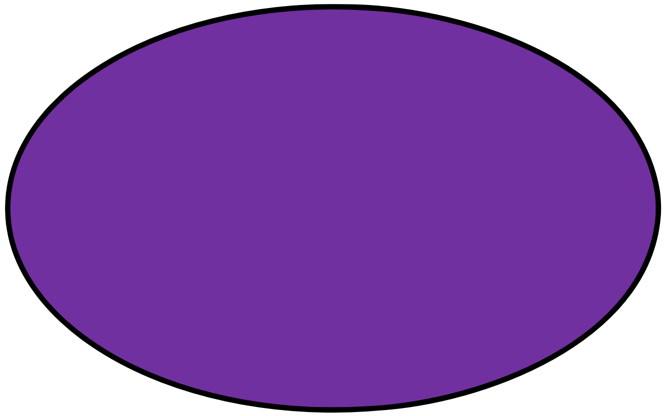
$$\max f(x)$$

$x \in$ 

Variational methods

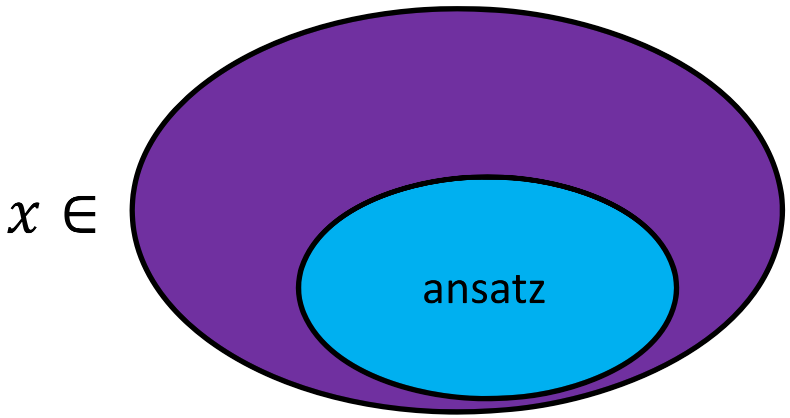
$$\max f(x)$$

$x \in$



Variational methods

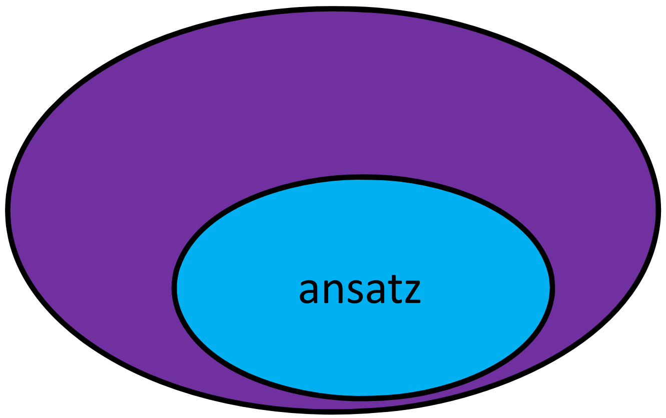
$$\max f(x)$$



Variational methods

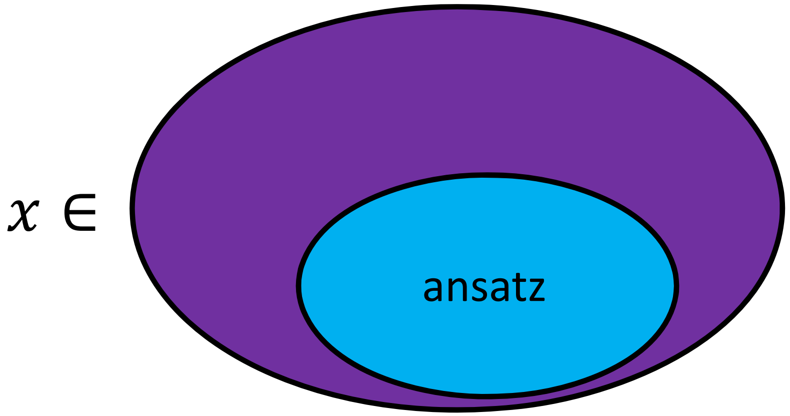
$$\max f(x) \geq \mu$$

$x \in$



Variational methods

$$\max f(x) \geq \mu$$



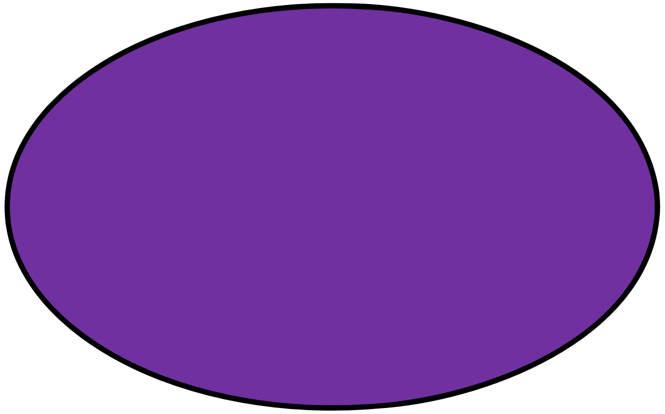
-usually, pretty straightforward
-usually, universal
} E.g.: Newton's method

-sometimes arise as intuitions gathered via numerical experiments. E.g.: MPS.

Relaxations

$$\max f(x)$$

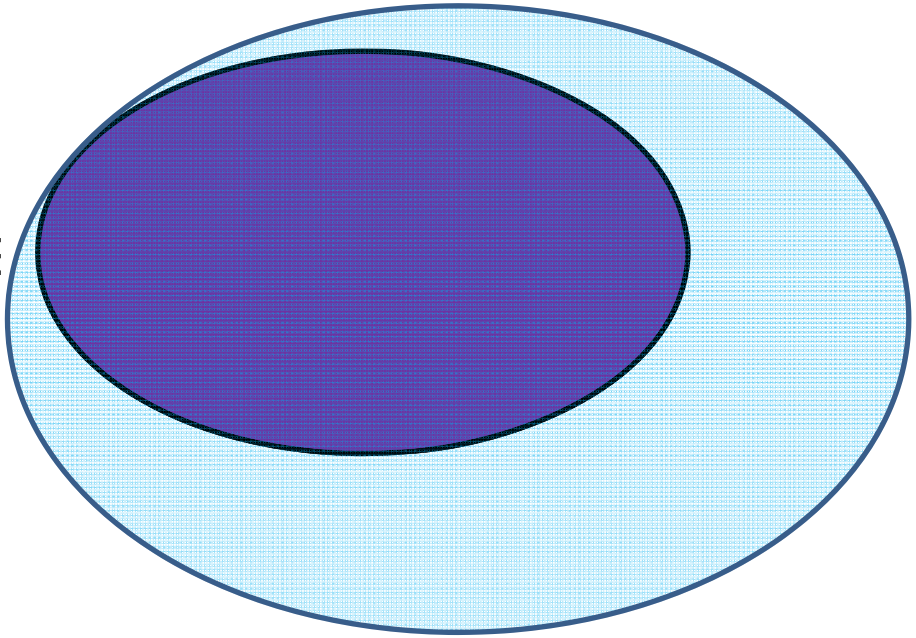
$x \in$



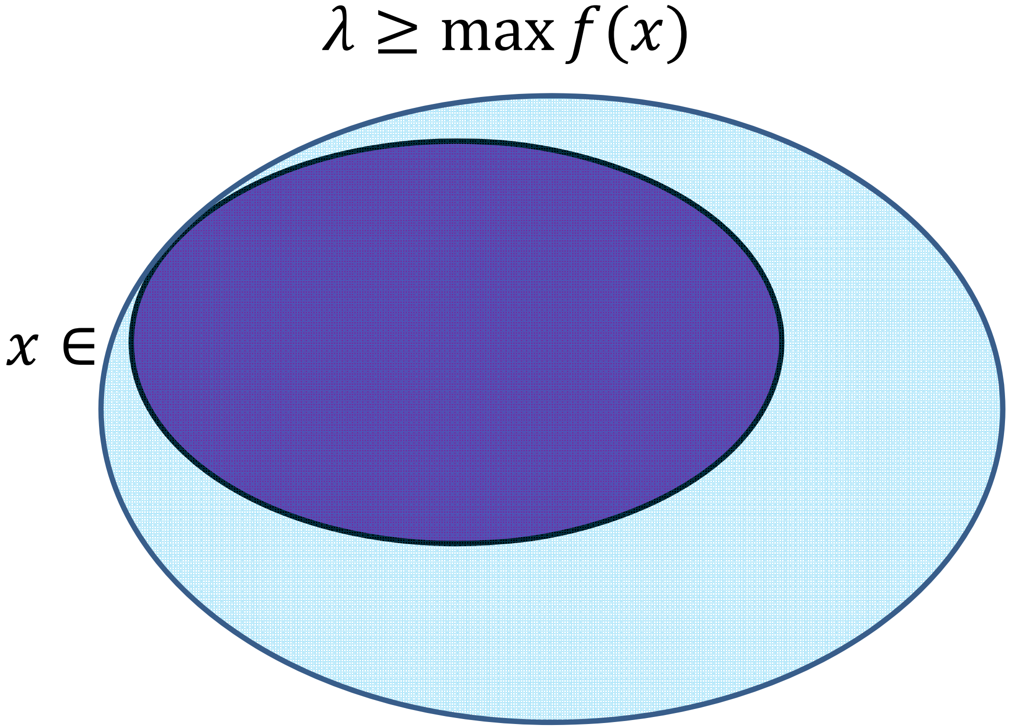
Relaxations

$$\max f(x)$$

$x \in$



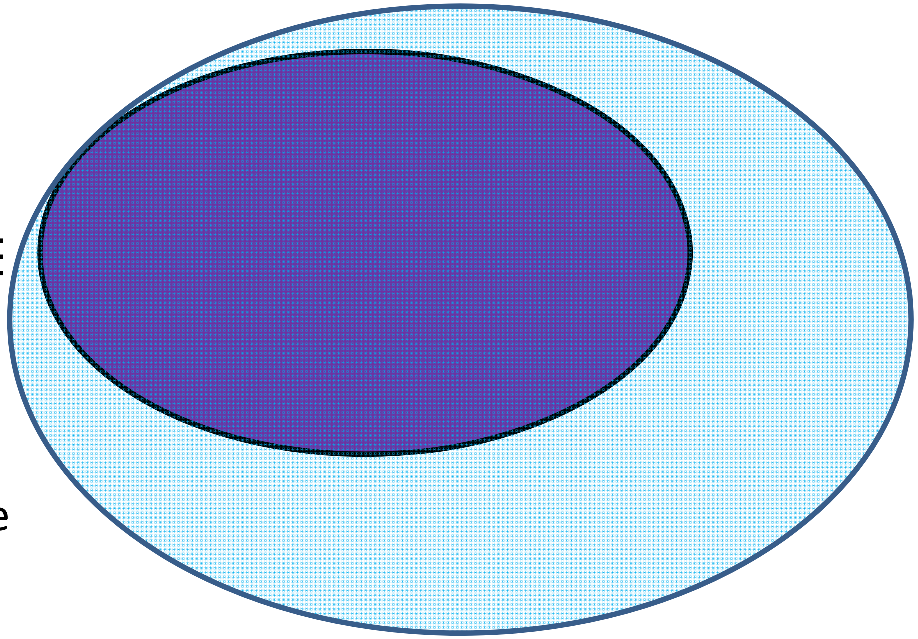
Relaxations



Relaxations

$$\lambda \geq \max f(x)$$

$x \in$

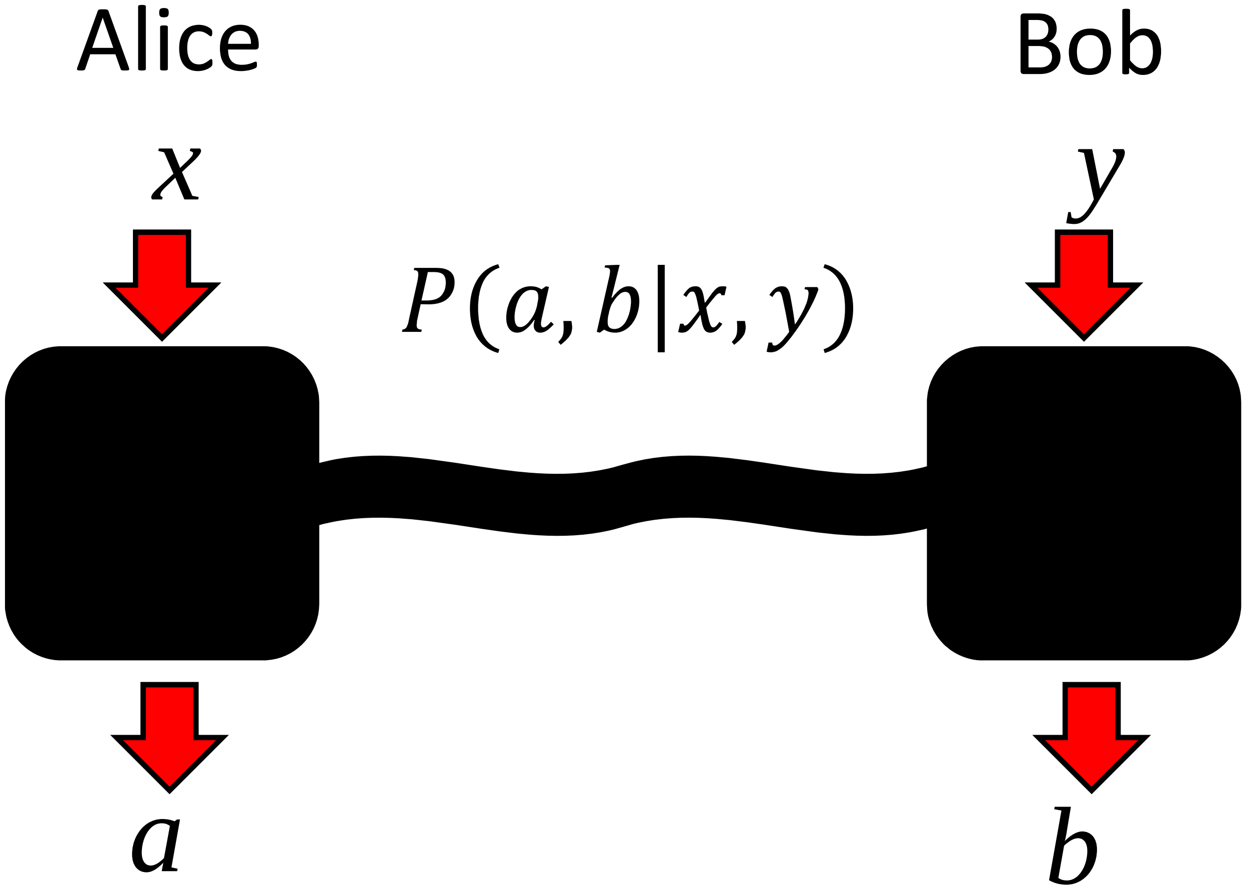


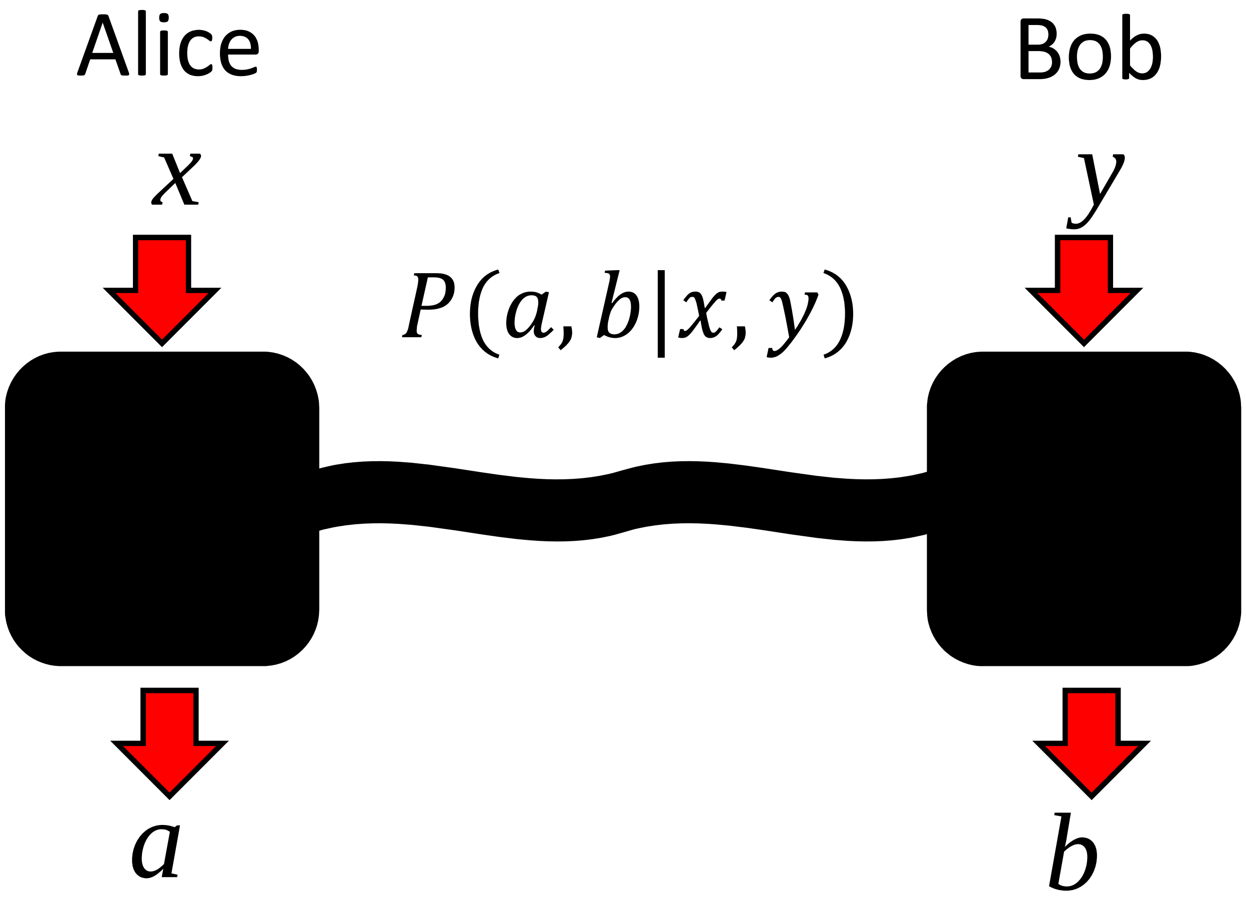
-Depend on the structure of the problem

-"Aha!" idea

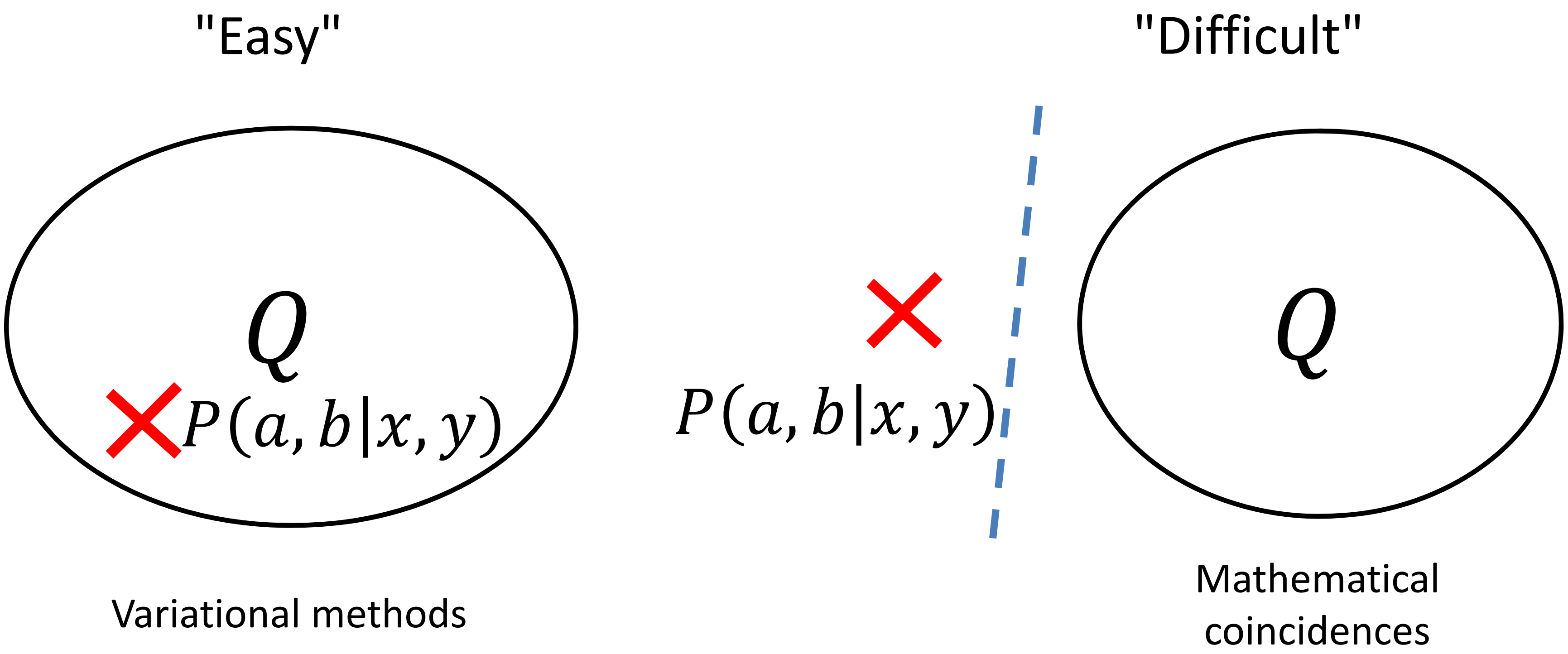
E.g.: the NPA hierarchy

MN, S. Pironio and A. Acín, Phys. Rev. Lett. 98, 010401 (2007).
MN, S. Pironio and A. Acín, New J. Phys. 10, 073013 (2008).

