*Kending, Taiwan, 01/16/11*

# Non-projective quantum measurement of solid-state qubits: Bayesian formalism

# (what is “inside” collapse)

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Outline:

- Very long introduction (incl. EPR, solid-state qubits)
- Basic Bayesian formalism for quantum measurement and its derivations
- Non-ideal detectors
- Bayesian formalism in circuit QED setup

Acknowledgements

Many useful discussions and collaborations

# Quantum mechanics is weird…

**Niels Bohr:**

“If you are not confused by quantum physics then you haven’t really understood it”

**Richard Feynman:**

“I think I can safely say that nobody understands quantum mechanics”

**Weirdest part is quantum measurement**

# Quantum mechanics =

Schrödinger equation (evolution)

+

collapse postulate (measurement)

1) Probability of measurement result $p_r = | \langle \psi | \psi_r \rangle |^2$

2) Wavefunction after measurement = $\psi_r$

- State collapse follows from common sense
- Does not follow from Schrödinger equation (contradicts, random vs. deterministic, "philosophy")

Collapse postulate is controversial since 1920s
(needs an observer, contradicts causality)

Our focus: what is "inside" collapse, but first discuss EPR

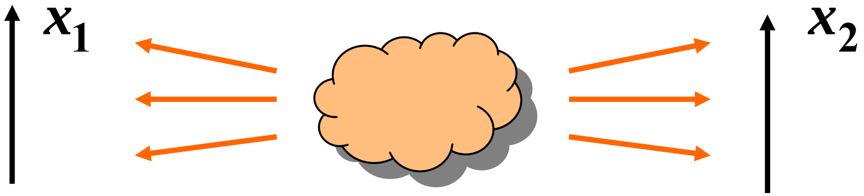
# Einstein-Podolsky-Rosen (EPR) paradox

**Phys. Rev., 1935**

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system.

$$\psi(x_1, x_2) = \sum_n \psi_n(x_2)\, u_n(x_1)$$ *(nowadays we call it entangled state)*

$$\psi(x_1, x_2) = \int_{-\infty}^{\infty} \exp[(i/\hbar)(x_1 - x_2)p]dp \sim \delta(x_1 - x_2)$$



**Measurement of particle 1 cannot affect particle 2, while QM says it affects (contradicts causality)**

**=> Quantum mechanics is incomplete**

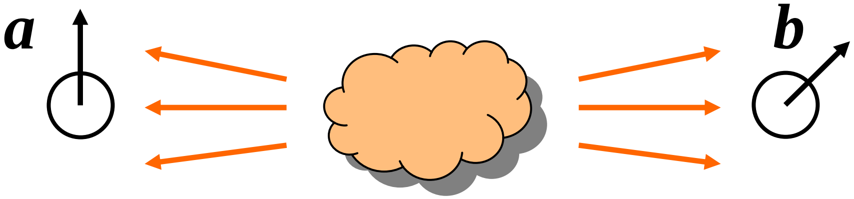
## Bohr's reply

**(Phys. Rev., 1935)** (seven pages, one formula: $\Delta p\,\Delta q \sim h$)
(except in footnotes)

It is shown that a certain "criterion of physical reality" formulated … by A. Einstein, B. Podolsky and N. Rosen contains an essential ambiguity when it is applied to quantum phenomena.

**Very crudely: No need to understand QM, just use the result**

# Bell’s inequality (John Bell, 1964)



(setup by David Bohm)

**Advantage: choice of meas. directions**

$$\psi = \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$$

Perfect anticorrelation of results for same meas. directions, $\vec{a} = \vec{b}$

**Is it possible to explain the QM result assuming local realism and hidden variables (without superluminal collapse)? No!!!**

**Assume:** $A(\vec{a},\lambda) = \pm 1, \quad B(\vec{b},\lambda) = \pm 1,$ perfect anticorr. for $(\vec{a},\vec{a})$ (deterministic result with hidden variable λ)

**Then:** $|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \le 1 + P(\vec{b},\vec{c})$

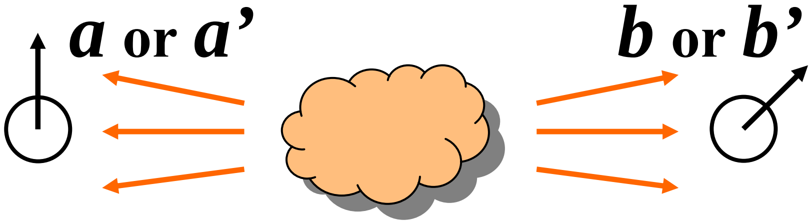
where $P \equiv P(++) + P(--) - P(+-) - P(-+)$

**QM:** $P(\vec{a},\vec{b}) = -\vec{a}\bullet\vec{b}$ For 0°, 90°, and 45°: $0.71 \not\le 1 - 0.71$ **violation!**

**Experiment** (Aspect et al., 1982; photons instead of spins, CHSH):

**yes, “spooky action-at-a-distance”**

# CHSH paper (Clauser, Horne, Shimony, Holt, 1969)



4 experiments instead of 3

**Problem with original Bell ineq.: need perfect anticorrelation for same directions ⇒ not practical!**

**In CHSH perfect anticorrelation not required ⇒ practical**

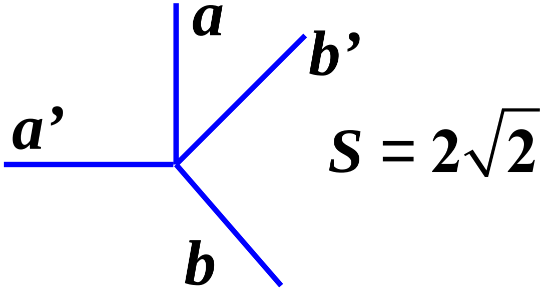
$$|S| \le 2, \ \text{where} \ \ S = P(a,b) - P(a,b') + P(a',b) + P(a',b')$$

(Aspect's version)

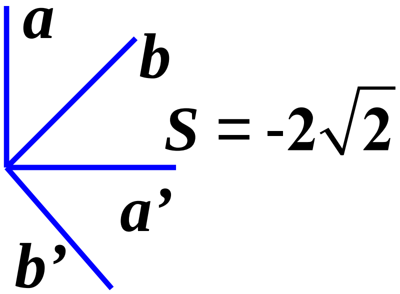
$$P \equiv p(++) + p(--) - p(+-) - p(-+)$$

Maximum violation by QM: $S = \pm 2\sqrt{2}$

$$P(a,b) = -\cos(a,b)$$



$a$=0°, $a'$=270°,
$b$=135°, $b'$=45°



$a$=0°, $a'$=90°,
$b$=45°, $b'$=135°

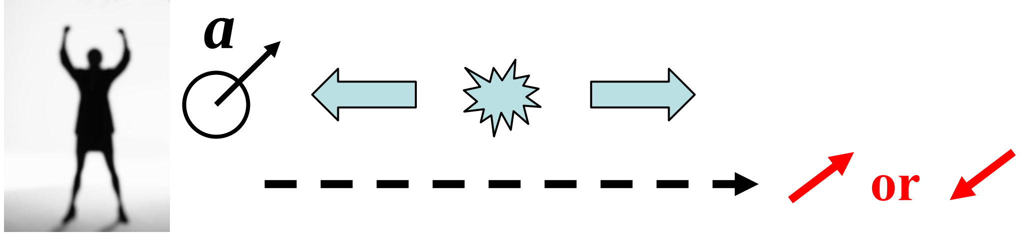
Easy derivation:

| a | a' | b | b' | S |
|---|---|---|---|---|
| + | + | + | + | 2 |
| + | + | + | - | 2 |
| + | + | - | + | -2 |
| + | + | - | - | -2 |
| .. | .. | .. | .. | .. |

Probab. by averaging

# What about causality?

Actually, not too bad: you cannot transmit your own information choosing a particular measurement direction $\boldsymbol{a}$



Result of the other measurement does not depend on direction $\boldsymbol{a}$

**Randomness saves causality**

Collapse is still instantaneous: OK, just our recipe, not an “objective reality”, not a “physical” process

**Consequence of causality: No-cloning theorem**

**Wootters-Zurek, 1982; Dieks, 1982; Yurke**

*You cannot copy an unknown quantum state*

**Proof:** Otherwise get information on direction $\boldsymbol{a}$ (and causality violated)

**Application: quantum cryptography**

**Information is an important concept in quantum mechanics**

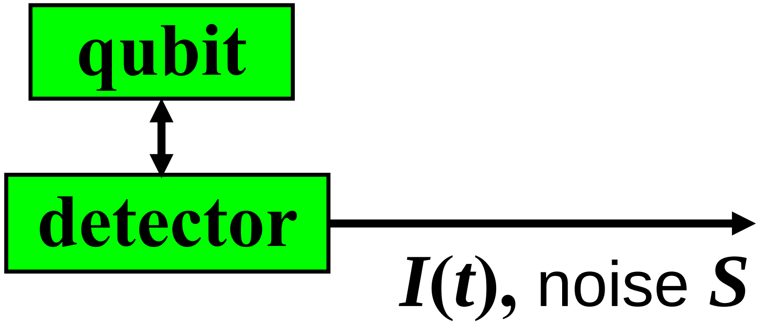
# Quantum measurement in solid-state systems

No violation of locality – too small distances

**However, interesting issue of continuous measurement (weak coupling, noise ⇒ gradual collapse)**

Same origin of paradoxes as in EPR (Schr. Eq. not enough)

**Starting point:**



**qubit**

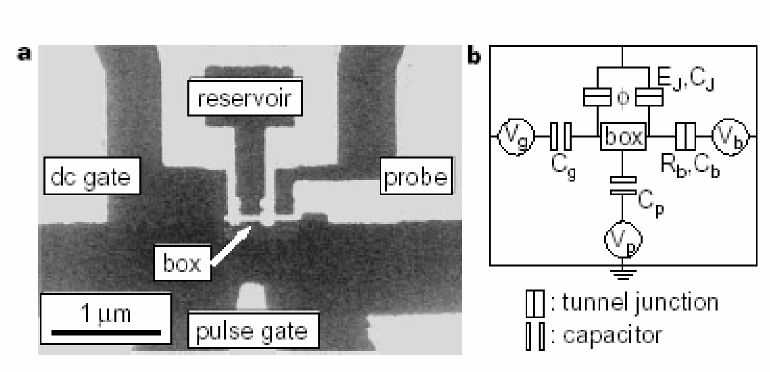
**detector**

$I(t)$, noise $S$

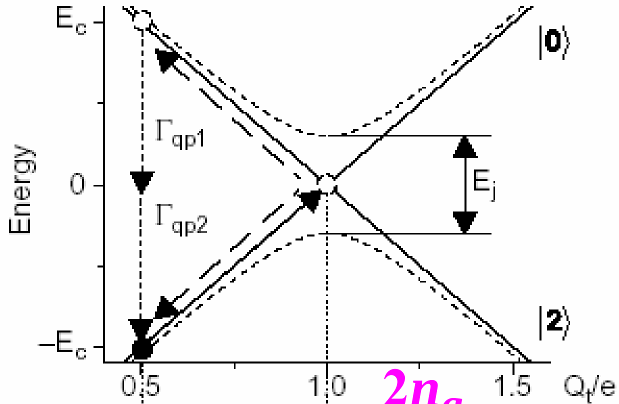
**What happens to a solid-state qubit (two-level system) during its continuous measurement by a detector?**

