Introduction to Transport in Nanostructures

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Prof. Dr. T. Brandes

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CONTENTS

1. INTRODUCTION TO SINGLE CHARGE TRANSPORT

Quantum transport deals with particles (electrons, holes, quasi-particles) moving in a more or less controlled way through artificial structures. I the following, I consider *quantum dots* as the conceptually simplest nanostructures, and I only discuss electrons and no other quasi-particles.

1.1 Quantum Dots

Electrons can be confined in small structures, e.g. metallic grains, or semiconductor-based structures. Examples for the latter are lateral dots (defined by metallic gates on top of a structure that supports a two-dimensional electron gas), and vertical dots. *Vertical dots* can have a high spatial symmetry (circular pillar) in analogy to real atoms. *Lateral dots* have a lower spatial symmetry which can be interesting for the study of, e.g., quantum chaos. The number of electrons in quantum dots can be well controlled (0,1,2,3,...) which is due to the Coulomb blockade effect to be discussed below.

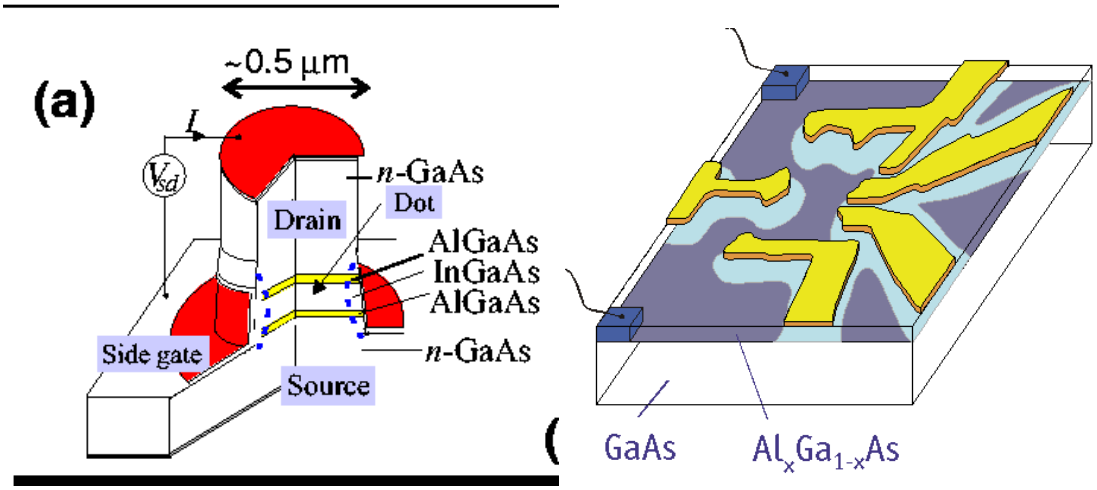

Fig. 1.1: LEFT: Vertical quantum dot, from: L. P. Kouwenhoven, D. G. Austing, and S. Tarucha, Rep. Prog. Phys. **64**, 701 (2001). RIGHT: lateral quantum dot (C. Marcus, Harvard; http://marcuslab.harvard.edu/research.html).

1.2 The Single Electron Transistor

The single electron transistor (SET) is a simple theoretical model for single electron transport (or more general: single charge transport) controlled by some external parameters (gate voltages etc.) It can be realized experimentally is many different ways - semiconductor quantum dots, superconducting Cooper pair boxes, molecules attached to external leads etc. The basic physics is the electrostatic charging energy, and some transfer mechanism (usually quantum mechanical tunneling).

1.3 Basic Properties

1.3.1 Electrostatic Energy

The model is based on classical electrostatics. This is sometimes called the 'orthodox theory' of Coulomb blockade due to Likarev et al., developed in the 80s. Let us consider the electrostatics of the system shown in the figure. This looks a bit 'electro-technical'

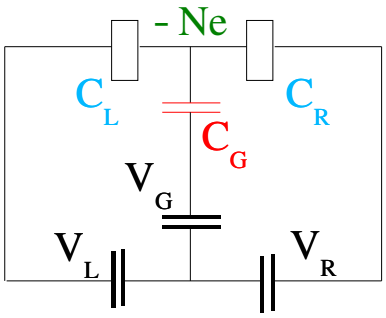

Fig. 1.2: Scheme of single electron transistor (SET).

and not very 'physics-like'. In the language of basic electrostatics (Maxwell's capacity coefficients, cf. textbooks like JACKSON or LANDAU II) this is, however, simply a system of four spatially separated metallic conductors: a central island (*'dot'* D) with charge $Q_D \equiv -Ne$ at potential V_D against three other conductors L (left), R (right), and G (gate) at potential V_L, V_R, and V_G with charges Q_L, Q_R, and Q_G.

From basic electrostatics, there is a linear relation

$$Q_i = \sum_{j=1}^{4} C_{ij} V_j, \tag{1.1}$$

with $i = L, D, R, G$ and the symmetric capacitance matrix C_{ij} that depends on the details of the geometry. For $i = D$ this means

$$Q_D = C_{DD} V_D + C_{DL} V_L + C_{DR} V_R + C_{DG} V_G \tag{1.2}$$

The three capacitors C_L, C_R and C_G in the circuit are parallel, and one has

$$C_{DD} = C_L + C_R + C_G \equiv C \tag{1.3}$$
$$C_{DL} = -C_L, \quad C_{DR} = -C_R, \quad C_{DG} = -C_G \tag{1.4}$$

which follows by considering potential differences, e.g. between the dot at potential V_D and all the other voltages zero. Therefore,

$$Q_D = CV_D - CV_{\text{ext}}, \quad V_{\text{ext}} \equiv \frac{C_L V_L + C_R V_R + C_G V_G}{C} \tag{1.5}$$

and the external voltage is a combination of the other three voltages.

The charge on the dot is

$$Q_D \equiv -Ne, \tag{1.6}$$

where $-e$ is the elementary charge and N the number of charged particles. We define the electrostatic energy of the central island (dot D) in the following way: the first contribution is the interaction energy among the N charges, when the external voltage V_{ext} is kept fixed at zero;

$$U_{\text{int}} = \frac{(Ne)^2}{2C}. \tag{1.7}$$

The second contribution is simply the potential energy of the total dot charge $-Ne$ in the finite, external potential V_{ext};

$$U_{\text{ext}} = -NeV_{\text{ext}}. \tag{1.8}$$

The electrostatic energy of the dot (central island) as a function of N therefore is

$$\boxed{U(N) = \frac{(Ne)^2}{2C} - NeV_{\text{ext}}.} \tag{1.9}$$

Another way to derive this expression is to 'continuously build up' the total dot charge by using

$$U(N) = \int_0^{-Ne} dQ_D V_D = \int_0^{-Ne} dQ_D \left(\frac{Q_D}{C} + V_{\text{ext}} \right) = \frac{(Ne)^2}{2C} - NeV_{\text{ext}}. \tag{1.10}$$

In the following, we assume that V_{ext} can be continuously changed. In general, V_{ext} only takes discrete values because microscopically, also the charges Q_L, Q_R, and Q_G only are discrete multiples of the elementary charge $-e$. However, this plays no role as long as $Q_L, Q_R, Q_G \gg Q_D$, which means that the charge Q_D in the dot is much smaller than the charge in the reservoirs (conductors) L, R, and G.

1.3.2 Chemical Potentials and Current Flow

In the following, we will assume electrons with charge $-e$ as the individual particles that charge the dot. The reservoirs (conductors L, R, G) are now modelled as zero temperature Fermi gases at chemical potentials $\mu_L \equiv eV_L$ and $\mu_R \equiv eV_R$. We assume that electrons flow from the left to the right reservoir but not into the gate reservoir. We find a condition for electrons flowing from left to right (but not backwards). First, we define the chemical potential of the dot,

Definition

$$\mu_{\text{dot}}(N+1) \equiv U(N+1) - U(N) = \frac{(N+1/2)e^2}{C} - eV_{\text{ext}}. \tag{1.11}$$

1. An electron from the left lead with energy E_L enters the dot. Energy conservation requires

   $$E_L + U(N) = U(N+1) \rightsquigarrow \mu_L > E_L = \mu_{\text{dot}}(N+1) \tag{1.12}$$

   The inequality is simply the definition of the chemical potential μ_L as the largest possible energy in the left reservoir.

2. Now there are two possibilities: another electron can flow from the left lead to the dot; for this to occur one must have $\mu_L > \mu_{\text{dot}}(N+2) > \mu_{\text{dot}}(N+1)$, i.e. a larger μ_L than in 1. is required. Alternatively, an electron leaves the dot into the right lead, where it enters at energy

   $$E_R = U(N+1) - U(N) \rightsquigarrow \mu_{\text{dot}}(N+1) > \mu_R. \tag{1.13}$$

   The last inequality follows from the fact that the electron can only enter *above* the Fermi energy of the right lead (Pauli principle).