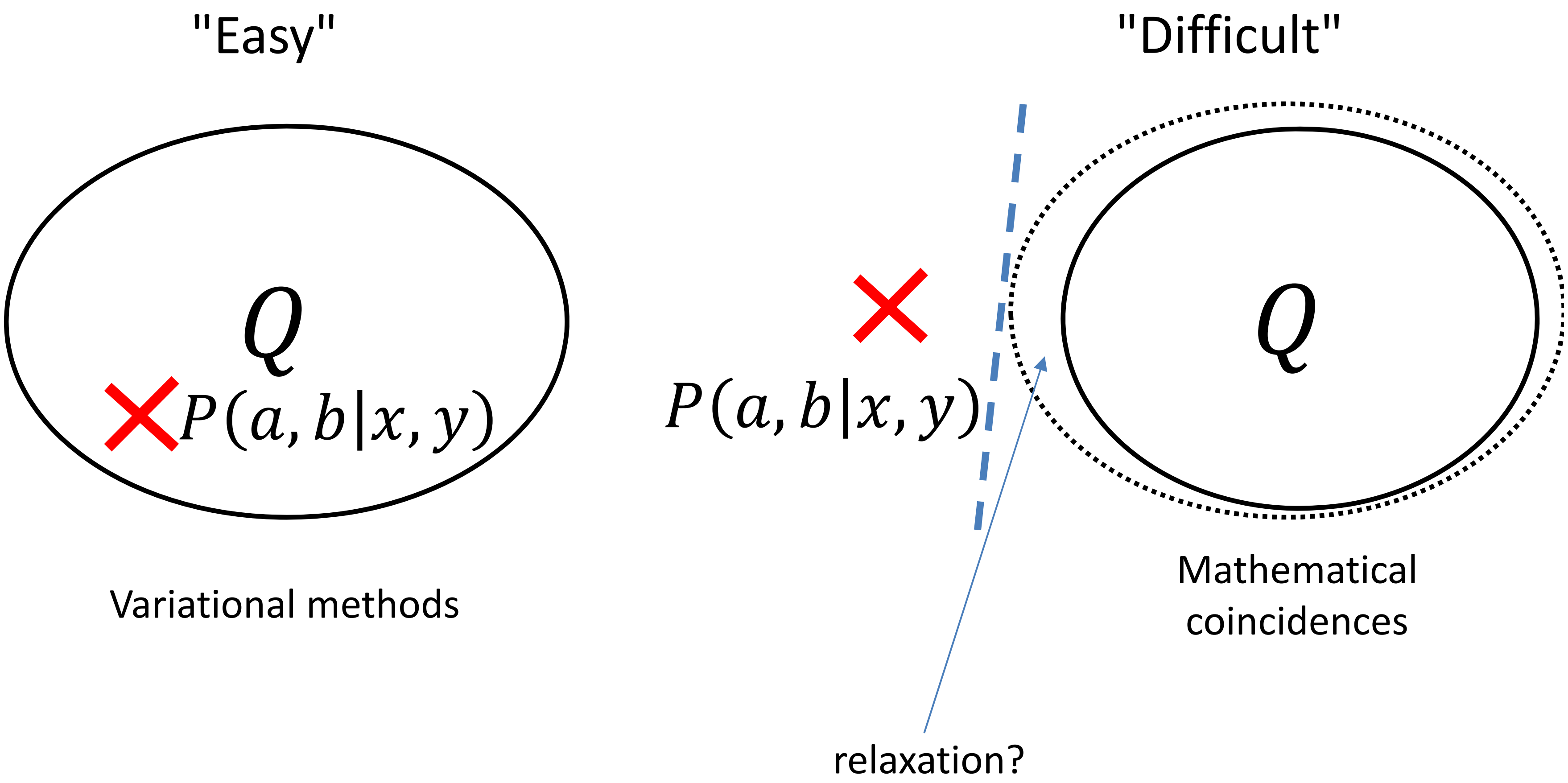

Alice
Bob
x
y
$P(a, b|x, y)$
a
b



Are the ouputs of this experiment compatible with quantum mechanics?



"Easy"
Q
$P(a, b|x, y)$
Variational methods
"Difficult"
$P(a, b|x, y)$
Q
Mathematical coincidences



"Easy"
"Difficult"
Q
$P(a,b|x,y)$
Variational methods
Q
$P(a,b|x,y)$
Mathematical coincidences
relaxation?

$P(a,b|x,y)$ is not quantum

$$\Updownarrow$$

There do not exist a Hilbert space $\mathcal{H}$, a quantum state $|\psi\rangle \in \mathcal{H}$ and projector operators $\{E_a^x, F_b^y\} \subset B(\mathcal{H})$, with $\sum_a E_a^x = \sum_b F_b^y = \mathbb{I}$, $[E_a^x, F_b^y] = 0$, such that $P(a,b|x,y) = \langle\psi|E_a^x F_b^y|\psi\rangle$.

The problem resists brute force approach!!

$P(a,b|x,y)$ is not quantum

$$\Updownarrow$$

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The problem resists brute force approach!!

What is a Hilbert space?

$$\psi = (\psi_1, \psi_2, \psi_3 \ldots)$$

$$\sum_i |\psi_i|^2 < \infty$$

$$P(a,b|x,y) = \langle \phi, E_a^x F_b^y \phi \rangle = \langle E_a^x F_b^y \rangle$$

visible averages

invisible averages

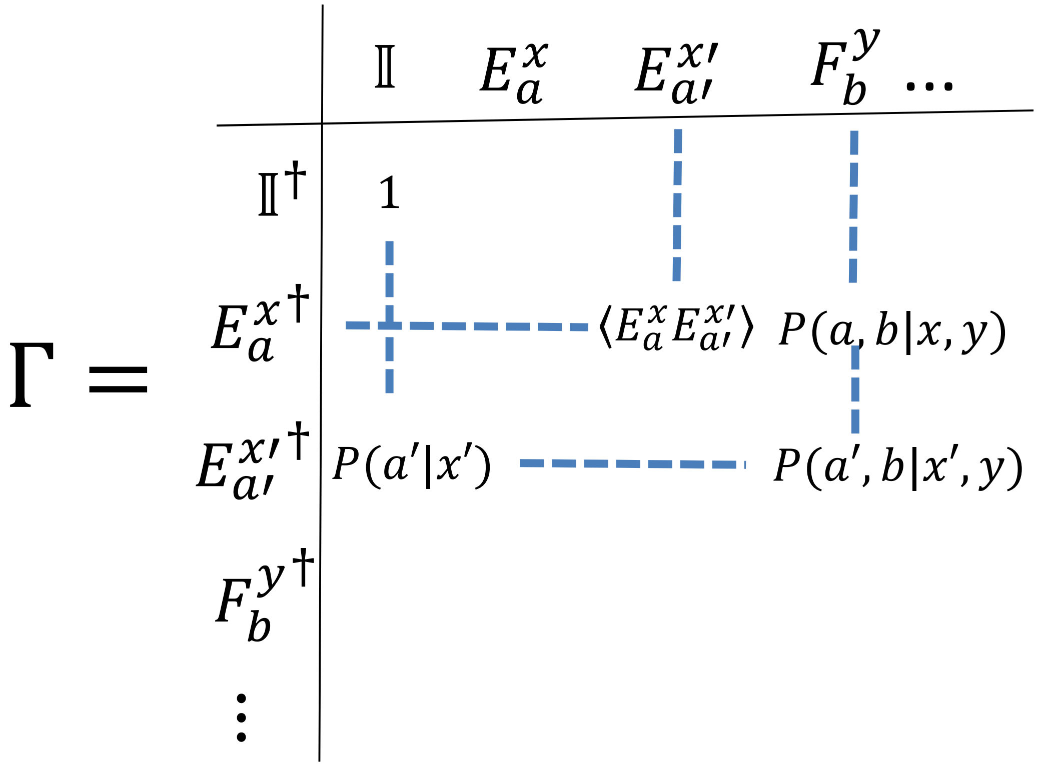
$$\langle E_c^z E_a^x \rangle$$

$$\langle F_c^z F_b^y \rangle$$

$$\langle E_c^z E_a^x F_b^y \rangle$$

$$\vdots$$

Moment matrix



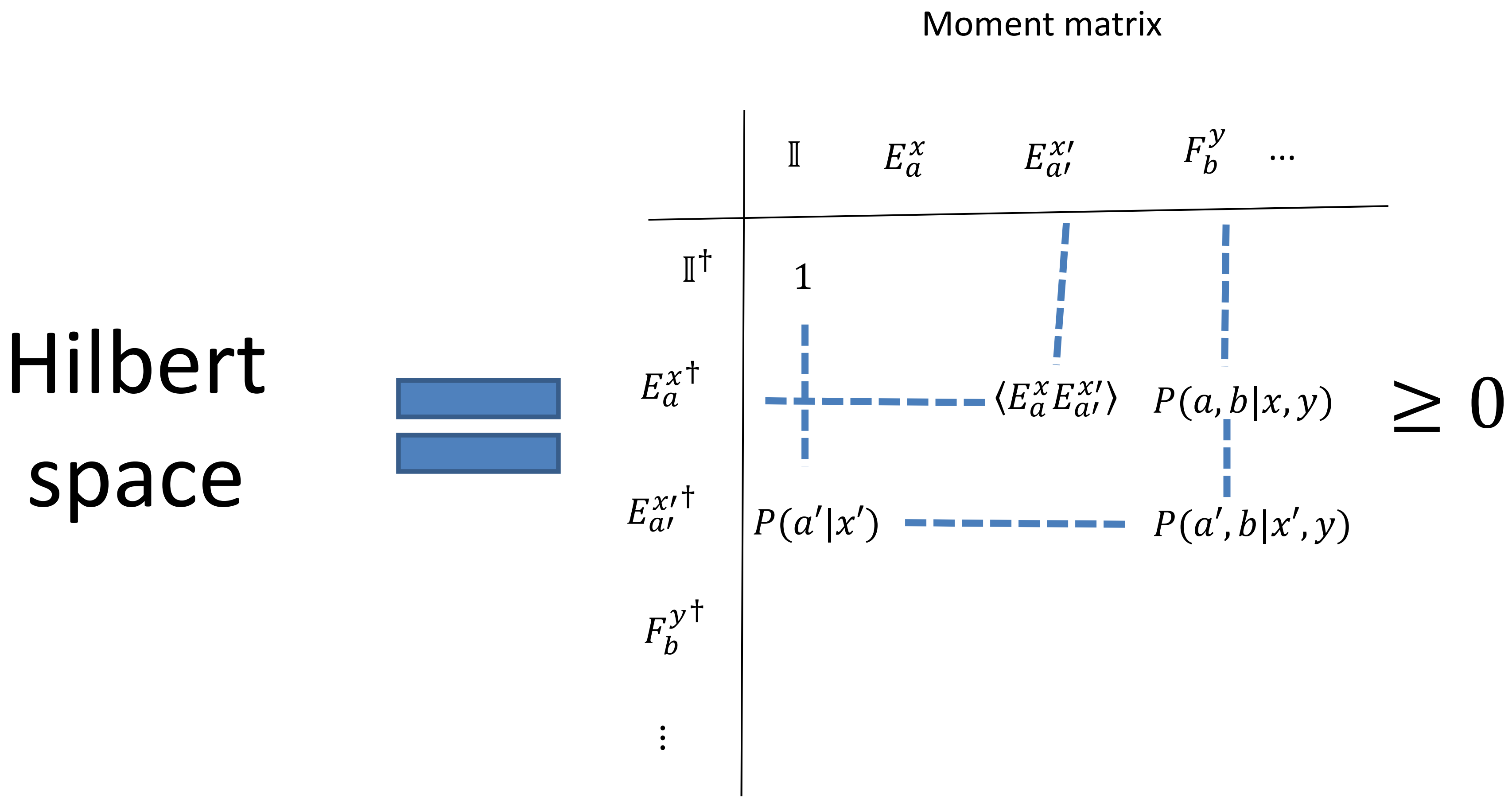
$$\Gamma_{u,v} = \langle u^\dagger v \rangle$$

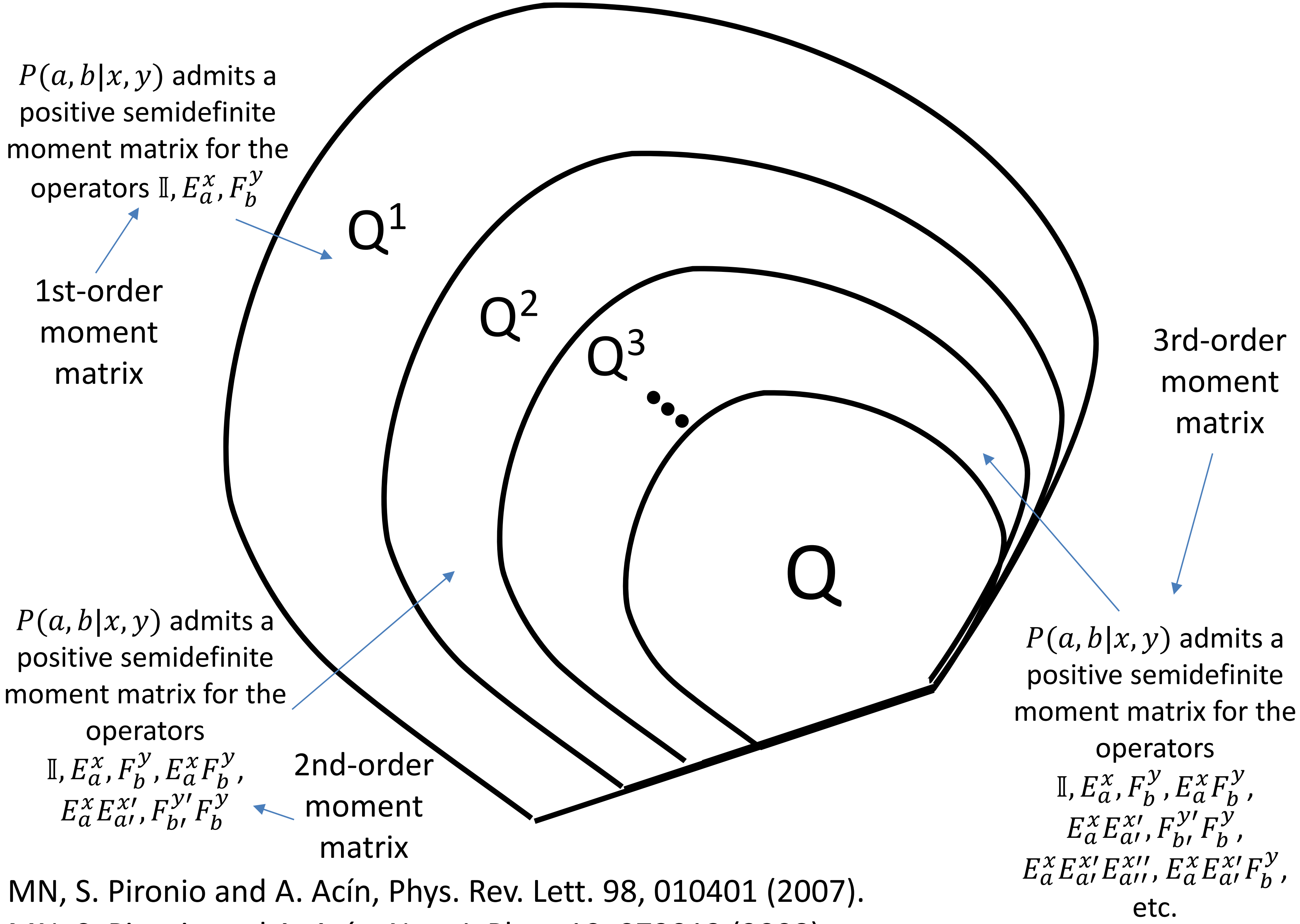
Moment matrix

$\Gamma =$

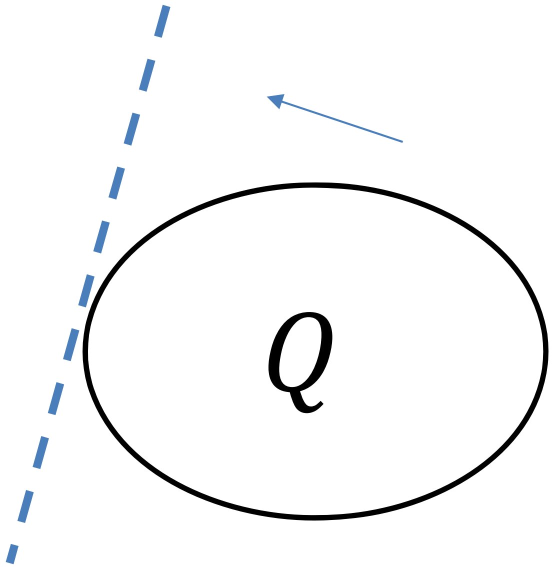
	$\mathbb{I}$	E_a^x	$E_{a'}^{x'}$	F_b^y	$\ldots$
$\mathbb{I}^\dagger$	1				
${E_a^x}^\dagger$			$\langle E_a^x E_{a'}^{x'} \rangle$	$P(a,b\|x,y)$	
${E_{a'}^{x'}}^\dagger$	$P(a'\|x')$			$P(a',b\|x',y)$	
${F_b^y}^\dagger$					
$\vdots$					

≥ 0


Hilbert space
Moment matrix
$\mathbb{I}$
E_a^x
$E_{a'}^{x'}$
F_b^y
$\cdots$
$\mathbb{I}^\dagger$
$E_a^{x\dagger}$
$E_{a'}^{x'\dagger}$
$F_b^{y\dagger}$
$\vdots$
1
$\langle E_a^x E_{a'}^{x'} \rangle$
$P(a,b|x,y)$
$P(a'|x')$
$P(a',b|x',y)$
≥ 0



MN, S. Pironio and A. Acín, Phys. Rev. Lett. 98, 010401 (2007).
MN, S. Pironio and A. Acín, New J. Phys. 10, 073013 (2008).



