Device-Independent Tests of Entropy

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with Rafael Chaves, Nicolas Brunner

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Device independence

Testing physical properties from experimental data
without detailed knowledge of the implementation.

Examples : Bell nonlocality, entanglement, dimension.

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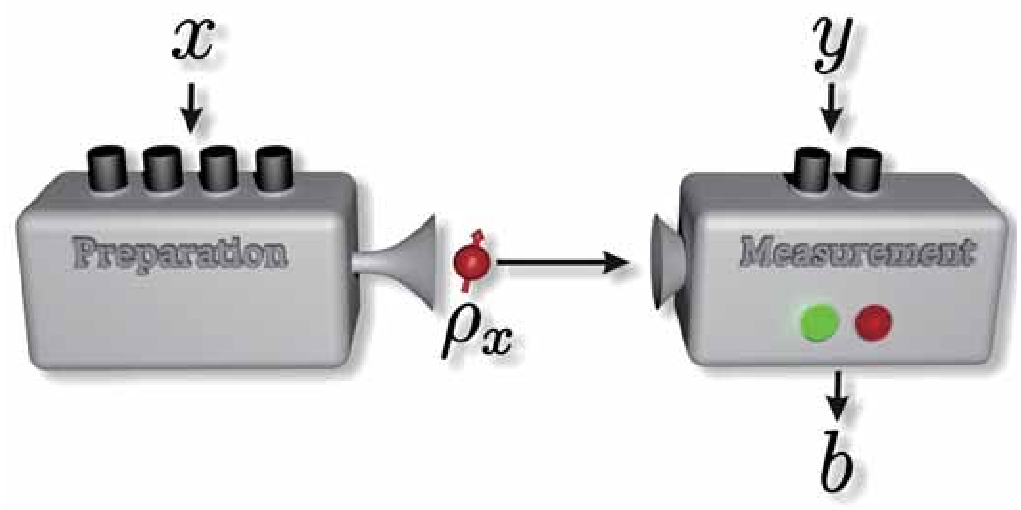
Examples : Bell nonlocality, entanglement, dimension.

In this talk: message entropy in prepare and measure scenario.



Bound minimal entropy $S(\rho)$

compatible with data $p(b|xy)$

Minimal entropy: *average* communication.

Minimal dimension: *worst case* communication.

Entropy witnesses

Want function of data and a bound such that

$$W(p(b|xy)) > L_s \Rightarrow S(\rho) > s$$

For the average message (we will assume uniform inputs).

$$\rho = \sum_x p(x)\rho_x$$

Diagonal for classical messages so von Neumann → Shannon.

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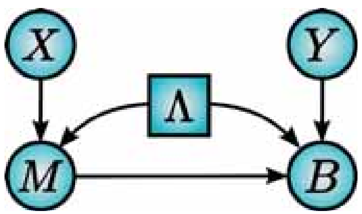
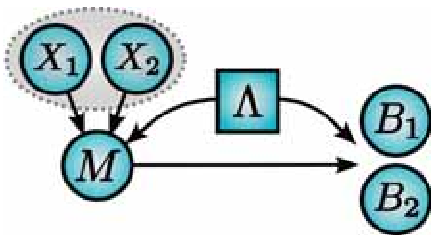
Causal inference graphs

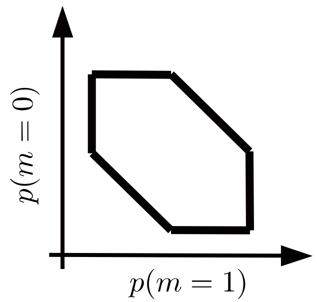
- Very general (arbitrary input/ouput)
- Generally not tight
- Does not distinguish classical/quantum.





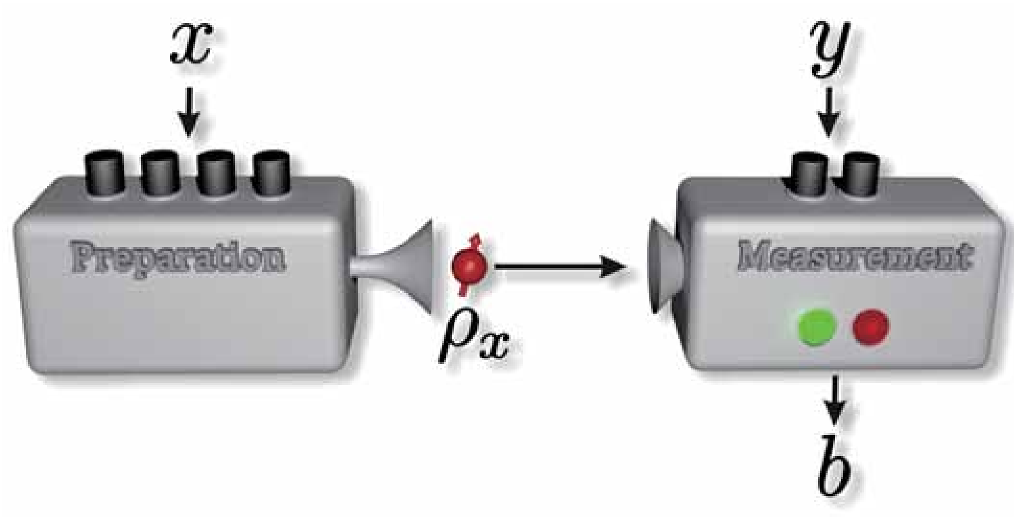
Convex optimisation

- Restricted numbers input/output
- Tight bounds
- Demonstrate quantum advantage.