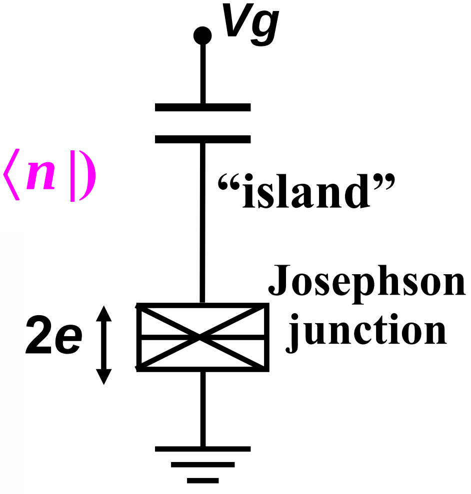
# Superconducting “charge” qubit

**Y. Nakamura, Yu. Pashkin, and J.S. Tsai (Nature, 1998)**

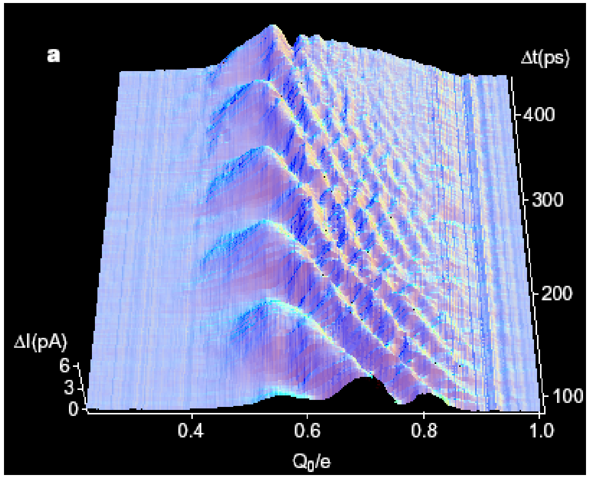


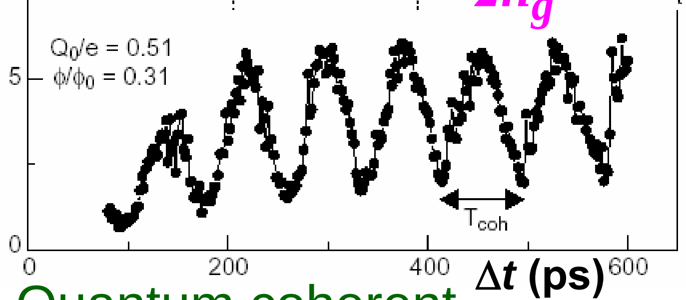
$$\hat{H} = \frac{(2e)^2}{2C}(\hat{n} - n_g)^2 - \frac{E_J}{2}(|n\rangle\langle n+1| + |n+1\rangle\langle n|)$$



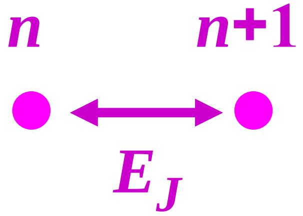


**Single Cooper pair box**





Quantum coherent (Rabi) oscillations

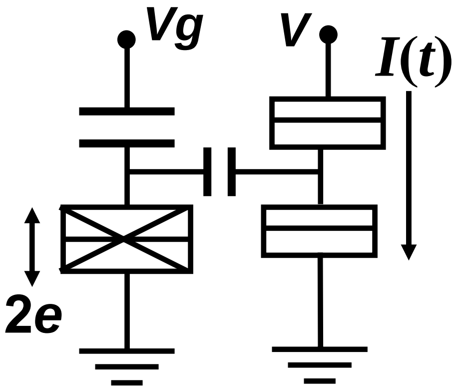


***n*: number of Cooper pairs on the island**

**Vion et al. (Devoret’s group); Science, 2002**

Q-factor of Ramsey oscillations = 25,000

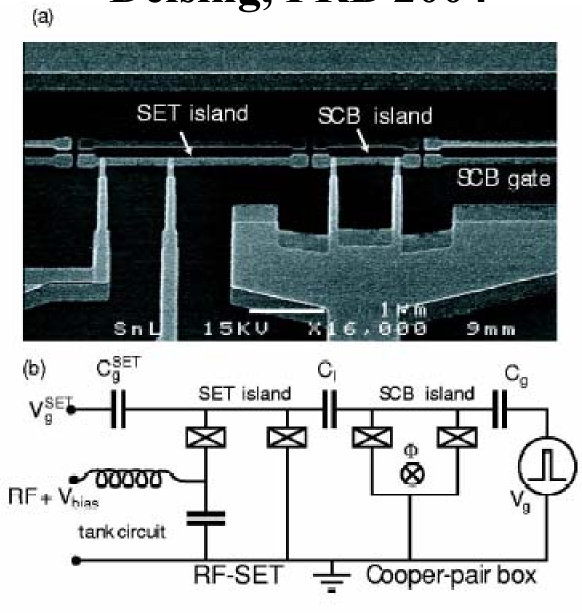
# Charge qubits with SET readout

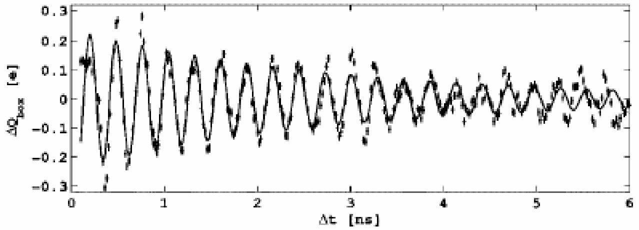


**Cooper-pair box measured by single-electron transistor (rf-SET)**

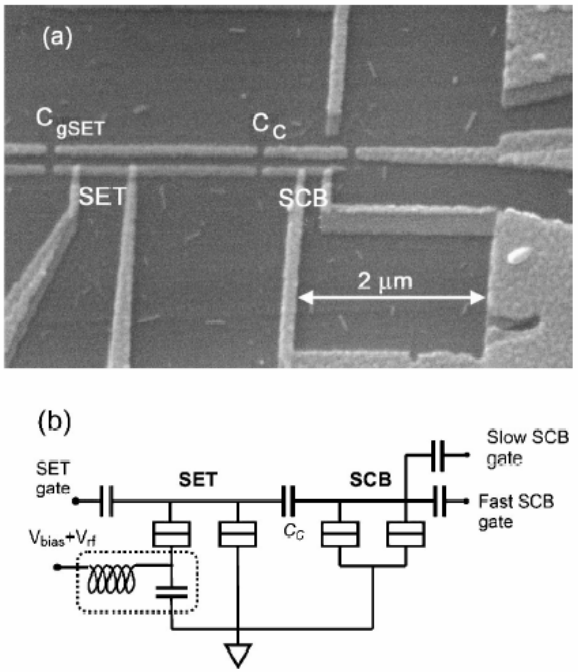
**Setup can be used for continuous measurements**

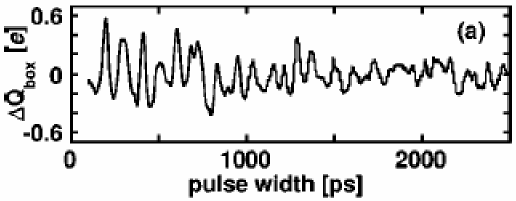
**Duty, Gunnarsson, Bladh, Delsing, PRB 2004**





**Guillaume et al. (Echternach's group), PRB 2004**



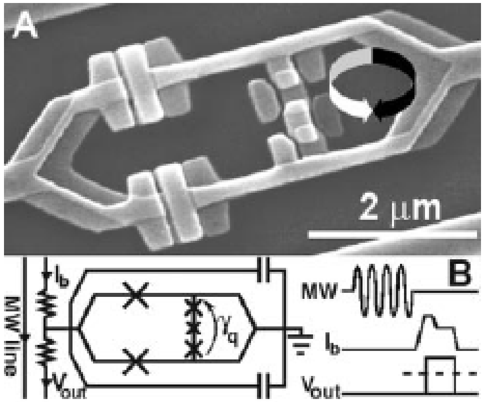


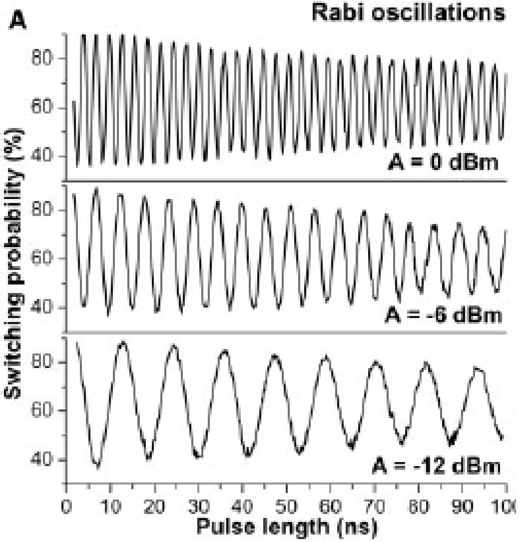
All results are averaged over many measurements (not "single-shot")

# Some other superconducting qubits

## Flux qubit

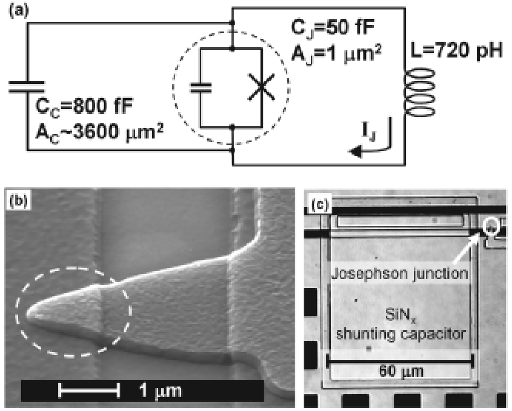
Mooij et al. (Delft)

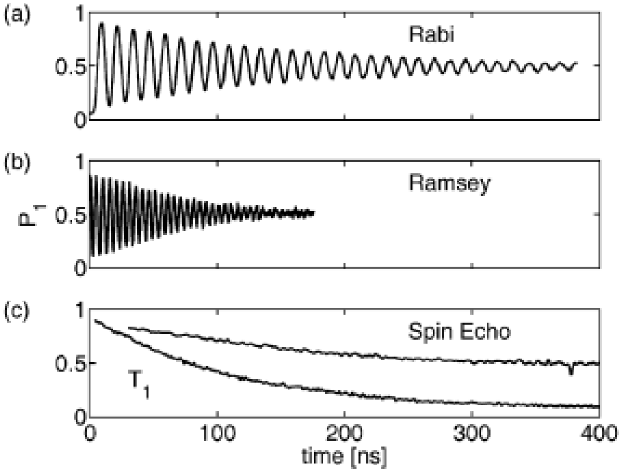




## Phase qubit

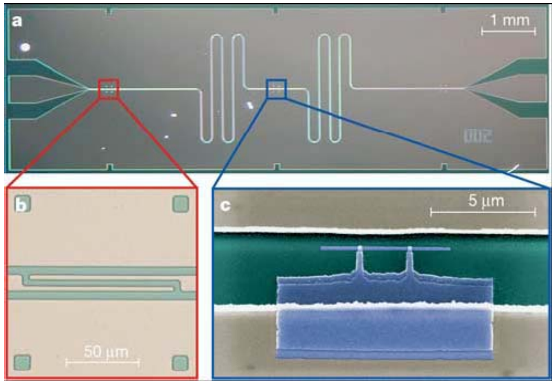
J. Martinis et al.
(UCSB and NIST)

