General relativistic issues on quantum information: Cloning theorem and non-universal Hawking decay of microscopic black holes

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Motivation

- **To understand black hole physics from quantum information theoretic perspectives: Quantum entanglement, unitary evolution, Hawking decay, safety of the LHC produced microscopic black holes**
- **To prove the Horowitz-Maldacena conjecture on the black hole evaporation [JHEP 02 (2004) 008]**
- **To investigate the quantum cloning in the presence of closed timelike curves**

Quantum Cloning

In quantum mechnics, copying an arbitrary quantum state is forbidden by "No cloning theorem"

$$A\left(|s\rangle \otimes |0\rangle\right) \rightarrow |s\rangle \otimes |s\rangle$$

$$A\left(\{\alpha|0\rangle + \beta|1\rangle\} \otimes |0\rangle\right) \rightarrow \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle$$

$$\neq \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes \left(\alpha|0\rangle + \beta|1\rangle\right)$$

Closed timelike curves (CTCs)

In principle, there is no theoretical barrier to the existence of closed time like curves

M. S. Morris et al., Phys. Rev. lett. 61, 1446 (1988)

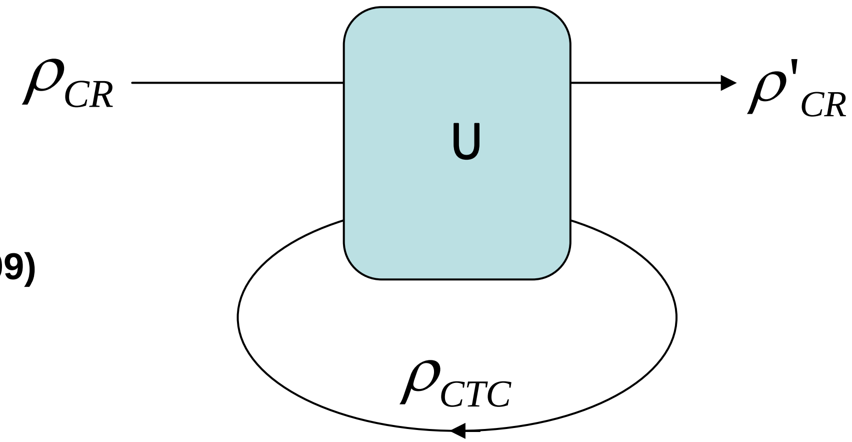
D. Deutsch, Phys. Rev. D 44, 3197 (1991)

S. W. Hawking, Phys. Rev. D 52, 5681 (1995)

T. C. Ralph, Phys. Rev. A76, 032309 (2007)

T. A. Brun et al., Phys. Rev. Lett. 102, 210402 (2009)

C. H. Bennett, Phys. Rev. Lett. 103, 170502 (2009)



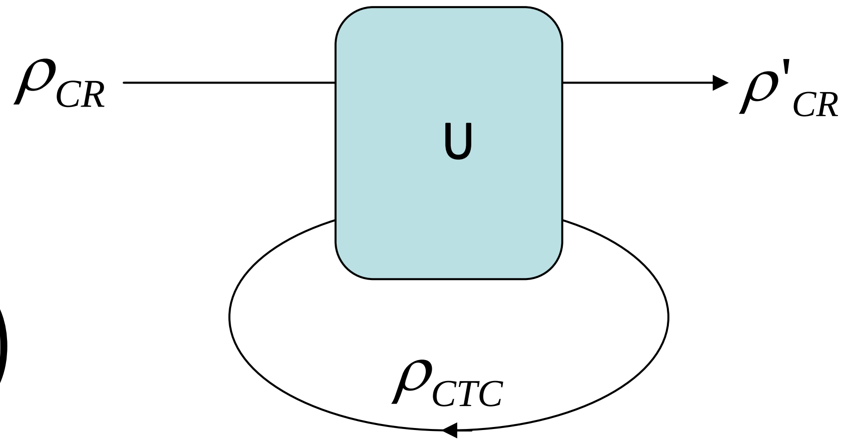
Closed timelike curves (CTCs)

Deutsch's Consistency condition

D. Deutsch, Phys. Rev. D 44, 3197 (1991)

$$\rho_{CTC} = Tr_{CR}\left(U\rho_{CR} \otimes \rho_{CTC} U^{\dagger}\right)$$

$$\rho'_{CR} = Tr_{CTC}\left(U\rho_{CR} \otimes \rho_{CTC} U^{\dagger}\right)$$



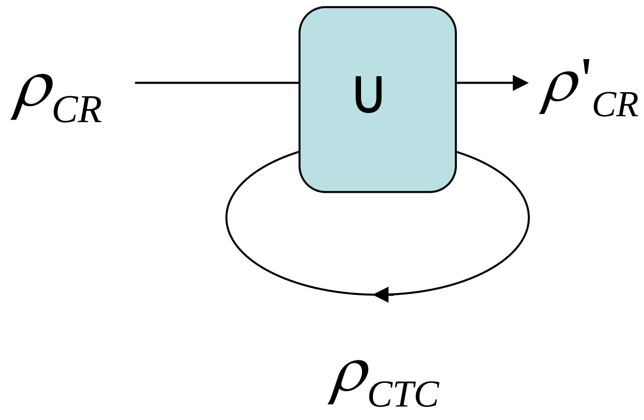
Closed timelike curves (CTCs)

Cloning of an arbitrary quantum states in CTC?

Then,

$$Tr_{CTC}\left(U\rho_s \otimes \Sigma \otimes \rho^s_{CTC} U^\dagger\right) = \rho_s \otimes \rho_s$$

$$\rho^s_{CTC} = Tr_{AB}\left(U\rho_s \otimes \Sigma \otimes \rho^s_{CTC} U^\dagger\right)$$

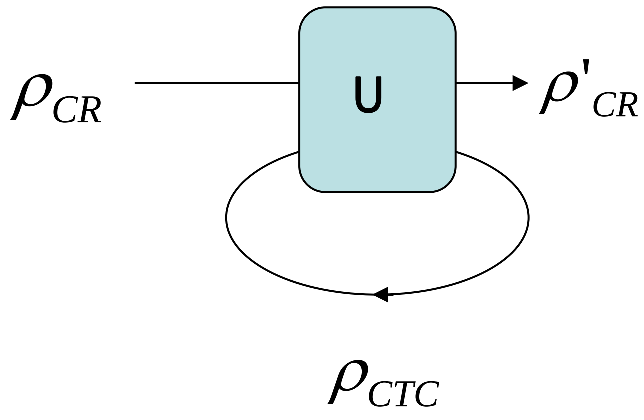


Closed timelike curves (CTCs)