Entropy and dimension are different quantities

Classical strategy for d^2 preparations
and d^2-1 measurements, binary output.



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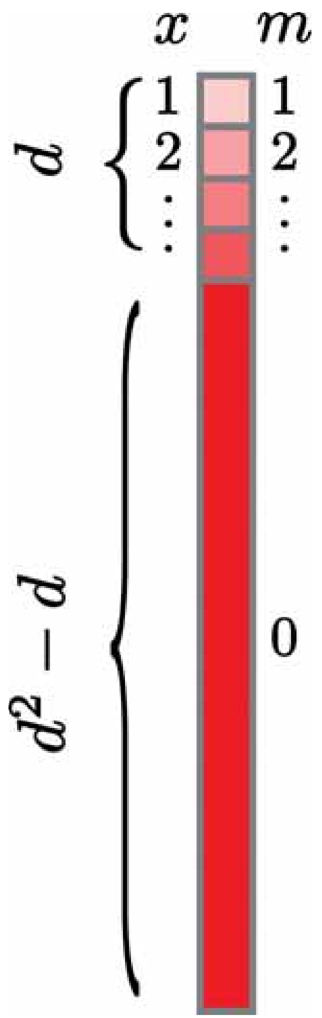

$$p(m) = \frac{1}{d^2} \to 0$$

$$p(m) = \frac{d^2 - d}{d^2} \to 1$$

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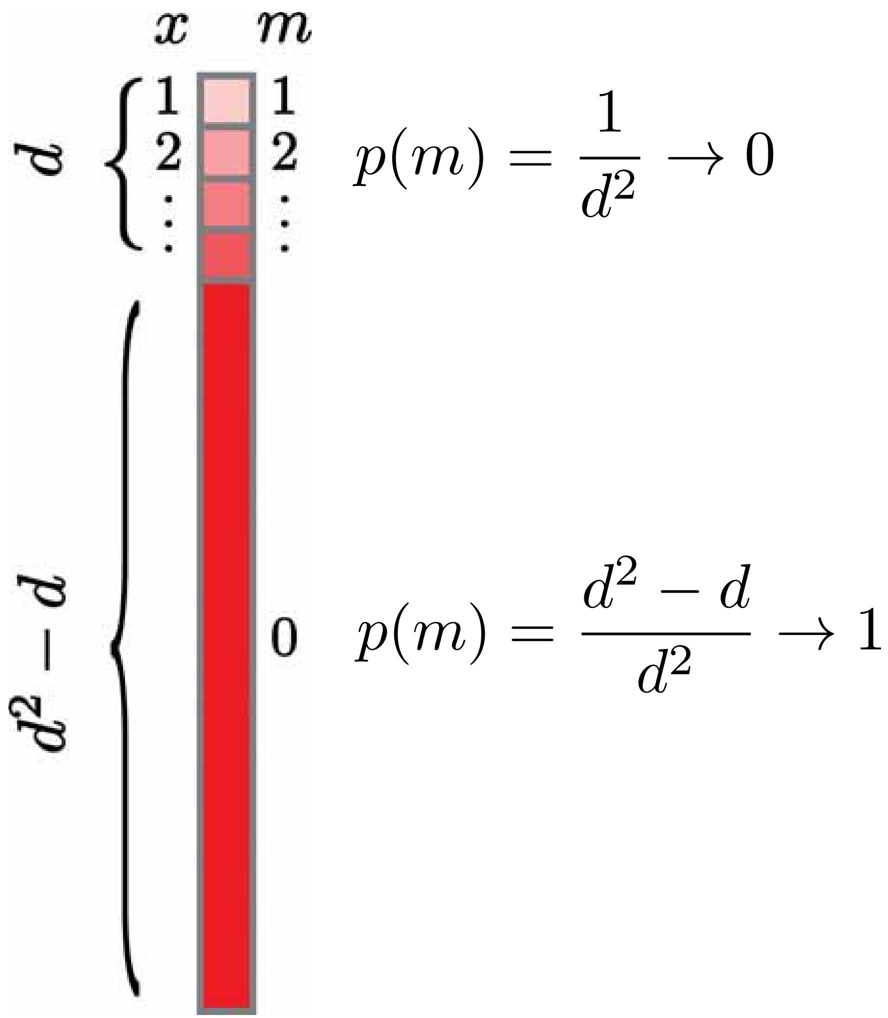
The entropy is

$$S(\rho) = -\sum_m \log(p(m))$$

Dimension witness of Gallego *et al.* (PRL'10)
→ Requires message dimension at least d+1



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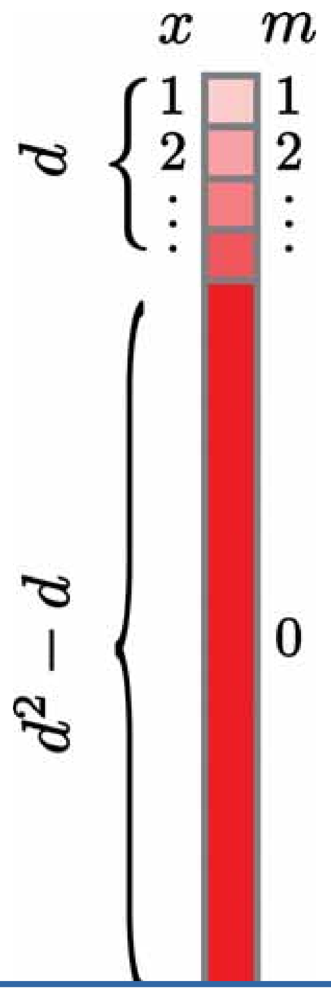