Note that the Pauli principle here really only comes into play when electrons *leave* the dot.

To summarize, the minimal requirement for an electron current is

$$eV_L \equiv \mu_L \quad > \quad \mu_{\text{dot}}(N+1) > \mu_R \equiv eV_R, \quad \text{current flow condition } (L \to R). \tag{1.14}$$

Definition If the current flow condition Gl. (1.14) is *not* fulfilled, the dot is in the *Coulomb blockade regime.*

The next step is to explore the regions in the space of parameters (μ_L, μ_R, V_G, and N) where Coulomb blockade occurs. We define

Definition

$$V_L - V_R, \quad \text{bias voltage} \tag{1.15}$$

$$V_G, \quad \text{gate voltage.} \tag{1.16}$$

1.3.3 Linear Transport $V_L - V_R \to 0$: Coulomb Blockade Oscillations

We first consider the limit of very small bias voltage $V_L - V_R \to 0^+$. Let us assume $V_R = 0$. The current flow condition Gl. (1.14) then becomes $O^+ > \mu_{\rm dot}(N+1) > 0$ such that current can only flow if

$$U(N+1) - U(N) = 0 \rightsquigarrow V_{\rm ext} = \frac{(N+1/2)e}{C}. \tag{1.17}$$

For $V_L = 0, V_R = 0$, $V_{\rm ext} \equiv \frac{C_G V_G}{C}$ is proportional to the gate voltage V_G. The current flow condition therefore is fulfilled at gate voltages

$$V_G = \frac{(N+1/2)e}{C_G}. \tag{1.18}$$

For other values no current flows. As a function of V_G, the current is zero, then finite, then again zero, finite, ... these transitions, which are are the so-called *Coulomb blockade oscillations*, occur for different values of N, and changing the gate voltage will thus change the number of electrons N in the dot one by one.

1.3.4 Non–Linear Transport: Coulomb Blockade Diamonds

We now consider the current flow condition Gl. (1.14) at finite bias. Again we start with a situation where N electrons are on the dot. The current flow condition is

$$eV_L > U(N+1) - U(N) > eV_R, \quad \text{from left to right} \tag{1.19}$$
$$eV_R > U(N+1) - U(N) > eV_L, \quad \text{from right to left.} \tag{1.20}$$

For convenience we take a symmetric situation $V_L = -V_R = V/2$. This means

$$eV/2 > U(N+1) - U(N) > -eV/2, \quad \text{from left to right} \tag{1.21}$$
$$e|V|/2 > U(N+1) - U(N) > -e|V|/2, \quad \text{from right to left.} \tag{1.22}$$

We evaluate these inequalities, using

$$U(N+1) - U(N) = \frac{(N+1/2)e^2}{C} - eV_{\rm ext} = \frac{(N+1/2)e^2}{C} - e\frac{C_G V_G}{C}, \tag{1.23}$$

where we assumed symmetric capacitances $C_L = C_R$. Therefore, the minimal condition for current to flow in either direction is given by

$$\frac{C|V|}{2e} > (N+1/2) - \frac{C_G V_G}{e} > -\frac{C|V|}{2e} \tag{1.24}$$

To visualize this in a diagram, let us define the dimensionless variables

$$y \equiv \frac{CV}{2e}, \quad x \equiv \frac{C_G V_G}{e} \rightsquigarrow |y| > N + 1/2 - x > -|y| \tag{1.25}$$

In the x-y-plane, this leads to a diagram with 'diamond' (rhombus) shaped areas. Inside these *Coulomb blockade diamonds* the current flow condition is *not* fulfilled and the dot is in a stable state with a fixed number N of electrons inside (FIGURE). We also recover the linear Coulomb blockade regime for $y = 0$, i.e. along the x-axis: the corners where the diamonds meet are the positions of current flow in the Coulomb blockade oscillations.

1.4 Rate Equations for the Anderson Impurity Model

Consider a simple dot model with two single particle states E_a and E_b with 0, 1, or 2 spin-polarized electrons (the spin quantum number playes no role). The dot is attached to a left and a right leads and can be in either of the four states $0, a, b, 2$ with energy 0, E_a, E_b, and $E_a + E_b + U$, where U is the (Hubbard like) interaction energy. We assume that the external voltage $V_{\rm ext} \equiv \Phi$ is directly controlled by the gate voltage V_G and shifts the energy levels according to

$$\varepsilon_a \equiv E_a - \Phi, \quad \varepsilon_b \equiv E_b - \Phi. \tag{1.26}$$

1.4.1 Rate Equations

We describe a time evolution of the system by introducing *rates*, i.e. probabilities per time, for internal transitions *between dot states* due to tunneling as $\gamma_{0\leftarrow a}$ (rate for internal transition from state a with 1 electron to state 0 with no electron), etc.

By considering the possibe transitions between states, we can immediately write down a rate equation (master equation) for probabilities as a function of time:

$$\begin{aligned}
\dot{p}_a &= \gamma_{a\leftarrow 0} p_0 - (\gamma_{2\leftarrow a} + \gamma_{0\leftarrow a}) p_a + \gamma_{a\leftarrow 2} p_2 \\
\dot{p}_b &= \gamma_{b\leftarrow 0} p_0 - (\gamma_{2\leftarrow b} + \gamma_{0\leftarrow b}) p_b + \gamma_{b\leftarrow 2} p_2 \\
\dot{p}_2 &= \gamma_{2\leftarrow a} p_a + \gamma_{2\leftarrow b} p_b - (\gamma_{a\leftarrow 2} + \gamma_{b\leftarrow 2}) p_2 \\
1 &= p_0 + p_a + p_b + p_2.
\end{aligned} \tag{1.27}$$

The rates are given by

$$\begin{aligned}
\gamma_{a\leftarrow 0} &= \Gamma_L f_L(\varepsilon_a) + \Gamma_R f_R(\varepsilon_a), \quad \gamma_{b\leftarrow 0} = \Gamma_L f_L(\varepsilon_b) + \Gamma_R f_R(\varepsilon_b) \\
\gamma_{0\leftarrow a} &= \Gamma_L \bar{f}_L(\varepsilon_a) + \Gamma_R \bar{f}_R(\varepsilon_a), \quad \gamma_{0\leftarrow b} = \Gamma_L \bar{f}_L(\varepsilon_b) + \Gamma_R \bar{f}_R(\varepsilon_b) \\
\gamma_{2\leftarrow a} &= \Gamma_L f_L(U + \varepsilon_b) + \Gamma_R f_R(U + \varepsilon_b), \quad \gamma_{2\leftarrow b} = \Gamma_L f_L(U + \varepsilon_a) + \Gamma_R f_R(U + \varepsilon_a) \\
\gamma_{a\leftarrow 2} &= \Gamma_L \bar{f}_L(U + \varepsilon_b) + \Gamma_R \bar{f}_R(U + \varepsilon_b), \quad \gamma_{b\leftarrow 2} = \Gamma_L \bar{f}_L(U + \varepsilon_a) + \Gamma_R \bar{f}_R(U + \varepsilon_a).
\end{aligned} \tag{1.28}$$

Here, the Fermi functions are

$$\begin{aligned}
f_{L/R}(\varepsilon) &\equiv f(\varepsilon - \mu_{L/R}), \quad f(\varepsilon) = \left[\exp\left(\frac{\varepsilon}{k_B T}\right) + 1\right]^{-1} \\
\bar{f}(\varepsilon) &\equiv 1 - f(\varepsilon).
\end{aligned} \tag{1.29}$$

We now want to calculate the current in the long time (stationary) limit, where initial conditions no longer play any role. The stationary current I through the left junction is obtained from the stationary solution of the master equation, i.e., the stationary values p_0, p_a, p_b, p_2, as

$$\begin{aligned}
I &= -e \left\{ \left[\gamma^L_{a\leftarrow 0} + \gamma^L_{b\leftarrow 0}\right] p_0 + \left[\gamma^L_{2\leftarrow a} - \gamma^L_{0\leftarrow a}\right] p_a \right. \\
&+ \left. \left[\gamma^L_{2\leftarrow b} - \gamma^L_{0\leftarrow b}\right] p_b - \left[\gamma^L_{a\leftarrow 2} + \gamma^L_{b\leftarrow 2}\right] p_2 \right\}.
\end{aligned} \tag{1.30}$$

Here, it does not matter whether we calculate the current through the left or the right junction: both currents are the same due to current conservation (Kirchhoff's first rule).

It is very easy to solve the system of linear equations Eq. (1.27). The important ingredients are the transition rates Eq. (1.28).