Check the possibility by using the fidelity

$$F(\rho_i,\rho_j)=Tr\sqrt{\rho_i^{1/2}\rho_j\rho_i^{1/2}}$$

$$F(\rho_i\otimes\sigma_i,\rho_j\otimes\sigma_j)=F(\rho_i,\rho_j)F(\sigma_i,\sigma_j)$$



$$\rho_s\rightarrow U\rho_s U^\dagger$$ ***F*** **is invariant**

if $\sigma=Tr_C(\tilde{\sigma})\qquad \tau=Tr_C(\tilde{\tau})$

$$F(\tilde{\sigma},\tilde{\tau})\le F(\sigma,\tau)$$ ***Partial trace property***

Closed timelike curves (CTCs)

Without CTC, quantum cloning condition becomes

$$Tr_C\left(U\rho_s \otimes \Sigma \otimes YU^\dagger\right) = \rho_s \otimes \rho_s$$

C: an auxiliary quantum system in some standard state *Y*

$$\boldsymbol{F(\rho_i,\rho_j) = F(\rho_i \otimes \rho_j, \rho_i \otimes \rho_j) \le F(\rho_i,\rho_j)^2}$$

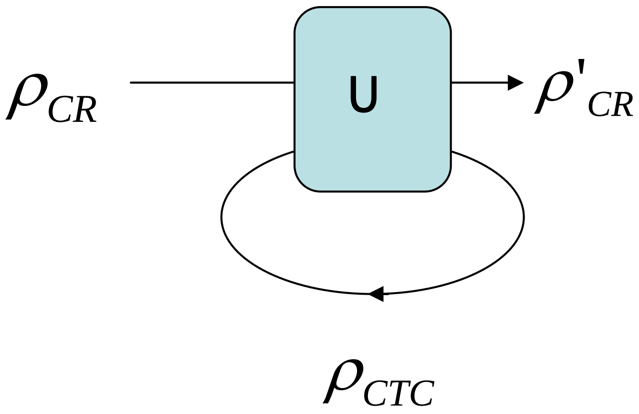
$$F(\rho_i,\rho_j) = 1 \quad or \quad 0$$

ρ_i and ρ_j are either identical or orthogonal

Closed timelike curves (CTCs)

With CTC, quantum cloning condition becomes

$$Tr_{CTC}\left(U\rho_s \otimes \Sigma \otimes \rho^s_{CTC} U^\dagger\right) = \rho_s \otimes \rho_s$$



$$F\left(U\rho_i \otimes \Sigma \otimes \rho^i_{CTC} U^\dagger, U\rho_j \otimes \Sigma \otimes \rho^j_{CTC} U^\dagger\right)$$

$$= F(\rho_i, \rho_j) F(\rho^i_{CTC}, \rho^j_{CTC})$$

$$F(\rho_i, \rho_j) F(\rho^i_{CTC}, \rho^j_{CTC}) \le F(\rho_i, \rho_j)^2$$

$$F(\rho_i, \rho_j) F(\rho^i_{CTC}, \rho^j_{CTC}) \le F(\rho^i_{CTC}, \rho^j_{CTC})$$

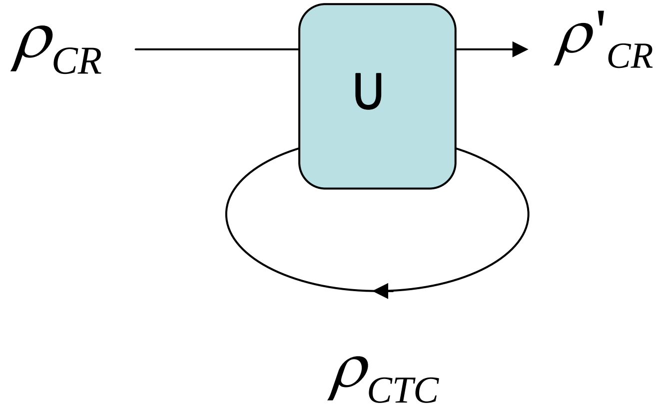
$$\longrightarrow \quad F(\rho_i, \rho_j) \ge 0 \qquad F(\rho^i_{CTC}, \rho^j_{CTC}) \le F(\rho_i, \rho_j)$$

Closed timelike curves (CTCs)

Example 1

$\left\{ |\psi_j\rangle \right\}_{j=0}^{N-1}$ ***N distinct states in a space of dimension N***
Not necessarily an orthonormal set

Brun's theorem: PRL 102, 210402 (2009)



$$U_j |\psi_j\rangle = |j\rangle; \left\{ |j\rangle \right\} \text{orthonormal set}$$

$$\rho_{CR} = |\psi_j\rangle_A \langle\psi_j| \otimes \Sigma, \quad \Sigma = |0\rangle\langle 0|$$

$$U = SVW$$

$$W : I_A \otimes SWAP$$

$$V = I_A \otimes C-SUM, \;\; C-SUM\left(|i\rangle \otimes |j\rangle\right) = |i\rangle \otimes |j + i (\text{mod } N)\rangle$$

$$S = I_A \otimes \sum_k U_k^\dagger \otimes |k\rangle\langle k|$$

$$\longrightarrow \quad \rho_{CTC} = |j\rangle\langle j|, \quad \rho'_{CR} = |\psi_j\rangle\langle\psi_j| \otimes |\psi_j\rangle\langle\psi_j|$$

Closed timelike curves (CTCs)

$$\rho^{-} = |\psi_j\rangle_A\langle\psi_j| \otimes |0\rangle_B\langle 0| \otimes |j\rangle_{CTC}\langle j|$$