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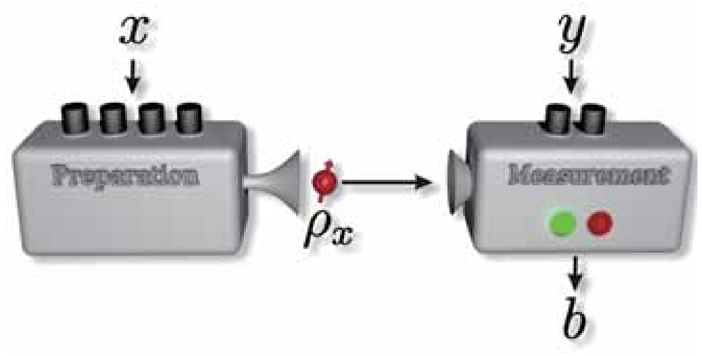
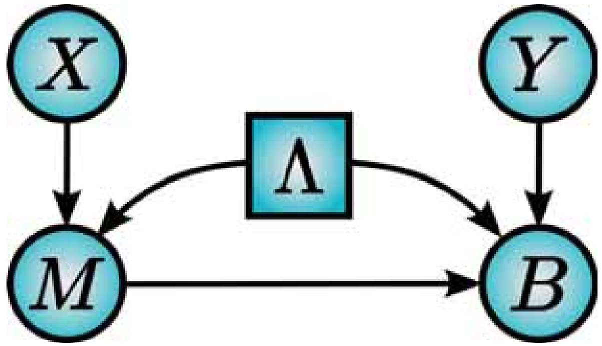
Dimension diverges Entropy vanishes	*worst case communication* vs. *average* communication

Causal inference method



Associate DAG



Causal relationships captured by linear equations in the entropies

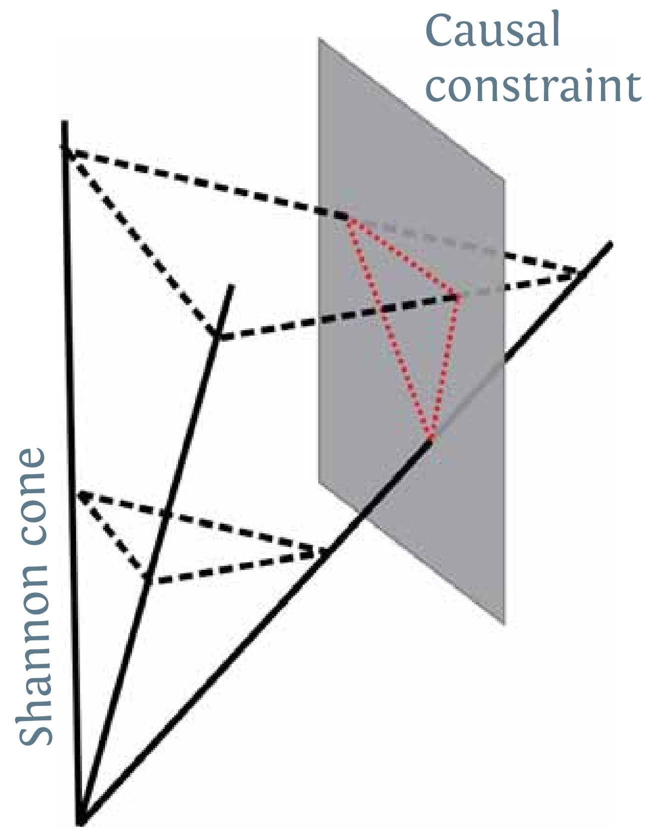
$$
\begin{aligned}
H(X,Y,\Lambda) &= H(X) + H(Y) + H(\Lambda) \\
H(M|X,\Lambda) &= 0 \\
H(B|Y,M,\Lambda) &= 0
\end{aligned}
$$

Form vectors of all joint entropies. E.g. for *n* variables:

$$[H(\emptyset), H(X_1), \ldots, H(X_1, X_2), \ldots, H(X_1, \ldots, X_n)] \in \mathbb{R}^{2^n}$$

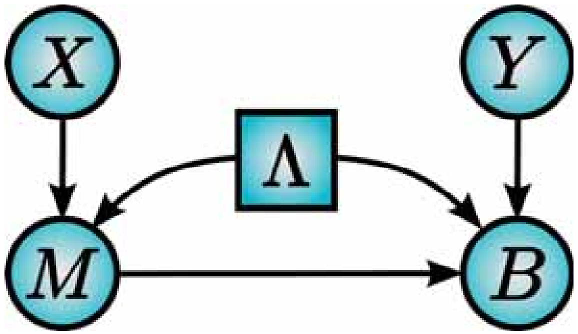
Entropy vectors are restricted by the causal constraints and by Shannon-type inequalities.

- Monotonicity (uncertainty of larger set is larger)
- Strong subadditivity (positivity of cond. info.)
- Positivity, normalisation.