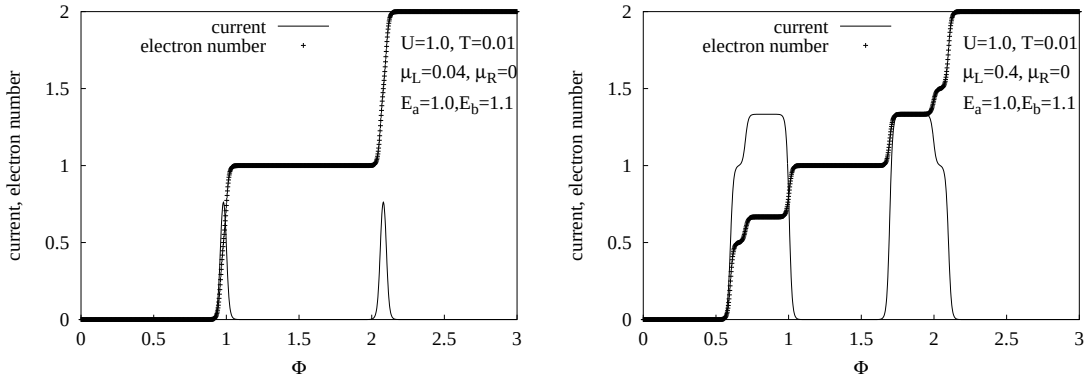

Fig. 1.3: Current and number of (spinless) electrons in the linear (left) and the non–linear (right) transport regime through a quantum dot, modeled as an Anderson impurity with two energy levels E_a and E_b. The current is in units of $-e\Gamma_L\Gamma_R/(\Gamma_L+\Gamma_R)$. The external voltage Φ shifts the energy levels, $\varepsilon_a \equiv E_a - \Phi$, $\varepsilon_b \equiv E_b - \Phi$. The transport (bias) voltage $V_{\rm transport} \equiv (\mu_L - \mu_R)/e$ is kept fixed here. The repulsion energy of two electrons is U.

1.4.2 Linear Transport

For $V_{\rm transport} \equiv (\mu_L - \mu_R)/e \to 0$, the Coulomb blockade oscillations discussed above are visible as two consecutive peaks at the chemical potentials

$$\begin{aligned} \mu_{\rm dot}(N=1) &= \varepsilon_a - 0 = \varepsilon_a \\ \mu_{\rm dot}(N=2) &= \varepsilon_a + \varepsilon_b + U - \varepsilon_a = \varepsilon_b + U. \end{aligned} \tag{1.31}$$

The distance between the peaks is given by the *addition energies* $\delta E(N) \equiv \mu(N) - \mu(N-1)$,

$$\begin{aligned} \delta E(N=1) &\equiv \mu_{\rm dot}(N=1) - \mu_{\rm dot}(N=0) = \varepsilon_a \\ \delta E(N=2) &\equiv \mu_{\rm dot}(N=2) - \mu_{\rm dot}(N=1) = \varepsilon_b - \varepsilon_a + U. \end{aligned} \tag{1.32}$$

We now consider the particular limit $\varepsilon_b, U \to \infty$. In this limit, things are much simplified as only 0 and a are involved, and we have

$$\dot{p}_a = \gamma_{a0} p_0 - \gamma_{0a} p_a, \quad 1 = p_0 + p_a \tag{1.33}$$

In the stationary case, this becomes

$$\begin{aligned} p_0 &= \frac{\gamma_{0a}}{\gamma_{0a}+\gamma_{a0}}, \quad p_a = \frac{\gamma_{a0}}{\gamma_{0a}+\gamma_{a0}} \\ \rightsquigarrow I &= -e\left\{\gamma^L_{a0}p_0 - \gamma^L_{0a}p_a\right\} = -e\frac{1}{\Gamma_L+\Gamma_R}\left\{\gamma^L_{a0}\gamma_{0a} - \gamma^L_{0a}\gamma_{a0}\right\} \\ &= -e\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\left\{f_L(\varepsilon_a) - f_R(\varepsilon_a)\right\}. \end{aligned} \tag{1.34}$$

In the linear transport regime, this expression can be further simplified: With

$$\begin{aligned} f_L(\varepsilon_a) - f_R(\varepsilon_a) &= f(\varepsilon_a-\mu_L) - f(\varepsilon_a-\mu_R) = f(\varepsilon_a-\mu_L) - f(\varepsilon_a-\mu_L+(\mu_L-\mu_R)) \\ &= (\mu_L-\mu_R)(-f'(\varepsilon_a-\mu_L)) + O(\mu_L-\mu_R)^2, \end{aligned} \tag{1.35}$$

we find

$$I = -e(\mu_L-\mu_R)\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\frac{1}{4k_BT\coth^2\left[\frac{\varepsilon_a-\mu_L}{2k_BT}\right]} + O(\mu_L-\mu_R)^2. \tag{1.36}$$

The *linear conductance* G is defined as the limit $\mu_L \to \mu_R \equiv \mu$ as

$$G \equiv \lim_{(\mu_L-\mu_R)\to 0}\frac{I}{-e(\mu_L-\mu_R)}, \tag{1.37}$$

so that we finally obtain

$$\boxed{G = \frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\frac{1}{4k_BT\coth^2\left[\frac{\varepsilon_a-\mu}{2k_BT}\right]}, \quad \text{linear conductance peak .}} \tag{1.38}$$

This expression describes a temperature–dependent resonance peak, when the voltage Φ and thus the energy $\varepsilon_a = E_a - \Phi$ is swept through the chemical potential μ or, alternatively, ε_a is kept fixed and μ is changed (see Fig. 1.4).

1.4.3 Non–linear transport regime

The interpretation of the results in the non–linear transport regime is a bit more involved: First , the chemical potentials are

$$\begin{aligned} \mu_{\rm dot}(N=1) &= \varepsilon_a - 0 = \varepsilon_a \\ \mu_{\rm dot}(N=2) &= \varepsilon_a + \varepsilon_b + U - \varepsilon_a = \varepsilon_b + U. \end{aligned} \tag{1.39}$$

for the transition $N \leftrightarrow N+1$ between groundstates, and

$$\begin{aligned} \mu'_{\rm dot}(N=1) &= \varepsilon_b - 0 = \varepsilon_b \\ \mu'_{\rm dot}(N=2) &= \varepsilon_a + \varepsilon_b + U - \varepsilon_b = \varepsilon_a + U. \end{aligned} \tag{1.40}$$

for the transitions $N \leftrightarrow N+1$ between ground and exited states. Now, increasing the gate voltage increases Φ and the following energies enter into the transport window $[\mu_R, \mu_L]$:

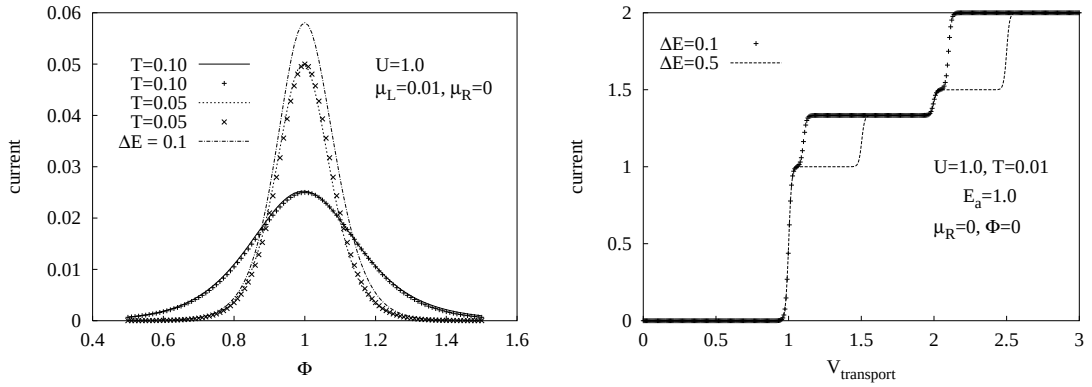

Fig. 1.4: LEFT: Current in the linear transport regime through a quantum dot (as above). Comparison is made between the exact solution of the master equation(solid lines) and the analytical form Eq. 1.38 which works well for $k_BT \ll \Delta E$, where Δ_E is the separation between the two levels. As above, the Coulomb repulsion is $U = 1$ here, and $\Delta E = 0.5$ for the two lower curves, where the agreement is perfect. For a smaller level separation, $\Delta E = 0.1$ and $T = 0.05$ (upper curve), the agreement with the analytical form (for $T = 0.05$) is not good. RIGHT: Coulomb staircase , i.e. current–voltage characteristics of the same system for fixed voltage Φ and variable transport voltage $V_{\rm transport} \equiv \mu_L - \mu_R$.