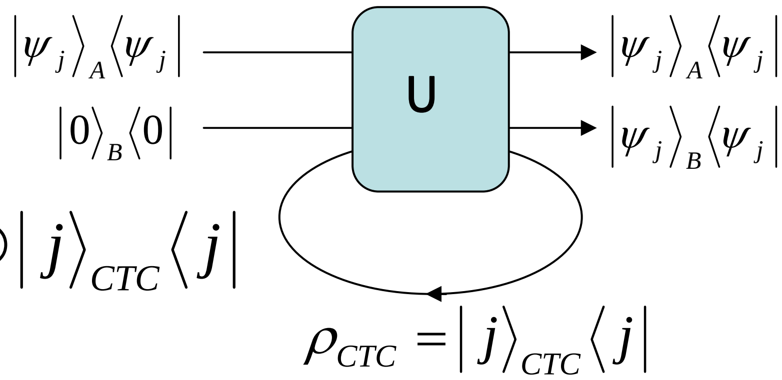
$$W\rho^{-}W^{\dagger} = |\psi_j\rangle\langle\psi_j| \otimes |j\rangle\langle j| \otimes |0\rangle\langle 0|$$

$$VW\rho^{-}W^{\dagger}V^{\dagger} = |\psi_j\rangle\langle\psi_j| \otimes |j\rangle\langle j| \otimes |j\rangle\langle j|$$

$$SVW\rho^{-}W^{\dagger}V^{\dagger}S^{\dagger} = |\psi_j\rangle_A\langle\psi_j| \otimes |\psi_j\rangle_B\langle\psi_j| \otimes |j\rangle_{CTC}\langle j|$$

$$Tr_{AB}\left(SVW\rho^{-}W^{\dagger}V^{\dagger}S^{\dagger}\right) = |j\rangle_{CTC}\langle j|$$

$$Tr_{CTC}\left(SVW\rho^{-}W^{\dagger}V^{\dagger}S^{\dagger}\right) = |\psi_j\rangle_A\langle\psi_j| \otimes |\psi_j\rangle_B\langle\psi_j|$$



Closed timelike curves (CTCs)

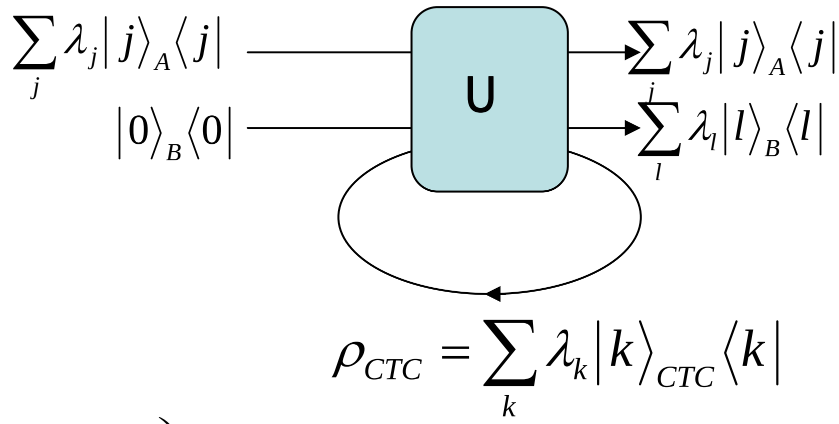
Example 2

$$\rho_A = \sum_j \lambda_j | j \rangle\langle j |$$

$$U = VW$$

$$Tr_{AB}\left(U\rho_A \otimes |0\rangle_B\langle 0| \otimes \rho_{CTC} U^{\dagger}\right) = \rho_{CTC}$$

$$\rightarrow \rho_{CTC} = \sum_k \lambda_k |k\rangle\langle k|$$



$$\rightarrow Tr_{CTC}\left(U\rho_A \otimes |0\rangle_B\langle 0| \otimes \rho_{CTC} U^{\dagger}\right) = \left(\sum_j \lambda_j | j\rangle\langle j|\right) \otimes \left(\sum_l \lambda_l |l\rangle\langle l|\right)$$

Quantum parallel processing: entanglement

$$|x\rangle \otimes |0\rangle \longrightarrow \boxed{U_f} \longrightarrow |x\rangle \otimes |f(x)\rangle$$

$$\left(\sum_x |x\rangle\right) \otimes |0\rangle \longrightarrow \boxed{U_f} \longrightarrow \sum_x \left(|x\rangle \otimes |f(x)\rangle\right)$$

$\Rightarrow$ **Massive parallelism !!!**

(2^N outputs in one query)

Quantum entanglement: universality

- ❑ **Entanglement in two-mode squeezed state (CV)**
- ❑ **Quantum black hole: Cosmological quantum computer?**

$$|\Psi\rangle_{AB} = (1-r^2)^{1/2}\sum_n r^n |n\rangle_A \otimes |n\rangle_B \quad 0 \le r < 1,$$

r: squeezing parameter

$$|\Phi_0\rangle_{in\otimes out} = (1-\lambda^2)^{1/2}\sum_n \lambda^n |n\rangle_{in} \otimes |n\rangle_{out}$$

$$\lambda = exp(-4\pi M\omega) \ \text{for HawkingRadiation}$$

With $\lambda \rightarrow r$ BH can be regarded as a cosmological quantum computer!

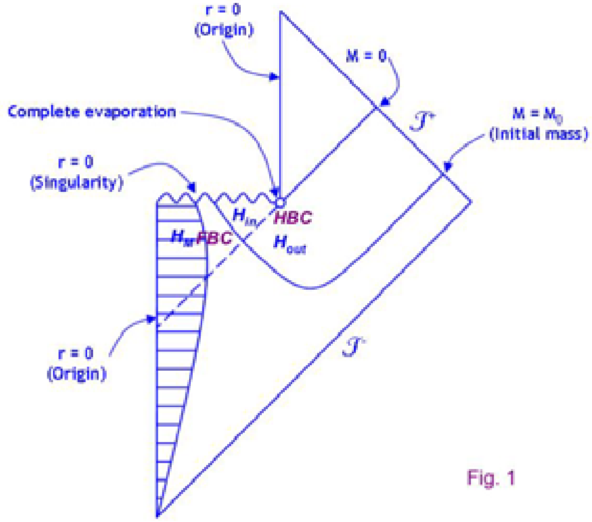


Fig. 1

Alsing & Milbrun, Phys. Rev. Lett. 91, 180404 (2003)
Fuentes-Schuller & Mann, Phys. Rev. Lett. 95, 020404 (2005)
D. Ahn, Phys. Rev. D 74, 084010 (2006)

Hawking radiation from Schwarzschild black hole

$$\left|\Phi_0\right\rangle_{in\otimes out} = \frac{1}{\cosh r_\omega}\sum_n e^{-4\pi M\omega n}\left|n\right\rangle_{in}\otimes\left|n\right\rangle_{out}$$

$$\left|\Phi_0\right\rangle_{in\otimes out} \in H_{in}\otimes H_{out}$$

$$\rho_{out} = tr_{in}\left(\left|\Phi_0\right\rangle_{in\otimes out}\left\langle\Phi_0\right|\right)$$