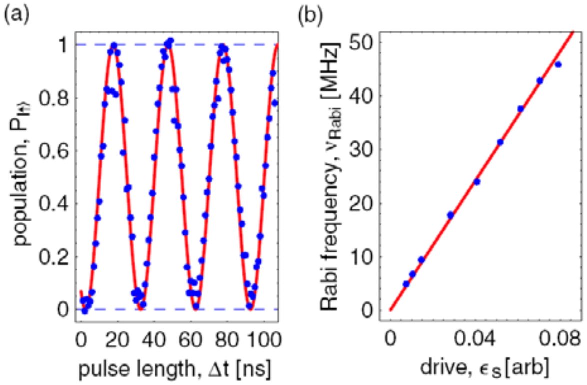




## Charge qubit with circuit QED

R. Schoelkopf et al. (Yale)

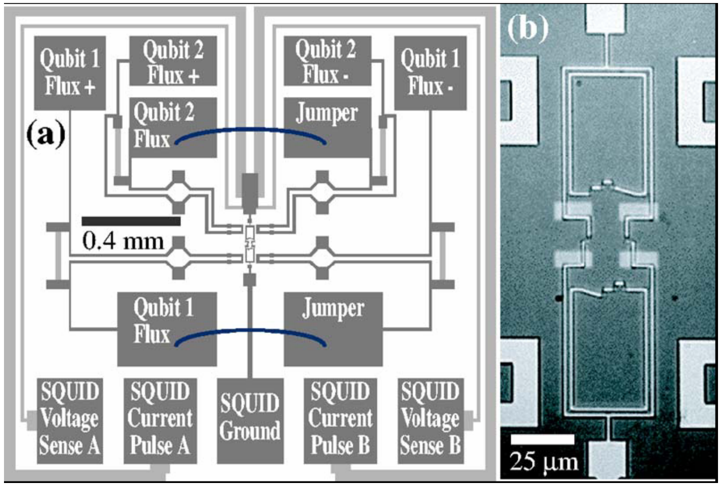


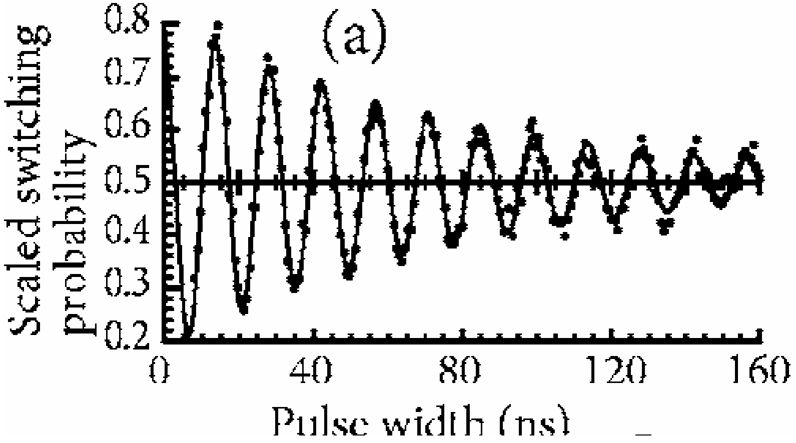


# Some other superconducting qubits

## Flux qubit

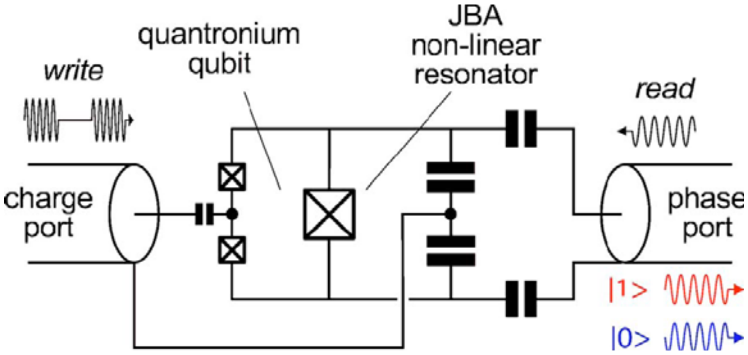
J. Clarke et al. (Berkeley)

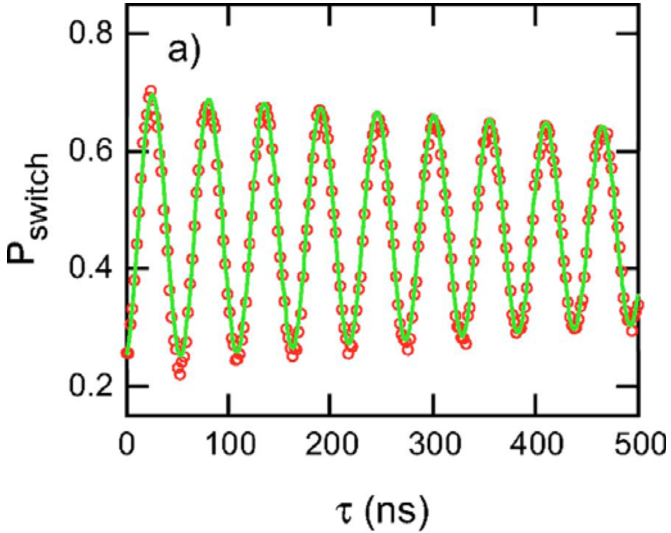




## “Quantronium” qubit

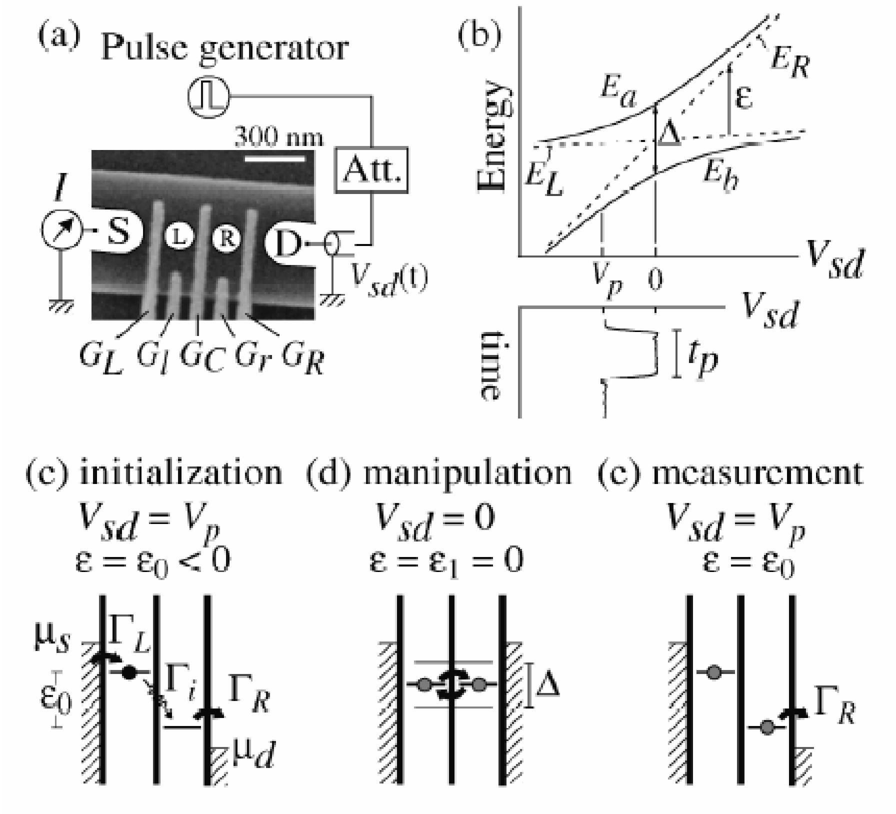
I. Siddiqi, R. Schoelkopf,
M. Devoret, et al. (Yale)

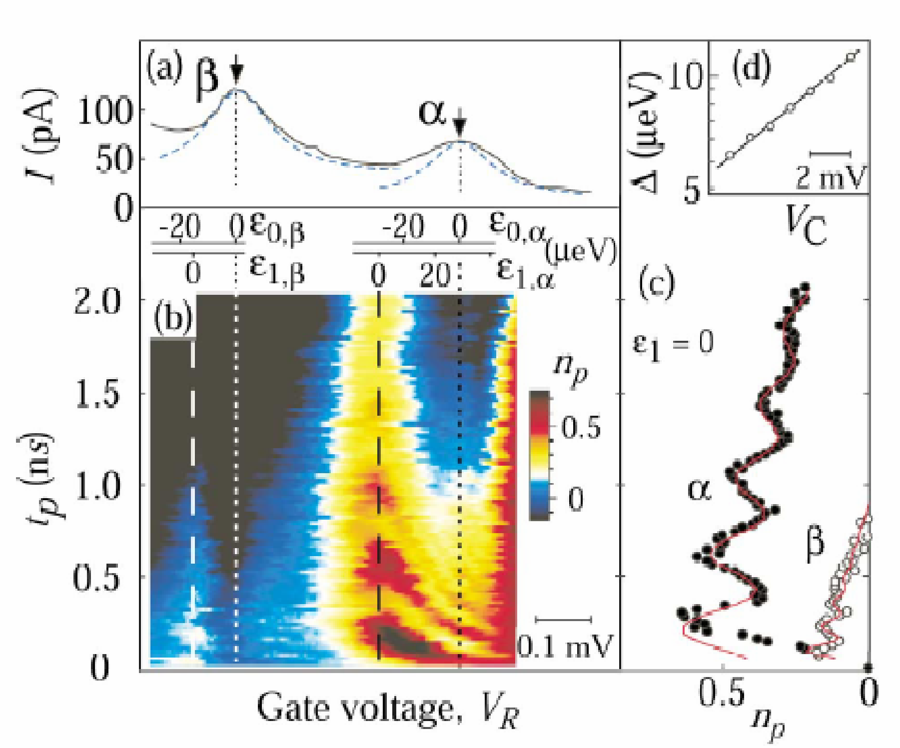




# Semiconductor (double-dot) qubit

**T. Hayashi et al. (NTT), PRL 2003**





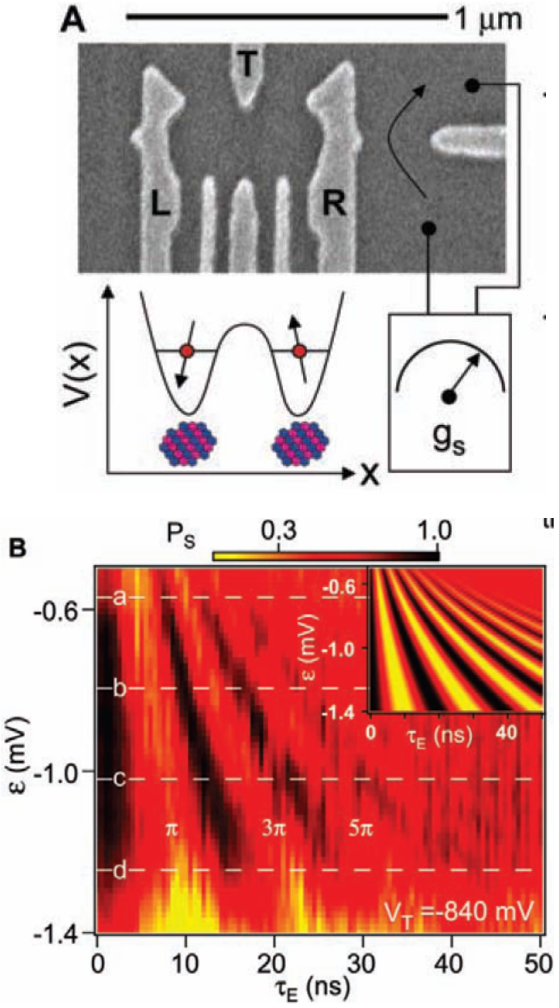
Rabi oscillations

Detector is not separated from qubit,
also possible to use a separate detector

# Some other semiconductor qubits

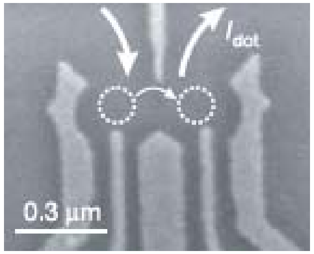
## Spin qubit (QPC meas.)

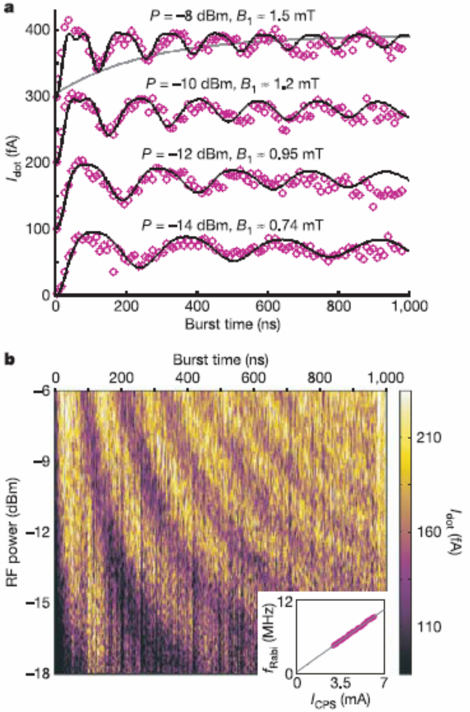
C. Marcus et al. (Harvard)



## Spin qubit

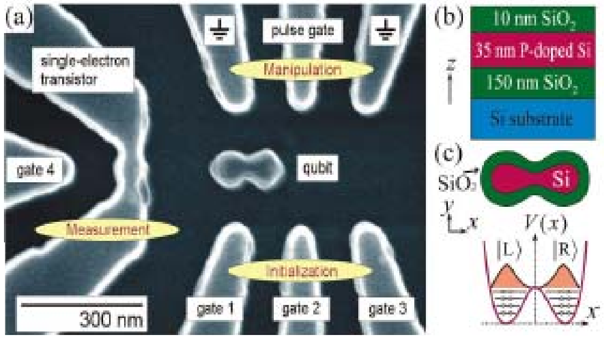
L. Kouwenhoven et al.
(Delft)