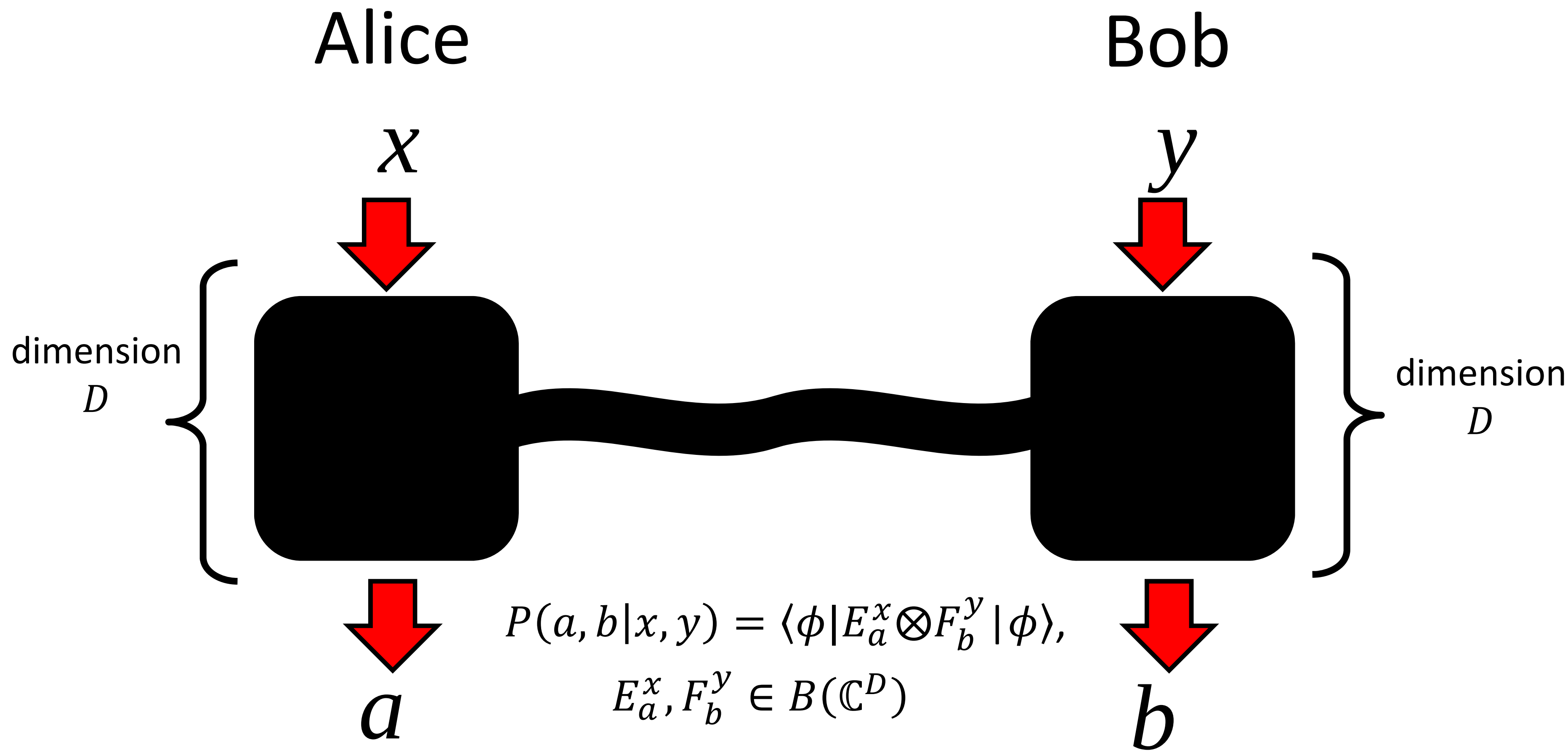
Variational methods

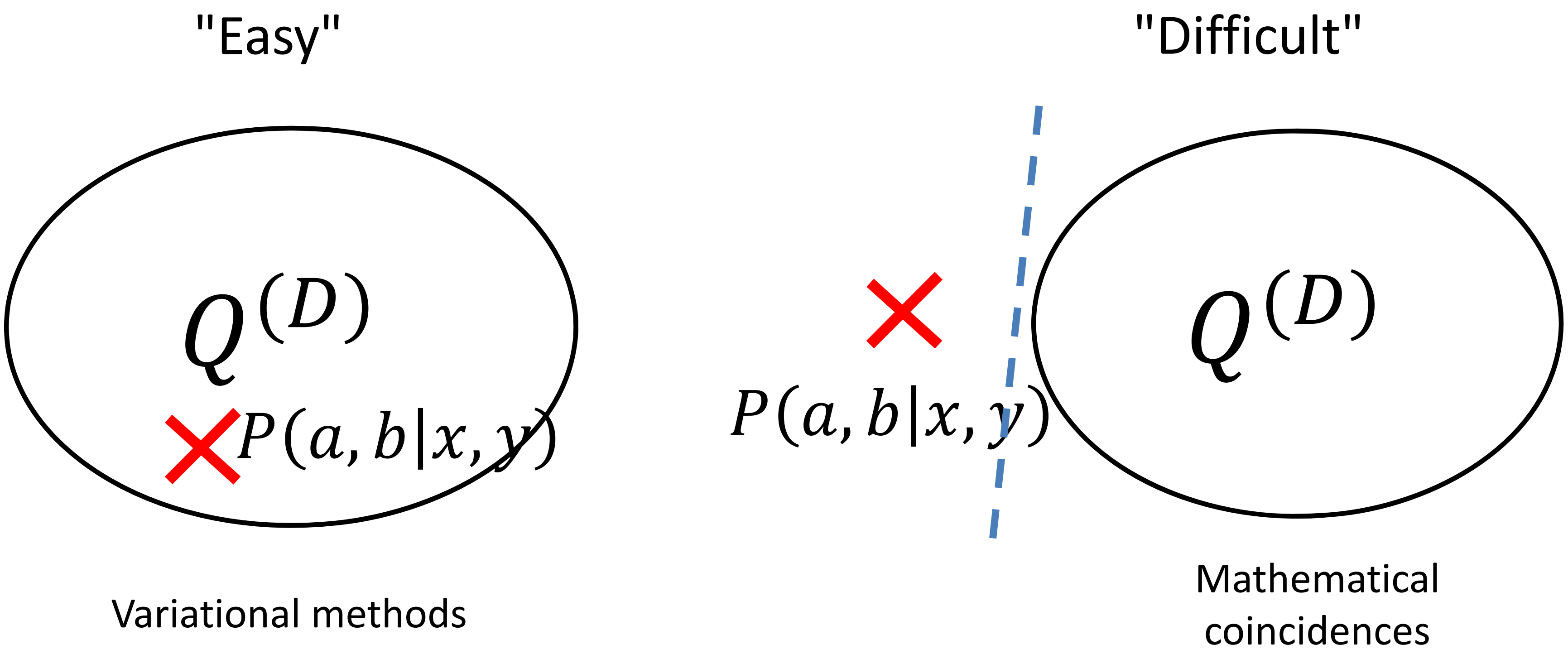
NPA

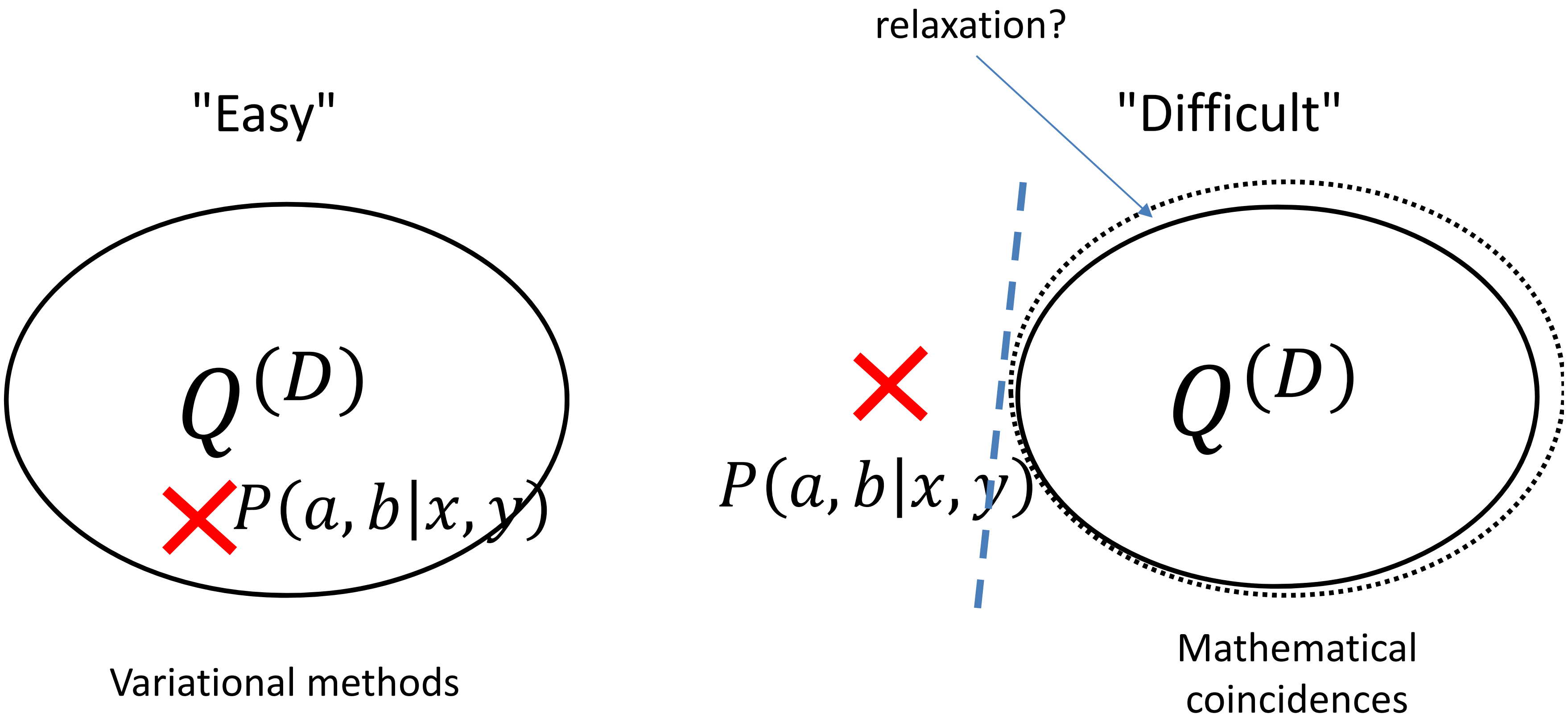
$$L \leq \max \sum c_{abxy} P(a, b|x, y) \leq U$$

Bounding the set of finite dimensional quantum correlations

MN and T. Vértesi, Phys. Rev. Lett. 115, 020501 (2015) .


Alice
Bob
x
y
dimension D
dimension D
a
b
$P(a,b|x,y) = \langle\phi|E_a^x \otimes F_b^y|\phi\rangle,$
$E_a^x, F_b^y \in B(\mathbb{C}^D)$


"Easy"
$Q^{(D)}$
$P(a,b|x,y)$
Variational methods
"Difficult"
$P(a,b|x,y)$
$Q^{(D)}$
Mathematical
coincidences



"Easy"
$Q^{(D)}$
$P(a,b|x,y)$
Variational methods
relaxation?
"Difficult"
$Q^{(D)}$
$P(a,b|x,y)$
Mathematical coincidences

Quantum communication complexity

Alice

Bob

H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).

Quantum communication complexity

$\bar{x} \in \{0,1\}^n$

Alice

$\bar{y} \in \{0,1\}^n$

Bob

H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).

Quantum communication complexity

$\bar{x} \in \{0,1\}^n$

$\bar{y} \in \{0,1\}^n$

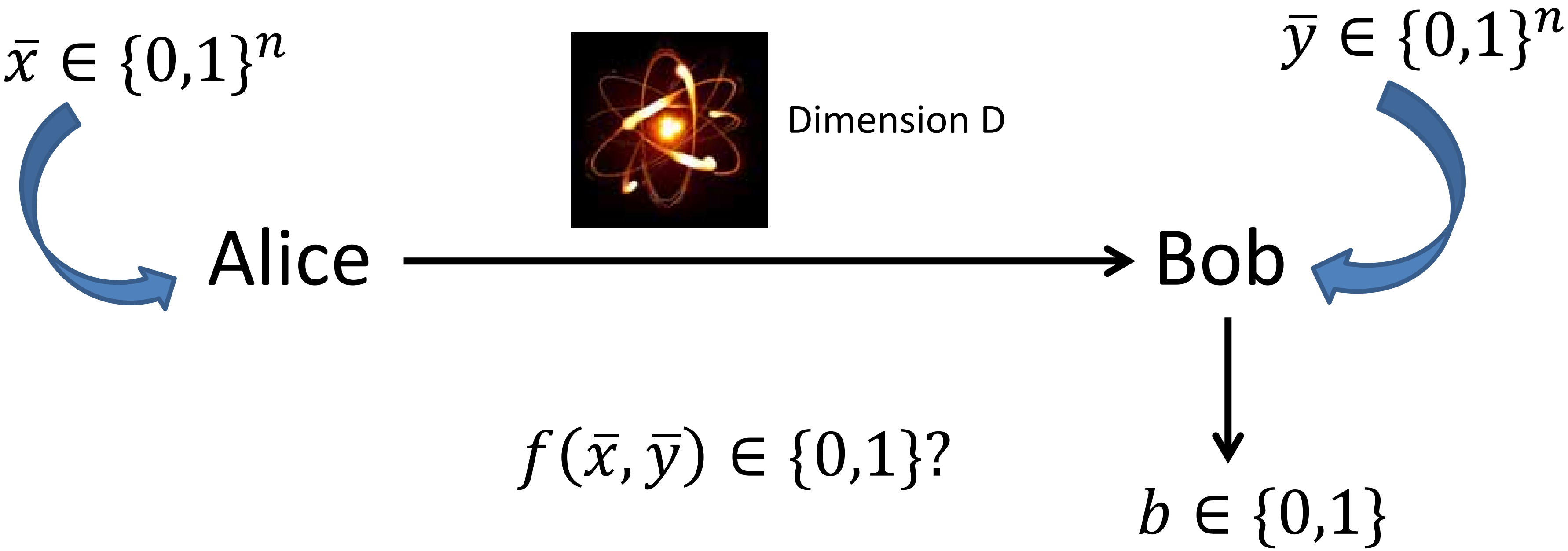
Alice

Bob

$f(\bar{x}, \bar{y}) \in \{0,1\}?$

H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).

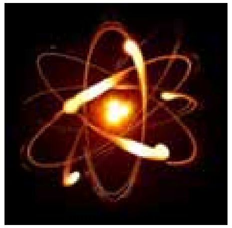
Quantum communication complexity



H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).

Quantum communication complexity

$\bar{x} \in \{0,1\}^n$



Dimension D

$\bar{y} \in \{0,1\}^n$

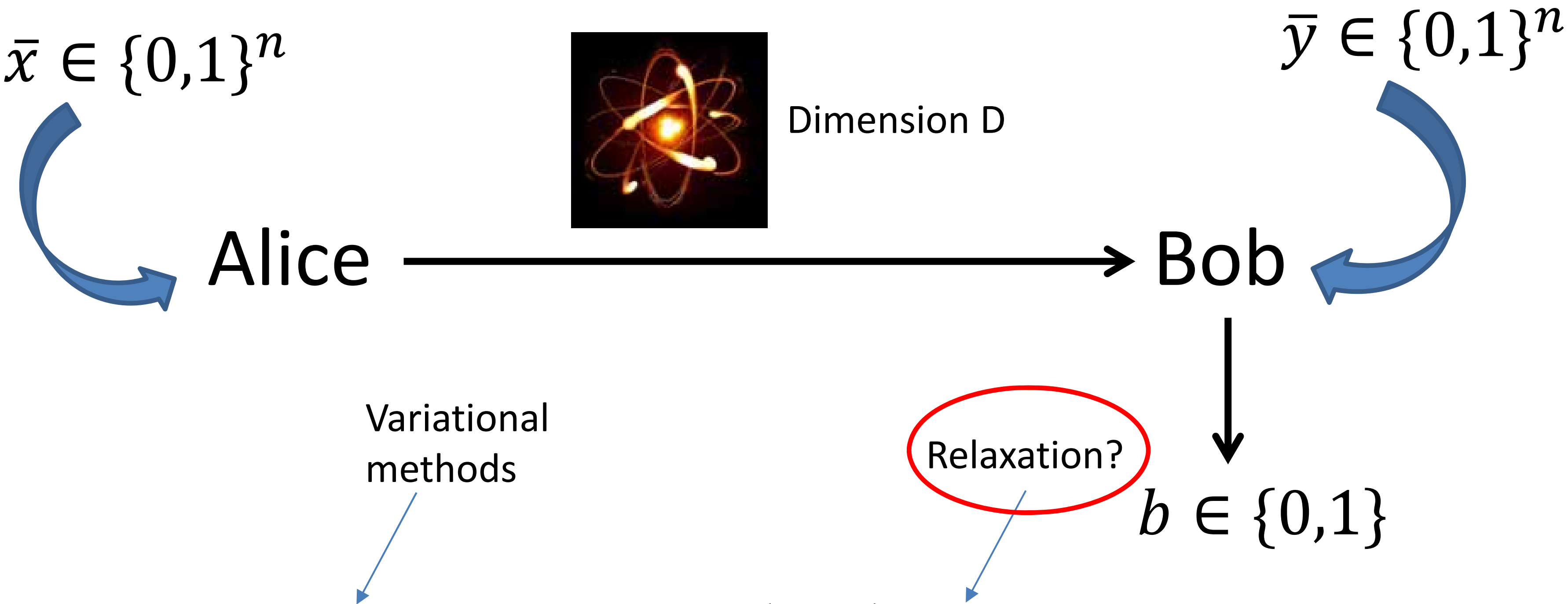
Alice → Bob

$b \in \{0,1\}$

Maximum probability that $b = f(\bar{x}, \bar{y})$?

H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).

Quantum communication complexity



$$L \leq \max \mathrm{P}[b = f(\bar{x}, \bar{y})] \leq U$$

H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).

$$p^* = \frac{1}{2^{2n}} \max \sum_{x,y} tr(F^y_{f(x,y)} \rho_x)$$

s.t. $F^y_b, \rho_x \in B(\mathbb{C}^D),$

$(F^y_b)^2 = F^y_b,$

$\rho_x \geq 0, tr(\rho_x) = 1$

$$\max \left\langle \phi, \sum_{a,b,x,y} C^{x,y}_{a,b} E^x_a \otimes F^y_b \phi \right\rangle$$

s.t. $\phi \in \mathcal{H}^{\otimes 2}, \langle \phi, \phi \rangle = 1$

$E^x_a,\ F^y_b \in \mathrm{B}(\mathcal{H})$, projectors,

$\sum_a E^x_a = \sum_b F^y_b = 1,$

$\dim(\mathcal{H}) \leq D$

Brute force theoretically posible, but impractical

What is a D-dimensional Hilbert space?

MN and T. Vértesi, Phys. Rev. Lett. 115, 020501 (2015) .

$$\psi = \overbrace{(\psi_1, \dots, \psi_D)}^{\text{D=\# of entries}}$$

$$X_i = \overbrace{\begin{pmatrix} x_{11}^i & \cdots & x_{1D}^i \\ \vdots & \ddots & \vdots \\ x_{D1}^i & \cdots & x_{DD}^i \end{pmatrix}}^{\text{D=\# of columns}}$$

Moment matrix

$\Gamma =$

	$\mathbb{I}$	E_a^x	$E_{a'}^{x'}$	F_b^y	$\dots$
$\mathbb{I}^\dagger$	1				
${E_a^x}^\dagger$			$\langle E_a^x E_{a'}^{x'} \rangle$	$P(a,b\|x,y)$	
${E_{a'}^{x'}}^\dagger$	$P(a'\|x')$			$P(a',b\|x',y)$	
${F_b^y}^\dagger$					
$\vdots$					

≥ 0

$$\Gamma_{u,v} = \langle u^\dagger v \rangle$$

How to incorporate dimension constraints?

Matrix polynomial identities

$$F(X) = 0, \forall\, X_1, \dots, X_n \in B\left(\mathbb{C}^d\right), d \leq D$$

Matrix polynomial identities

$$F(X) = 0, \forall\, X_1, \dots, X_n \in B\left(\mathbb{C}^d\right), d \leq D$$

E.g.: $\left\{ D = 1 \longrightarrow [\, X_1, X_2 \,] \right.$

Matrix polynomial identities

$$F(X) = 0, \forall\, X_1, \dots, X_n \in B\left(\mathbb{C}^d\right), d \leq D$$

E.g.:

$$\begin{cases} D = 1 \longrightarrow [\, X_1, X_2\,] \\ D = 2 \longrightarrow [[\, X_1, X_2\,]^2, X_3\,] \end{cases}$$

Matrix polynomial identities

$$F(X) = 0, \forall\, X_1, \dots, X_n \in B\left(\mathbb{C}^d\right), d \leq D$$

E.g.:

$$\begin{cases} D = 1 \longrightarrow [\, X_1, X_2\,] \\ D = 2 \longrightarrow [[\, X_1, X_2\,]^2, X_3\,] \end{cases}$$

Central polynomial (commutes with everything)