Deriving inequalities

1) List Shannon-type inequalities.
2) List causal constraints.
3) Marginalise to observable variables.



Quantum: some joint entropies not physical.

$$\cancel{S(\rho, B)}$$

→ Replace constraints by data processing.
(Chaves, Majenz, Gross, Nat. Comm.'15).

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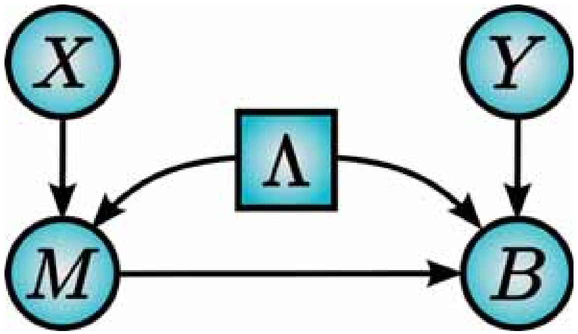

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→ Replace constraints by data processing. (Chaves, Majenz, Gross, Nat. Comm.'15).

$$H(X, Y, \Lambda) = H(X) + H(Y) + H(\Lambda)$$

$$H(M|X, \Lambda) = 0$$

$$H(B|Y, M, \Lambda) = 0$$

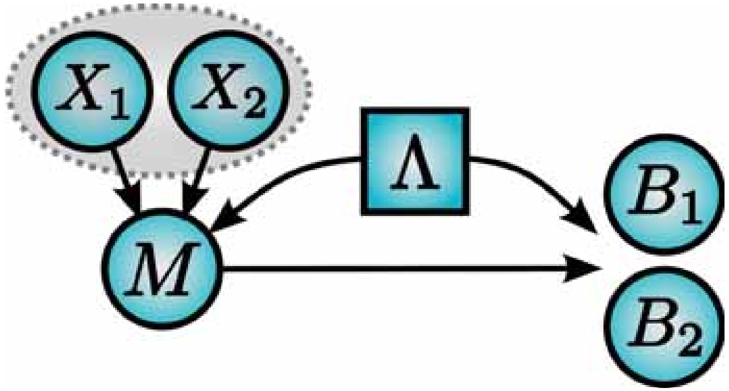
Only non-trivial inequality

$$S(\rho) \geq I(X : Y, B)$$

Also follows directly from the Holevo bound.

Fine-graining a bit more

We can fine-grain by adapting the graph to a fixed number of measurements



Get the nontrivial inequality

$$S(\rho) \geq I(X_1 : B_1) + I(X_2 : B_2) + I(X_1 : X_2|B_1) - I(X_1 : X_2)$$

Generalising

$$S(\rho) \geq \sum_{i=1}^{l} I(X_i : B_i) + \sum_{i=2}^{l} I(X_1 : X_i|B_i) - \sum_{i=1}^{l} H(X_1) + H(X_1, \ldots, X_l)$$

Reminiscent of Information Causality, but here: classical corr. / quantum comm.
IC: quantum corr. / classical comm.

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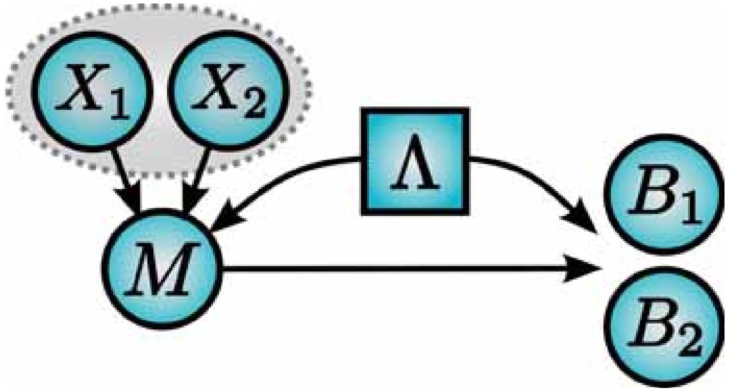

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Valid for arbitrary number of inputs/outputs
but does not distinguish quantum from classical.

Generalising

$$S(\rho) \geq \sum_{i=1}^{l} I(X_i : B_i) + \sum_{i=2} I(X_1 : X_i|B_i) - \sum_{i=1} H(X_1) + H(X_1, \ldots, X_l)$$

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classical
I AM NOT A DOG

Convex optimisation method

Decompose observed data over deterministic strategies

$$m = g_\lambda(x)$$
$$b = f_\lambda(y, m)$$

$$p(b|xy) = \sum_{\lambda,m} q_\lambda \delta_{b,f_\lambda(y,m)} \delta_{m,g_\lambda(x)}$$

Enough to consider message dimension = number of preparations → finite no. of strategies

Convex optimisation method

Decompose observed data over deterministic strategies

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Enough to consider message dimension = number of preparations → finite no. of strategies

$$\boxed{\min_{\mathbf{q}} H(M) \text{ subject to } \mathbf{A}\mathbf{q} = \mathbf{p},\ q_\lambda \geq 0,\ \sum_\lambda q_\lambda = 1}$$

$$\left.\begin{array}{r} H(M) \text{ concave in } \mathbf{q} \\ \mathbf{q} \text{ lives in polytope} \end{array}\right\}$$ Enough to check extremal points