- First, $\mu_{\rm dot}(N = 1)$ enters which amounts to *one* transport channel.
- Then, $\mu'_{\rm dot}(N = 1)$ enters which amounts to *two* channels which is why the current becomes even larger.
- For the next current peak to occur, one has to increase Φ until $\mu_{\rm dot}(N = 2)$ (not $\mu'_{\rm dot}(N = 2)$) enters the transport window: this leads to two transport channels (large current).
- A further increase of Φ shifts $\mu'_{\rm dot}(N = 2)$ out of the window and we are again left with only one channel.
- A further increase ($\mu_{\rm dot}(N = 2) < \mu_R$) then finally leads to zero current.

2. INTRODUCTION TO COUNTING STATISTICS

2.1 Markovian Master Equation for Classical Systems

Let us assume transitions of a system between states $n = 0, 1, 2, ...$, for example the population of a species. The transitions shall occur at certain rates,

$$\begin{aligned} &\Gamma_n, \quad \text{forward transition from state } n \text{ to } n+1 \\ &\mu_n, \quad \text{backward transition from state } n \text{ to } n-1 \ . \end{aligned} \tag{2.1}$$

We denote by $p(n,t)$ the probablity to find the system in state n at time t. This probability changes with time due to the transitions,

$$\begin{aligned} \dot{p}(n,t) &= -\Gamma_n p(n,t) + \Gamma_{n-1} p(n-1,t), \quad \text{due to forward trans.} \\ &- \mu_n p(n,t) + \mu_{n+1} p(n+1,t), \quad \text{due to backward transitions} \\ &= -(\Gamma_n + \mu_n) p(n,t) + \Gamma_{n-1} p(n-1,t) + \mu_{n+1} p(n+1,t), \quad n \geq 1 \\ \dot{p}(0,t) &= -\Gamma_0 p(0,t) + \mu_1 p(1,t). \end{aligned} \tag{2.2}$$

Such a process is a continuous **Markov process** (no memory effects), and we call the differential equation (more precisely: the system of equations) for the probabilities the **Master equation**.

The particular case of our simple Master equation Gl. (2.2) is often called a *birth-death process*. We can write the equations in matrix form,

$$\dot{\mathbf{p}}(t) = M\mathbf{p}(t), \quad M \equiv \begin{pmatrix} -\Gamma_0 & \mu_1 & 0 & ... \\ \Gamma_0 & -\Gamma_1 - \mu_1 & \mu_2 & 0... \\ 0 & \Gamma_1 & -\Gamma_2 - \mu_2 & \mu_3.... \\ .. & ... & ... & ... \end{pmatrix} \mathbf{p}(t). \tag{2.3}$$

If n runs through all integers, this is an infinite matrix. If the process stops at a finite $n = N$, e.g. $N = 2$, this looks like

$$\dot{\mathbf{p}}(t) = M\mathbf{p}(t), \quad M \equiv \begin{pmatrix} -\Gamma_0 & \mu_1 & 0 \\ \Gamma_0 & -\Gamma_1 - \mu_1 & \mu_2 \\ 0 & \Gamma_1 & -\mu_2 \end{pmatrix} \mathbf{p}(t). \tag{2.4}$$

Note that there is no $-\Gamma_2$ in the right lower corner as with $N = 2$, there is no transition from state 2 to state 3. We have tridiagonal matrices M of dimension $N+1$. The upper and lower secondary diagonals contain the positive backwards and forwards transitions

rates, respectively. The diagonal with the negative sum compensates the two secondary diagonals: the matrix M in fact is singular, the sum of all rows yields zero. This is a consequence of probability conservation, since we must have

$$\sum_n p(n,t) = 1 \rightsquigarrow \frac{d}{dt}\sum_n p(n,t) = 0. \tag{2.5}$$

We can express this by defining

$$\langle\langle\tilde{0}| \equiv (1,1,1,1,...) \tag{2.6}$$

$$\rightsquigarrow \langle\langle\tilde{0}|\mathbf{p}(t) = 1 \tag{2.7}$$

$$\langle\langle\tilde{0}|M \equiv (1,1,1,1,...)\begin{pmatrix} -\Gamma_0 & \mu_1 & 0 & ... \\ \Gamma_0 & -\Gamma_1-\mu_1 & \mu_2 & 0... \\ 0 & \Gamma_1 & -\Gamma_2-\mu_2 & \mu_3.... \\ .. & ... & ... & ... \end{pmatrix} = 0, \tag{2.8}$$

M therefore has an eigenvalue zero, the corresponding eigenvector(s) $\mathbf{p}_{\rm st}$ are stationary solutions of the process,

$$0 = \dot{\mathbf{p}}_{\rm st} = M\mathbf{p}_{\rm st}. \tag{2.9}$$

2.2 Forward process with constant transition rates. Generating function

With constant rates,

$$\Gamma_i = \Gamma, \quad \mu_i = \mu, \tag{2.10}$$

the solution of Gl. (2.2) becomes particularly simple.

We discuss the case of only forward transitions, i.e., the backward transition rate

$$\mu = 0, \quad \text{forward process (unidirectional).} \tag{2.11}$$

An example would be the transitions of particles from an infinite particle reservoir (source) into an initially empty second reservoir (drain) at rate Γ. The Master equation then reads

$$\begin{aligned} \dot{p}(0,t) &= -\Gamma p(0,t) \\ \dot{p}(n,t) &= -\Gamma p(n,t) + \Gamma p(n-1,t), \quad n = 1,2,... \end{aligned} \tag{2.12}$$

where we set $p(-1) = 0$. This can be most easily solved by introducing a *generating function*

$$G(s,t) \equiv \sum_{n=0}^{\infty} s^n p(n,t), \quad s \in \mathbb{C}, \tag{2.13}$$

where s is a complex variable (later we will consider values of $s = e^{i\chi}$ on the unit circle). Note that due to normalisation,

$$G(1,t) = 1, \quad \text{normalisation .} \tag{2.14}$$

Summing Gl. (2.12) over n yields

$$\begin{aligned}\sum_{n=0}^{\infty} s^n \dot{p}(n,t) &= \sum_{n=0}^{\infty} s^n \left[-\Gamma p(n,t) + \Gamma p(n-1,t)\right] \\ &= -\Gamma \sum_{n=0}^{\infty} s^n p(n,t) + \Gamma s \sum_{n=0}^{\infty} s^{n-1} \Gamma p(n-1,t) \\ &= \Gamma(s-1) \sum_{n=0}^{\infty} s^n p(n,t). \end{aligned} \tag{2.15}$$

Note that we used the fact that $p(-1,t) = 0$: there is no state -1. We therefore obtain a differential equation for $G(s,t)$,

$$\frac{\partial}{\partial t} G(s,t) = \Gamma(s-1) G(s,t) \rightsquigarrow G(s,t) = e^{(s-1)\Gamma t} G(s,t=0). \tag{2.16}$$

We need to specifiy the initial condition $G(s,0)$ in order to have an explicit solution. A typical choice of initial condition is

$$\begin{aligned} p(n,0) &= \delta_{n,0}, \quad \text{start with 0 particles in the drain reservoir.} \\ \rightsquigarrow G(s,0) &= 1, \end{aligned} \tag{2.17}$$

but one could also imagine other choices, for example $p(n,0) = \delta_{n,m}$ with $m > 0$.

From $G(s,t)$, we obtain the probabilities $p(n,t)$:

$$G(s,t) \equiv \sum_{n=0}^{\infty} s^n p(n,t) \rightsquigarrow p(n,t) = \frac{1}{n!} \frac{d^n}{ds^n} G(s,t)\Big|_{s=0}. \tag{2.18}$$

In our example with $G(s,0) = 1$, we obtain

$$G(s,t) = e^{(s-1)\Gamma t} \tag{2.19}$$

$$\rightsquigarrow p(n,t) = \frac{(\Gamma t)^n}{n!} e^{-\Gamma t}, \quad \text{Mandel formula.} \tag{2.20}$$

In the context of the theory of the photodetector where the formula describes photons emitted from a source and detected at rate Γ, it is often called the Mandel formula. In the context of transport, for example of electrons tunneling between two metals, this is a simple model for a tunnel junction. The tunnel rate Γ then can be calculated, e.g. in perturbation theory (Fermi's Golden Rule), from a microscopic Hamiltonian describing the tunneling of electrons between two metals.

Note that Gl. (2.19) describes a **Poisson process** with parameter Γt. In fact, Γt is the average $\langle n \rangle$ of n that can be most easily obtained as follows:

$$\langle n \rangle \equiv \sum_{n=0}^{\infty} n p(n,t) = s \frac{d}{ds} G(s,t) \Big|_{s=1} = s \frac{d}{ds} e^{(s-1)\Gamma t} \Big|_{s=1} = \Gamma t. \tag{2.21}$$

One can therefore write the Mandel formula Gl. (2.19) as

$$p(n,t) = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle}, \quad \langle n \rangle = \Gamma t. \tag{2.22}$$

In the following, we discuss sightly more complex situations that lead to other distribution functions $p(n,t)$. Experimentally, one would study $p(n,t)$ in order to learn more about system parameters, dynamics etc. This is sometimes called *transport spectroscopy*, in analogy to (optical) spectroscopy.