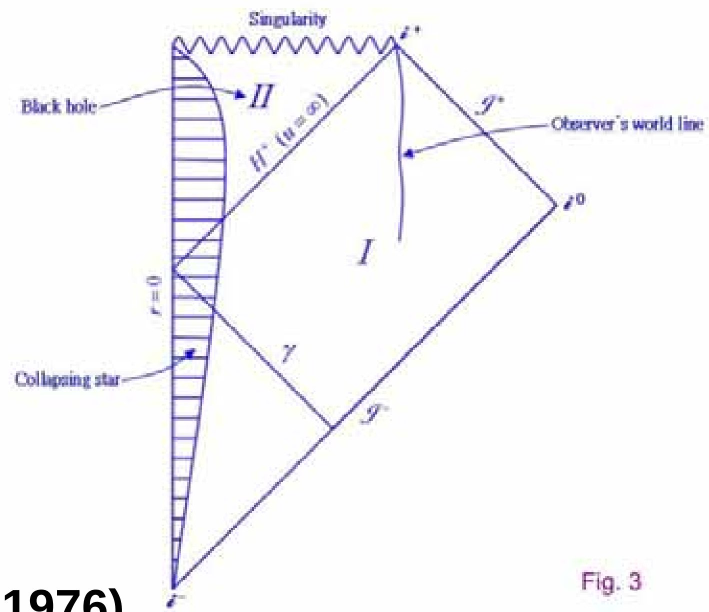


Fig. 3

$\Rightarrow$ **Non-unitary evaporation (Hawking, 1976)**

Conflicts with ADS/CFT

- **Quantum Gravity should be unitary to outside observers!**

  (Madacena & Strominger 1997)

Remedy: Black hole final state

Horowitz & Maldacena, JHEP (2004)

• Microcanonical form:

$$|\Phi_0\rangle_{in\otimes out} = \frac{1}{\sqrt{N}}\sum_i |i\rangle_{in} \otimes |i\rangle_{out}$$

$$|\Phi_0\rangle_{M\otimes in} = \frac{1}{\sqrt{N}}\sum_i |i\rangle_{M} \otimes |i\rangle_{in}$$

$$|\Psi\rangle_{M\otimes in} = \frac{1}{\sqrt{N}}\sum_l (S\otimes I)|l\rangle_{M} \otimes |l\rangle_{in} : \mathrm{FBC}$$

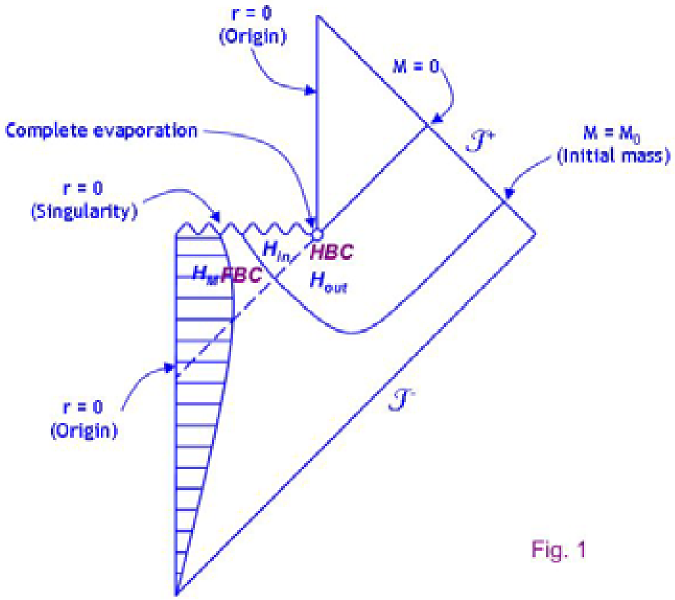


Fig. 1

HM Conjecture

$|\psi\rangle_M$: initial collapsing matter state

$$|\psi\rangle_M \rightarrow |\Psi_0\rangle_{M\otimes in\otimes out} = |\psi\rangle_M \otimes |\Phi_0\rangle_{in\otimes out}$$

$\Rightarrow$ **Final State Projection**

$$|\phi_0\rangle_{out} = {}_{M\otimes in}\langle\Psi\|\Psi_0\rangle_{M\otimes in\otimes out}$$

$$= \sum_i {}_M\langle i|S|\psi\rangle_M |i\rangle_{out}$$

$\Rightarrow$ **Unitary**

HM Conjecture

- **Evaporation is unitary (no information is lost)**
- **Non-local process is involved**
  (information transfer over the event horizon)
- **What about the effects of state evolution?**

Quantum entanglement: teleportation by twist

$$|\vec{\Psi}\rangle\rangle_{1,2} = \frac{1}{\sqrt{N}} \sum_n e^{-i\phi_n} |n\rangle_1 \otimes |n\rangle_2, \; |\overleftarrow{\Psi}\rangle\rangle_{1,2} = \frac{1}{\sqrt{N}} \sum_n e^{-i\phi_n} |n\rangle_2 \otimes |n\rangle_1$$

$$\tau_{b,a} = \sum_n |n\rangle_{ba}\langle n| : \textit{transfer operator}$$

$$_{1,2}\langle\langle \vec{\Psi} | \overleftarrow{\Psi} \rangle\rangle_{2,3} = \frac{1}{N}\tau_{3,1}, \; _{1,2}\langle\langle \overleftarrow{\Psi} | \vec{\Psi} \rangle\rangle_{2,3} = \frac{1}{N}\tau_{3,1}$$

$|\vec{\Phi}\rangle\rangle_{2,3}$: entangled state shared by Alice & Bob

$|\varphi\rangle_1$: unkown state to be teleported from Alice to Bob

$|\vec{\Psi}\rangle\rangle_{1,2}$: result of generalized measurement by Alice

$$_{1,2}\langle\langle \vec{\Psi} | \left[|\varphi\rangle_1 \otimes |\overleftarrow{\Psi}\rangle\rangle_{2,3} \right] = \frac{1}{N} |\varphi\rangle_3 : \textit{Quantum teleportation}$$