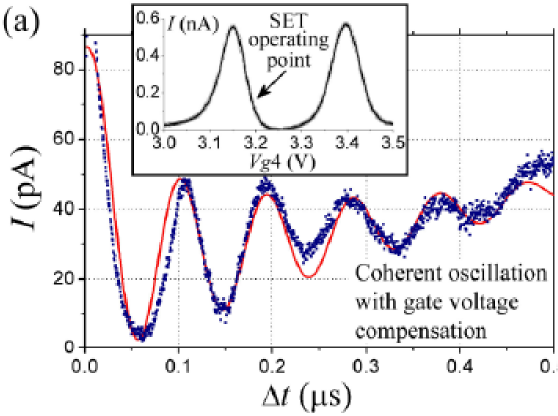




## Double-dot qubit

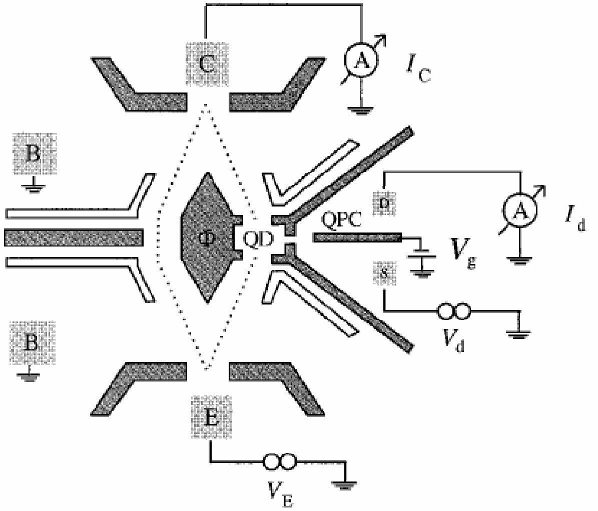
Gorman, Hasko, Williams
(Cambridge)

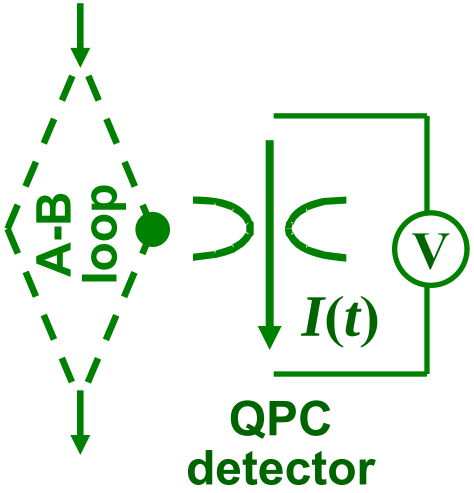


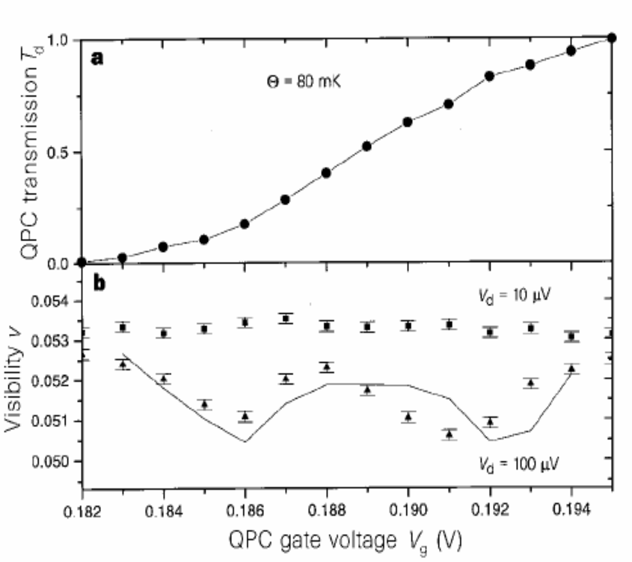


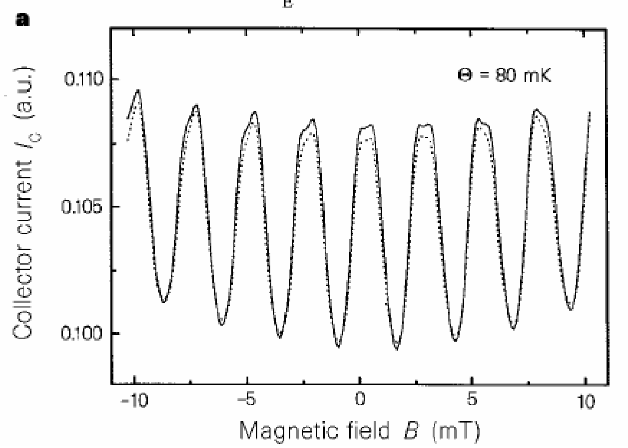
# “Which-path detector” experiment

**Buks, Schuster, Heiblum, Mahalu, and Umansky, Nature 1998**









**Dephasing rate:** $$\Gamma = \frac{eV}{h}\frac{(\Delta T)^2}{T(1-T)} = \frac{(\Delta I)^2}{4S_I}$$

$\Delta I$ – detector response, $S_I$ – shot noise

**The larger noise, the smaller dephasing!!!**

$(\Delta I)^2/4S_I$ ~ rate of “information flow”

**Theory: Aleiner, Wingreen, and Meir, PRL 1997**

# What is the evolution due to measurement? (What is "inside" collapse?)

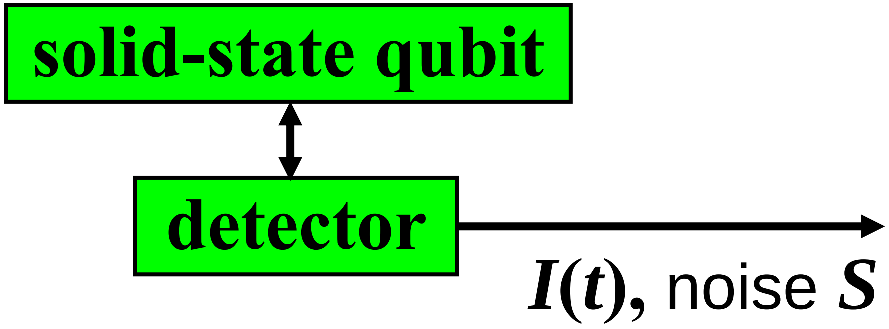
- controversial for last 80 years, many wrong answers, many correct answers
- solid-state systems are more natural to answer this question

## Various approaches to non-projective (weak, continuous, partial, generalized, etc.) quantum measurements

**Names:** Davies, Kraus, Holevo, Mensky, Caves, Knight, Plenio, Walls, Carmichael, Milburn, Wiseman, Gisin, Percival, Belavkin, etc. (very incomplete list)

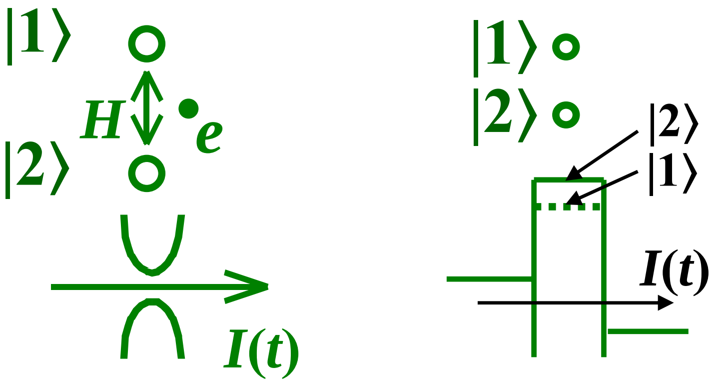
**Key words**: POVM, restricted path integral, quantum trajectories, quantum filtering, quantum jumps, stochastic master equation, etc.

**Our limited scope:**
**(simplest system, experimental setups)**



# "Typical" setup: double-quantum-dot (DQD) qubit + quantum point contact (QPC) detector

Gurvitz, 1997



$H = H_{\mathbf{QB}} + H_{\mathbf{DET}} + H_{\mathbf{INT}}$

$$H_{QB} = \frac{\varepsilon}{2}\sigma_Z + H\sigma_X$$

$$I(t) = I_0 + \frac{\Delta I}{2} z(t) + \xi(t)$$

const + signal + noise

Two levels of average detector current: $I_1$ for qubit state $|1\rangle$, $I_2$ for $|2\rangle$

Response: $\Delta I = I_1 - I_2$ Detector noise: white, spectral density $S_I$

For low-transparency QPC

$$H_{DET} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} T(a_r^\dagger a_l + a_l^\dagger a_r)$$

$$H_{INT} = \sum_{l,r} \Delta T\,(c_1^\dagger c_1 - c_2^\dagger c_2)(a_r^\dagger a_l + a_l^\dagger a_r)$$

$$S_I = 2eI$$

# What happens to a qubit state during measurement?

Start with density matrix **evolution due to measurement only** ($H=\varepsilon=0$)

**"Orthodox" answer**

$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \begin{matrix}\nearrow & \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\\ \searrow & \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\end{matrix}$$

|1> or |2>, depending on the result

**"Decoherence" answer**

$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \rightarrow \begin{pmatrix}\frac{1}{2} & \frac{\exp(-\Gamma t)}{2}\\ \frac{\exp(-\Gamma t)}{2} & \frac{1}{2}\end{pmatrix} \rightarrow \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$$

no measurement result! (ensemble averaged)

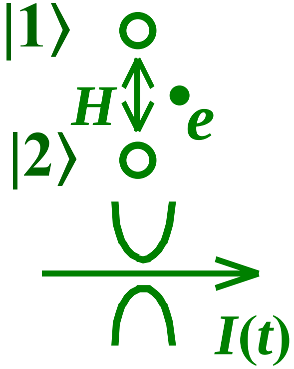
**Decoherence has nothing to do with collapse!**

| applicable for: | single quant. system | continuous meas. |
|---|---|---|
| Orthodox | yes | no |
| Decoherence (ensemble) | no | yes |
| Bayesian, POVM, quant. traject., etc. | yes | yes |

Bayesian (POVM, quant. traj., etc.) formalism describes gradual collapse of a single quantum system, **taking into account measurement result**

# Bayesian formalism for DQD-QPC system

$H_{QB} = 0$



**Qubit evolution due to measurement (quantum back-action):**

$$\psi(t) = \alpha(t)\,|1\rangle + \beta(t)\,|2\rangle \quad \text{or} \quad \rho_{ij}(t)$$

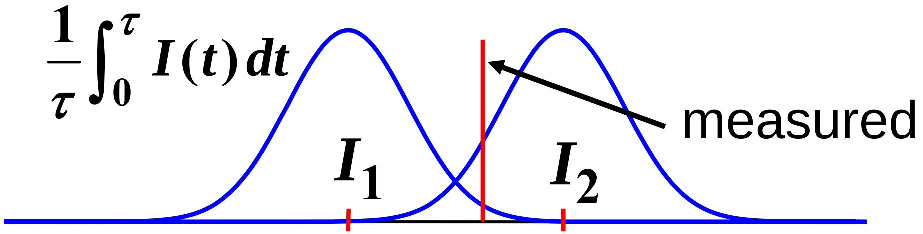
1) $|\alpha(t)|^2$ and $|\beta(t)|^2$ evolve as probabilities, i.e. according to the **Bayes rule** (same for $\rho_{ii}$)

2) phases of $\alpha(t)$ and $\beta(t)$ do not change (**no dephasing!**), $\rho_{ij}/(\rho_{ii}\,\rho_{jj})^{1/2} = \text{const}$

**(A.K., 1998)**

**Bayes rule (1763, Laplace-1812):**

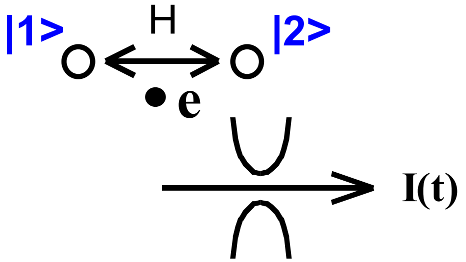
$$\overbrace{P(A_i\,|\,\text{res})}^{\text{posterior probability}} = \frac{\overbrace{P(A_i)}^{\text{prior probab.}}\;\overbrace{P(\text{res}\,|\,A_i)}^{\text{likelihood}}}{\sum_k P(A_k)\,P(\text{res}\,|\,A_k)}$$