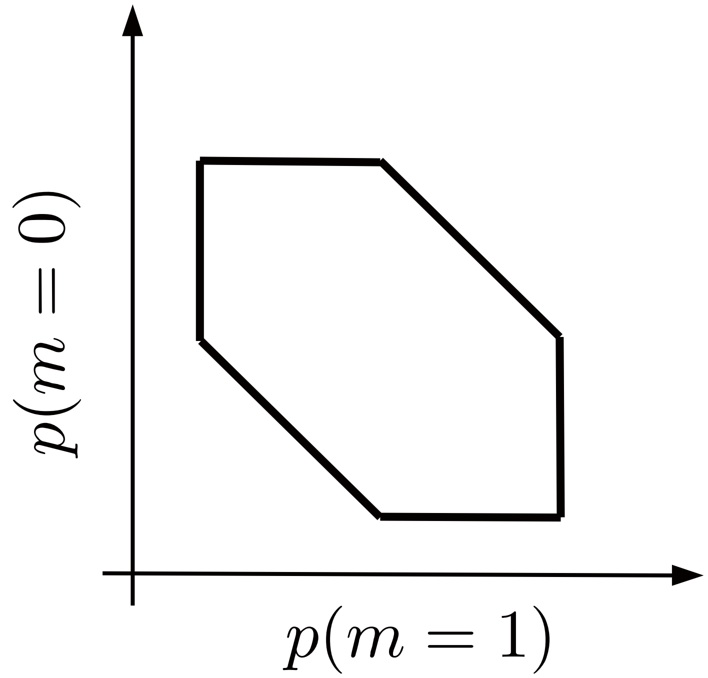
To reduce complexity

Size of polytope is intractable
→ Note: to evaluate the entropy we only need the distribution

$$p(m) = \sum_{\lambda,x} p(m|x,\lambda)p(x)q_\lambda = \frac{1}{n}\sum_{\lambda,x} p(m|x,\lambda)q_\lambda$$

Observed data implies linear constraints on this.
Find polytope by a sequence of linear programs.

→ significantly reduces complexity.



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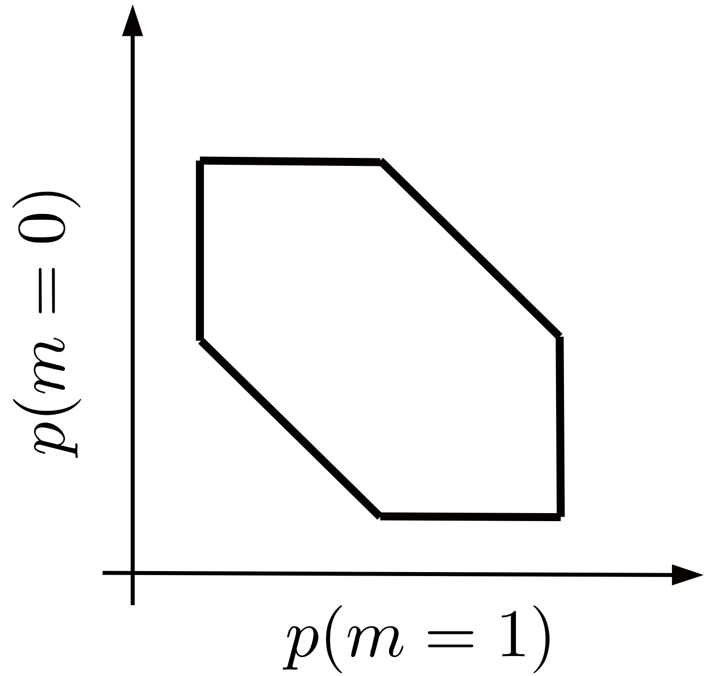


In addition, consider only linear functions of data : dimension witnesses.