1: State to teleport
2: Alice
3: Bob

S. L. Braunstein et al., Phys. Rev. Lett. 84,3486 (2000)

Quantum entanglement: teleportation & evaporation

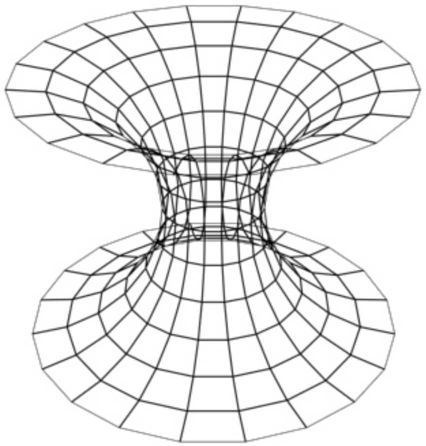
- ❑ **Quantum teleportation analogy of the BH final state by twist**

$$|\vec{\Psi}\rangle\rangle_{1,2} = \frac{1}{\sqrt{N}}\sum_n e^{-i\phi_n}|n\rangle_1 \otimes |n\rangle_2$$

$$|\vec{\Psi}\rangle\rangle_{1,2} = \frac{1}{\sqrt{N}}\sum_n e^{-i\phi_n}|n\rangle_2 \otimes |n\rangle_1$$

$${}_{1,2}\langle\langle\vec{\Psi}|\left[|\varphi\rangle_1 \otimes |\vec{\Psi}\rangle\rangle_{2,3}\right] = \frac{1}{N}|\varphi\rangle_3 \quad \textit{CV \& DV teleportation}$$

$${}_{M\otimes in}\langle\langle\vec{\Psi}|\left[|\varphi\rangle_M \otimes |\vec{\Psi}\rangle\rangle_{in\otimes out}\right] = \frac{S(g)}{N}|\varphi\rangle_{out} \quad \textit{BH evaporation}$$



D. Ahn & M. S. Kim, Phys. Rev. D78, 064025 (2008)

HM Conjecture

- Need to check that BH internal state is in maximal entangled state

$$\left|\Phi_0\right\rangle_{M\otimes in} = \frac{1}{\sqrt{N}}\sum_l \left|l\right\rangle_M \otimes \left|l\right\rangle_{in}$$

- Find the BH internal state for the gravitational collapse

Revisit to Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Massless scalar field:

$$(-g)^{1/2}\frac{\partial}{\partial x^\mu}\left[g^{\mu\nu}(-g)^{1/2}\frac{\partial}{\partial x^\nu}\right]\phi = 0$$

$$\rightarrow \phi_{\omega lm} = (2\pi|\omega|)^{-1/2}e^{-i\omega t}f_{\omega l}(r)Y_{lm}(\theta,\varphi)$$

W. G. Unruh (1976)

$$\frac{\partial^2 f_{\omega l}}{\partial r^{*2}} + \omega^2 f_{\omega l} - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) f_{\omega l} = 0$$

$$r^* = r + 2M \ln(r/2M - 1)$$

$$f^-_{\omega l}(r) \approx e^{i\omega r^*} + A^-_{\omega l} e^{-i\omega r^*}$$

~Wave from the past horizon of a black hole

Kruskal spacetime

$$ds^2 = -2M\frac{e^{-r/2M}}{r}d\bar{u}d\bar{v} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2,$$

$$\bar{u} = -4Me^{-u/4M},\ \bar{v} = 4Me^{v/4M},$$

$$u = t - r^*,\ v = t + r^*,$$

$$r^* = r + 2M\ln(r/2M - 1).$$

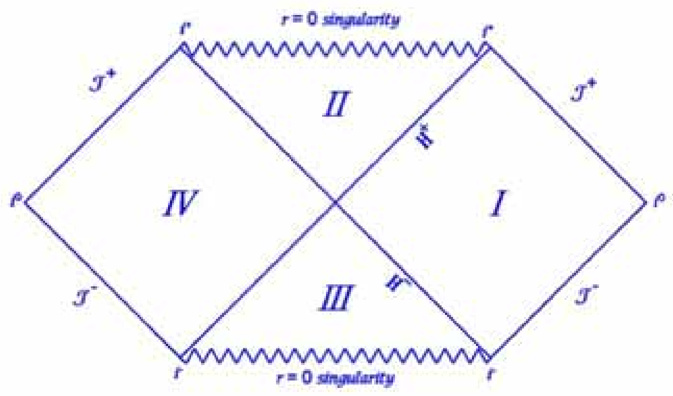


Fig. 2

$$\text{Killing vector}: \frac{\partial}{\partial\bar{\mathbf{u}}}$$

$$\rightarrow \bar{\phi}_{\varpi lm} = (2\pi|\varpi|)^{-1/2}e^{-i\varpi\bar{u}}Y_{lm}(\theta,\varphi)$$

Matching boundary conditions

on H^-

$$\phi^-_{\omega lm} = (2\pi|\omega|)^{-1/2}(e^{-i\omega u} + A^-_{\omega l} e^{-i\omega v})Y_{lm}(\theta,\varphi)$$

$$e^{-i\omega u} = (|\bar{u}|/4M)^{i4M\omega}$$

$$e^{-i\omega v} = (\bar{v}/4M)^{-i4M\omega} = 0\ (\because \bar{v} = 0 \text{ on } H^-)$$

$$\rightarrow \phi^-_{\omega lm} = (2\pi|\omega|)^{-1/2}(|\bar{u}|/4M)^{i4M\omega} Y_{lm}(\theta,\varphi)$$

$$\rightarrow \phi^-_{\omega lm} = (e^{2\pi M\omega}\ {}_{out}\phi_{\omega lm} + e^{-2\pi M\omega}\ {}_{in}\phi_{\omega lm})/(2\sinh(4\pi M\omega))^{1/2}$$

$$\bar{u} < 0\,(I)\ \&\ \bar{u} > 0\,(II)$$

$$(-1)^{i4M\omega} = e^{4\pi M\omega}$$

$$\left(|\bar{u}|\right)^{i4M\omega} = \begin{cases} \left(\bar{u}\right)^{i4M\omega} e^{4\pi M\omega} & (I) \\ \left(\bar{u}\right)^{i4M\omega} & (II) \end{cases}$$

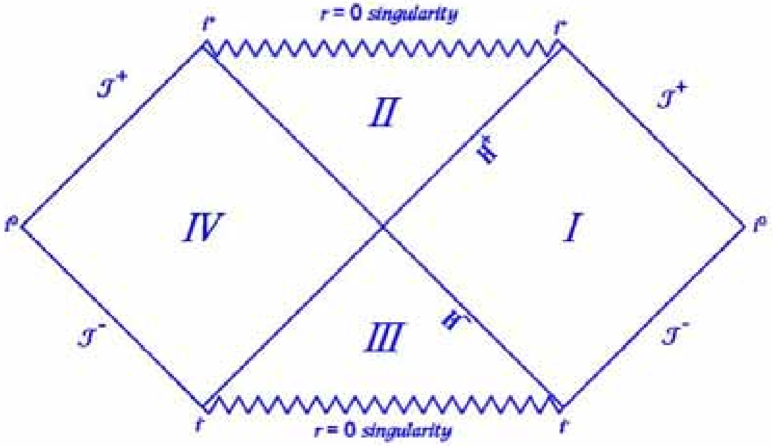


Fig. 2

Bogoliubov transformation

$$a_{K,\omega lm} = \cosh r_\omega b_{out,\omega lm} - \sinh r_\omega b^\dagger{}_{in,\omega lm}$$

$$a^\dagger{}_{K,\omega lm} = \cosh r_\omega b^\dagger{}_{out,\omega lm} - \sinh r_\omega b_{in,\omega lm}$$

$$\tanh r_\omega = e^{-4\pi M\omega},\ \cosh r_\omega = (1 - e^{-8\pi M\omega})^{-1/2}$$

$$\rightarrow \left|\Phi_0\right\rangle_{in\otimes out} = \frac{1}{\cosh r_\omega} \sum_n e^{-4\pi M\omega n} \left|n\right\rangle_{in} \otimes \left|n\right\rangle_{out}$$

Same as two-mode squeezed state in Quantum Optics!