So simple because:

1) no entaglement at large QPC voltage
2) QPC happens to be an ideal detector
3) no Hamiltonian evolution of the qubit

# "Quantum Bayes theorem" (ideal detector assumed)



$$H = \varepsilon = 0$$

("frozen" qubit)

Initial state: $\begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}$

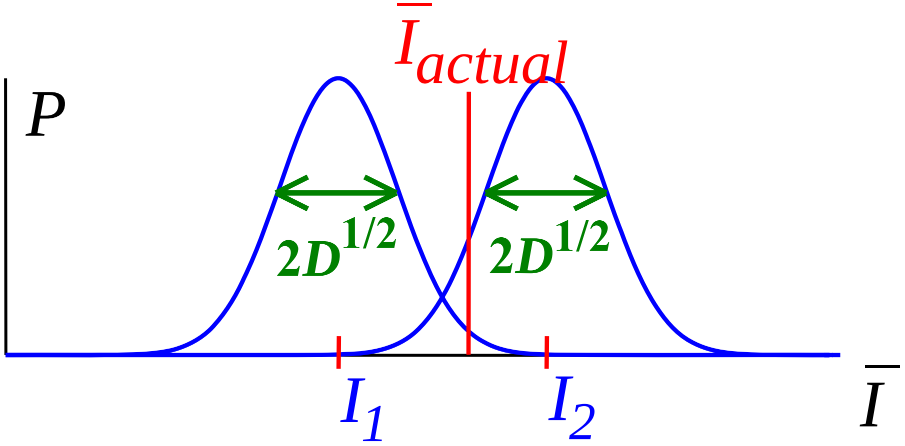
Measurement (during time τ):

$$\bar I \equiv \frac{1}{\tau}\int_0^\tau I(t)\,dt$$

$$P(\bar I,\tau) = \rho_{11}(0)\,P_1(\bar I,\tau) + \rho_{22}(0)\,P_2(\bar I,\tau)$$

$$P_i(\bar I,\tau) = \frac{1}{\sqrt{2\pi D}}\exp[-(\bar I - I_i)^2/2D],$$

$$D = S_I/2\tau, \quad |I_1 - I_2| \ll I_i, \quad \tau \gg S_I/I_i^2$$



After the measurement during time τ, the probabilities should be updated using the standard Bayes formula:

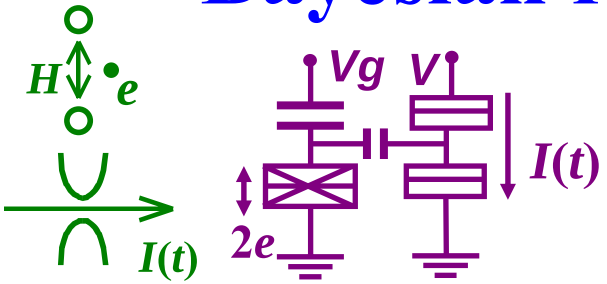
$$P(B_i\,|\,A) = \frac{P(B_i)P(A\,|\,B_i)}{\sum_k P(B_k)P(A\,|\,B_k)}$$

Quantum Bayes formulas:

$$\rho_{11}(\tau) = \frac{\rho_{11}(0)\,\exp[-(\bar I - I_1)^2/2D]}{\rho_{11}(0)\,\exp[-(\bar I - I_1)^2/2D] + \rho_{22}(0)\,\exp[-(\bar I - I_2)^2/2D]}$$

$$\frac{\rho_{12}(\tau)}{[\rho_{11}(\tau)\,\rho_{22}(\tau)]^{1/2}} = \frac{\rho_{12}(0)}{[\rho_{11}(0)\,\rho_{22}(0)]^{1/2}}\,, \qquad \rho_{22}(\tau) = 1 - \rho_{11}(\tau)$$

# Bayesian formalism for a single qubit



- Time derivative of the quantum Bayes rule
- Add unitary evolution of the qubit
- Add decoherence γ (if any)

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = -2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}\frac{2\Delta I}{S_I}[\underline{\underline{I(t)}} - I_0]$$

$$\dot{\rho}_{12} = i\varepsilon\rho_{12} + i\frac{H}{\hbar}(\rho_{11} - \rho_{22}) + \rho_{12}(\rho_{11} - \rho_{22})\frac{\Delta I}{S_I}[\underline{\underline{I(t)}} - I_0] - \gamma\rho_{12}$$

$\Delta I = I_1 - I_2$, $I_0 = (I_1 + I_2)/2$, $S_I$ – **detector noise**

**(A.K., 1998)**

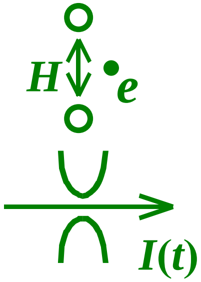
$\gamma = 0$ for QPC

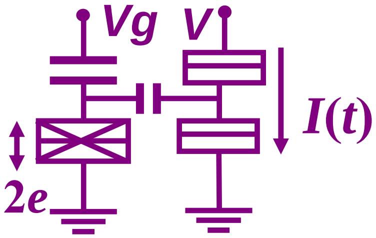
For simulations: $I = I_0 + \dfrac{\Delta I}{2}(\rho_{11} - \rho_{22}) + \xi$

noise $S_\xi = S_I$

Evolution of qubit *wavefunction* can be monitored if γ=0 (quantum-limited)

Natural generalizations:
- add classical back-action
- entangled qubits





# Relation to "conventional" master equation

$$\dot\rho_{11} = -\dot\rho_{22} = -2H\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}\frac{2\Delta I}{S_I}[I(t)-I_0]$$

$$\dot\rho_{12} = i\varepsilon\rho_{12} + iH(\rho_{11}-\rho_{22}) + \rho_{12}(\rho_{11}-\rho_{22})\frac{\Delta I}{S_I}[I(t)-I_0] - \gamma\rho_{12}$$

**$\Delta I$ – detector response, $S_I$ – detector noise** $\hbar = 1$

Averaging over result $I(t)$ leads to conventional master equation:

$$\dot\rho_{11} = -\dot\rho_{22}/dt = -2H\,\mathrm{Im}\,\rho_{12}$$

$$\dot\rho_{12} = i\varepsilon\rho_{12} + iH(\rho_{11}-\rho_{22}) - \Gamma\rho_{12}$$

$\Gamma$ – ensemble decoherence, $\Gamma = \gamma + (\Delta I)^2/4S_I$

Ensemble averaging includes averaging over measurement result

Quantum efficiency: $\eta = \frac{(\Delta I)^2/4S_I}{\Gamma} = \frac{\textbf{quantum}}{\textbf{total}} = 1 - \frac{\textbf{decoherence}}{\textbf{total}}$

# Assumptions needed for the Bayesian formalism:

- Detector voltage is much larger than the qubit energies involved

$eV \gg \hbar\Omega,\ eV \gg \hbar\Gamma,\ \hbar/eV \ll (1/\Omega, 1/\Gamma),\ \ \Omega=(4H^2+\varepsilon^2)^{1/2}$

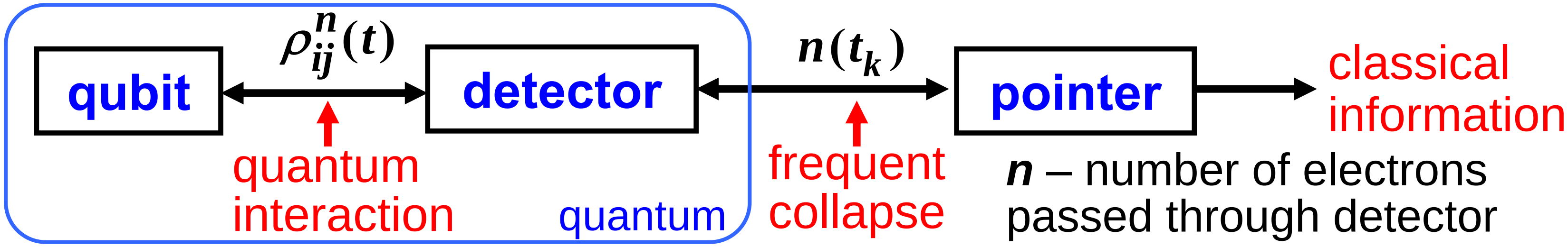
(no coherence in the detector, **classical output**, Markovian approximation)

- Simpler if weak response, $|\Delta I| \ll I_0$, (coupling $C \sim \Gamma/\Omega$ is arbitrary)

# Derivations:

1) "logical": via correspondence principle and comparison with decoherence approach (A.K., 1998)

2) "microscopic": Schr. eq. + collapse of the detector (A.K., 2000)



**qubit**

$\rho_{ij}^{n}(t)$

**detector**

quantum interaction

quantum

$n(t_k)$

frequent collapse

**pointer**

classical information

***n*** – number of electrons passed through detector

3) from "quantum trajectory" formalism developed for quantum optics (Goan-Milburn, 2001; also: Wiseman, Sun, Oxtoby, etc.)

4) from POVM formalism (Jordan-A.K., 2006)

5) from Keldysh formalism (Wei-Nazarov, 2007)

# “Informational” derivation of the Bayesian formalism

(A.K., 1998)

**Step 1.** Assume $H = \varepsilon = 0$ (“frozen” qubit).
Since $\rho_{12}$ is not involved, evolution of $\rho_{11}$ and $\rho_{22}$ should be the same as in the classical case, i.e. Bayes formula (correspondence principle).

**Step 2.** Assume $H = \varepsilon = 0$ and pure initial state: $\rho_{12}(0) = [\rho_{11}(0)\,\rho_{22}(0)]^{1/2}$.
For any realization $|\rho_{12}(t)| \le [\rho_{11}(t)\,\rho_{22}(t)]^{1/2}$ . Then averaging over realizations gives $|\rho_{12}^{av}(t)| \le \rho_{12}^{av}(0)\exp[-(\Delta I^2/4S_I)\,t]$.
Compare with conventional (ensemble) result (Gurvitz-1997, Aleiner et al.-97) for QPC: $\rho_{12}^{av}(t) = \rho_{12}^{av}(0)\exp[-(\Delta I^2/4S_I)\,t]$. Exactly the upper bound!
**Therefore, pure state remains pure**: $\rho_{12}(t) = [\rho_{11}(t)\,\rho_{22}(t)]^{1/2}$.

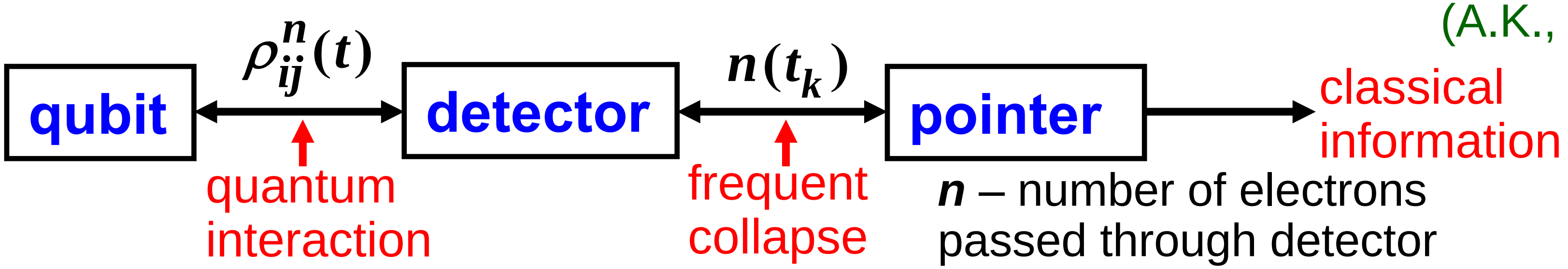
**Step 3.** Account of a mixed initial state
Result: the degree of purity $\rho_{12}(t)/[\rho_{11}(t)\,\rho_{22}(t)]^{1/2}$ is conserved.