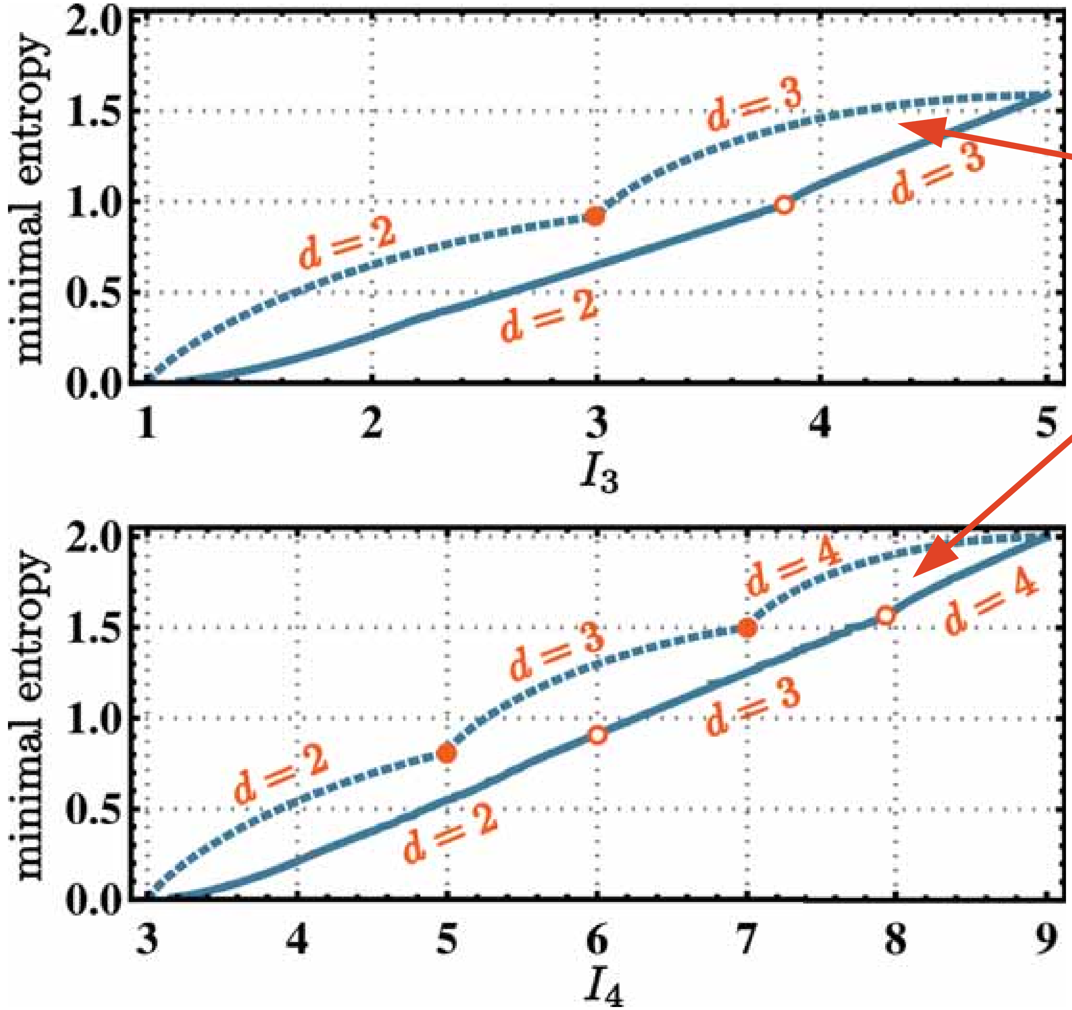
$$\mathbf{p} = \mathbf{A}\mathbf{q} \to \mathbf{I}\mathbf{p} = \mathbf{I}\mathbf{A}\mathbf{q}$$

Use witness of Gallego *et al.*, PRL'10, for *n* preparations, *n-1* measurements, binary ouputs.

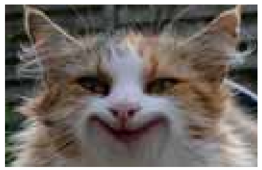
Convex optimisation result

Compare bound for classical messages with numerical optimisation for quantum messages.

The classical bound is tight.



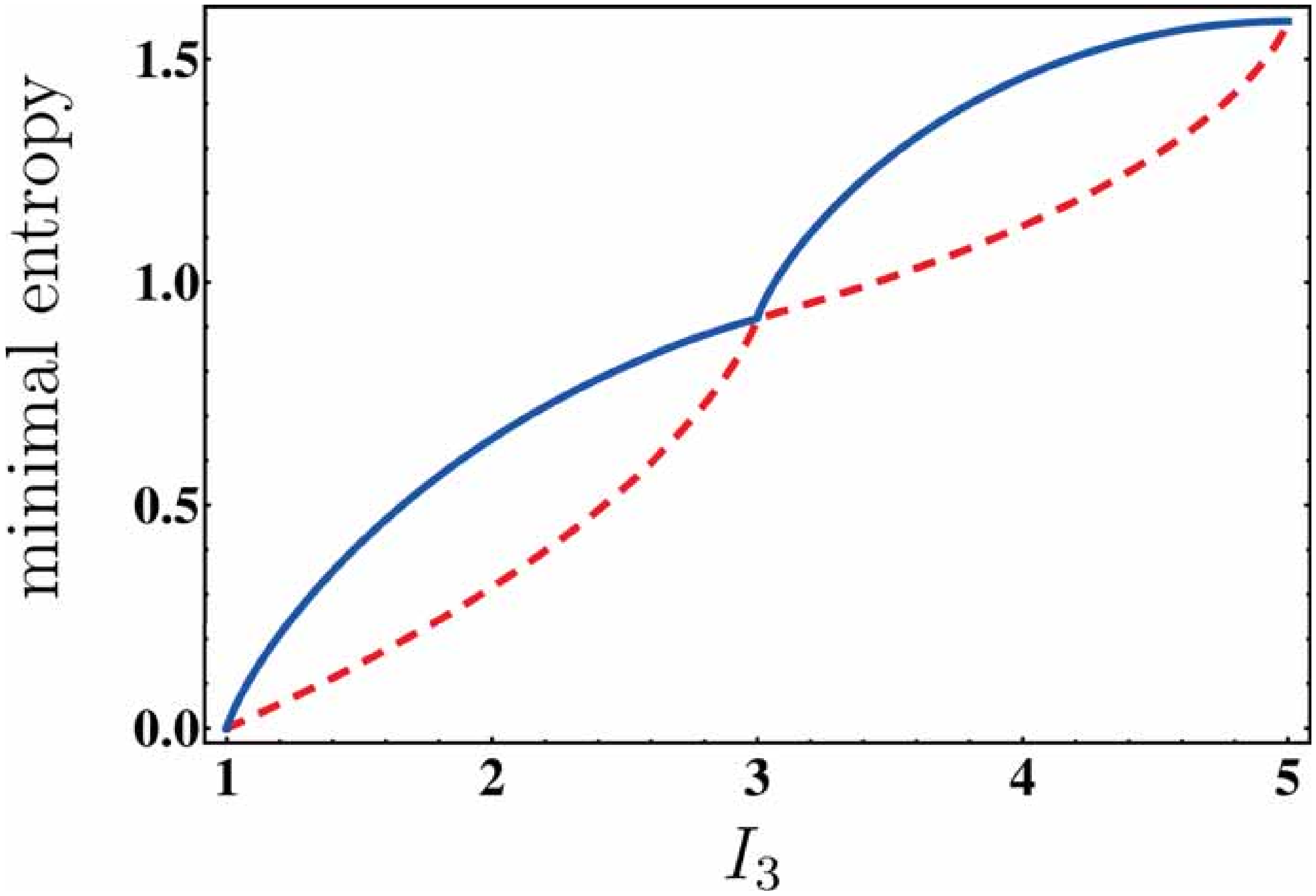
Quantum advantage!