Two-mode squeezed state

$$a_{K,\omega lm} = a,\, b_{out,\omega lm} = b,\, b_{in,\omega lm} = \bar{b}\ ,\, r_\omega = r.$$

$$a = \cosh r\, b - \sinh r\, \bar{b}^\dagger$$

$$a\left|\Phi_0\right\rangle_{in\otimes out} = 0$$

$$\to (\cosh r\, b - \sinh r\, \bar{b}^\dagger)\left|\Phi_0\right\rangle_{in\otimes out} = 0$$

$$\text{or } (b - s\bar{b}^\dagger)\left|\Phi_0\right\rangle_{in\otimes out} = 0,\, s = \tanh r$$

Two-mode squeezed state

$$\left|\Phi_0\right\rangle_{in\otimes out} : \text{Kruskal vacuum at } J_+$$

$$\left|0\right\rangle_S : \text{Schwarzschild vacuum at } J_-$$

$$\left|\Phi_0\right\rangle_{in\otimes out} = F(b^\dagger, \bar{b}^\dagger)\left|0\right\rangle_S$$

$$\left[b, b^\dagger\right] = 1, \left[b, \left(b^\dagger\right)^2\right] = 2b^+ \rightarrow \left[b, \left(b^\dagger\right)^m\right] = \frac{\partial}{\partial b^\dagger}\left(b^\dagger\right)^m$$

$$b\left|0\right\rangle_S = 0 \ \& \ \bar{b}\left|0\right\rangle_S = 0$$

$$bF(b^\dagger, \bar{b}^\dagger)\left|0\right\rangle_S = \left[b, F(b^\dagger, \bar{b}^\dagger)\right]\left|0\right\rangle_S = \frac{\partial}{\partial b^\dagger}F(b^\dagger, \bar{b}^\dagger)\left|0\right\rangle_S$$

Two-mode squeezed state

$$\left(b - s\bar{b}^\dagger\right)\left|\Phi_0\right\rangle_{in\otimes out}$$

$$= \left(b - s\bar{b}^\dagger\right)F\left(b^\dagger, \bar{b}^\dagger\right)\left|0\right\rangle_S$$

$$= \left(\frac{\partial}{\partial \bar{b}^\dagger} - s\bar{b}^\dagger\right)F\left(b^\dagger, \bar{b}^\dagger\right)\left|0\right\rangle_S = 0$$

$$\therefore \left(\frac{\partial}{\partial b^\dagger} - s\bar{b}^\dagger\right)F\left(b^\dagger, \bar{b}^\dagger\right) = 0$$

$$\rightarrow F \propto exp\left(sb^\dagger\bar{b}^\dagger\right)$$

Two-mode squeezed state

$$\exp\left(s b^{\dagger} \bar{b}^{\dagger}\right)|0\rangle_S$$

$$= \sum_n \frac{s^n}{n!}\left(b^{\dagger}\right)^n\left(\bar{b}^{\dagger}\right)^n|0\rangle_S$$

$$= \sum_n s^n |n\rangle_{in} \otimes |n\rangle_{out} \quad \because |n\rangle = \frac{\left(b^{\dagger}\right)^n}{\sqrt{n!}}|0\rangle$$

$$\therefore |\Phi_0\rangle_{in\otimes out} = N \sum_n s^n |n\rangle_{in} \otimes |n\rangle_{out}$$

$$_{in\otimes out}\langle\Phi_0\|\Phi_0\rangle_{in\otimes out} = N^2 \sum_n s^{2n} = \frac{N^2}{1-s^2} = 1$$

$$\therefore N = \sqrt{1-s^2}$$

Two-mode squeezed state

$$N = \sqrt{1 - s^2}$$

$$= \left(1 - \tanh^2 r\right)^{1/2}$$

$$= \frac{1}{\cosh r}$$

$$\therefore \left|\Phi_0\right\rangle_{in \otimes out} = \frac{1}{\cosh r} \sum_n \tanh^n r \left|n\right\rangle_{in} \otimes \left|n\right\rangle_{out}$$

$$\tanh r = \exp\left(-4\pi M\omega\right)$$

Gravitational collapse & black hole Internal state

D.Ahn, Phys. Rev. D74,084010 (2006)

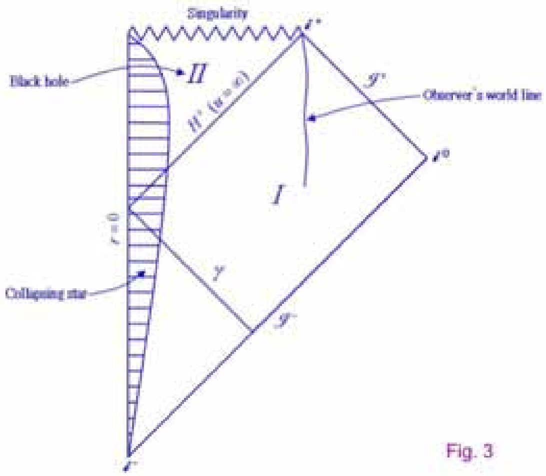


Fig. 3

$$ds^2 = \begin{cases} -d\tau^2 + dr^2, \; r < R(\tau) \\ -\left(1 - \dfrac{2M}{r}\right)dt^2 + \dfrac{dr^2}{1 - \dfrac{2M}{r}}, \; r > R(\tau), \end{cases}$$

$$R(\tau) = \begin{cases} R_o, \; \tau < 0 \\ R_o - v\tau, \; \tau > 0. \end{cases}$$