2.3 Process with internal states

Let us now assume that we can characterise the states of the system with a multi-index (α, n), where α denotes the *internal* state and n the *external* state of the system.

We consider the example of passengers passing one-by-one from an entrance through a turn-style onto a station platform. They enter the turn-style at rate Γ_{in} and leave the turn-style at rate Γ_{out}. We assume that all passengers move 'forwards' onto the platform and no-one 'backwards' from the platform to the entrance.

We use the indices as follows: index $\alpha = 0$: turn-style is empty; $\alpha = 1$ turnstile is occupied; n number of passengers already passed through onto the platform. The probabilities $p(n,t)$ now acquire an additional index α. We have

$p_0(n,t)$, probability for turnstile empty and n passengers on platform at time t

$p_1(n,t)$, probability for turnstile occupied and n passengers on platform at time t .

To find a master equation for the probabilities, we re-call what we did for the birth-death process. We can proceed in an analogous way here;

$$\dot{p}_0(n,t) = -\Gamma_{\text{in}} p_0(n,t) + \Gamma_{\text{out}} p_1(n-1,t) \tag{2.23}$$

$$\dot{p}_1(n,t) = \Gamma_{\text{in}} p_0(n,t) - \Gamma_{\text{out}} p_1(n,t) \tag{2.24}$$

Note that the argument $n-1$ only occurs once in the first line of these equations. We write these equations in a more compact form by introducing the column vector

$$\begin{aligned} \rho(n,t) &\equiv \begin{pmatrix} p_0(n,t) \\ p_1(n,t) \end{pmatrix} \\ \dot{\rho}(n,t) &= \mathcal{L}_0 \rho(n,t) + \mathcal{J} \rho(n-1,t) \\ \mathcal{L}_0 &\equiv \begin{pmatrix} -\Gamma_{\text{in}} & 0 \\ \Gamma_{\text{in}} & -\Gamma_{\text{out}} \end{pmatrix}, \quad \mathcal{J} \equiv \begin{pmatrix} 0 & \Gamma_{\text{out}} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.25}$$

Definition The matrix $\mathcal{J}$ is called **jump-operator**. Master equations for probabilities resolved with respect to a number n of jumps are called **n-resolved Master equations**.

The jump-operator describes the transitions from $n-1$ to n, i.e., the 'jumps' of the passengers onto the platform at rate Γ_{out}.

Before we solve the n-resolved Master equation Gl. (2.23), we introduce another

Definition The probabilities $p_0(t)$ and $p_1(t)$ in

$$\rho(t) \equiv \sum_{n=0}^{\infty} \rho(n,t) = \begin{pmatrix} p_0(t) \\ p_1(t) \end{pmatrix}, \quad \text{reduced probabilities} \tag{2.26}$$

are called **reduced probabilities** of the system. Furthermore, we define the **Full Counting Statistics (FCS)** of the process,

$$p(n,t) \equiv p_0(n,t) + p_1(n,t) \equiv \text{Tr}\rho(n,t), \quad \text{Full Counting Statistics (FCS)} . \tag{2.27}$$

Here, the **trace** over a probability vector is defined by the sum over the probabilities. In particular,

$$\text{Tr}\rho(t) \equiv \langle\langle\tilde{0}|\rho(t) = p_0(t) + p_1(t) = 1. \tag{2.28}$$

The reduced probabilities contain information on the internal states only, in our example therefore on the internal state (empty or busy) of the turnstile but not on the number n of passengers on the platform. In contrast, the FCS contains no information on the internal state α but information on the external state n.

From Gl. (2.23), we find by summation over n

$$\dot{\rho}(t) = (\mathcal{L}_0 + \mathcal{J})\rho(t), \quad \text{Master equation for reduced probabilities.} \tag{2.29}$$

This linear system of ODE with constant coefficients can be easily solved (EXERCISE). We can do even better and solve the full system Gl. (2.23), again by introducing a generating function

$$g(s,t) \equiv \sum_{n=0}^{\infty} s^n \rho(n,t), \quad s \in \mathbb{C} \tag{2.30}$$

whereby we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} s^n \dot{\rho}(n,t) &= \sum_{n=0}^{\infty} s^n \mathcal{L}_0 \rho(n,t) + \sum_{n=0}^{\infty} s^n \mathcal{J} \rho(n-1,t) \\ \rightsquigarrow \dot{g}(s,t) &= (\mathcal{L}_0 + s\mathcal{J})\, g(s,t). \end{aligned} \tag{2.31}$$

Again, this linear system of ODE with constant coefficients can be easily solved, but now we have the advantage of finding the full information on the entire process contained in

one single (vector valued) function $g(s,t)$. We list the quantities of interest,

$$\begin{aligned}
\rho(t) &= g(1,t), \quad \text{reduced probabilities} && (2.32)\\
\rho(n,t) &= \frac{1}{n!}\frac{d^n}{ds^n} g(s,t)\Big|_{s=0} = \int_{-\pi}^{\pi} \frac{d\chi}{2\pi} e^{-in\chi} g\left(s=e^{i\chi},t\right) && (2.33)\\
p(n,t) &= \mathrm{Tr}\rho(n,t). && (2.34)
\end{aligned}$$

In the second line, we used the generating function on the unit circle $g\left(s=e^{i\chi},t\right)$ as a Fourier series representation of $\rho(n,t)$.

We formally solve

$$\dot{g}(s,t) = \left(\mathcal{L}_0 + s\mathcal{J}\right) g(s,t) \rightsquigarrow g(s,t) = e^{(\mathcal{L}_0+s\mathcal{J})t} g(s,t=0) \tag{2.35}$$

by a matrix exponential. We therefore calculate the eigenvalues of

$$\begin{aligned}
\mathcal{L}_0 + s\mathcal{J} &= \begin{pmatrix} -\Gamma_{\mathrm{in}} & s\Gamma_{\mathrm{out}} \\ \Gamma_{\mathrm{in}} & -\Gamma_{\mathrm{out}} \end{pmatrix} && (2.36)\\
\rightsquigarrow 0 &= (\Gamma_{\mathrm{in}}+\lambda)(\Gamma_{\mathrm{out}}+\lambda) - s\Gamma_{\mathrm{out}}\Gamma_{\mathrm{in}} \\
\rightsquigarrow \lambda_{\pm}(s) &= -\frac{\Gamma_{\mathrm{out}}+\Gamma_{\mathrm{in}}}{2}\left(1 \pm \sqrt{1 + 4\frac{\Gamma_{\mathrm{in}}\Gamma_{\mathrm{out}}}{(\Gamma_{\mathrm{out}}+\Gamma_{\mathrm{in}})^2}(s-1)}\right) && (2.37)
\end{aligned}$$

We recognize that for $s=1$ one of the eigenvalues, $\lambda_-(s)$, is zero as must be since the matrix $M = \mathcal{L}_0 + \mathcal{J}$ must be singular (conservation of probability). At large times $t \to \infty$, this eigenvalue determines the solution $g(s,t) = e^{(\mathcal{L}_0+s\mathcal{J})t} g(s,t=0)$ also for $s \neq 1$. By taking the trace, asymptotically one thus has

$$\mathrm{Tr}g(s,t) = \mathrm{Tr}e^{(\mathcal{L}_0+s\mathcal{J})t} g(s,t=0) \sim e^{t\lambda_-(s)}\mathrm{Tr}g(s,t=0), \quad t\to\infty, \tag{2.38}$$

This becomes particularly simple if we choose the initial condition

$$\begin{aligned}
\rho(n,t=0) &= \rho(t=0)\delta_{n,0} \rightsquigarrow g(s,t=0) = \rho(t=0) \\
\rightsquigarrow \mathrm{Tr}g(s,t) &\sim e^{t\lambda_-(s)}, \quad t\to\infty && (2.39)
\end{aligned}$$

because $\mathrm{Tr}g(s,t=0) = \mathrm{Tr}\rho(t=0) = p_0(0) + p_1(0) = 1$.

In order to calculate the FCS $p(n,t)$ at large times, one has to evaluate the Fourier integral

$$p(n,t) = \mathrm{Tr}\rho(n,t) = \int_{-\pi}^{\pi} \frac{d\chi}{2\pi} e^{-in\chi} \mathrm{Tr}g\left(s=e^{i\chi},t\right) \sim \int_{-\pi}^{\pi} \frac{d\chi}{2\pi} e^{-in\chi} e^{t\lambda_-\left(e^{i\chi}\right)}, \quad t\to\infty \tag{2.40}$$

In general, this must be done numerically. For large n, one can evaluate the integral by a stationary phase approximation.