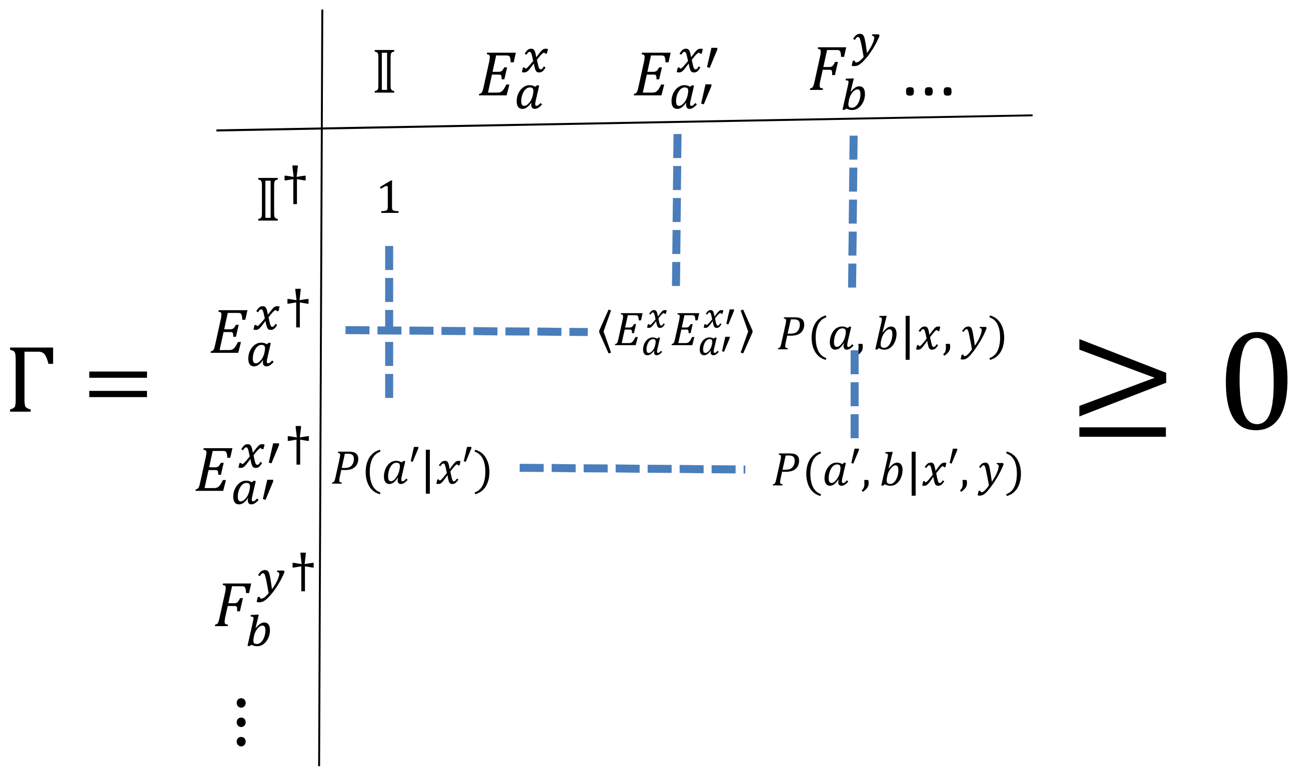
Standard Identity

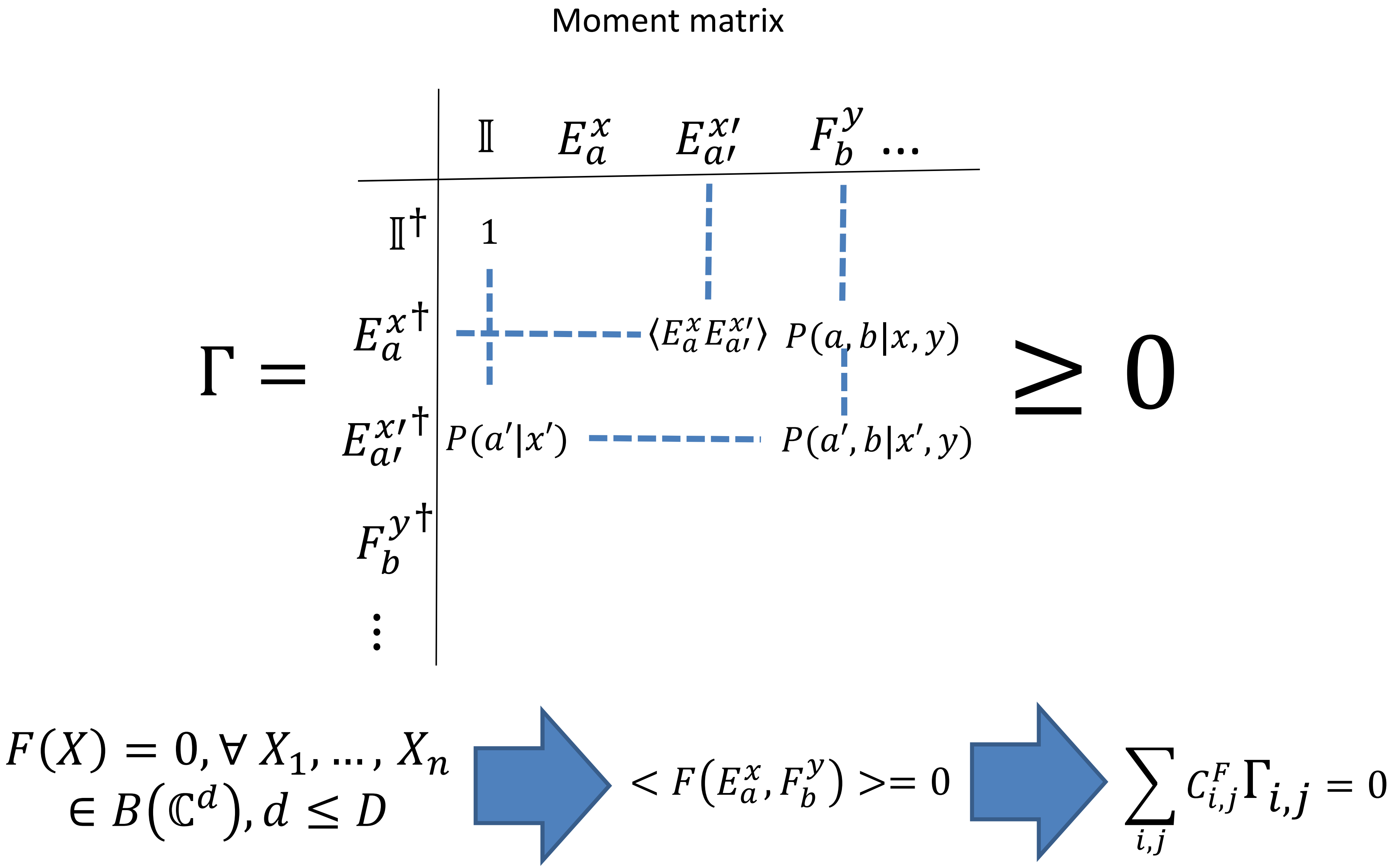
$X_1, \ldots, X_{2D}, D \times D$ matrices

$$F_D(X) = \sum_{\pi \in S_{2D}} sgn(\pi) X_{\pi(1)} \ldots X_{\pi(2D)} = 0$$

C. Procesi. Rings with polynomial identities. Marcel Dekker, New York, 1973

Moment matrix





Moment matrix
$\Gamma =$
$\mathbb{I}$ E_a^x $E_{a'}^{x'}$ F_b^y …
$\mathbb{I}^\dagger$
1
${E_a^x}^\dagger$
$\langle E_a^x E_{a'}^{x'} \rangle$
$P(a,b|x,y)$
${E_{a'}^{x'}}^\dagger$
$P(a'|x')$
$P(a',b|x',y)$
${F_b^y}^\dagger$
⋮
≥ 0
$F(X) = 0, \forall\, X_1, \dots, X_n \in B(\mathbb{C}^d), d \leq D$
$< F(E_a^x, F_b^y) >= 0$
$\sum_{i,j} c_{i,j}^F \Gamma_{i,j} = 0$

Moment matrix

$\Gamma =$

	$\mathbb{I}$	E_a^x	$E_{a'}^{x'}$	F_b^y	$\dots$
$\mathbb{I}^\dagger$	1				
${E_a^x}^\dagger$			$\langle E_a^x E_{a'}^{x'} \rangle$	$P(a,b\|x,y)$	
${E_{a'}^{x'}}^\dagger$	$P(a'\|x')$			$P(a',b\|x',y)$	
${F_b^y}^\dagger$					
$\vdots$					

≥ 0

Moment matrices of D-level quantum systems are subject to non-trivial linear constraints!!!

D-dimensional Hilbert space

=

$$\Gamma \geq 0$$

$$\Gamma \in \mathcal{S}^D$$

D-dimensional
Hilbert space

=

$\Gamma \geq 0$

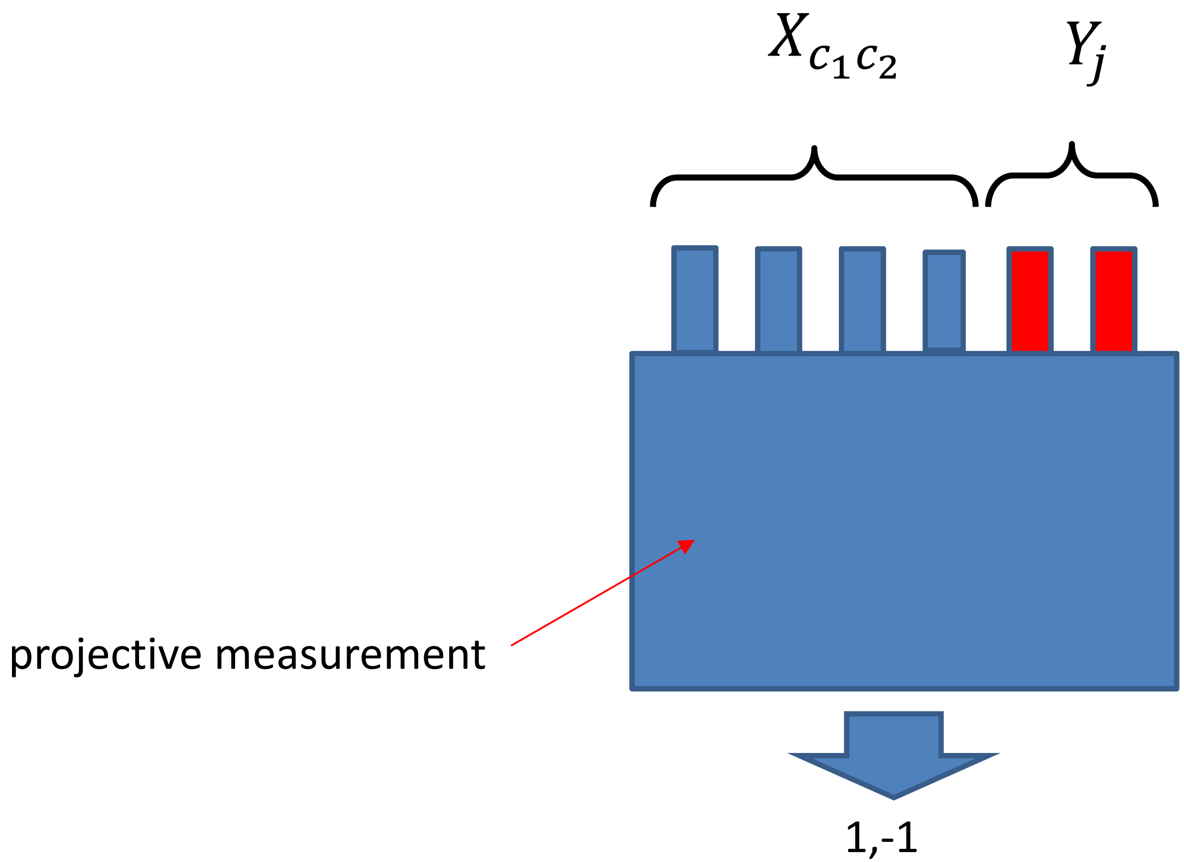
$\Gamma \in \mathcal{S}^D$?

Toy problem

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \langle X_{c_1 c_2} Y_j X_{c_1 c_2} \rangle$$

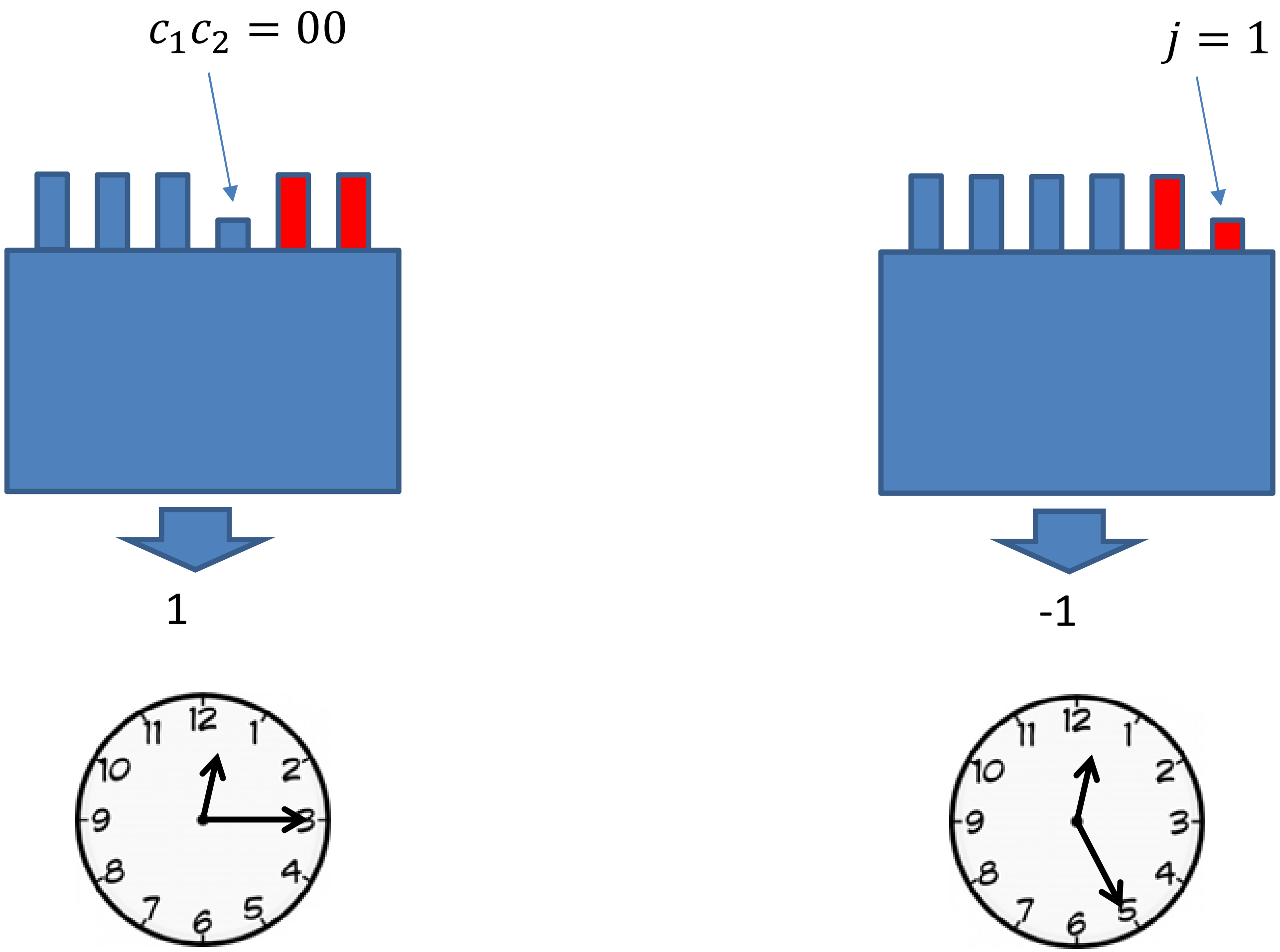
$$\text{s.t.} \quad X_{c_1 c_2}^2 = Y_j^2 = 1$$

Temporal quantum correlations



C. Budroni, T. Moroder, M. Kleinmann, O. Gühne, Phys. Rev. Lett. 111, 020403 (2013)

Temporal quantum correlations



C. Budroni, T. Moroder, M. Kleinmann, O. Gühne, Phys. Rev. Lett. 111, 020403 (2013)

Toy problem

$$p^* = \max\langle\phi| \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} X_{c_1c_2} Y_j X_{c_1c_2} |\phi\rangle$$

s.t. $X_{c_1c_2}^2 = Y_j^2 = 1,$

$\langle\phi|\phi\rangle = 1$

Toy problem

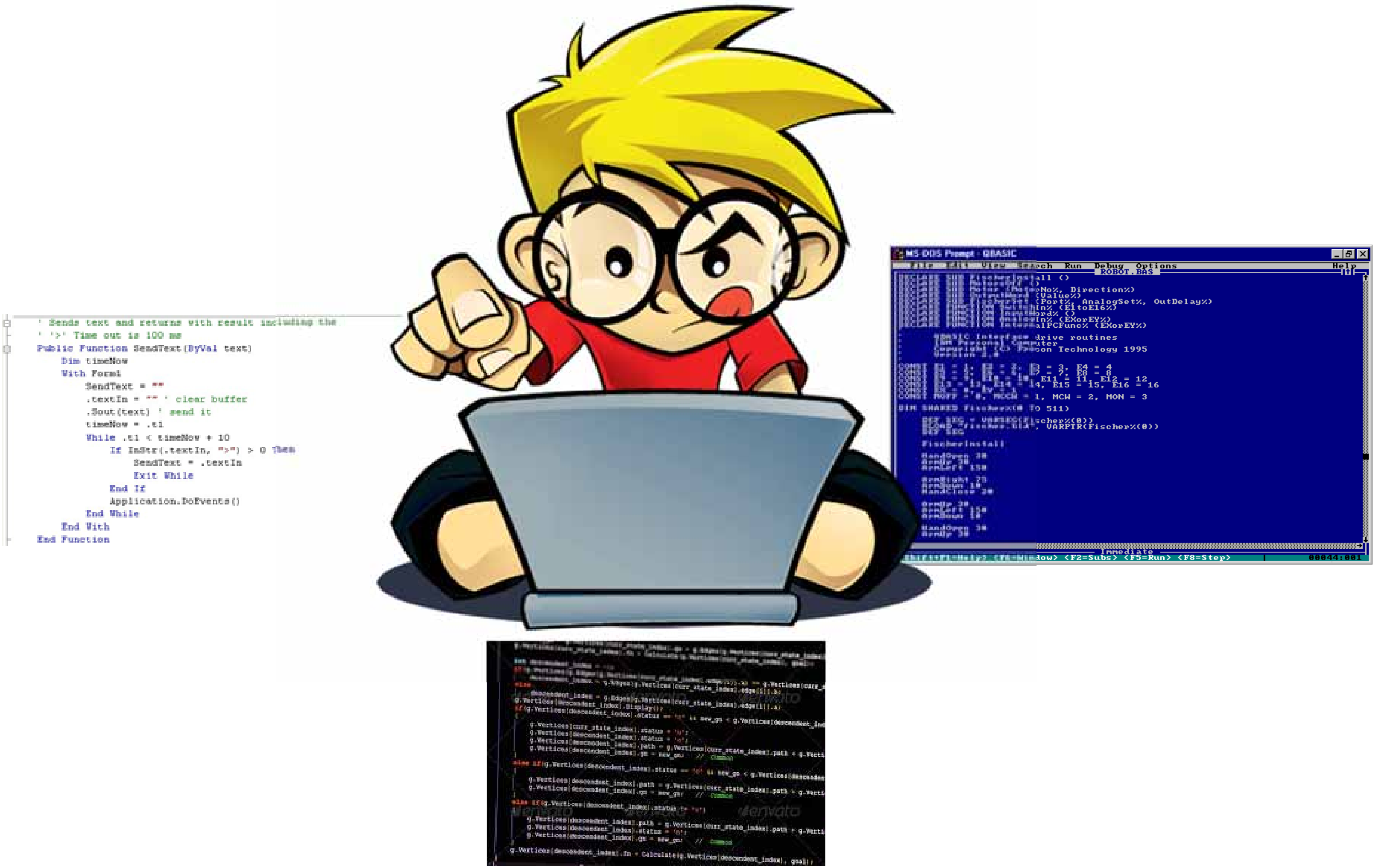
$$p^* = \max\langle\phi|\sum_{j=1,2}\sum_{c_1,c_2=0,1}(-1)^{c_j}X_{c_1c_2}Y_jX_{c_1c_2}|\phi\rangle$$

$$\text{s.t.}\quad X_{c_1c_2}^2 = Y_j^2 = 1,$$

$$\langle\phi|\phi\rangle = 1$$

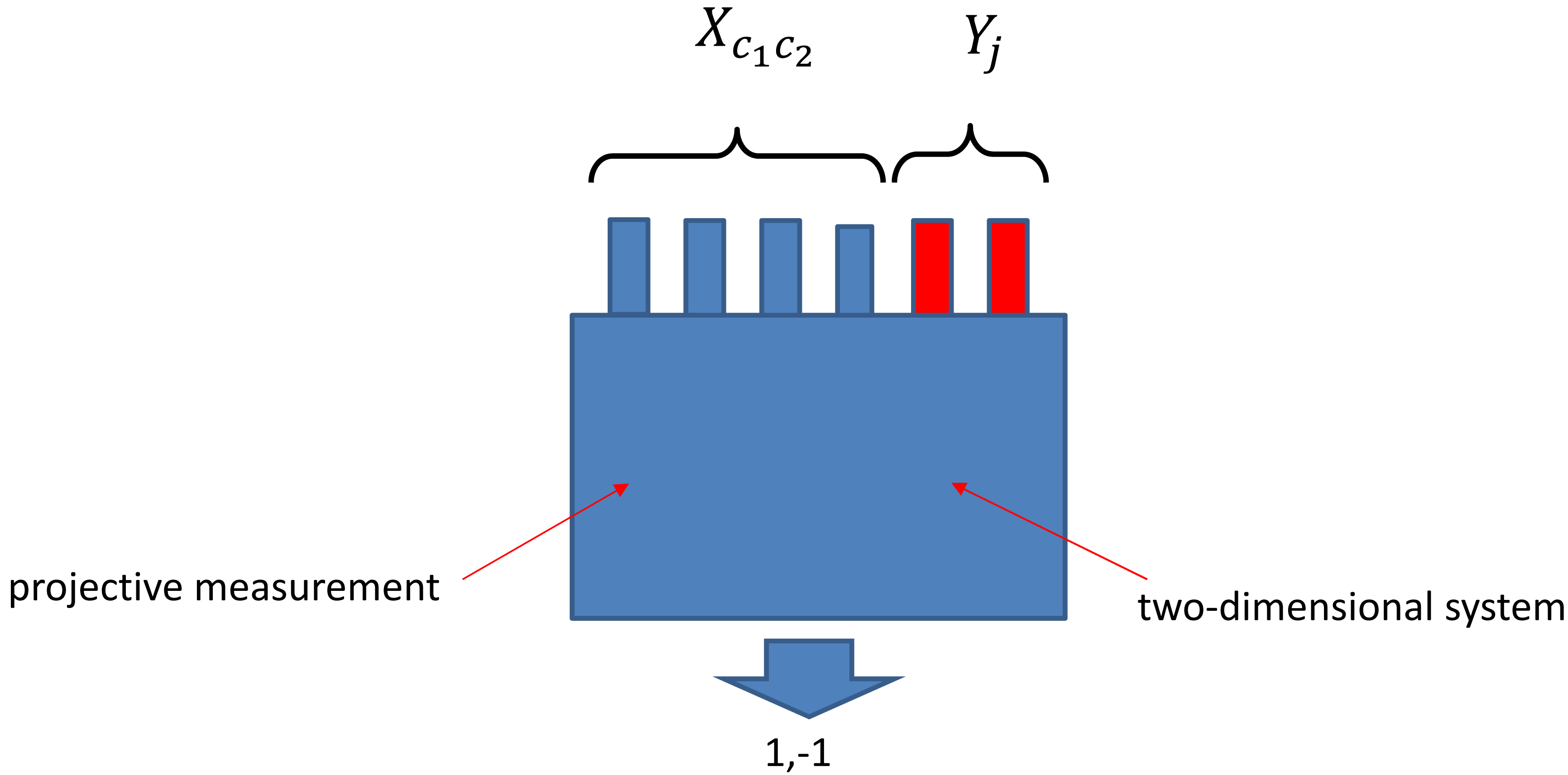
In the dimension-free case, this kind of problems can be solved with a single SDP