W. G. Unruh, Phys. Rev. D 14, 870 (1976)

Advanced & retarded null coordinates

$$V = \tau + r - R_o, \; U = \tau - r + R_o$$

$$v^* = t + r - R_o^*, \; u^* = t - r^* + R_o^*$$

$$R_o^* = R_o + 2M \ln(R_o / 2M - 1)$$

$$r^* = r + 2M \ln(r / 2M - 1)$$

$$\text{at } \tau = 0 : \; U = V = u^* = v^* = 0$$

$$\tau < 0 : \; U < 0 \;\&\; V < 0$$

Collapsing shell

$$\frac{dt}{d\tau} = \begin{cases} \left(1 - 2M / R_0\right)^{-1/2} \quad \text{for} \quad \tau < 0 \\ \left[\dfrac{R_0 - V\tau}{\left(R_0 - V\tau - 2M\right)^2} \left(R_0 - 2M - V\tau + 2MV^2\right) \right]^{1/2} \quad \text{for } \tau > 0 \end{cases}$$

for $\tau > 0$, near the shell surface

$$v^* \approx 4M \, ln\left(1 - \frac{\nu V}{(1-\nu)(R_o - 2M)} \right)$$

$$u^* \approx -4M \, ln\left(1 - \frac{\nu U}{(1+\nu)(R_o - 2M)} \right)$$

Incoming wave from future null infinity

$$\phi^{+}_{\omega lm} = (2\pi|\omega|)^{-1/2}(e^{-iwv} - A^{+}_{\omega l}e^{-i\omega u})Y_{lm}(\theta,\varphi)$$

$$\phi^{+}_{\omega lm}|_{r=0} = 0$$

$$\text{at } H^{+} : e^{-i\omega u} = \left(\frac{|\bar{u}|}{4\pi}\right)^{i4M\omega} = 0$$

$$\text{Kruskal : Killing vector} = \frac{\partial}{\partial \bar{v}} \text{ on } H^{+}$$

$$\therefore \bar{\phi}^{+}_{\omega lm} = (2\pi|\omega|)^{-1/2}(e^{-iw\bar{v}})Y_{lm}(\theta,\varphi)$$

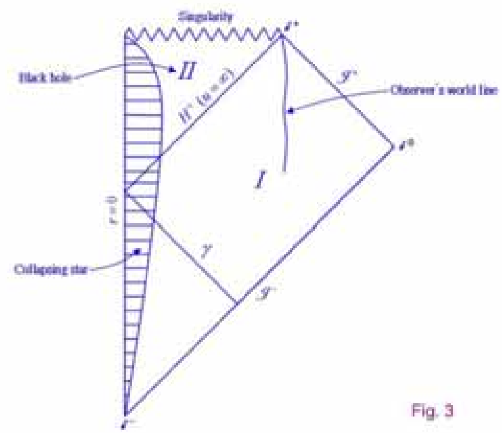


Fig. 3

Incoming wave from future null infinity

$$v = v^* - R_0^*$$

$$e^{-i\omega v} = e^{i\omega R_0^*} e^{-i\omega v^*}$$

$$\text{from } v^* = 4M\, ln\left(1 - \frac{vV}{(1-v)(R_0 - 2M)}\right)$$

$$e^{-i\omega v^*} = \left|1 - \frac{vV}{(1-v)(R_0 - 2M)}\right|^{-i4M\omega}$$

Matching boundary conditions at the shell

$$\phi^{+}_{\omega lm} = (2\pi|\omega|)^{-1/2} e^{i\omega R_o^*} \left| 1 - \frac{\nu V}{(1-\nu)(R_o - 2M)} \right|^{-i4M\omega} Y_{lm}(\theta,\varphi)$$

$$= \left(e^{2\pi M\omega} {}_M\phi_{\omega lm} + e^{-2\pi M\omega} {}_{in}\phi_{\omega lm} \right) / (2\sinh(4\pi M\omega))^{-1/2},$$

${}_M\phi_{\omega lm}$: vanishes outside the shell

${}_{in}\phi_{\omega lm}$: vanishes inside the shell

$\rightarrow$ Matching solutions on H^+ : Killing vector $= \partial / \partial \bar{v}$

$$\bar{\phi}^{+}_{\omega lm} = (2\pi|\bar{\omega}|)^{-1/2} e^{-i\bar{\omega}\bar{v}} Y_{lm}(\theta,\varphi)$$

Bogoliubov transformations

$$c_{K,\omega lm} = \cosh r_\omega b_{M,\omega lm} - \sinh r_\omega b^{\dagger}{}_{in,\omega lm},$$

$$c^{\dagger}{}_{K,\omega lm} = \cosh r_\omega b^{\dagger}{}_{M,\omega lm} - \sinh r_\omega b_{in,\omega lm},$$

$$\tanh r_\omega = e^{-4\pi M\omega}, \; \cosh r_\omega = (1 - e^{-8\pi M\omega})^{-1/2},$$

$$\rightarrow \left|\Phi_0\right\rangle_{M\otimes in} = \frac{1}{\cosh r_\omega} \sum_n e^{-4\pi M\omega n} \left|n\right\rangle_M \otimes \left|n\right\rangle_{in}$$

D. Ahn, Phys. Rev. D 74, 084010 (2006)

Final state boundary condition

$${}_{M\otimes in}\langle\Psi| = {}_{M\otimes in}\langle\Phi_0|(S\otimes I)$$

$$= \frac{1}{\cosh r_\omega}\sum_n e^{-4\pi M\omega n}\left({}_M\langle n|S\right)\otimes{}_{in}\langle n|,$$

S : random unitary transformation

$$|\Psi_0\rangle_{M\otimes in\otimes out} = |\psi\rangle_M \otimes |\Phi_0\rangle_{in\otimes out}$$

$$= |\psi\rangle_M \otimes \left(\frac{1}{\cosh r_\omega}\sum_n e^{-4\pi M\omega n}|n\rangle_{in}\otimes|n\rangle_{out}\right).$$

$|\psi\rangle_M$: initial matter state

Black hole evaporation: final state projection

$$\left|\phi\right\rangle_{out} = {}_{M\otimes in}\left\langle\Psi\|\Psi_0\right\rangle_{M\otimes in\otimes out}$$

$$= \frac{1}{\cosh^2 r_\omega}\sum_{n,m} e^{-4\pi M\omega(n+m)}\ {}_M\left\langle m\left|S\right|\Psi\right\rangle_M\ {}_{in}\left\langle m\|n\right\rangle_{in} \otimes \left|n\right\rangle_{out}$$

$$= \frac{1}{\cosh^2 r_\omega}\sum_{n} e^{-8\pi M\omega n}\ {}_M\left\langle n\left|S\right|\Psi\right\rangle_M \left|n\right\rangle_{out}.$$

$$\left|\tilde{\phi}\right\rangle_{out} = \left(\sum_n e^{-8\pi M\omega n}\left|n\right\rangle_{out\ M}\left\langle n\left|S\right|\psi\right\rangle_M\right) / \sqrt{Z(\beta,\omega)}$$

$$Z(\beta,\omega) = \sum_n e^{-\beta\omega n}\left|{}_M\left\langle n\left|S\right|\psi\right\rangle_M\right|^2$$