**Step 4.** Add qubit evolution due to $H$ and $\varepsilon$.

**Step 5.** Add extra dephasing due to detector nonideality (i.e., for SET).

# “Microscopic” derivation of the Bayesian formalism

(A.K., 2000)



***n*** – number of electrons passed through detector

Schrödinger evolution of “qubit + detector”

for a low-*T* QPC as a detector (Gurvitz, 1997)

$$\frac{d}{dt}\rho_{11}^{n} = -\frac{I_1}{e}\rho_{11}^{n} + \frac{I_1}{e}\rho_{11}^{n-1} - 2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12}^{n}$$

$$\frac{d}{dt}\rho_{22}^{n} = -\frac{I_2}{e}\rho_{22}^{n} + \frac{I_2}{e}\rho_{22}^{n-1} + 2\frac{H}{\hbar}\,\mathrm{Im}\,\rho_{12}^{n}$$

$$\frac{d}{dt}\rho_{12}^{n} = i\frac{\varepsilon}{\hbar}\rho_{12}^{n} + i\frac{H}{\hbar}(\rho_{11}^{n} - \rho_{22}^{n}) - \frac{I_1+I_2}{2e}\rho_{12}^{n} + \frac{\sqrt{I_1 I_2}}{e}\rho_{12}^{n-1}$$

Detector collapse at $t=t_k$

Particular $n_k$ is chosen at $t_k$

$$P(n) = \rho_{11}^{n}(t_k) + \rho_{22}^{n}(t_k)$$

$$\rho_{ij}^{n}(t_k+0) = \delta_{n,nk}\,\rho_{ij}(t_k+0)$$

$$\rho_{ij}(t_k+0) = \frac{\rho_{ij}^{nk}(t_k)}{\rho_{11}^{nk}(t_k) + \rho_{22}^{nk}(t_k)}$$

If $H=\varepsilon=0$, this leads to

$$\rho_{11}(t) = \frac{\rho_{11}(0)P_1(n)}{\rho_{11}(0)P_1(n) + \rho_{22}(0)P_2(n)}, \quad \rho_{22}(t) = \frac{\rho_{22}(0)P_2(n)}{\rho_{11}(0)P_1(n) + \rho_{22}(0)P_2(n)}$$

$$\rho_{12}(t) = \rho_{12}(0)\frac{[\rho_{11}(t)\rho_{22}(t)]^{1/2}}{[\rho_{11}(0)\rho_{22}(0)]^{1/2}}, \quad P_i(n) = \frac{(I_i t/e)^n}{n!}\exp(-I_i t/e),$$

**which are exactly quantum Bayes formulas**

# Derivation via POVM

(Jordan, A.K., 2006)

Quantum measurement in POVM formalism (Nielsen-Chuang, p. 85,100):

Measurement (Kraus) operator $M_r$ (any linear operator in H.S.) : $\psi \rightarrow \dfrac{M_r \psi}{\| M_r \psi \|}$ or $\rho \rightarrow \dfrac{M_r \rho M_r^\dagger}{\mathrm{Tr}(M_r \rho M_r^\dagger)}$

Probability : $P_r = \| M_r \psi \|^2$ or $P_r = \mathrm{Tr}(M_r \rho M_r^\dagger)$

Completeness : $\sum_r M_r^\dagger M_r = 1$

(People often prefer linear evolution and non-normalized states)

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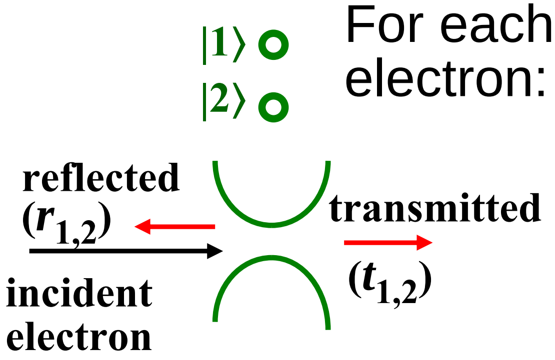


For each incident electron:

$$|in\rangle(\alpha|1\rangle + \beta|2\rangle) \rightarrow \alpha(r_1|L\rangle + t_1|R\rangle)|1\rangle + \beta(r_2|L\rangle + t_2|R\rangle)|2\rangle$$

$$M_{\mathrm{refl}} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad M_{\mathrm{trans}} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

For many incident electrons ⇒ Bayesian formalism

---

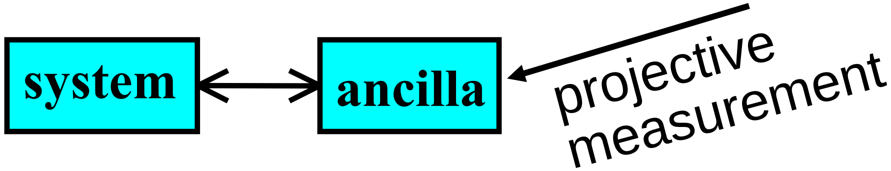
Relation between POVM and quantum Bayesian formalism:

decomposition $M_r = \underset{\text{unitary}}{U_r} \underbrace{\sqrt{M_r^\dagger M_r}}_{\text{Bayes}}$

(almost equivalent)

# Where POVM measurement comes from

Initial state $|\psi_{in}\rangle = \sum_k c_k\,|k\rangle$



Interaction with ancilla

$$|k\rangle|0\rangle \to \sum_{l,a} U_{k,la}\,|l\rangle|a\rangle$$

(*U* comes from unitary transformation in combined Hilbert space system+ancilla)

$$|\psi_{in}\rangle|0\rangle \to \sum_{k,l,a} c_k U_{k,la}\,|l\rangle|a\rangle = \sum_l |l\rangle\left(\sum_{k,a} c_k U_{k,la}\,|a\rangle\right)$$

Project ancilla onto $|r\rangle = \sum_a r_a\,|a\rangle$

$$|\psi_{in}\rangle|0\rangle \to \frac{1}{\text{Norm}}\sum_l |l\rangle\left(\sum_{k,a} c_k U_{k,la}\langle r|a\rangle\right)|r\rangle = \frac{1}{\text{Norm}}\sum_l |l\rangle\sum_k c_k\left(\underbrace{\sum_a r_a^* U_{k,la}}_{M_{r,lk}}\right)|r\rangle$$

So, as a result:

ancilla $|0\rangle \to |r\rangle$

system $\sum_k c_k\,|k\rangle \to \dfrac{\sum_{k,l} M_{r,lk}\,c_k\,|l\rangle}{\text{Norm}}$ i.e. $|\psi_{in}\rangle \to \dfrac{M_r\,|\psi_{in}\rangle}{\text{Norm}}$

# Quantum trajectory formalism for the same system

Goan, Milburn, Wiseman, Sun, 2000
Goan, Milburn, 2001

Ito form

$$\dot{\rho}_c(t) = -\frac{i}{\hbar}[\mathcal{H}_{CQD}, \rho_c(t)] + \mathcal{D}[\mathcal{T} + \mathcal{X} n_1]\rho_c(t)$$
$$+ \xi(t)\frac{\sqrt{\zeta}}{|\mathcal{T}|}[\mathcal{T}^* \mathcal{X} n_1 \rho_c(t) + \mathcal{X}^* \mathcal{T} \rho_c(t) n_1$$
$$- 2\,\mathrm{Re}(\mathcal{T}^* \mathcal{X})\langle n_1\rangle_c(t)\rho_c(t)].$$

$$\mathcal{D}[B]\rho = \mathcal{J}[B]\rho - \mathcal{A}[B]\rho,$$
$$\mathcal{J}[B]\rho = B\rho B^\dagger,$$
$$\mathcal{A}[B]\rho = (B^\dagger B\rho + \rho B^\dagger B)/2.$$

$$|\mathcal{T}_\pm|^2 = D_\pm = 2\pi e|T_{00}|^2 g_L g_R V_\pm/\hbar,$$
$$|\mathcal{T}_\pm + \mathcal{X}_\pm|^2 = D'_\pm = 2\pi e|T_{00} + \chi_{00}|^2 g_L g_R V_\pm/\hbar$$

$$\dot{\rho}_{aa}(t) = i\Omega[\rho_{ab}(t) - \rho_{ba}(t)] - \sqrt{8\Gamma_d}\rho_{aa}(t)\rho_{bb}(t)\xi(t),$$
$$\dot{\rho}_{ab}(t) = i\varepsilon\rho_{ab}(t) + i\Omega[\rho_{aa}(t) - \rho_{bb}(t)] - \Gamma_d\rho_{ab}(t)$$
$$+ \sqrt{2\Gamma_d}\rho_{ab}(t)[\rho_{aa}(t) - \rho_{bb}(t)]\xi(t).$$

$\xi(t)$ - white noise

Looks different, but equivalent to Bayesian formalism

# Stratonovich and Ito forms for nonlinear stochastic differential equations

Definitions of the derivative:

$$\frac{df(t)}{dt}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t/2)-f(t-\Delta t/2)}{\Delta t}$$ (Stratonovich)

$$\frac{df(t)}{dt}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$$ (Ito)

**Why matters?** Usually $(f+df)^2\approx f^2+2f\,df,\ (df)^2\ll df$

But if $df=\xi\,dt$ (white noise ξ), then $(df)^2=\xi^2dt^2\approx\frac{S_\xi}{2}dt$

**Simple translation rule:**

$$\frac{d}{dt}x_i(t)=G_i(\vec{x},t)+F_i(\vec{x},t)\,\xi(t)$$ (Stratonovich)

$$\frac{d}{dt}x_i(t)=G_i(\vec{x},t)+F_i(\vec{x},t)\,\xi(t)+\frac{S_\xi}{4}\sum_k\frac{\partial F_i(\vec{x},t)}{dx_k}F_k(\vec{x},t)$$ (Ito)

Advantage of Stratonovich: usual calculus rules (intuition)

Advantage of Ito: simple averaging

# Methods for calculations

## Monte Carlo

- “Ideologically” simplest
- In many cases most efficient

Idea:
- use finite time step $\Delta t$
- find probability distribution for $\bar{I}(\Delta t)$
- pick a random number for $\bar{I}(\Delta t)$
- do quantum Bayesian update

## Analytics (or non-random numerics)

- Be very careful about Ito-Stratonovich issue
- Use Stratonovich form for derivations (derivatives, etc.)
- Convert into Ito for averaging over noise
- Very good idea to compare with Monte Carlo and/or check second order terms in *dt*

# Fundamental limit for ensemble decoherence

$$\Gamma = (\Delta I)^2/4S_I + \gamma$$

ensemble decoherence rate

~ information flow [bit/s]

single-qubit decoherence

“measurement time” (S/N=1)

$$\tau_m = 2S_I/(\Delta I)^2$$

$$\Gamma \tau_m \geq \frac{1}{2}$$

**A.K., 1998, 2000**
**S. Pilgram et al., 2002**
**A. Clerk et al., 2002**
**D. Averin, 2000,2003**

$$\gamma \geq 0 \;\Rightarrow\; \boxed{\Gamma \geq (\Delta I)^2/4S_I}$$

$$\eta = 1 - \frac{\gamma}{\Gamma} = \frac{(\Delta I)^2/4S_I}{\Gamma}$$

detector ideality (quantum efficiency)
$\eta \leq 100\%$

Translated into energy sensitivity: $(\varepsilon_O\, \varepsilon_{BA})^{1/2} \geq \hbar/2$

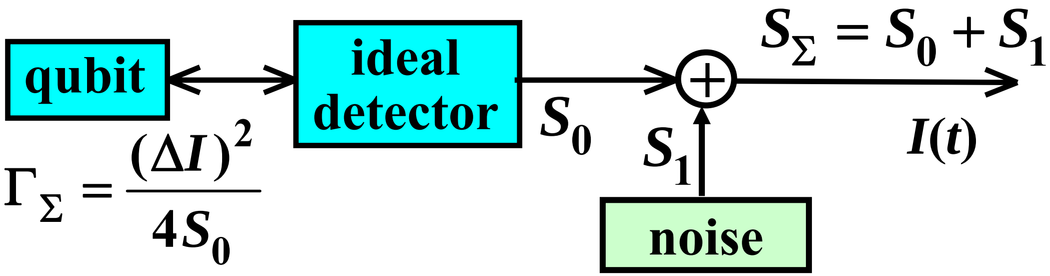
Danilov, Likharev, Zorin, 1983

$\varepsilon_O$, $\varepsilon_{BA}$: sensitivities [J/Hz] limited by output noise and back-action

Known since 1980s (Caves, Clarke, Likharev, Zorin, Vorontsov, Khalili, etc.)

$$(\varepsilon_O\, \varepsilon_{BA} - \varepsilon_{O,BA}{}^2)^{1/2} \geq \hbar/2 \;\Leftrightarrow\; \Gamma \geq (\Delta I)^2/4S_I + K^2 S_I/4$$