S. Pironio, M. Navascués and A. Acín, SIAM J. Optim. 20, 5, 2157-2180 (2010).
C. Budroni, T. Moroder, M. Kleinmann, O. Gühne, Phys. Rev. Lett. 111, 020403 (2013)


' Sends text and returns with result including the
' '>' Time out is 100 ms
Public Function SendText(ByVal text)
Dim timeNow
With Form1
SendText = ""
.textIn = "" ' clear buffer
.Sout(text) ' send it
timeNow = .t1
While .t1 < timeNow + 10
If InStr(.textIn, ">") > 0 Then
SendText = .textIn
Exit While
End If
Application.DoEvents()
End While
End With
End Function
MS DOS Prompt - QBASIC
File Edit View Search Run Debug Options Help
ROBOT.BAS
DIM SHARED FischerX(0 To 511)
FischerInstall
Immediate

$$p^* = 8$$

Temporal quantum correlations



C. Budroni, T. Moroder, M. Kleinmann, O. Gühne, Phys. Rev. Lett. 111, 020403 (2013)

Toy problem

$$p^* = \max \langle \phi | \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} X_{c_1 c_2} Y_j X_{c_1 c_2} | \phi \rangle$$

s.t. $X_{c_1 c_2}^2 = Y_j^2 = 1,$

$\langle \phi | \phi \rangle = 1,$

$|\phi\rangle \in \mathbb{C}^2, X_{c_1 c_2}, Y_j \in B(\mathbb{C}^2)$

Solving the toy problem (I): divide the problem into classes

$$X^2 = 1$$

- $X = 1 \Rightarrow rank(X+1) = 2$ — Trivial
- $X = -1 \Rightarrow rank(X+1) = 0$ — Trivial
- $X = 2\frac{|u\rangle\langle u|}{\langle u|u\rangle} - 1 \Rightarrow rank(X+1) = 1$ — non-trivial

A random instance can be generated easily

Solving the toy problem (I): divide the problem into classes

729 classes

Each class labeled by a vector $\vec{r} \in \{0,1,2\}^6$

$$rank(X_{c_1 c_2} + 1) = r_{c_1 c_2}$$

$$rank(Y_j + 1) = r_j$$

Classed toy problem

$$p^*_{\vec{r}} = \max\langle\phi| \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} X_{c_1c_2} Y_j X_{c_1c_2} |\phi\rangle$$

s.t. $X_{c_1c_2}^2 = Y_j^2 = 1,$

$\langle\phi|\phi\rangle = 1,$

$|\phi\rangle \in \mathbb{C}^2, X_{c_1c_2}, Y_j \in B(\mathbb{C}^2),$

$rank\left(X_{c_1c_2} + 1\right) = r_{c_1c_2},$

$rank\left(Y_j + 1\right) = r_j$

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t.} \quad \Gamma_{1,1} = 1$$

$$\Gamma \geq 0,$$

$$\Gamma \in S_{D,\vec{r}}$$

Solving the toy problem (II): Identify $S_{D,\vec{r}}$

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t.} \quad \Gamma_{1,1} = 1$$

$$\Gamma \geq 0,$$

$$\Gamma \in S_{D,\vec{r}}$$

High level description

$$i = 1$$

High level description

$i = 1$

Generate random dichotomic operators $X^i_{c_1 c_2}, Y^i_j \in \mathrm{B}(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$

High level description

$$i = 1$$

Generate random dichotomic operators $X^i_{c_1 c_2}, Y^i_j \in \mathrm{B}(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$

Build moment matrix Γ^j

$$\Gamma^i_{\mathbb{I},\mathbb{I}} = \langle \phi^i | \mathbb{I}_d | \phi^i \rangle = 1,$$

$$\Gamma^i_{X_{c_1 c_2}, Y_j} = \langle \phi^i | X^i_{c_1 c_2}, Y^i_j | \phi^i \rangle,$$

$$\vdots$$

High level description

$i = 1$

Generate random dichotomic operators $X^i_{c_1 c_2}, Y^i_j \in \mathrm{B}(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$

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$\vdots$

$i = i + 1$

Repeat until the moment matrix $\Gamma^{(N+1)}$ is a linear combination of $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$

High level description

$i = 1$

Generate random dichotomic operators $X^i_{c_1 c_2}, Y^i_j \in \mathrm{B}(\mathbb{C}^2)$ (with app. ranks) and vector $\phi^i \in \mathbb{C}^2$

Build moment matrix Γ^j

$\Gamma^i_{\mathbb{I},\mathbb{I}} = \langle \phi^i | \mathbb{I}_d | \phi^i \rangle = 1,$

$\Gamma^i_{X_{c_1 c_2}, Y_j} = \langle \phi^i | X^i_{c_1 c_2}, Y^i_j | \phi^i \rangle,$

$\vdots$

$i = i + 1$

At that point, *any* feasible moment matrix Γ must be a linear combination of $\Gamma^{(1)}, \dots, \Gamma^{(N)}$

$\mathcal{S}^D_r = span(\Gamma^{(1)}, \dots, \Gamma^{(N)})$

SDP relaxation

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t.} \quad \Gamma_{1,1} = 1$$

$$\Gamma \geq 0,$$

$$\Gamma = \sum_{s=1}^{N} c_s \Gamma^s$$

SDP relaxation

$$\max \sum_{j=1,2} \sum_{c_1,c_2=0,1} (-1)^{c_j} \Gamma_{X_{c_1 c_2} Y_j X_{c_1 c_2}}$$

$$\text{s.t.} \quad \Gamma_{1,1} = 1$$

$$\Gamma \geq 0,$$

Lack of strict feasibility, i.e., there are no $\Gamma > 0$

Problems with SDP solvers

$$\Gamma = \sum_{s=1}^{N} c_s \Gamma^s$$

$|c_s| \gg 1$

Implementation tips

1

$$\frac{1}{N}\sum_{s=1}^{N}\Gamma^{s} = \Gamma_{mix} \geq 0 \quad \Rightarrow \quad \Gamma \geq 0 \Rightarrow \bar{V}^{\dagger}\Gamma\bar{V} \geq 0$$

Isometry $\bar{V}$ to the support of Γ_{mix}

Why?

$$\bar{V}^{\dagger}\Gamma_{mix}\bar{V} > 0$$

Strict positivity

Implementation tips

2

Use (modified) Gram-Schmidt as you generate the matrices Γ^1, Γ^2 ...

Sequence of orthogonal matrices $\widetilde{\Gamma^1}, \widetilde{\Gamma^2}$ ...

at s=N+1, $\widetilde{\Gamma^s} = 0$, up to computer precision

$$\Gamma = \sum_{s=1}^{N} c_s \Gamma^s \quad \Rightarrow \quad \Gamma = \sum_{s=1}^{N} \widetilde{c_s} \frac{\widetilde{\Gamma}_s}{\sqrt{tr(\widetilde{\Gamma}_s{}^2)}}$$

SDP relaxation

$$\min \sum_{a,b,x,y} C_{a,b}^{x,y} \Gamma_{E_a^x, F_b^y}$$

$$\text{s.t.} \quad \Gamma_{1,1} = 1,$$

$$\Gamma \geq 0,$$

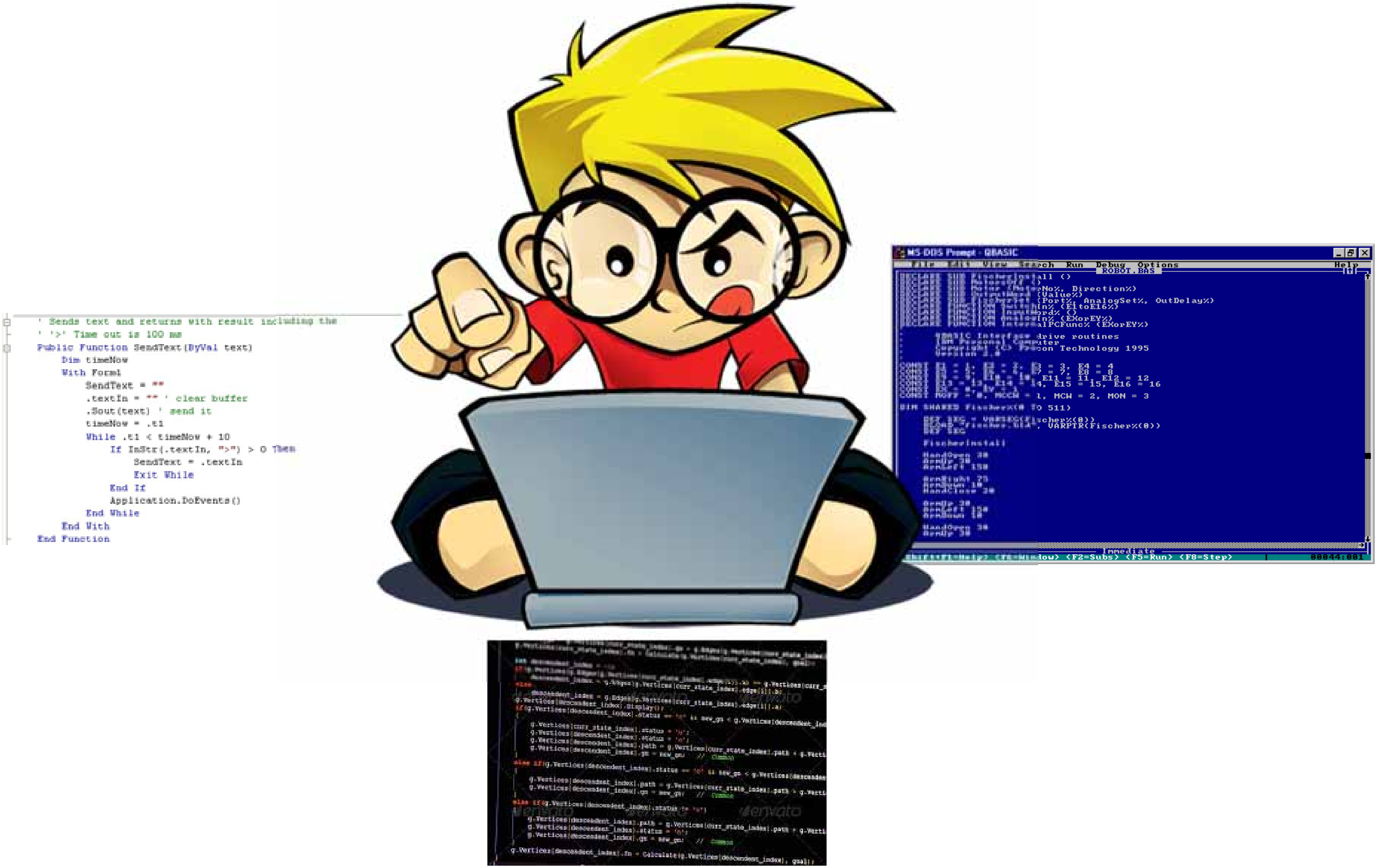
$$\Gamma = \sum_{s=1}^{N} c_s \Gamma^s$$

$$\min \sum_{a,b,x,y} C_{a,b}^{x,y} \Gamma_{E_a^x, F_b^y}$$

$$\text{s.t.} \quad \Gamma_{1,1} = 1,$$

$$\bar{V}^\dagger \Gamma \bar{V} \geq 0,$$

$$\Gamma = \sum_{s=1}^{N} \widetilde{c_s} \tilde{\Gamma}^s$$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End With
End Function
MS-DOS Prompt - QBASIC
File Edit View Search Run Debug Options Help
ROBOT.BAS
DIM SHARED Fischerx(0 To 511)
Immediate

Free dimensionality

$$p^* = 8$$

Second relaxation, D=2

$$p^2 = 5.656854\ldots$$

Generalization

NPO problem with dimension constraints

$$p^* = \min_{\mathcal{H}, X, \phi} \langle \phi, p(X)\phi \rangle$$

$$\begin{aligned} \text{s.t.}\quad & X_1, \dots, X_n \in B(\mathcal{H}), \\ & q_i(X) \geq 0, i = 1, \dots, s \\ & \phi \in \mathcal{H} \\ & \langle \phi, \phi \rangle = 1 \\ & \dim(\mathcal{H}) \leq D \end{aligned}$$

MN, A. Feix, M. Araújo and T. Vértesi, Phys. Rev. A 92, 042117 (2015).

$$
\begin{aligned}
p_k^* = \min & \sum_w p_w y_w \\
\text{s.t.} \quad & y_1 = 1, \\
& M_k(y) \succeq 0, \\
& M_{k-\lceil d_i/2 \rceil}(q_i y) \succeq 0, \\
& y = \textstyle\sum_{s=1}^{j-1} c_s y^{(s)}, \; c_s \in \mathbb{C}.
\end{aligned}
$$

$$p_k^* \le p^*$$

$$p_k^* = \min \sum_w p_w y_w$$

$$\text{s.t.} \quad y_1 = 1,$$

$$M_k(y) \geq 0,$$

$$M_{k-\lceil d_i/2 \rceil}(q_i y) \geq 0,$$

$$y = \sum_{s=1}^{j-1} c_s y^{(s)}, \; c_s \in \mathbb{C}.$$

Archimedean condition

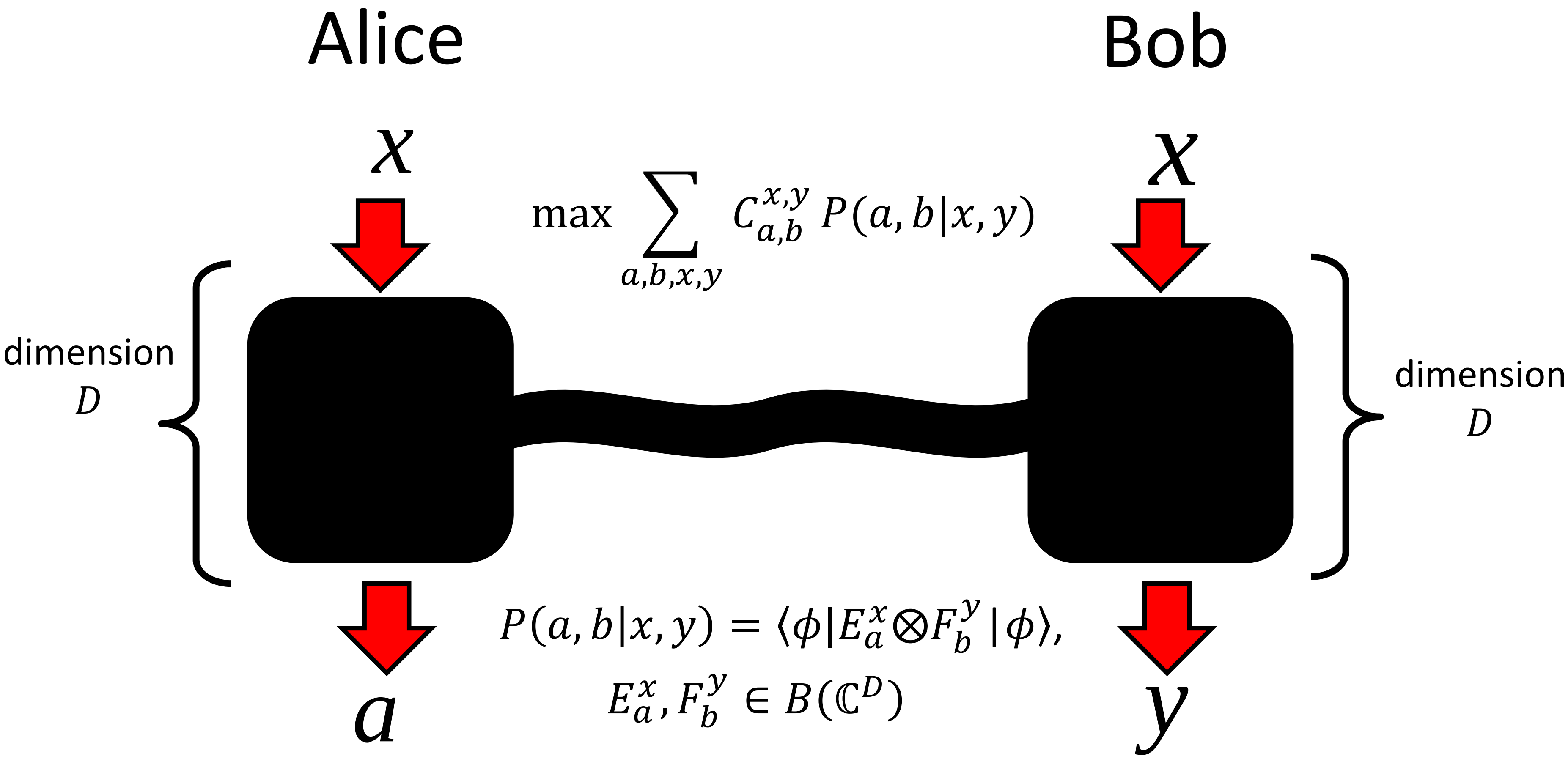
$$C - \sum_r X_r^\dagger X_r + X_r X_r^\dagger = \sum_s f_s f_s^\dagger + \sum_{s,i} g_{s,i} q_i g_{s,i}^\dagger$$

⇩

$$\lim_{k\to\infty} p_k^* = p^*$$

Related hierarchies

SDP hierarchy for quantum non-locality under dimension constraints



$$\max \left\langle \phi, \sum_{a,b,x,y} C_{a,b}^{x,y} E_a^x \otimes F_b^y \phi \right\rangle$$

$$\begin{aligned} \text{s.t.} \quad & \phi \in \mathcal{H}^{\otimes 2}, \langle \phi, \phi \rangle = 1 \\ & E_a^x,\ F_b^y \in \mathrm{B}(\mathcal{H}), \text{projectors}, \\ & \sum_a E_a^x = \sum_b F_b^y = 1, \\ & \dim(\mathcal{H}) \le D \end{aligned}$$

$$\max \left\langle \phi, \sum_{a,b,x,y} C_{a,b}^{x,y} E_a^x \otimes F_b^y \phi \right\rangle$$

s.t. $\phi \in \mathcal{H}^{\otimes 2}, \langle \phi, \phi \rangle = 1$

$E_a^x,\ F_b^y \in \mathrm{B}(\mathcal{H})$, projectors,

$$\sum_a E_a^x = \sum_b F_b^y = 1,$$

$\dim(\mathcal{H}) \leq D$,

$rank(E_a^x) = r_a^x, rank(F_b^y) = t_b^y$

High level description

$$i = 1$$

Generate random projectors

$E_a^{i,x},\ F_b^{i,y} \in \mathrm{B}(\mathbb{C}^D)$ (with app. ranks)

and vector $\phi^i \in \mathbb{C}^{D^2}$

Build moment matrix Γ^i

$$\Gamma^i_{\mathbb{I},\mathbb{I}} = \langle \phi^i | \mathbb{I}_d \otimes \mathbb{I}_d | \phi^i \rangle = 1,$$

$$\Gamma^i_{E_a^x, F_b^y} = \left\langle \phi^i \middle| E_a^{i,x} \otimes F_b^{i,y} \middle| \phi^j \right\rangle,$$

$$\vdots$$

$$i = i + 1$$

Repeat until the moment matrix $\Gamma^{(N+1)}$ is a linear combination of $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$