Notice: No advantage by increasing quantum message dimension.

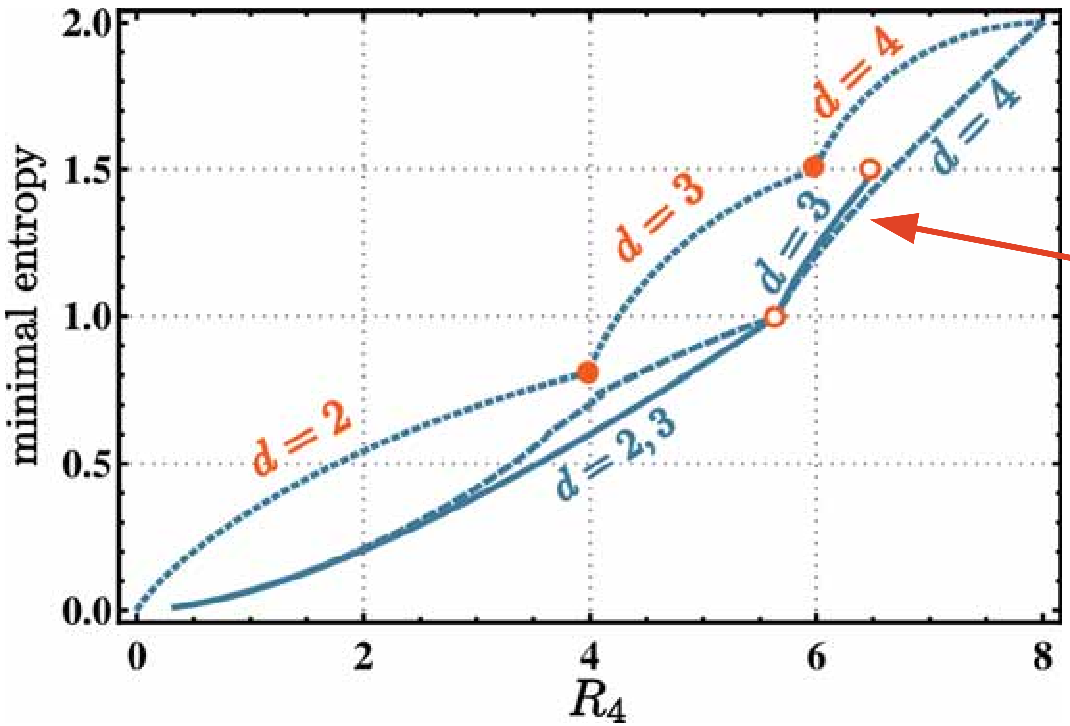
Compare the entropic and convex optimisation approaches

Comparison for a <u>specific</u> observed distribution (saturating the convex opt. bound).
→ the entropic approach is clearly not tight.



Possible advantage of higher dimensions

Random Access Code for 4 preparation and 2 measurements, binary output

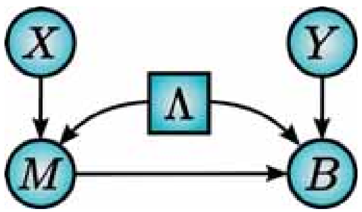


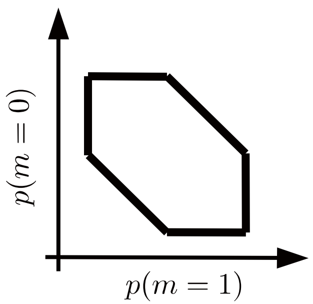
Possible advantage of dim. 4 over dim. 3.