However, instead of calculating the full distribution $p(n,t)$, we are often satisfied with knowing some of its *moments* μ_k or *cumulants* C_k. For example, the first two moments and cumulants are defined as

$$\begin{aligned}
\mu_1(t) &\equiv \sum_{n=0}^{\infty} np(n,t), \quad \mu_2(t) \equiv \sum_{n=0}^{\infty} n^2 p(n,t) && (2.41)\\
C_1(t) &\equiv \mu_1(t), \quad C_2(t) \equiv \mu_2(t) - \mu_1^2(t). && (2.42)
\end{aligned}$$

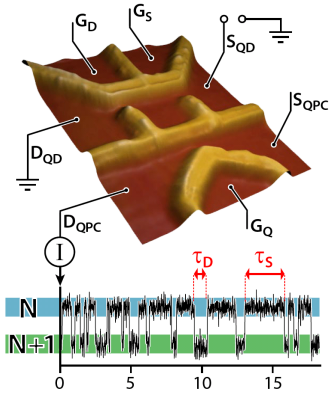

Fig. 2.1: A quantum point contact (QPC) detects single electrons tunneling through a single quantum dot. The time-dependent signal is used to construct the probability distribution $p(n,t)$ of the number n of electgrons tunneled after time t, and its corresponding cumulants $C_k(t)$, cf. Gl. (2.47). From **??**.

Recalling the definition of the generating function, this works for $\mu_1(t)$ as

$$\mathrm{Tr} g(s,t) = \sum_{n=0}^{\infty} s^n p(n,t) \tag{2.43}$$

$$\mu_1(t) = s\frac{\partial}{\partial s}\Big|_{s=1} \mathrm{Tr} g(s,t) = -i\,\frac{\partial}{\partial \chi}\mathrm{Tr}\left(e^{i\chi},t\right)\Big|_{\chi=0} \tag{2.44}$$

At large times, we use Gl. (2.39) to find

$$\mu_1(t) = t\lambda'_-(s=1) = t\frac{\Gamma_{\mathrm{in}}\Gamma_{\mathrm{out}}}{\Gamma_{\mathrm{in}}+\Gamma_{\mathrm{out}}}, \quad t\to\infty \tag{2.45}$$

We define the *stationary current* of passengers through the turnstile as

$$I_{\mathrm{st}} \equiv \lim_{t\to\infty}\frac{\mu_1(t)}{t} = \frac{\Gamma_{\mathrm{in}}\Gamma_{\mathrm{out}}}{\Gamma_{\mathrm{in}}+\Gamma_{\mathrm{out}}} \tag{2.46}$$

It turns out that the cumulants C_k, rather than the moments μ_k, are usually a much more useful way to represent the information contained in the FCS $p(n,t)$. The cumulants are obtained from

$$F(s,t) \equiv \ln \mathrm{Tr} g(s,t) = \ln \mathrm{Tr}\sum_{n=0}^{\infty} s^n p(n,t), \quad \text{Cumulant Generating Function} \tag{2.47}$$

$$C_k(t) = \left(s\frac{\partial}{\partial s}\right)^k F(s,t)\Big|_{s=1} = (-i)^k \frac{\partial^k}{\partial \chi^k} F\left(e^{i\chi},t\right)\Big|_{\chi=0} \tag{2.48}$$

$$= t\times(-i)^k\frac{\partial^k}{\partial\chi^k}\lambda_-\left(e^{i\chi},t\right)\Big|_{\chi=0}, \quad t\to\infty. \tag{2.49}$$

In particular, all cumulants are proportional to the time t at large t: this scaling is a consequence of the law of large numbers and is in analogy to the corresponding scaling of cumulants with volume for the grand-canonical ensemble of thermodynamics (see my lecture notes Thermodynamics and Statistics).

An example where high order cumulants $C_k(t)$ (up to $k = 15$) have been measured in a quantum transport experiment is shown in the figure.

EXERCISE: calculate the second cumulant for the two-state process discussed in this section.

2.4 Transport in Quantum Systems

Let us recall the master equation for the passenger transfer process,Gl. (2.25)

$$\rho(n,t) \equiv \begin{pmatrix} p_0(n,t) \\ p_1(n,t) \end{pmatrix}, \quad \dot{\rho}(n,t) = \mathcal{L}_0\rho(n,t) + \mathcal{J}\rho(n-1,t). \tag{2.50}$$

The state of this classical system at time t is described by a (vector) of probabilities.

In quantum mechanics, the state of a system is described by a *density operator* $\hat{\rho}(t)$ at time t that usually is represented as a matrix

$$\rho_{\alpha\beta}(t) \equiv \langle\alpha|\hat{\rho}(t)|\beta\rangle \tag{2.51}$$

in some basis $\{|\alpha\rangle\}$ of the Hilbert space (we assume finite-dimensional Hilbert spaces in the following). The diagonal elements of $\hat{\rho}(t)$ are probabilities,

$$p_\alpha(t) \equiv \rho_{\alpha\alpha}(t). \tag{2.52}$$

Contrary to the classical case, there are in general also non-diagonal elements of the density operator. They are sometimes called *coherences* and reflect the superposition principle of quantum mechanics: linear combination of Hilbert space vectors are again Hilbert space vectors.

The classical n-resolved master equations like Gl. (2.50) can be written in a completely analogous fashion for the quantum case, however with the vector ρ replaced by the density matrix $\hat{\rho}(n,t)$ and the operators (matrices) $\mathcal{L}_0$ and $\mathcal{J}$ replaced by **superoperators**, for example

$$\frac{d}{dt}\hat{\rho}(n,t) = \mathcal{L}_0\hat{\rho}(n,t) + \mathcal{J}\hat{\rho}(n-1,t) \tag{2.53}$$

for unidirectional transport ('forward' processes only). Here again, a splitting of the total system into an 'internal system' and an 'external system' is assumed, with n referring to the external system (e.g., the number of particles tunneled into a reservoir) and α, β etc. refering to the internal (quantum) system.

For practical reasons, however, one tries to avoid the tensorial notation with 4th rank tensorial superoperators and 2nd rank density operators from the very beginning

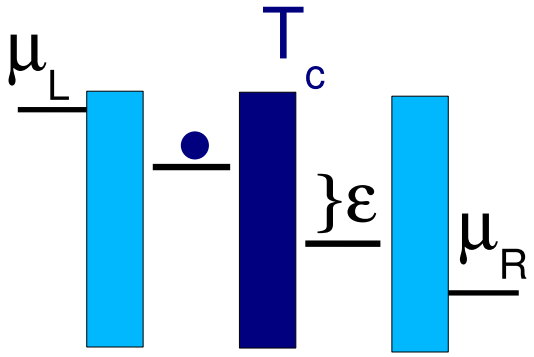

Fig. 2.2: Double quantum dot

by introducing a *vectorial representation of the density operator* $\hat{\rho}$,

$$\rho_{\alpha\beta} \rightarrow \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ .. \\ \rho_{dd} \\ \Re\rho_{12} \\ \Im\rho_{12} \\ ... \end{pmatrix} . \tag{2.54}$$

Here, the first d elements of the column vector represent the diagonal elements of $\hat{\rho}$ (d is the dimension of the internal system Hilbert space), the rest denotes the real and imaginary parts of the coherences. For example, the trace of the density operator is the sum over the first d entries of the vector. Correspondingly, the superoperators become usual square matrices. In contrast to the classical case, in general they now operate not only on the probabilities but also on the coherences. This leads to a mixing of coherences and probabities. A simple example are the *Bloch equations* in nuclear magnetic resonance (NMR), quantum optics, or semiconductor optics. In quantum transport, similar equations can be derived from microscopic Hamiltonians for electronic transport through *solid state qubits*, e.g. double quantum dots.