SDP hierarchy

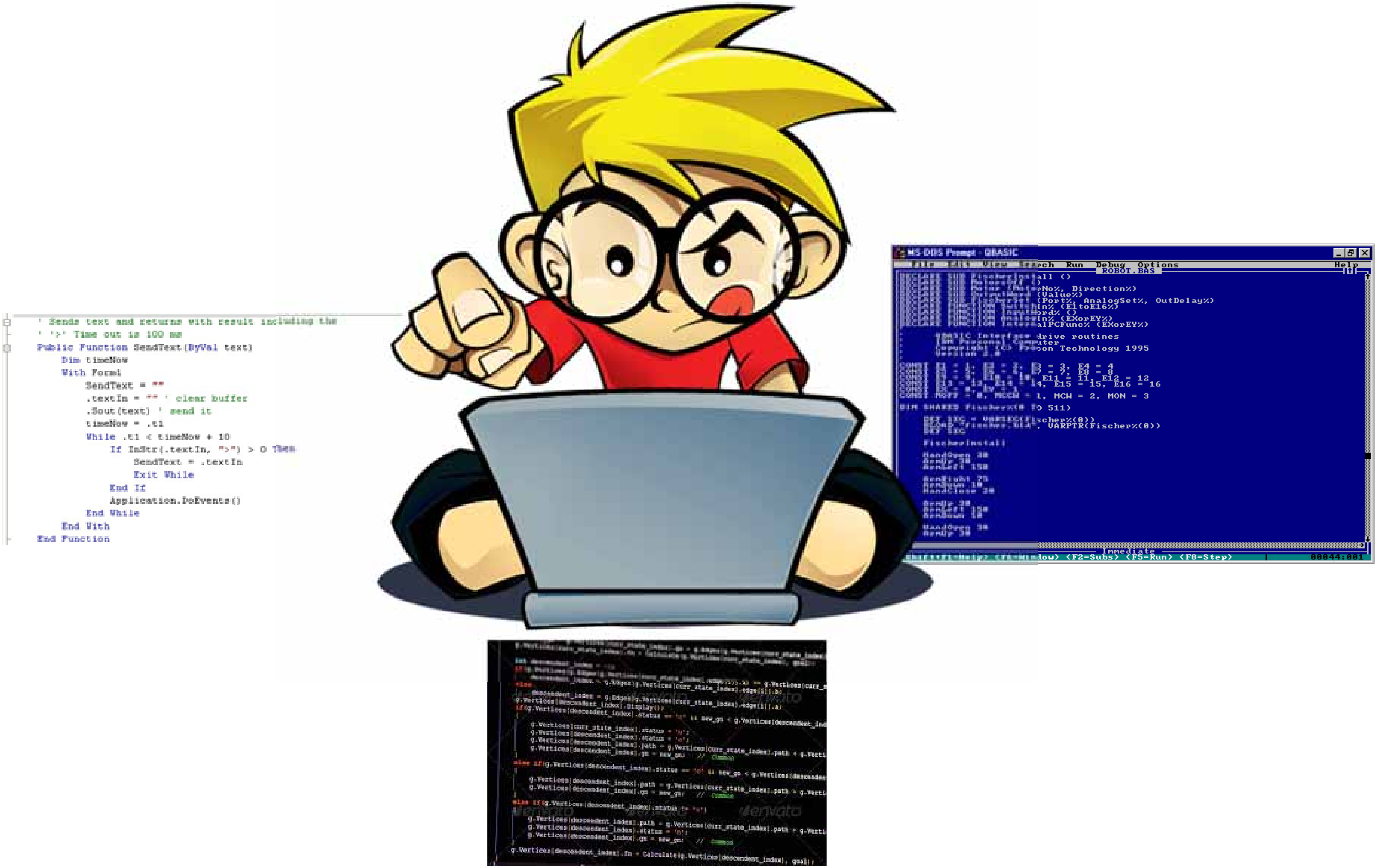
$$\max \sum_{a,b,x,y} C_{a,b}^{x,y} \Gamma_{E_a^x, F_b^y}$$

s.t. $\Gamma_{1,1} = 1$

$$\Gamma \geq 0, \Gamma = \sum_{s=1}^{j} c_s \Gamma^s$$

Convergence is guaranteed

MN, A. Feix, M. Araújo and T. Vértesi, Phys. Rev. A 92, 042117 (2015).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Application.DoEvents()
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MS-DOS Prompt - QBASIC
QBASIC Interface drive routines
Fischer Technology 1995
DIM SHARED Fischer%(0 To 511)
FischerInstall

NPA

Pironio-Bell$\leq$ 0.2532

Pironio-Bell$\leq$ 0.2071

D=2

NPA

$$I_{3322} \leq 0.250875$$

$$I_{3322} \leq 0.25$$

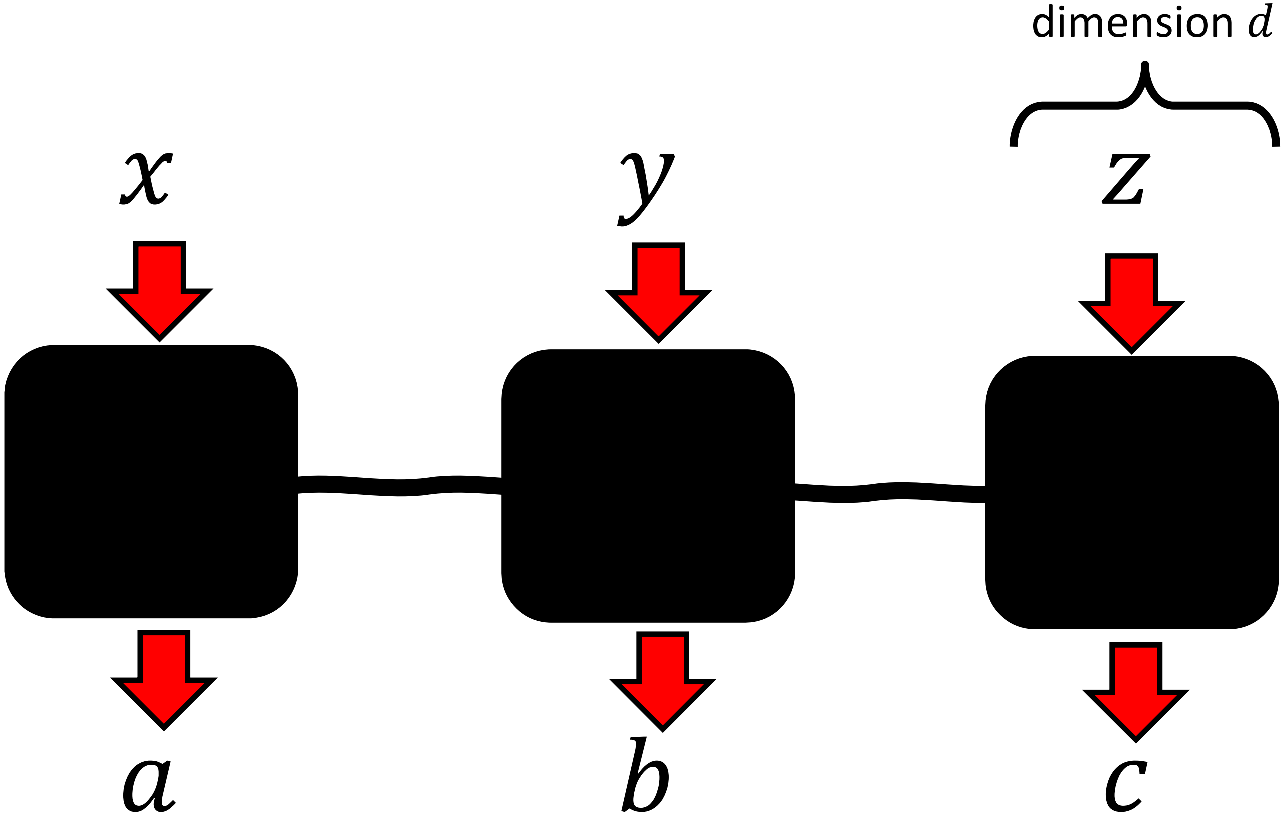
D=2, 3

$$I_{4,4} = E_1^A + E_{1,1} + E_{1,2} + E_{2,1} - E_{2,2} + E_{3,3} + E_{3,4} + E_{4,3} - E_{4,4} \leq 5.9907$$

quantum mechanics

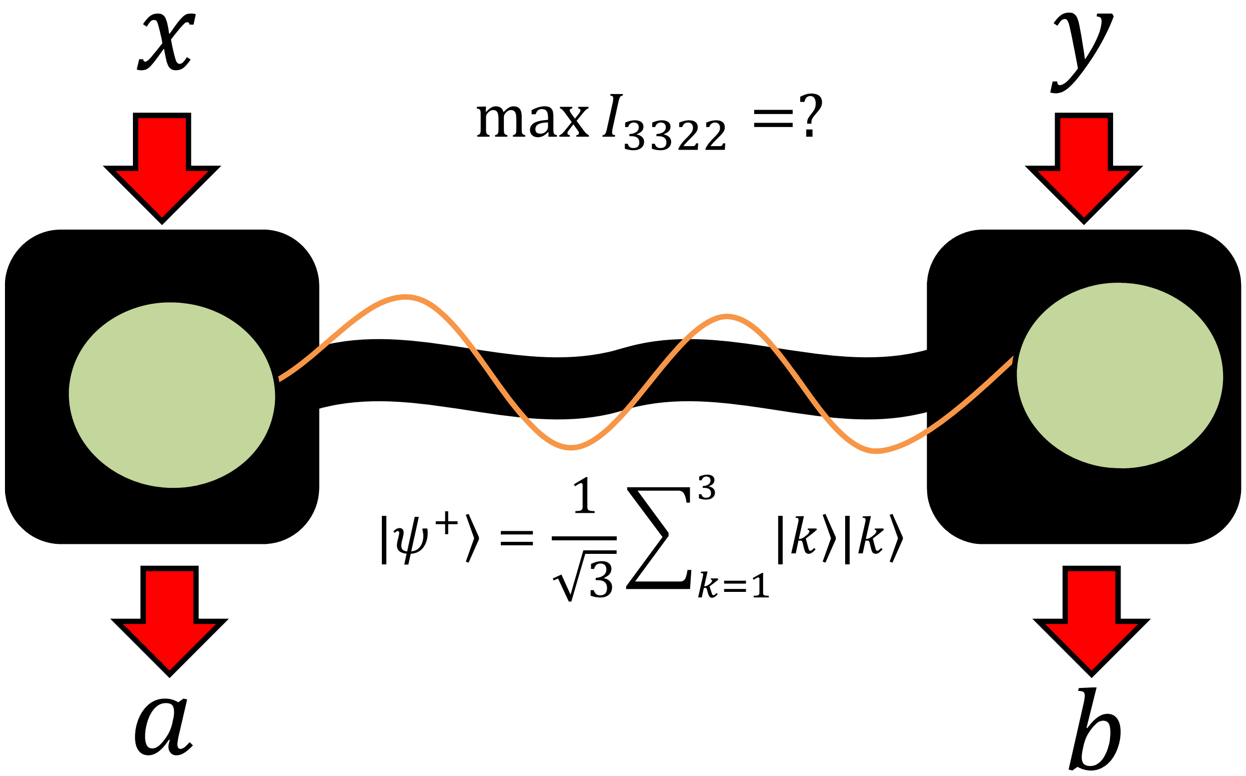
$$I_{4,4} = E_1^A + E_{1,1} + E_{1,2} + E_{2,1} - E_{2,2} + E_{3,3} + E_{3,4} + E_{4,3} - E_{4,4} \leq 5.8310$$

D=2, 3



$$P(a, b, c|x, y, z)$$

MN, A. Feix, M. Araújo and T. Vértesi, Phys. Rev. A 92, 042117 (2015).


x
y
$\max I_{3322} = ?$
$|\psi^{+}\rangle = \frac{1}{\sqrt{3}}\sum_{k=1}^{3}|k\rangle|k\rangle$
a
b

High level description

$i = 1$

Generate random projectors

$E_a^{i,x},\ F_b^{i,y} \in \mathrm{B}(\mathbb{C}^D)$ (with app. ranks)

Build moment matrix Γ^i

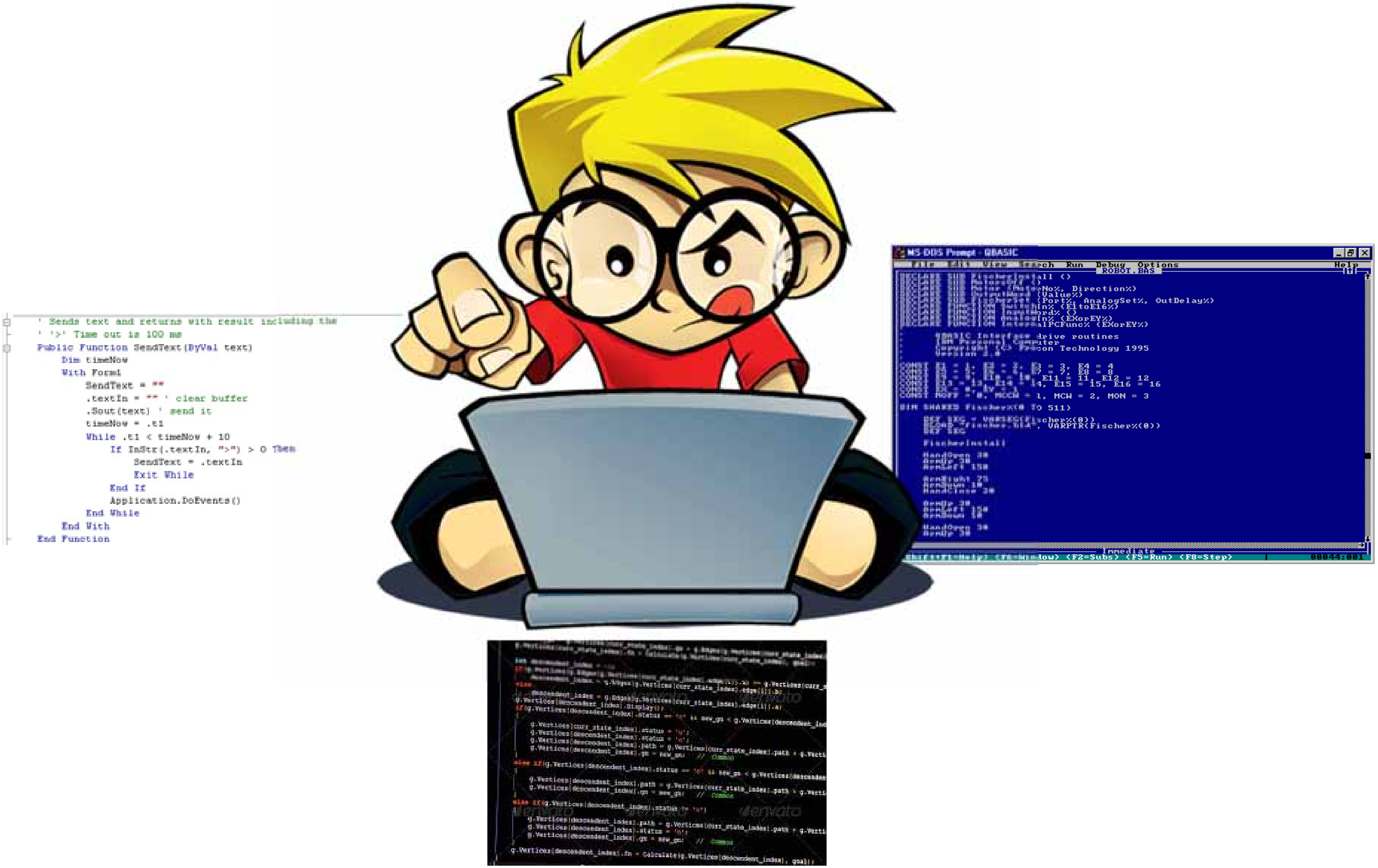
$\Gamma^i_{\mathbb{I},\mathbb{I}} = \langle \psi^+ | \mathbb{I}_d \otimes \mathbb{I}_d | \psi^+ \rangle = 1,$

$\Gamma^i_{E_a^x, F_b^y} = \left\langle \psi^+ \middle| E_a^{i,x} \otimes F_b^{i,y} \middle| \psi^+ \right\rangle,$

⋮

$i = i + 1$

Repeat until the moment matrix $\Gamma^{(N+1)}$ is a linear combination of $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$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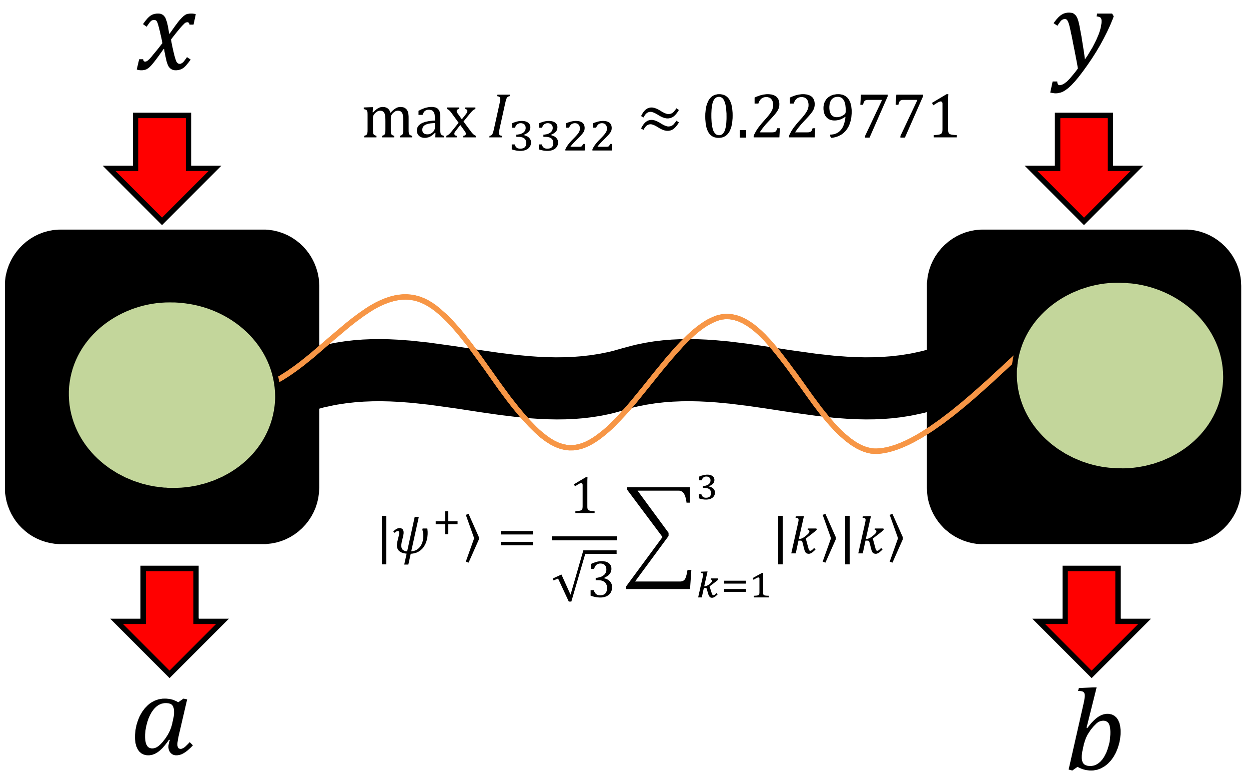
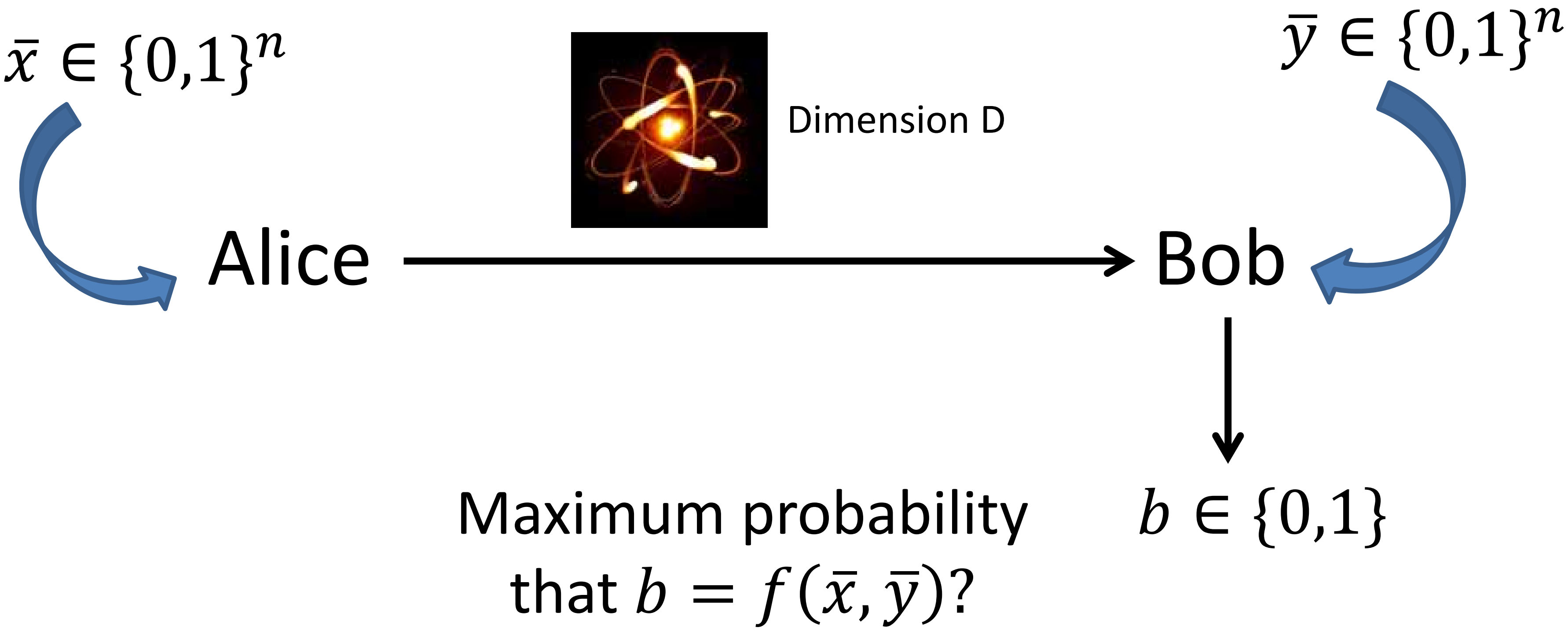


x
y
$\max I_{3322} \approx 0.229771$
$|\psi^+\rangle = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} |k\rangle|k\rangle$
a
b

SDP hierarchy for quantum communication complexity



$$p^* = \frac{1}{2^{2n}} \max \sum_{x,y} tr(F^{y}_{f(x,y)} \rho_x)$$

such that $F^y_b, \rho_x \in B(\mathbb{C}^D)$,

$$(F^y_b)^2 = F^y_b,$$

$$\rho_x \geq 0, tr(\rho_x) = 1$$

High level description

$i = 1$

Generate random projectors $\rho_x^i, F_b^{i,y} \in \mathrm{B}(\mathbb{C}^D)$ (with app. ranks)

Build moment matrix Γ^i

$\Gamma^i_{\mathbb{I},\mathbb{I}} = tr(\mathbb{I}) = D,$

$\Gamma^i_{\rho_x, F_b^y} = tr(\rho_x^i F_b^{i,y}),$

$\vdots$

$i = i + 1$

Repeat until the moment matrix $\Gamma^{(N+1)}$ is a linear combination of $\Gamma^{(1)}, \dots, \Gamma^{(N)}$

SDP hierarchy

$$\max \frac{1}{2^{2n}} \sum_{b,x,y} \Gamma_{\rho_x, F^y_{f(x,y)}}$$

$$\text{s.t.} \quad \Gamma_{1,1} = 1$$

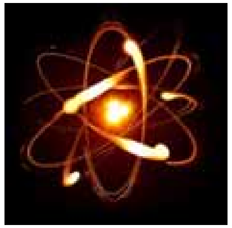
$$\Gamma \geq 0, \Gamma = \sum_{s=1}^{j} c_s \Gamma^s$$

No proof of convergence!!!