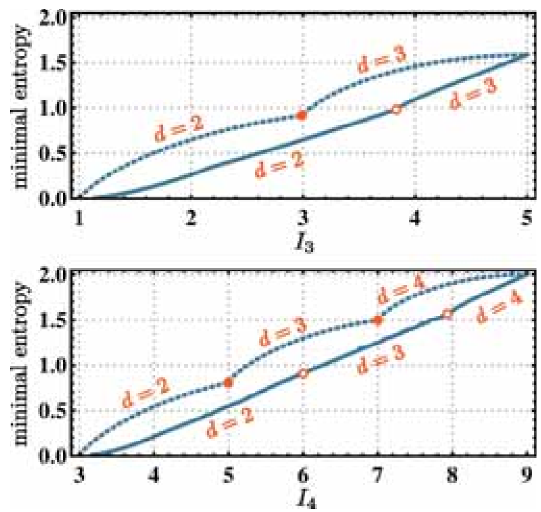
$$R_4 = E_{11} + E_{12} + E_{21} - E_{22} - E_{31} + E_{32} - E_{41} - E_{42}$$

Summary

- Device-independent tests of entropy in prepare & measure scenario.
- Two approaches: entropic based on causal inference / convex optimisation.
- Entropic approach : general but non-tight.
- Convex optimisation approach : fixed numbers of inputs, output, but tight.
- Quantum strategies show advantage over classical:
  achieve same dimension witness value with less entropy.







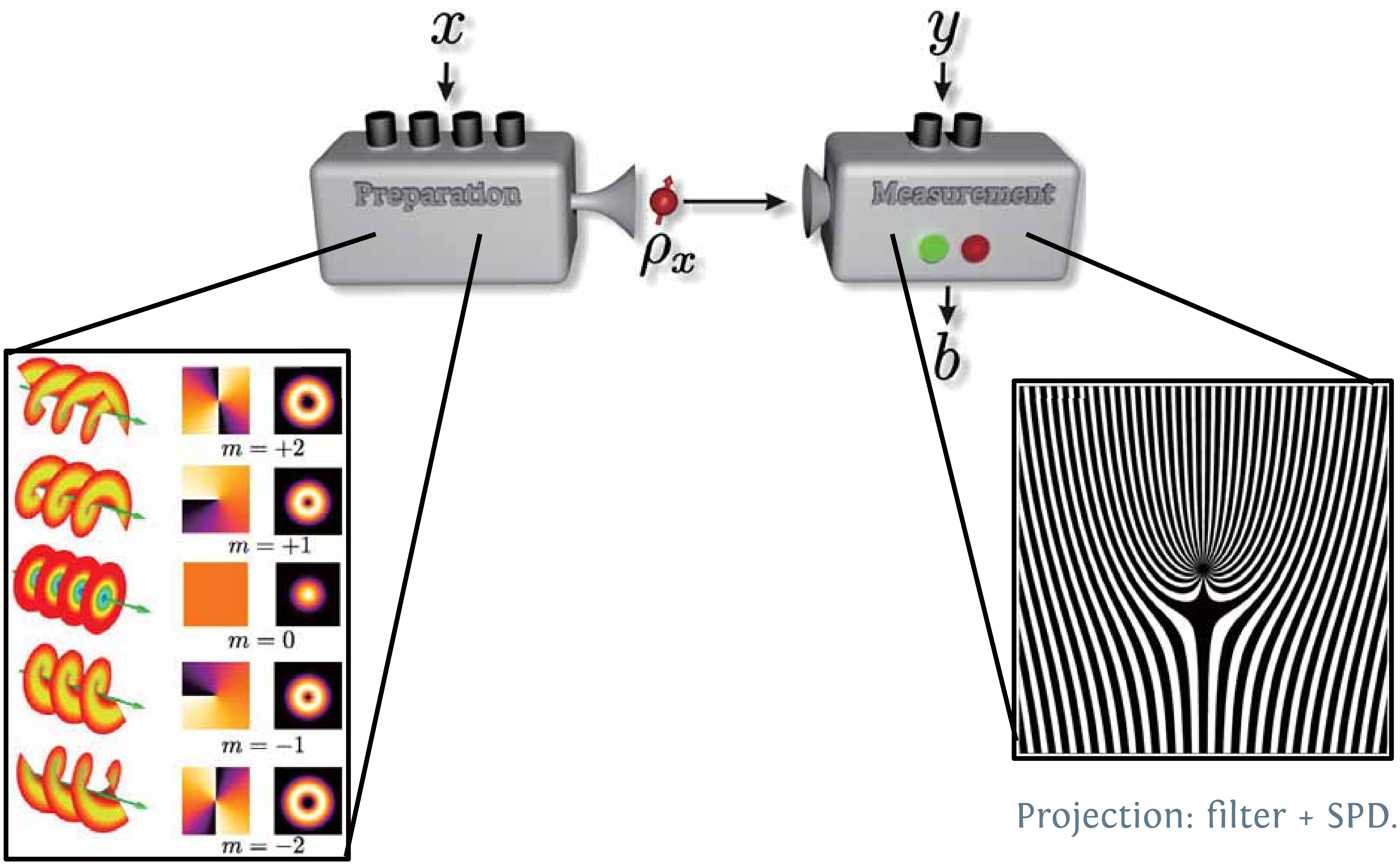
Is there a killer app?

Secure communication?

Randomness generation?

Experiment under way...

Stephen Walborn group, Rio de Janeiro, Brazil.
Optical implementation with single photons.



Orbital angular momentum

Projection: filter + SPD.

Thanks for your attention!

Strategy saturating the convex optimisation bound for witness I_n:

$$\text{for } x \leq d-2 \text{ send } m = x$$

$$\text{for } x = d-1 \text{ send } m = \begin{cases} 0 \text{ with prob. } p \\ x \text{ with prob. } 1-p \end{cases}$$

$$\text{otherwise send } m = 0$$

$$\text{where } p = \frac{1}{2}(L_d - I_n)$$

L_d : classical bound for dimension d

I_n : witness value