Black hole evaporation: statistical properties

$$\hat{A}|n\rangle = A_n|n\rangle$$

$$_{out}\langle\tilde{\phi}|\hat{A}|\tilde{\phi}\rangle_{out} = \sum_n A_n p_n(\beta,\omega)$$

$$p_n(\beta,\omega) = e^{-\beta\omega n}\left|{}_M\langle n|S|\psi\rangle_M\right|^2 / Z(\beta,\omega)$$

$$\langle E\rangle = \sum_n n\omega p_n(\beta,\omega)$$

$$= \frac{\sum_n n\omega e^{-\beta\omega}\left|\langle n|S|\psi\rangle\right|^2}{Z(\beta,\omega)}$$

$$= -\frac{\partial}{\partial\beta}\log Z(\beta,\omega).$$

Black hole evaporation: statistical properties

Average number of emitted Hawking particle

$$\langle N \rangle = \sum_n n p_n(\beta,\omega)$$

$$= \frac{\sum_n n e^{-\beta\omega n} \left|\langle n|S|\psi\rangle\right|^2}{\sum_n e^{-\beta\omega n} \left|\langle n|S|\psi\rangle\right|^2}$$

$$= -\frac{\partial}{\beta\partial\omega} \log Z(\beta,\omega).$$

$$\delta S_{rad} = \sum_n \frac{e^{-\beta\omega n}}{Z(\beta,\omega)} \left|\langle n|S|\psi\rangle\right|^2 \left\{\beta\omega n - 2\log\left|\langle n|S|\psi\rangle\right| + \log Z(\beta,\omega)\right\}$$

Black hole evaporation: statistical properties

For random unitary evolution S

$$\left|\langle n|S|\psi\rangle\right|^2 \approx \text{constant}$$

$$Z_{Hawk}(\beta,\omega) \approx \sum_n e^{-\beta\omega n}$$

$$\langle N\rangle_{Hawk} = -\frac{\partial}{\beta\partial\omega} Z_{Hawk}(\beta,\omega) = \frac{1}{e^{\beta\omega}-1}$$

$$\delta S_{rad} = \left(\langle N\rangle_{Hawk}+1\right)\log\left(\langle N\rangle_{Hawk}+1\right) - \langle N\rangle_{Hawk}\log\langle N\rangle_{Hawk} \geq 0$$

S. W. Hawking, *Commun. Math. Phys.* 43, 199-220 (1975)

Black hole evaporation: coherent state

$$|\psi\rangle_M = |0\rangle_M \text{ at far past infinity } J^-$$

$$S(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right)$$

$$S(\alpha)|\psi\rangle_M = |\alpha\rangle_M = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle_M,$$

$$_M\langle n|S(\alpha)|\psi\rangle_M = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}},$$

$$Z_{CH}(\beta,\omega) = e^{-|\alpha|^2}\sum_n \frac{1}{n!}\left(|\alpha|^2 e^{-\beta\omega}\right)^n.$$

Black hole evaporation: coherent state

$$\langle N \rangle_{CH} = -\frac{\partial}{\beta \partial \omega} \log Z_{CH}(\beta, \omega) = |\alpha|^2 e^{-\beta\omega}$$

$$\langle N \rangle_{CH} / \langle N \rangle_{Hawk} = |\alpha|^2 (1 - e^{-\beta\omega})$$

$$\hat{x}_\lambda = \left(a e^{-i\lambda} + a^\dagger e^{i\lambda} \right) / \sqrt{2}$$ **quadrature operator**

$$\langle x_\lambda | n \rangle = \pi^{-1/4} \left(2^n n! \right)^{-1/2} e^{-x_\lambda^2 - in\lambda} H_n(x_\lambda)$$

$$\langle x_\lambda^2 \rangle = \langle 0 | S^\dagger(\alpha) \hat{x}_\lambda^2 S(\alpha) | 0 \rangle \approx |\alpha|^2$$ **measure of wave function spread in the black hole**

Black hole evaporation: coherent state

TeV microscopic black holes

$\sqrt{\langle x_\lambda^2 \rangle} \approx R_H$ **Schwarzschild radius**

$$R_H = l_P \frac{M_P}{M_D}\left(\frac{M}{M_D}\right) \approx 2\times 10^{-19}\ \mathrm{m} \qquad D = 4 + d \quad (d = 5) \qquad M \approx M_D$$

Giddings & Mangano, Phys Rev D 78, 035509 (2008)

$$M_D \approx 1\,\mathrm{TeV}$$

M_P **Planck mass**

$$\tau_{Hawk} = C\left(\frac{M}{M_D}\right)^{\frac{2D-4}{D-3}} \frac{1}{M} \qquad \tau_{CH} = \langle N \rangle_{Hawk} / \langle N \rangle_{CH}\, \tau_{Hawk}$$

$$\langle N \rangle_{CH} / \langle N \rangle_{Hawk} = |\alpha|^2 (1 - e^{-\beta\omega}) \approx \langle x_\lambda^2 \rangle (1 - e^{-\beta\omega}) \approx 2.53\times 10^{-38}$$

$$t_D^{CH} \approx \langle N \rangle_{Hawk} / \langle N \rangle_{CH} / M_D \approx 4\times 10^{8} s \approx 10 yr$$

Black hole evaporation: Assay

The coherent states saturate minimum uncertainty bound and hence are good candidates for semi-classical states.

Formation of a black hole with internal state described by a coherent state is possible when the short distance feature of the particle is much smaller than the impact parameter *b* in high energy particle collisions.

In principle, one can avoid the catastrophic growth of a black hole or the formation of a black hole with internal coherent state by adjusting the impact parameter such that *r* is comparable with *b*.

Summary

- Internal state of a Schwarzschild black hole is an entangled state in $H_M \otimes H_{in}$ (proof of HM conjecture)
- Final state boundary condition and final state projection allow the evaporation process
- When black hole state is described by a coherent state, Hawking decay is suppressed
- Presence of a closed timelike curve allows cloning of arbitrary quantum states