MN, A. Feix, M. Araújo and T. Vértesi, Phys. Rev. A 92, 042117 (2015).

Random Access Codes (RAC)

Alice



Dimension D

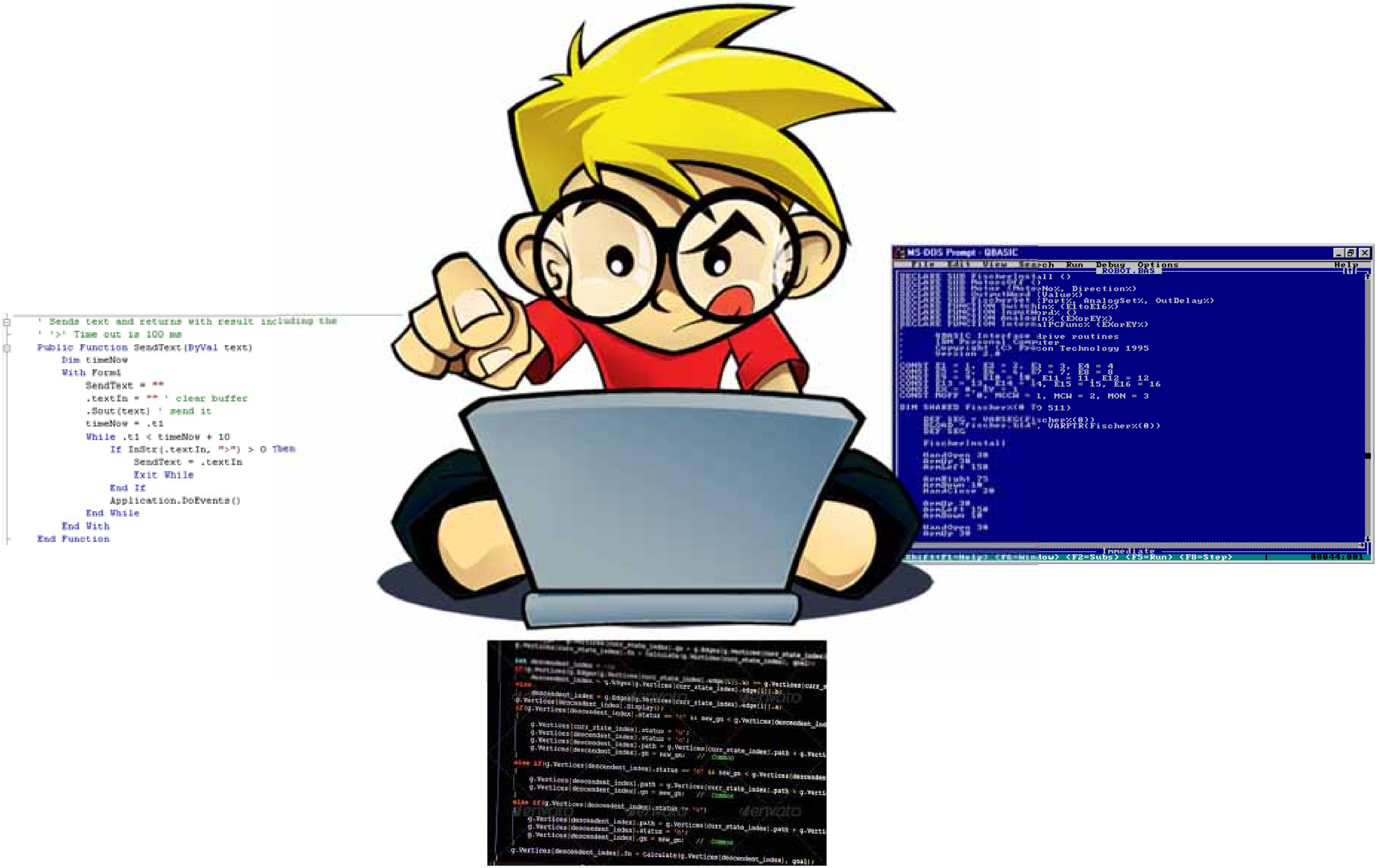
Bob

$\bar{x} \in \{0,1\}^n$ → $y \in 1, \ldots, n$

$f(\bar{x}, y) = x_y$

↓

$b \in \{0,1\}$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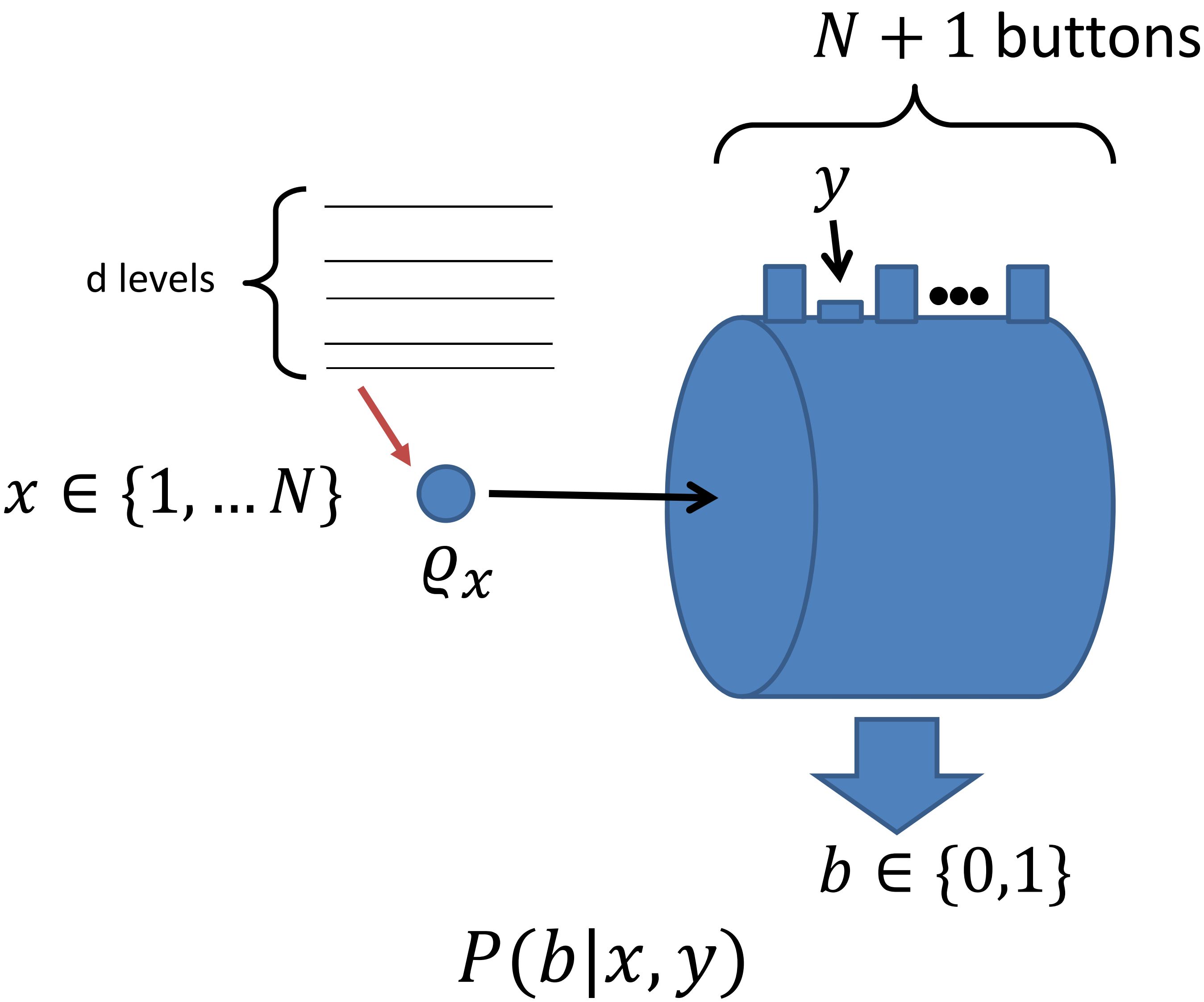
MS-DOS Prompt - QBASIC
Run Debug Options Help
ROBOT.BAS
Immediate
<F2=Subs> <F5=Run> <F8=Step>
00044:001

Random Access Codes (RAC)

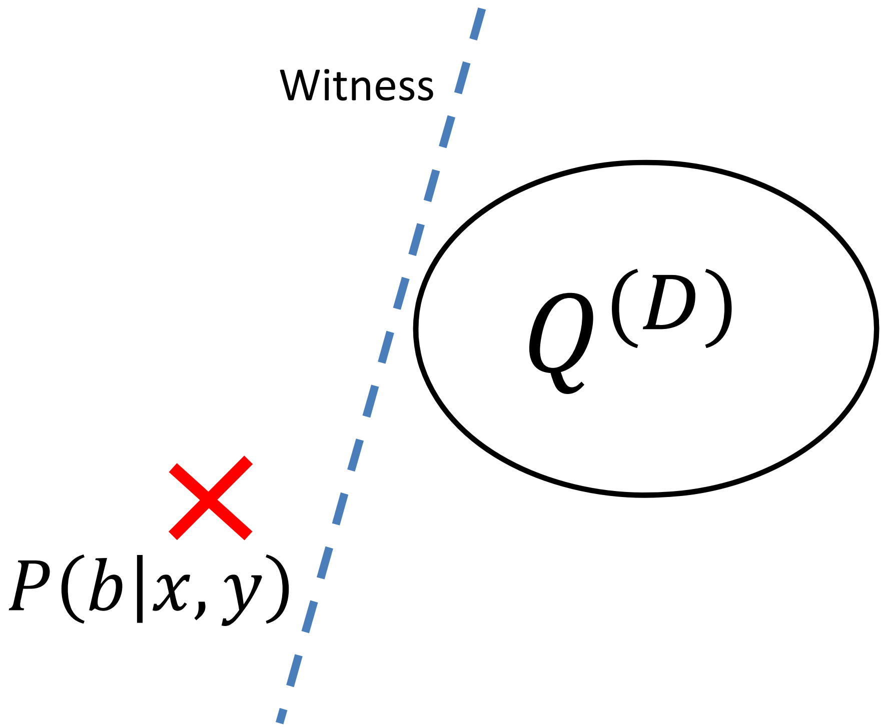
D	2	3	4	5	6	7
LB	0.788675	0.832273	0.908248	0.924431	0.951184	0.969841
UB	0.788675	0.832273	0.908248	0.924445	0.954123	0.969841

TABLE I: Lower and upper bounds on $P_{\max}(3 \to \log_2(D))$.



R. Gallego, N. Brunner, C. Hadley and A. Acín, Phys. Rev. Lett. 105, 230501 (2010).

Prepare-and-measure dimension witnesses



Your quantum technology allows you to manipulate systems of dimensión at least D+1

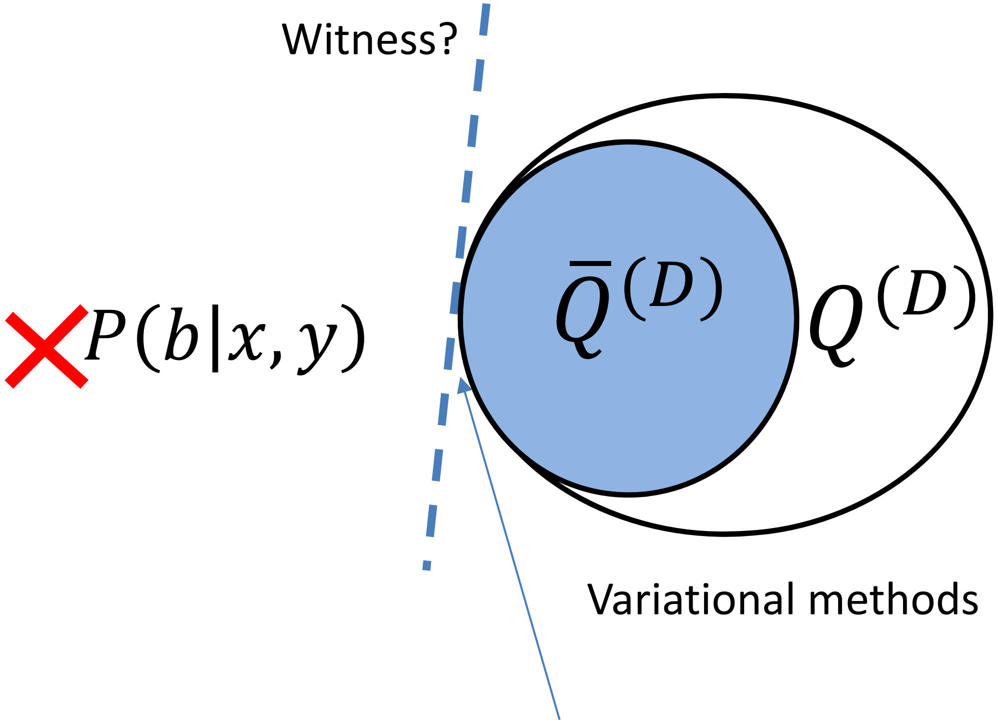
R. Gallego, N. Brunner, C. Hadley and A. Acín, Phys. Rev. Lett. 105, 230501 (2010).

	C_2 (bit)	Q_2 (qubit)	C_3 (trit)	Q_3 (qutrit)	C_4 (quat)
I_3	3	$1+2\sqrt{2}$	5	5	5
I_4	5	6	7	7.9689	9

TABLE I: Classical and quantum bounds for the dimension witnesses I_3 and I_4. Notably, these witnesses can distinguish classical and quantum systems of given dimensions.

R. Gallego, N. Brunner, C. Hadley and A. Acín, Phys. Rev. Lett. 105, 230501 (2010).

In the lab



hope: the inequality is satisfied by the bigger set

R. Gallego, N. Brunner, C. Hadley and A. Acín, Phys. Rev. Lett. 105, 230501 (2010).

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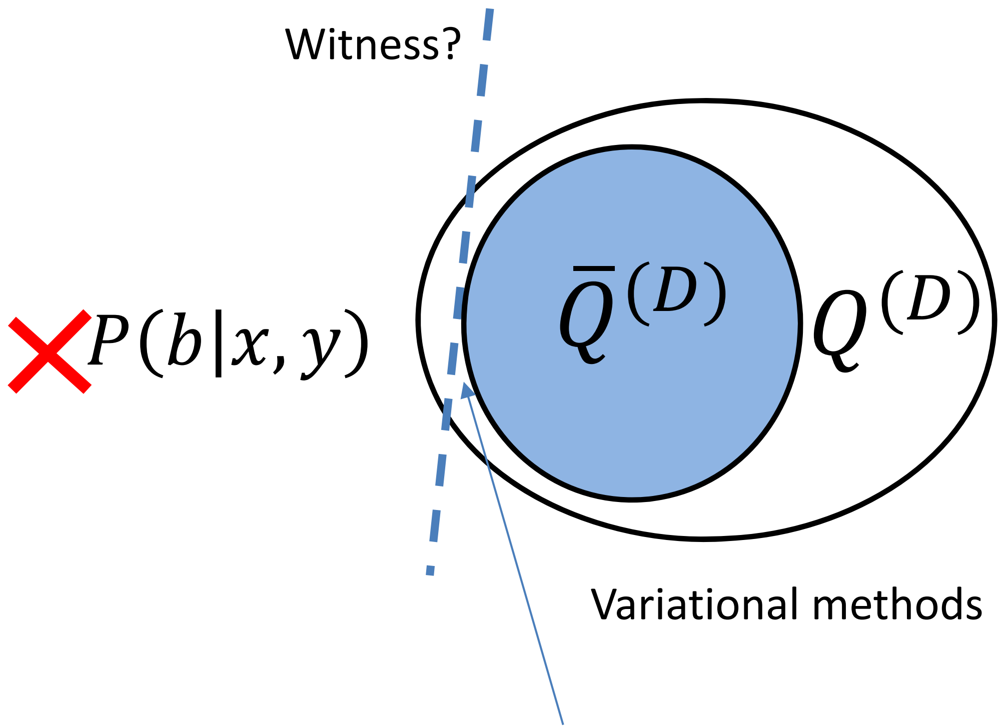

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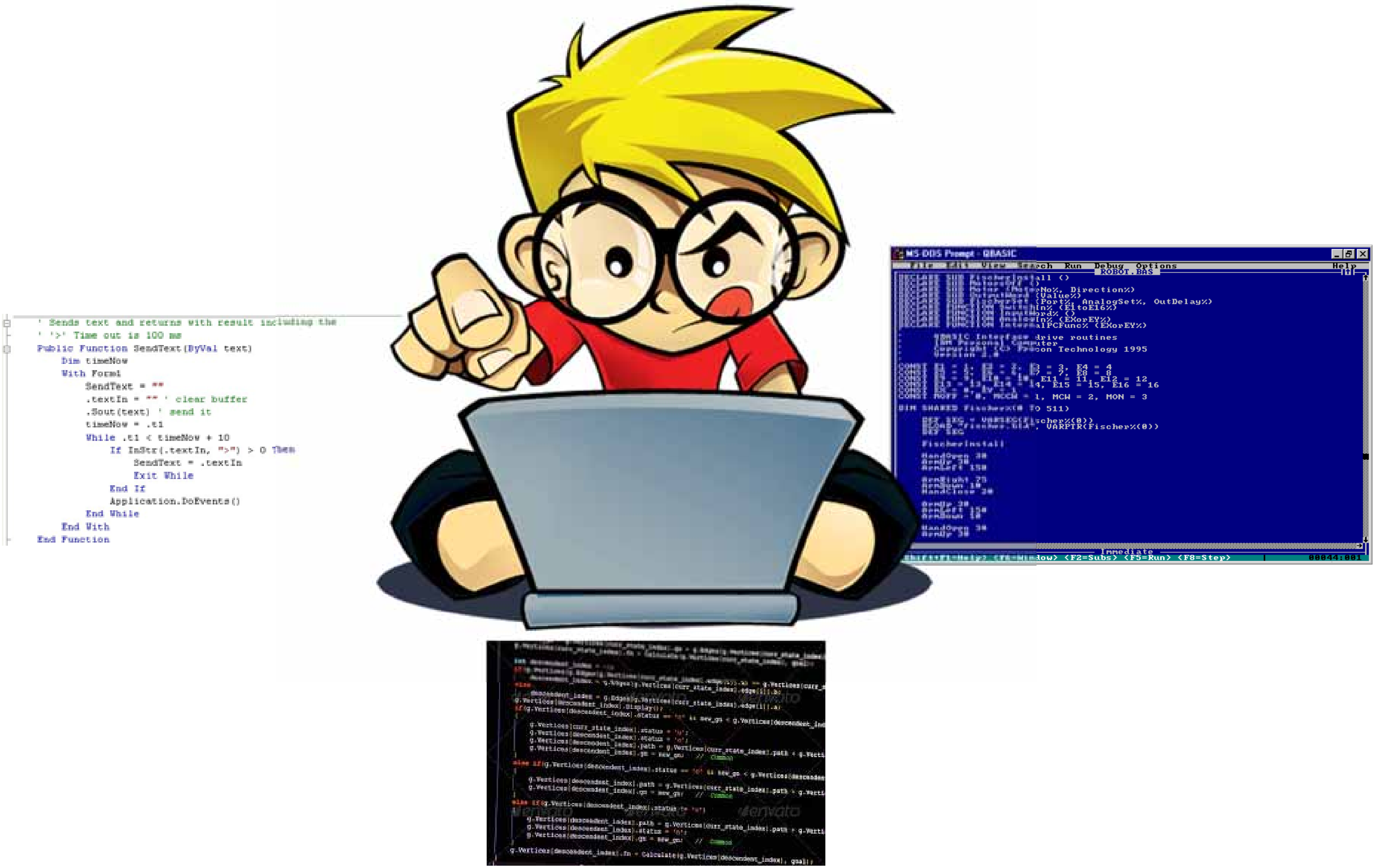
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I_4	5	6	7	7.9689	9

TABLE I: Classical and quantum bounds for the dimension witnesses I_3 and I_4. Notably, these witnesses can distinguish classical and quantum systems of given dimensions.