# Two ways to think about a non-ideal detector ($\eta$<1)



qubit

ideal detector

$S_0$

$S_1$

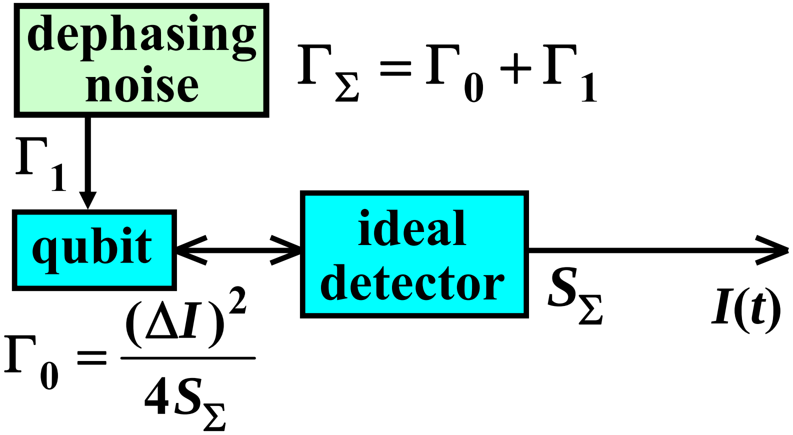
noise

$S_\Sigma = S_0 + S_1$

$I(t)$

$$\Gamma_\Sigma = \frac{(\Delta I)^2}{4S_0}$$

$$\eta = \frac{(\Delta I)^2/4S_\Sigma}{\Gamma_\Sigma}$$



dephasing noise

$\Gamma_1$

qubit

ideal detector

$S_\Sigma$

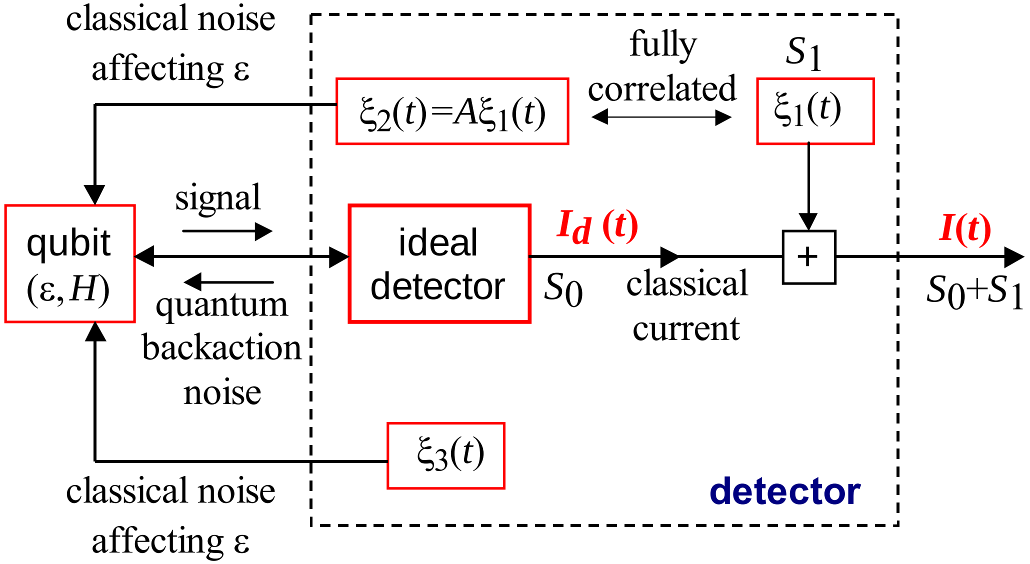
$I(t)$

$$\Gamma_\Sigma = \Gamma_0 + \Gamma_1$$

$$\Gamma_0 = \frac{(\Delta I)^2}{4S_\Sigma}$$

These ways are equivalent
(same results for any expt.)
⇒ **matter of convenience**

# Nonideal detectors with input-output noise correlation



A.K., 2002

$$K = \frac{AS_1 + \theta S_0}{\hbar S_I}, \quad S_I = S_0 + S_1$$

***K*** – correlation between output and ε–backaction noises

$$\frac{d}{dt}\rho_{11} = -\frac{d}{dt}\rho_{22} = -2H\,\mathrm{Im}\,\rho_{12} + \rho_{11}\rho_{22}\frac{2\Delta I}{S_I}[I(t) - I_0]$$

$$\frac{d}{dt}\rho_{12} = i\varepsilon\rho_{12} + iH(\rho_{11} - \rho_{22}) + \rho_{12}(\rho_{11} - \rho_{22})\frac{\Delta I}{S_I}[I(t) - I_0] + \underline{\underline{iK[I(t) - I_0]\rho_{12}}} - \tilde{\gamma}\rho_{12}$$

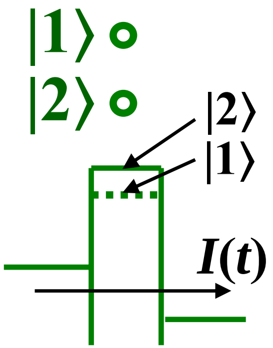
quantum efficiency :

$$\tilde{\eta} = 1 - \frac{\tilde{\gamma}}{\Gamma} = \frac{(\Delta I)^2/4S_I + K^2 S_I/4}{\Gamma}$$

or

$$\eta = \frac{(\Delta I)^2/4S_I}{\Gamma}$$

# A simple general form for a broadband linear detector (QPC, SET, etc.)

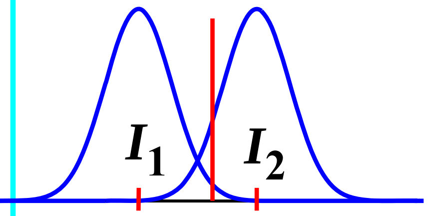


$\Delta I = I_1 - I_2$

noise $S_I$

$$\bar I \equiv \frac{1}{\tau}\int_0^\tau I(t)\,dt$$

$$D = S_I / 2\tau$$



$$H_{qb} = 0$$

quantum backaction (non-unitary, “spooky”, “unphysical”)

$$\begin{cases} \dfrac{\rho_{11}(\tau)}{\rho_{22}(\tau)} = \dfrac{\rho_{11}(0)}{\rho_{22}(0)} \boxed{\dfrac{\exp[-(\bar I - I_1)^2/2D]}{\exp[-(\bar I - I_2)^2/2D]}} \\ \rho_{12}(\tau) = \rho_{12}(0) \boxed{\sqrt{\dfrac{\rho_{11}(\tau)\,\rho_{22}(\tau)}{\rho_{11}(0)\,\rho_{22}(0)}}} \exp(iK\bar I\tau)\exp(-\gamma\tau) \end{cases}$$

no self-evolution of qubit assumed

decoherence

classical backaction (unitary)

## Example of classical (“physical”) backaction:

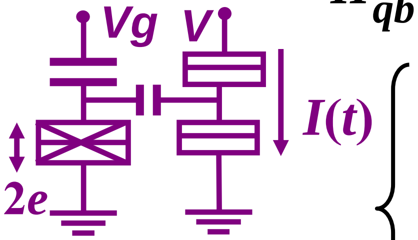
Each electron passed through QPC rotates qubit (sensitivity of tunneling phase for an asymmetric barrier)

$$\arg(T^*\Delta T) \neq 0$$

$$H_{DET} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} T(a_r^\dagger a_l + a_l^\dagger a_r)$$

$$H_{INT} = \sum_{l,r} \Delta T\,(c_1^\dagger c_1 - c_2^\dagger c_2)(a_r^\dagger a_l + a_l^\dagger a_r)$$

# A simple general form for a broadband linear detector (QPC, SET, etc.)



$$H_{qb} = 0$$

$$\begin{cases} \dfrac{\rho_{11}(\tau)}{\rho_{22}(\tau)} = \dfrac{\rho_{11}(0)}{\rho_{22}(0)} \dfrac{\exp[-(\bar{I} - I_1)^2/2D]}{\exp[-(\bar{I} - I_2)^2/2D]} \\ \rho_{12}(\tau) = \rho_{12}(0) \sqrt{\dfrac{\rho_{11}(\tau)\,\rho_{22}(\tau)}{\rho_{11}(0)\,\rho_{22}(0)}} \exp(iK\bar{I}\tau)\exp(-\gamma\tau) \end{cases}$$

quantum backaction (non-unitary, "spooky", "unphysical")

no self-evolution of qubit assumed

decoherence

classical backaction (unitary)

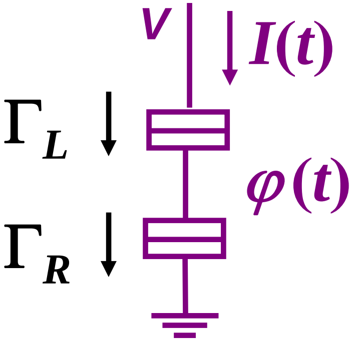
$\Delta I = I_1 - I_2$

noise $S_I$

$$\bar{I} \equiv \frac{1}{\tau}\int_0^\tau I(t)\,dt$$

$$D = S_I / 2\tau$$

Another example of classical backaction:



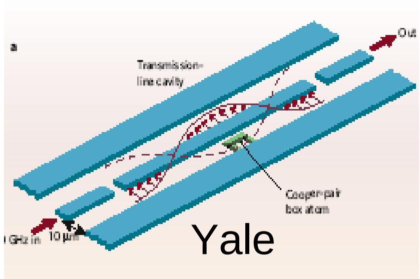
Correlation between voltage and current noises in SET

$$S_{I\varphi} \neq 0 \qquad \frac{S_{I\varphi}(0)}{\sqrt{S_{II}(0)S_{\varphi\varphi}(0)}} = \frac{\Gamma_L - \Gamma_R}{\sqrt{2(\Gamma_L^2 + \Gamma_R^2)}}$$

A.K., 1994

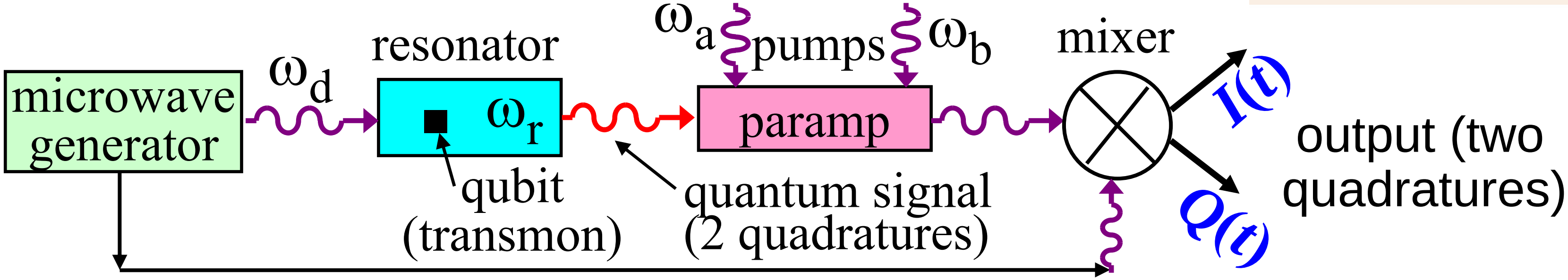
(easy to understand when $\Gamma_L \ll \Gamma_R$)

# Narrowband linear measurement



Difference from broadband: two quadratures

System: qubit in cQED setup + parametric amplifier



microwave generator

$\omega_d$

resonator

$\omega_r$

qubit (transmon)

$\omega_a$ pumps $\omega_b$

paramp

quantum signal (2 quadratures)

mixer

$I(t)$

$Q(t)$

output (two quadratures)