2.5 Double Quantum Dot

The double quantum dot (charge qubit) is the simplest model where quantum coherence becomes visible in transport. Spin-polarized electrons move between two tunnel-coupled levels $|L\rangle$ (left) and $|R\rangle$ (right) attached to fermionic reservoirs. The Hamiltonian is a

transport version of the spin-boson model ($\hbar = 1$),

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_S + \mathcal{H}_{\rm res} + \mathcal{H}_T + \mathcal{H}_{ep} + \mathcal{H}_p \\
\mathcal{H}_S &= \frac{\varepsilon}{2}\hat{\sigma}_z + T_c\hat{\sigma}_x, \quad \mathcal{H}_{\rm res} = \sum_{k,\alpha=L,R} \varepsilon_k c^\dagger_{k,\alpha} c_{k,\alpha} \\
\mathcal{H}_T &= \sum_{k,\alpha=L,R} (V_k^\alpha c^\dagger_{k,\alpha}|0\rangle\langle\alpha| + H.c.) \\
\mathcal{H}_{ep} &= \hat{\sigma}_z \sum_Q \frac{g_Q}{2}\left(a_{-Q} + a_Q^\dagger\right), \quad \mathcal{H}_p = \sum_Q \omega_Q a_Q^\dagger a_Q,
\end{aligned} \tag{2.55}$$

with pseudo-spin $\hat{\sigma}_z \equiv |L\rangle\langle L| - |R\rangle\langle R|$, $\hat{\sigma}_x \equiv |L\rangle\langle R| + |R\rangle\langle L|$, the 'empty' state $|0\rangle$, the standard tunnel Hamiltonian $\mathcal{H}_T$ for coupling to the reservoirs $\mathcal{H}_{\rm res}$, and coupling of the transport electron in the double dot to a phonon bath $\mathcal{H}_p$ via $\mathcal{H}_{ep}$. One can derive a generalized Master equation in the limit of infinite source-drain bias and in the regime of strong Coulomb blockade, i.e. with only one additional transport electron in the double dot.

The Liouvillian in the basis $\rho = (\rho_0, \rho_L, \rho_R, \Re\rho_{RL}, \Im\rho_{RL})$ has the form

$$\mathcal{L} = \begin{pmatrix}
-\Gamma_L & 0 & \Gamma_R & 0 & 0 \\
\Gamma_L & 0 & 0 & 0 & 2T_c \\
0 & 0 & -\Gamma_R & 0 & -2T_c \\
0 & \gamma_+ & -\gamma_- & -\frac{\Gamma_R}{2} - \gamma & -\varepsilon \\
0 & -T_c & T_c & \varepsilon & -\frac{\Gamma_R}{2} - \gamma
\end{pmatrix} \tag{2.56}$$

with tunnel rates $\Gamma_\alpha = 2\pi \sum_{k_\alpha} |V_k^\alpha|^2 \delta(\varepsilon - \varepsilon_{k_\alpha})$, $\alpha = L/R$ (assumed as energy-independent), rates for electron-phonon interaction

$$\begin{aligned}
\gamma &= \frac{g\pi}{\Delta^2}\left[\frac{\varepsilon^2}{\beta} + 2T_c^2\Delta e^{-\Delta/\omega_c}\coth\left(\frac{\beta\Delta}{2}\right)\right] \\
\gamma_\pm &= g\frac{\pi T_c}{\Delta^2}\left[\frac{\varepsilon}{\beta} - \frac{\varepsilon}{2}\Delta e^{-\Delta/\omega_c}\coth\left(\frac{\beta\Delta}{2}\right) \mp \frac{\Delta^2}{2}e^{-\Delta/\omega_c}\right]
\end{aligned} \tag{2.57}$$

with a dimensionless coupling constant g, a Debye cutoff ω_c, the level splitting

$$\Delta = \sqrt{\varepsilon^2 + 4T_c^2}, \tag{2.58}$$

and the inverse temperature $\beta = (k_B T)^{-1}$. These electron-phonon rates correspond to a bosonic environment with Ohmic spectral density

$$\rho(\omega) = g\omega e^{-\omega/\omega_c}\Theta(\omega). \tag{2.59}$$

Quantum coherence enters via T_c in the off-diagonal terms outside the upper left three-by-three block in Gl. (2.56).

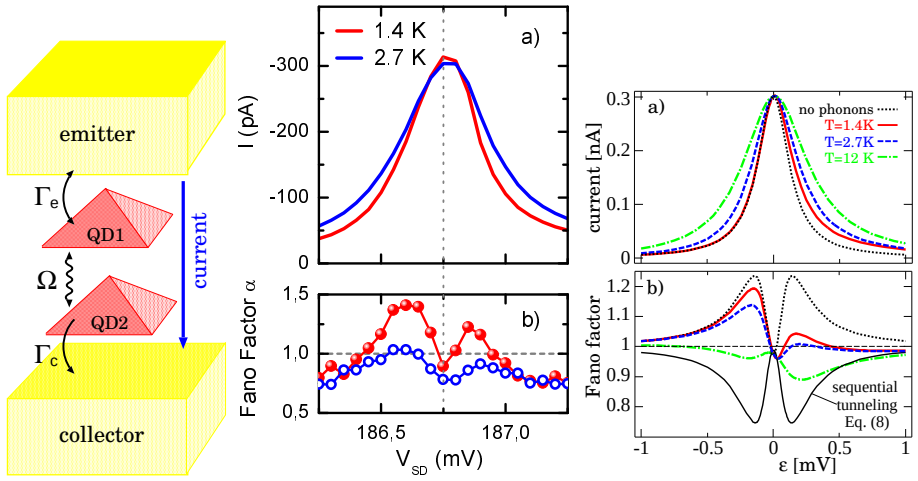

Fig. 2.3: LEFT: Double quantum dot ($\Gamma_e = \Gamma_L$, $\Gamma_c = \Gamma_R$); CENTER: Measured current I and Fano factor F; RIGHT: Calculated I and F. From G. Kießlich, E. Schöll, T. Brandes, F. Hohls, and R. J. Haug, Phys. Rev. Lett. **99**, 206602 (2007).

In the stationary limit $t \to \infty$, one can now easily calculate the cumulants $C_k(t \to \infty)$ that we introduced above. Of particular interest are C_1 and C_2, as these can be compared to experimental results for the stationary current I and the stationary Fano factor F,

$$I \equiv -e \lim_{t\to\infty} \frac{C_1(t)}{t}, \quad F \equiv \lim_{t\to\infty} \frac{C_2(t)}{C_1(t)}. \tag{2.60}$$

Without electron-phonon interaction, the current is given by

$$I = -e \frac{T_c^2 \Gamma_R}{\Gamma_R^2/4 + \varepsilon^2 + T_c^2(2 + \Gamma_R/\Gamma_L)}, \quad \text{Stoof-Nazarov formula.} \tag{2.61}$$

(T. H. Stoof and Yu. V. Nazarov, Phys. Rev. B **53**, 1050 (1996)). As a function of the internal bias ε, this simply is Lorentzian. Some results from a recent comparison with an experiment are shown in the Figure.

A particular limit (not realised experimentally, though) is

$$\lim_{\Gamma_R\to\infty} I = 0, \quad \text{quantum Zeno effect}, \tag{2.62}$$

which can be interpreted as a quantum Zeno effect for permanent observation of the charge qubit by the drain (right) reservoir. The electron is localised on the *left* dot in this limit.

3. INTRODUCTION INTO FEEDBACK CONTROL OF QUANTUM TRANSPORT

3.1 Trajectories and n-resolved Master Equation

In the following, we will always assume a Markovian Master equation. For simplicity, we discuss a situation with a single jump superoperator $\mathcal{J}$, e.g. as in the double quantum dot case in the infinite bias limit discussed above. The Master equation thus is

$$\dot{\rho}(t) \quad = \quad (\mathcal{L}_0 + \mathcal{J})\,\rho(t). \tag{3.1}$$

This can be formally solved as follows: we define

$$\begin{aligned}
\bar{\rho}(t) &\equiv e^{-\mathcal{L}_0 t}\rho(t), \quad \bar{\mathcal{L}}_1(t) \equiv e^{-\mathcal{L}_0 t}\mathcal{J}e^{\mathcal{L}_0 t} \qquad (3.2)\\
\rightsquigarrow \frac{d}{dt}\bar{\rho}(t) &= -\mathcal{L}_0\bar{\rho}(t) + e^{-\mathcal{L}_0 t}\,(\mathcal{L}_0+\mathcal{J})\,e^{\mathcal{L}_0 t}\bar{\rho}(t) = \bar{\mathcal{L}}_1(t)\bar{\rho}(t)\\
\rightsquigarrow \bar{\rho}(t) &= \rho(0) + \int_0^t dt_1 \bar{\mathcal{J}}(t_1)\bar{\rho}(t_1)\\
&= \rho(0) + \int_0^t dt_1 \bar{\mathcal{J}}(t_1)\rho(0) + \int_0^t dt_1 \int_0^{t_1} dt_2 \bar{\mathcal{J}}(t_1)\bar{\mathcal{J}}(t_2)\bar{\rho}(t_2)\\
&...\\
&= \rho(0) + \sum_{n=1}^{\infty}\int_0^t dt_1 ... \int_0^{t_n} dt_n \bar{\mathcal{J}}(t_1)...\bar{\mathcal{J}}(t_n)\rho(0). \qquad (3.3)
\end{aligned}$$