We proved that all these witnesses were sound,
hence validating of the conclusions of the
experimental papers

M. Hendrych, R. Gallego, M. Micuda, N. Brunner, A. Acín and J. P. Torres, Nat. Phys. 8, 588 (2012).
J. Ahrens, P. Badziag, A. Cabello, M. Bourennane, Nat. Phys. 8, 592 (2012).

Hierarchy for real quantum systems

$i = 1$

Generate *real* random projectors $\rho_x^i, F_b^{i,y} \in \mathrm{B}(\mathbb{C}^D)$ (with app. ranks)

Build moment matrix Γ^i

$\Gamma^i_{\mathbb{I},\mathbb{I}} = tr(\mathbb{I}) = 1,$

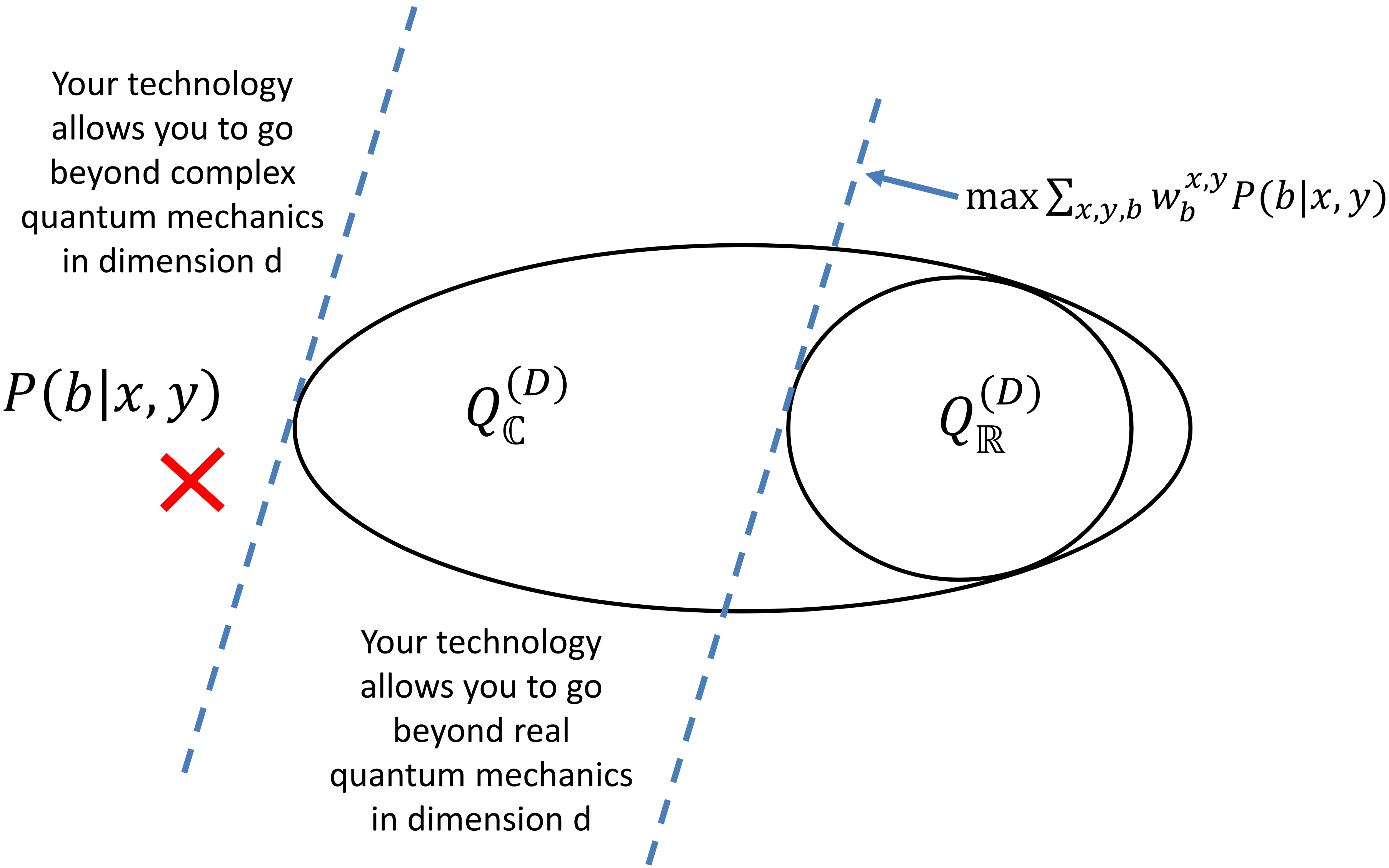
$\Gamma^i_{\rho_x, F_b^y} = tr(\rho_x^i F_b^{i,y}),$

$\vdots$

$i = i + 1$

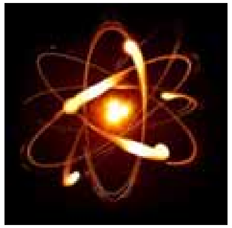
Repeat until the moment matrix $\Gamma^{(N+1)}$ is a linear combination of $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$

Real vs. Complex quantum mechanics



Random Access Codes (RAC)

Alice



Dimension D

Bob

$y \in 1, \ldots, n$

$$f(\bar{x}, y) = x_y$$

$b \in \{0,1\}$

complex qubits	real qubits
$P_{max}(3 \to 1) \approx 0.788675$	$P_{max}(3 \to 1) \approx 0.7696723$

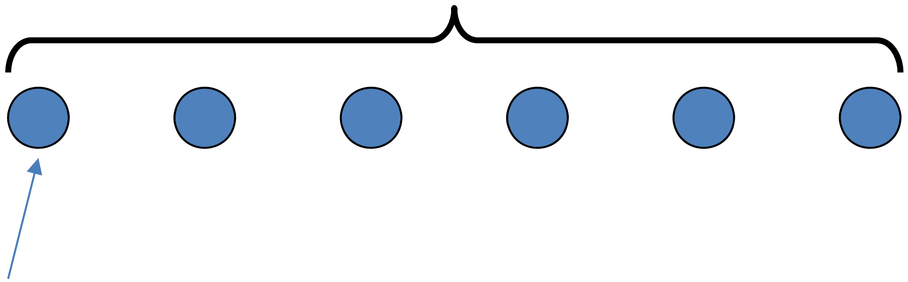
The structure of Matrix Product States

MN and T. Vértesi, arXiv:1509.04507.

s-local term
(acts on particles
j,…,j+s-1)

$$H = \sum_{j=1}^{n-s} h_j$$

k particles



physical
dimension d

ground state $|\varphi\rangle \in \mathbb{C}^{d^k}$ ⇒ Impossible to even *store* for high k

Matrix product states

boundary condition

$A_1 \dots A_d \in B(\mathbb{C}^D)$ physical dimension d

bond dimension D

$$|\psi\rangle = \sum_{i_{1,\dots,}i_k} tr(\sigma A_{i_1} \dots A_{i_k}) |i_1\rangle \otimes \dots \otimes |i_k\rangle$$

D. Perez-Garcia, F. Verstraete, M.M. Wolf and J.I. Cirac, Quantum Inf. Comput. 7, 401 (2007).

Matrix product states

Features:

a) Efficient computation of expectation values

b) Calculations in the thermodynamical limit $k \to \infty$

D. Perez-Garcia, F. Verstraete, M.M. Wolf and J.I. Cirac, Quantum Inf. Comput. 7, 401 (2007).

Correspondence between polynomials and states

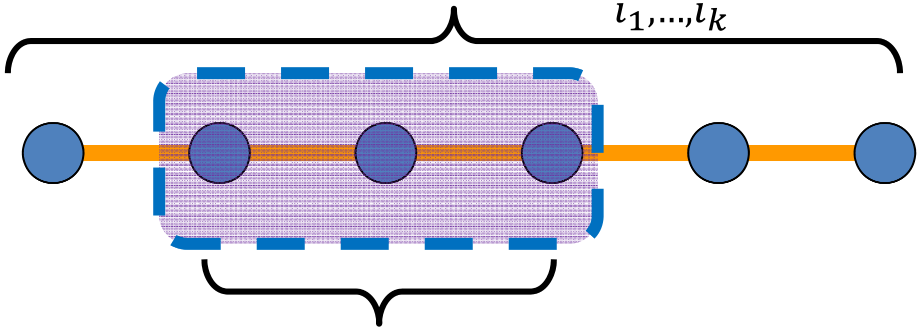
$P(X) = P(X_1, \ldots, X_d)$,
homogeneous polynomial
of degree m

$$P(X) = \sum_{\vec{i}} p_{\vec{i}} X_{i_1} \ldots X_{i_m} \quad \Rightarrow \quad |P(X)\rangle = \sum_{\vec{i}} p_{\vec{i}}^* |i_1, \ldots, i_m\rangle$$

MN and T. Vértesi, arXiv:1509.04507.

Overlap between a polynomial and a MPS

$$|\psi\rangle = \sum_{i_1,\ldots,i_k} tr(\sigma A_{i_1} \ldots A_{i_k})|i_1\rangle \otimes \ldots \otimes |i_k\rangle$$



$$\langle P(X)|\psi\rangle = \sum_{\substack{i_1,\ldots,i_s,\\ i_{s+m+1},\ldots,i_k}} tr(\sigma A_{i_1} \ldots A_{i_s} P(A) A_{i_{s+m+1}} \ldots A_{i_k})|i_1, \ldots, i_s, i_{s+m+1}, \ldots i_k\rangle$$

"defect"

MN and T. Vértesi, arXiv:1509.04507.

Existence of annihilation operators

$$\langle P(X)|\psi\rangle = \sum_{\substack{i_1,\dots,i_s,\\ i_{s+m+1},\dots,i_k}} tr(\sigma A_{i_1}\dots A_{i_s} P(A) A_{i_{s+m+1}}\dots A_{i_k})|i_1,\dots,i_s,i_{s+m+1},\dots i_k\rangle$$

$P(X)$, MPI for dimension D $\Longrightarrow$ $\langle P(X)|\psi\rangle = 0$, for *all* MPS with bond dimension D!!

There exist local operators h which annihilate all MPS of bond dimension D or smaller

MN and T. Vértesi, arXiv:1509.04507.

Actually, for high m, almost all m-local operators are annihilation operators, because

Local dimension of m particles d^m

Local dimension of MPS subspace $poly(m)$

Existence of cut-and-glue operators

$$\langle P(X)|\psi\rangle = \sum_{\substack{i_1,\ldots,i_s,\\ i_{s+m+1},\ldots,i_k}} tr(\sigma A_{i_1} \ldots A_{i_s} P(A) A_{i_{s+m+1}} \ldots A_{i_k}) |i_1, \ldots, i_s, i_{s+m+1}, \ldots i_k\rangle$$

scalar

$P(X)$, central polynomial for dimension D ⇒ $\langle P(X)|\psi\rangle = \mathrm{p(A)}\,|\psi'\rangle$

$$|\psi'\rangle = \sum_{\substack{i_1,\ldots,i_s,\\ i_{s+m+1},\ldots,i_k}} tr(\sigma A_{i_1} \ldots A_{i_s} A_{i_{s+m+1}} \ldots A_{i_k}) |i_1, \ldots, i_s, i_{s+m+1}, \ldots i_k\rangle$$

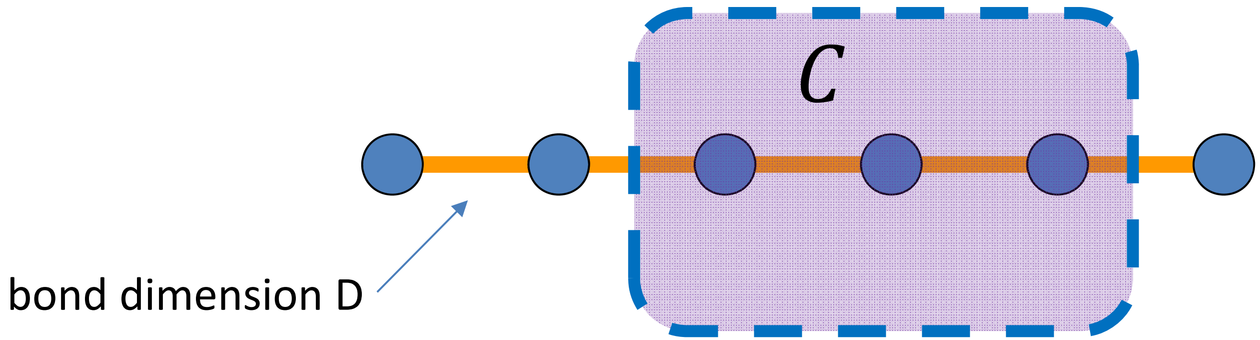
MPS with m particles less

MN and T. Vértesi, arXiv:1509.04507.

$\{P_j(X)\}$, basis of homogeneous central polynomials for dimension D of degree m

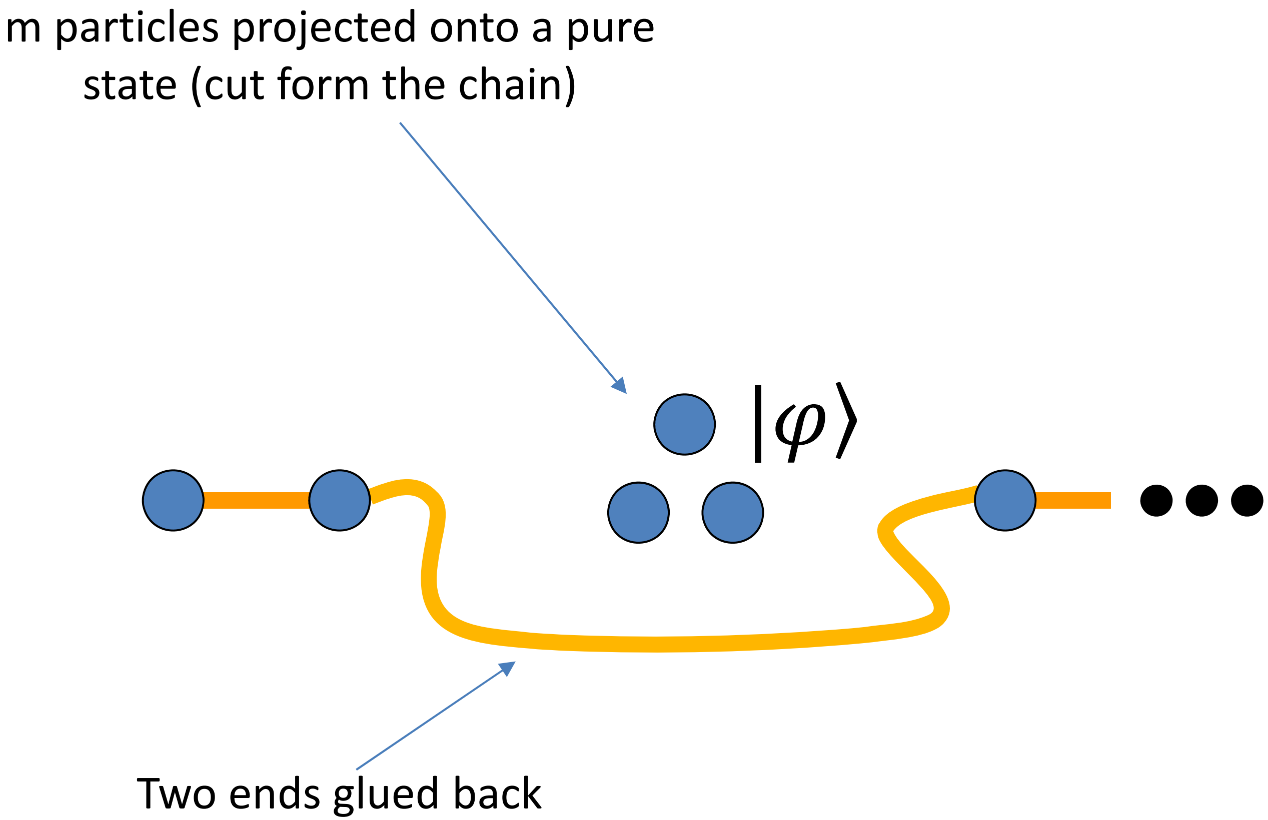
$$C = \sum_j |j\rangle\langle P_j(X)|$$

MN and T. Vértesi, arXiv:1509.04507.

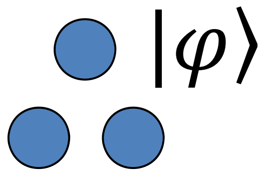


$$|\psi\rangle = \sum_{i_1,\ldots,i_k} tr(\sigma A_{i_1} \ldots A_{i_k}) |i_1\rangle \otimes \ldots \otimes |i_k\rangle$$

C, cut-and-glue operator



C, cut-and-glue operator

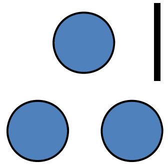


$$|\psi\rangle = \sum_{i_1,\dots,i_k} tr(\sigma A_{i_1} \dots A_{i_{k-m}})|i_1\rangle \otimes \dots \otimes |i_{k-m}\rangle$$

C, cut-and-glue operator for dimension D

ρ_k, k-site MPS of bond dimension D

$$(C \otimes 1)\rho_k\left(C^\dagger \otimes 1\right), \text{ separable}$$



$|\varphi\rangle$

$$|\psi\rangle = \sum_{i_1,\ldots,i_k} tr(\sigma A_{i_1} \ldots A_{i_{k-m}})|i_1\rangle \otimes \ldots \otimes |i_{k-m}\rangle$$

Theoretical complexity of the SDP hierarchy and connection with MPS

MN and T. Vértesi, arXiv:1509.04507.

Complexity of implementing the kth relaxation

SDP constraint

$$\Gamma_k = \left(\qquad\qquad \right) \Big\} \exp(k)$$

$$|c^k| = \exp(k)$$ free variables

Free-dimensional problems have the same complexity as dimension-constrained ones!?

Connection with MPSs

$$\Gamma_k = \sum_{i_1,\ldots,i_k} \sum_{j_1,\ldots,j_k} \mathrm{tr}(\tilde{X}_{j_n}\ldots\tilde{X}_{j_1}|\psi\rangle\langle\psi|\tilde{X}_{i_1}\ldots\tilde{X}_{i_k})|\vec{i}\rangle\langle\vec{j}|$$

Feasible moment matrices are extendible MPSs of bond dimension D!!!

MN and T. Vértesi, arXiv:1509.04507.

Complexity of implementing the kth relaxation

SDP constraint

$$\Gamma_k = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \Bigg\} \exp(k)$$

Supp∈span{poly(k) k-site MPS of bond dimension D}

$$|c^k| = \exp(k)$$ free variables

c^k ∈span{poly(k) 2k-site MPS of bond dimension D}

Complexity of implementing the kth relaxation

$$\tilde{\Gamma}^k = \left(\quad \right) \Big\} \, poly(k)$$

$$|\widetilde{c^k}| = \text{poly}(k)$$

MN and T. Vértesi, arXiv:1509.04507.

Finite dimensions are exponentially easier to characterize than infinite dimensions!!!

Extra PSD conditions to boost speed of convergence

$$\Gamma^k = \sum p_j \rho_{MPS}^j$$

MN and T. Vértesi, arXiv:1509.04507.

Extra PSD conditions to boost speed of convergence

$$\Gamma^k = \sum p_j \rho_{MPS}^j$$

cut-and-glue-operator

$C\Gamma^k C^\dagger$, separable operator

MN and T. Vértesi, arXiv:1509.04507.

Extra PSD conditions to boost speed of convergence

$$\Gamma^k = \sum p_j \rho_{MPS}^j$$

cut-and-glue-operator

$C\Gamma^k C^\dagger$, separable operator

New PSD conditions $C\Gamma^k C^\dagger$, PPT

MN and T. Vértesi, arXiv:1509.04507.

Conclusions

-Simple, easy-to-program technique to enforce dimension constraints in noncommutative polynomial optimization.

-Plenty of numerical evidence suggests that it is effective.

-It can be combined with other hierarchies, such as MLHG or BFS.

T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann and O. Gühne, Phys. Rev. Lett. 111, 030501 (2013).

M. Berta, O. Fawzi and V. B. Scholz, arXiv:1506.08810.


THE END