Paramp traditionally discussed in terms of noise temperature

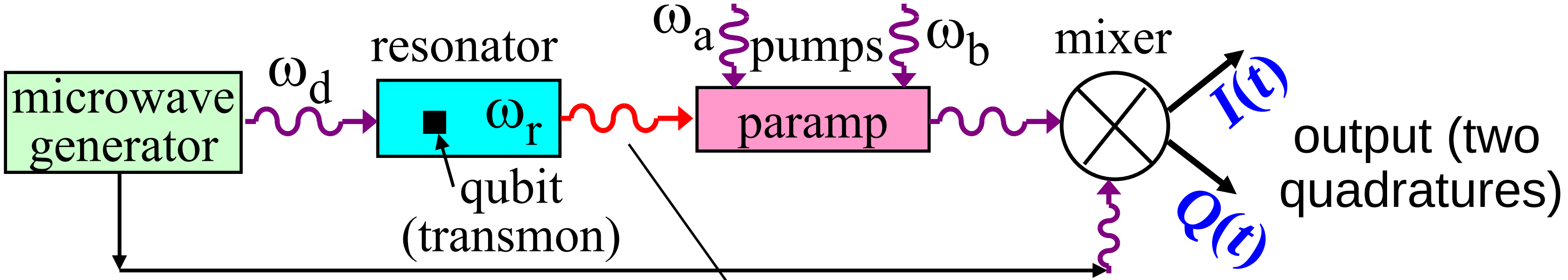
$\theta \geq 0$ for phase-sensitive (degenerate, homodyne) paramp

$\theta \geq \dfrac{\hbar\omega}{2}$ for phase-preserving (non-degenerate, heterodyne) paramp

Haus, Mullen, 1962
Giffard, 1976

Ackn.: Likharev,
Devoret

We will discuss it in terms of qubit evolution due to measurement



# Simplest case

$$H = \frac{\hbar\tilde{\omega}_{qb}}{2}\sigma_z + \hbar\omega_r a^\dagger a + \hbar\chi a^\dagger a \sigma_z$$ (dispersive)

$$\frac{\omega_r}{Q} = \kappa \gg \max(\Gamma, \Omega_R)$$ (Markovian, “bad cavity”)

$\kappa_{out} = \kappa$ (everything collected; i.e. reflection)

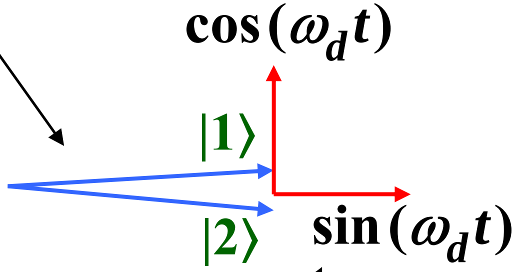
$\chi \ll \kappa$ (weak response)

$\omega_d = \omega_r$ (center of resonance, only phase change if transmission)

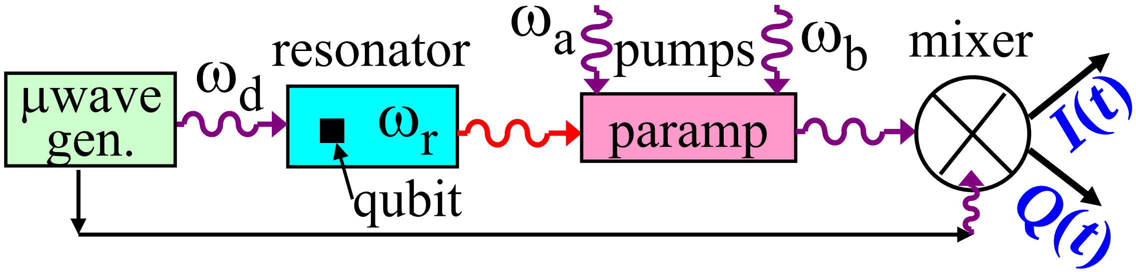
assume everything most ideal

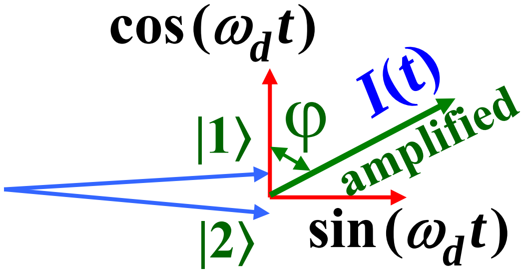
Blais et al., 2004
Gambetta et al., 2006, 2008

carries information about qubit ($\sigma_z$) (quantum back-action)



carries information about fluctuating photon number in the resonator (classical back-action)





# Phase-sensitive (degenerate) paramp

pumps $\omega_a+\omega_b=2\omega_d$

quadrature $\cos(\omega_d t+\varphi)$ is amplified,
quadrature $\sin(\omega_d t+\varphi)$ is suppressed

Assume $I(t)$ measures $\cos(\omega_d t+\varphi)$, then $Q(t)$ not needed

get some information ($\sim\cos^2\varphi$) about qubit state and
some information ($\sim\sin^2\varphi$) about photon fluctuations

$$\begin{cases} \dfrac{\rho_{gg}(\tau)}{\rho_{ee}(\tau)} = \dfrac{\rho_{gg}(0)}{\rho_{ee}(0)} \dfrac{\exp[-(\bar{I} - I_g)^2/2D]}{\exp[-(\bar{I} - I_e)^2/2D]} \\ \rho_{ge}(\tau) = \rho_{ge}(0)\sqrt{\dfrac{\rho_{gg}(\tau)\,\rho_{ee}(\tau)}{\rho_{gg}(0)\,\rho_{ee}(0)}}\,\exp(iK\bar{I}\tau) \end{cases}$$

unitary

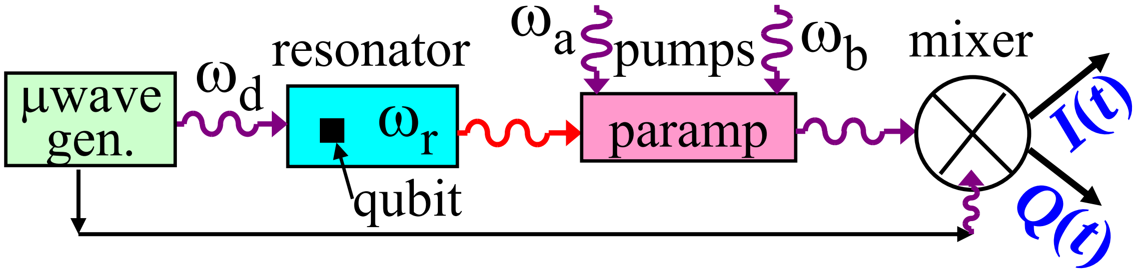
(rotating frame)

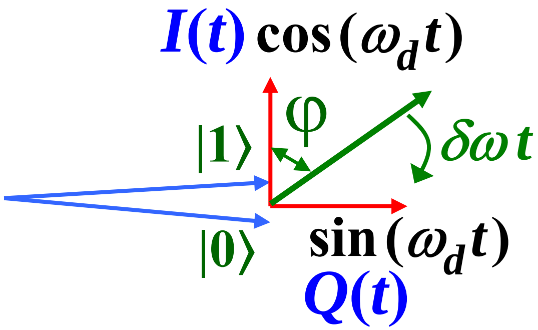
$$\bar{I} \equiv \frac{1}{\tau}\int_0^\tau I(t)\,dt \qquad D = S_I/2\tau$$

$$I_g - I_e = \Delta I\cos\varphi \qquad K = \frac{\Delta I}{S_I}\sin\varphi$$

$$\Gamma = \frac{(\Delta I\cos\varphi)^2}{4S_I} + K^2\frac{S_I}{4} = \frac{\Delta I^2}{4S_I} = \frac{8\chi^2\bar{n}}{\kappa}$$

Same as for QPC/SET, but trade-off ($\varphi$)
between quantum & classical back-actions





# Phase-preserving (nondegenerate) paramp

pumps $\omega_a+\omega_b=2(\omega_d+\delta\omega)$ $\varphi=\delta\omega t$

Now information in both $I(t)$ and $Q(t)$.

Choose
$I(t) \leftrightarrow \cos(\omega_d t)$ (qubit information)
$Q(t) \leftrightarrow \sin(\omega_d t)$ (photon fluct. info)

Small $\delta\omega \Rightarrow$ can follow $\varphi(t)$
Large $\delta\omega$ ($>>\Gamma$) $\Rightarrow$ averaging over $\varphi$ (phase-preserving)

$$\begin{cases} \dfrac{\rho_{gg}(\tau)}{\rho_{ee}(\tau)} = \dfrac{\rho_{gg}(0)}{\rho_{ee}(0)}\,\dfrac{\exp[-(\bar{I}-I_g)^2/2D]}{\exp[-(\bar{I}-I_e)^2/2D]} \\ \rho_{ge}(\tau) = \rho_{ge}(0)\sqrt{\dfrac{\rho_{gg}(\tau)\,\rho_{ee}(\tau)}{\rho_{gg}(0)\,\rho_{ee}(0)}}\exp(iK\bar{Q}\tau) \end{cases}$$

(rotating frame)

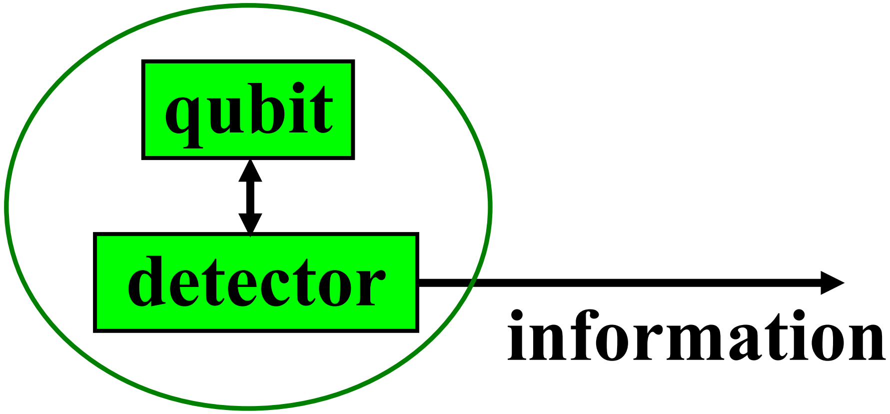
$$\bar{I} \equiv \frac{1}{\tau}\int_0^\tau I(t)\,dt \qquad \bar{Q} \equiv \frac{1}{\tau}\int_0^\tau Q(t)\,dt \qquad D=\frac{S_I}{2\tau}$$

$$I_g - I_e = \frac{\Delta I}{\sqrt{2}} \qquad K = \frac{\Delta I}{\sqrt{2S_I}}$$

$$\Gamma = \frac{\Delta I^2}{8S_I} + \frac{\Delta I^2}{8S_I} = \frac{8\chi^2\bar{n}}{\kappa}$$

Equal contributions to ensemble dephasing from quantum & classical back-actions

# Why not just use Schrödinger equation for the whole system?



qubit

detector

information

## Impossible in principle!

Technical reason: Outgoing information (measurement result) makes it an open system

Philosophical reason: Random measurement result, but deterministic Schrödinger equation

Einstein: God does not play dice

Heisenberg: unavoidable quantum-classical boundary

# Measurement vs. decoherence

Widely accepted point of view:

**measurement = decoherence (environment)**

**Is it true?**

- **Yes**, if not interested in information from detector (ensemble-averaged evolution)
- **No**, if take into account measurement result (single quantum system)

Measurement result obviously gives us more information about the measured system, so we know its quantum state better (ideally, a pure state instead of a mixed state)

# Can we verify the Bayesian formalism experimentally?

**Direct way:**

**prepare** → **partial measur.** → **control (rotation)** → **projective measur.**

A.K.,1998

However, difficult: bandwidth, control, efficiency
(expt. realized only for supercond. phase qubits)

Tricks are needed for real experiments

# Experimental predictions and proposals from Bayesian formalism

- Direct experimental verification (1998)
- Measured spectrum of Rabi oscillations (1999, 2000, 2002)
- Bell-type correlation experiment (2000)
- Quantum feedback control of a qubit (2001, 2004, 2009)
- Entanglement by measurement (2002)
- Measurement by a quadratic detector (2003)
- Squeezing of a nanomechanical resonator (2004)
- Violation of Leggett-Garg inequality (2005)
- Partial collapse of a phase qubit (2005)
- Undoing of a weak measurement (2006, 2008)
- Decoherence suppression by uncollapsing (2010)
- Persistent Rabi revealed in noise (2010)

**3 solid-state experiments realized so far**

# Conclusions

- Quantum measurement is the most controversial and fascinating part of quantum mechanics
- It is easy to see what is "inside" collapse: simple Bayesian formalism works for many solid-state setups
- Classical information plays a very important part in quantum measurement
- Measurement backaction necessarily has a "spooky" part ("unphysical", informational, without a mechanism); it may also have a "classical" part (with a physically understandable mechanism)