Transforming back to $\rho(t)$, we can explicitely write this as

$$\begin{aligned}
\rho(t) &= e^{\mathcal{L}_0 t}\rho(0)\\
&+ \sum_{n=1}^{\infty}\int_0^t dt_1 ... \int_0^{t_n} dt_n e^{\mathcal{L}_0 t}e^{-\mathcal{L}_0 t_1}\mathcal{J}e^{\mathcal{L}_0 t_1}e^{-\mathcal{L}_0 t_2}\mathcal{J}e^{\mathcal{L}_0 t_2}...e^{-\mathcal{L}_0 t_n}\mathcal{J}e^{\mathcal{L}_0 t_n}\rho(0)\\
&= e^{\mathcal{L}_0 t}\rho(0)\\
&+ \sum_{n=1}^{\infty}\int_0^t dt_1 ... \int_0^{t_n} dt_n \underline{e^{\mathcal{L}_0(t-t_1)}\mathcal{J}e^{\mathcal{L}_0(t_1-t_2)}\mathcal{J}e^{\mathcal{L}_0(t_2-t_3)}...e^{\mathcal{L}_0(t_{n-1}-t_n)}e^{\mathcal{L}_0 t_n}\rho(0)}\\
&\equiv e^{\mathcal{L}_0 t}\rho(0) + \sum_{n=1}^{\infty}\int_0^t dt_1 ... \int_0^{t_n} dt_n \underline{\rho_c(t;t_1,...,t_n)}, \qquad (3.4)
\end{aligned}$$

where we defined the un-normalised, *conditioned* 'density matrix' $\rho_c(t;t_1,...,t_n)$ at time t with n quantum jumps occuring at times $t_1,...,t_n$. This object (the underlined term

in Eq.(3.4)) indeed corresponds to the original density matrix $\rho(0)$, 'freely' time-evolved with the effective Hamiltonian H_{eff} during the time intervals $(0, t_n]$, $(t_n, t_{n-1}]$,... interrupted by n 'jumps' at times t_n, t_{n-1}, ..., t_1. The total density matrix $\rho(t)$ at time t then is the sum over all possible 'trajectories' with $n = 0, ..., \infty$ jumps occuring in between a 'free', effective time evolution. This defines the Full Counting Statistics (FCS), i.e. the probability distribution $p(n, t)$;

$$\rho(t) = \sum_{n=0}^{\infty} \rho^n(t) \quad p(n,t) \equiv \text{Tr}\rho^n(t). \tag{3.5}$$

We mention that the first ideas for this *n-resolved* Master equation were already developed in the early 1980s by Cook and others in their analysis of resonance fluorescence from single atoms.

The n-resolved Master equation (without feedback)

$$\dot{\rho}^n(t) = \mathcal{L}_0\rho^n(t) + \mathcal{J}\rho^{n-1}(t) \tag{3.6}$$

is formally solved by introducing a *counting field* χ and a χ-dependent generalization $\rho(\chi, t)$ of the density matrix $\rho(t)$,

$$\rho(\chi,t) \equiv \sum_{n=0}^{\infty} e^{in\chi}\rho^n(t) \tag{3.7}$$

$$\dot{\rho}(\chi,t) = (\mathcal{L}_0 + e^{i\chi}\mathcal{J})\rho(\chi,t), \quad \rho(0,t) = \rho(t). \tag{3.8}$$

In the following, we will present three routes for introducing feedback control into Markovian Master equations. All of them will be based on the notion of a 'quantum jump', i.e., the jump superoperator $\mathcal{J}$ will play a central role in each of them.

3.2 Earlier Form

One of the historically earlier forms of feedback control in Master equations is due to G. J. Milburn, J. Mod. Opt **38** (10), 1973 (1991), in which a kind of saturation effect in photodetection was introduced in a theory of a photodetector with rates that depended on the number n of detected photons. We will not discuss this model in detail but rather extract the main idea: Essentially, the number n in Gl. (3.7) is a *bath* variable - it is even the only variable of the bath that is continuously monitored. Summing over all n amounts to fully tracing out this information, which yields the usual, not-n-resolved Master equation. Therefore, a natural way to introduce some specific form of control is to assume some backaction from the bath onto the system that depends on the number n, i.e. on the number of quantum jumps (photons, phonons, electrons or other, depending on the form of the jump operator $\mathcal{J}$). The generic form of such a feedback control Master equation then becomes

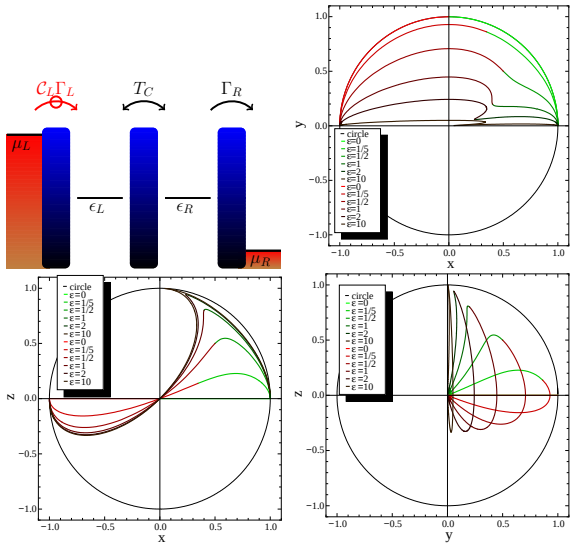

Fig. 3.1: Wiseman-Milburn feedback in transport through a double quantum dot (C. Emary, C. Pöltl, unpublished, 2010).

$$\dot{\rho}^n(t) = \mathcal{L}_n\rho^n(t) + \mathcal{J}_n\rho^{n-1}(t), \qquad (3.9)$$

where the jump- and non-jump operators now depend on the index n. This form of Master equation can, at least in principle, be derived from a microscopic system-bath Hamiltonian by assuming some specific kind of interactions involving a bath number operator $\hat{N}$ with eigenvalues n. The interaction parameters would then lead to, e.g., a functional dependence of transition rates or energies appearing in Gl. (3.9) on n. Note that the form Gl. (3.9) includes a wide range of possibilities. One primitive control mechanism could consist, e.g., in the strong suppression of every second quantum jump by making transitions rates depend on parity (odd or even n).

3.3 Wiseman-Milburn Feedback Scheme

Wiseman and Milburn introduced a feedback scheme that relies on the concept individual quantum jumps. Essentially, their idea is as follows: after each quantum jump, as described by the operation $\mathcal{J}\rho$ on the density matrix, a certain unitary operation is performed on the system, for example the immediate rotation of a qubit about a fixed angle in qubit space after the emission of a particle, cf. the example discussed below. Delays can also be built in but lead to more complicated forms. The Wiseman-Milburn feedback scheme can be formulated in a Master equation

$$\dot{\rho}(t) = \mathcal{L}_0\rho(t) + e^{\mathcal{K}}\mathcal{J}\rho(t). \qquad (3.10)$$

Here, $\rho(t)$ is the trajectory-averaged density operator at time t, and $e^{\mathcal{K}}$ is the new superoperator that acts right after the quantum jump. This is done in such a way

that Gl. (3.10) is still a 'good' Master equation (usually of Lindblad form), but with qualitatively new features as compared with the case $\mathcal{K} = 0$. This allows one, in quite a transparent and direct way, to model an external operation conditioned on the occurence of a quantum jump. As the result of this operation depends on the state $\rho(t)$ at time t, it is justified call this feedback control.

From the point of view of counting statistics, there exists a further motivation of this approach. We recall how the counting field χ was introduced in the eom $\dot{\rho}(\chi, t) = (\mathcal{L}_0 + e^{i\chi}\mathcal{L}_0)\rho(\chi, t)$, cf. Gl. (3.7). The Wiseman-Milburn form Gl. (3.10) thus upgrades the (complex) counting variable to a *super-operator* $\mathcal{K}$. The detector that previously did nothing else but counting now becomes an active device that controls the system.

A recent application of these ideas to transport through a double quantum dot (charge quibit) is shown in the Figure. This is a 'transport version' of similar work by J. Wang, H. M. Wiseman (2001) in an optical two-level system. In the transport case, an interesting observation is the following: parameters can be chosen such that one can generate nearly pure charge qubit state (on the surface of the Bloch sphere that is obtained by projecting out the empty state $|0\rangle$, e.g. in the $\Gamma_L \to \infty$ limit.)

3.4 Continuous Feedback in Quantum Transport

Here I refer to some recent work: T. Brandes, Phys. Rev. Lett. 105, 060602 (2010).

4. FURTHER READING

For an introduction into quantum transport, there is the recent textbook by Blanter and Nazarov. For some more specific material related to these lectures, also see my web-page at TU Berlin. There is a review article, Physics Reports 408/5-6, pp. 315-474 (2005), and there are also some lecture notes by Clive Emary on quantum transport (see his web-page).