Dephasing

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I. BACKGROUND

Quantum information protocols rely upon imperfect physical systems to implement certain idealized operations. In this lecture we examine the types of noise that commonly enter into consideration, and develop some techniques to combat this noise.

Consider a quantum bit, described by pauli matrices, σ_ν. One common scenario is the two-level system with a large energy gap, $\hbar\omega_0$. The system Hamiltonian is:

$$H = \hbar\omega_0(\sigma_z/2) \tag{1}$$

You may recall how coupling to a bath of harmonic oscillators (such as photons) leads to dephasing and decay of the qubit. That approach relied upon the Born and Markov approximations, i.e., that the bath degrees of freedom have a short correlation time. Within those approximations, one can derive a *master equation*, describing the equations of motion for the density matrix of the system, without having to know the exact bath dynamics. This leads to the Lindblad form of the master equation:

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}\left(\rho\sigma_+\sigma_- + \sigma_+\sigma_-\rho - 2\sigma_-\rho\sigma_+\right) \tag{2}$$

As an exercise, you may consider the decay of the excited state population $\rho_{\uparrow\uparrow}$ and the off-diagonal density matrix component, $\rho_{\uparrow\downarrow}$. You should find

$$\dot{\rho}_{\uparrow\uparrow} = -\gamma\rho_{\uparrow\uparrow} \tag{3}$$

$$\dot{\rho}_{\uparrow\downarrow} = -(i\omega_0 + \gamma/2)\rho_{\uparrow\downarrow} \tag{4}$$

We can associate a time scale, $T_1 = 1/\gamma$ with the decay of excited states to ground state and another scale $T_2 = 2T_1$ to the decay of the off-diagonal term of the density matrix, which is called the coherence.

II. LOW FREQUENCY NOISE

There are many scenarios where the Born and Markov assumptions used in the derivation of Eq. 2 are patently false: for example, 1/f noise in control electronics or in the evolution of microscopic degrees of freedom in a solid-state system. We will show that this leads to an additional pure dephasing term, which can cause the coherence to decay on a time scale faster than T_1. This time scale is often called T_2, though I hope to show you that there is rarely a single time scale that encompasses all of these dephasing dynamics. After developing a dephasing formalism, I will focus on how to combat noise with long correlation times, along with motivating several specific models of such noise.

A. Classical random noise

Let us start with the pedagogical case: classical noise acting on the system. Say we have some random noise induced by a classical (external) control:

$$H_{SB} = \hbar\vec{A}(t)\cdot\vec{\sigma}/2 \tag{5}$$

where $\vec{A}(t)$ is a classical, gaussian process . For example, a stray laser field leads to an AC Stark shift of the qubit's frequency by $|\Omega|^2/\Delta \propto I$. Intensity fluctuations in the laser lead to a fluctuation in the effective qubit's frequency, $\omega_0 \to \omega_0 + A_z(t)$.

Without loss of generality, we take $\langle \vec{A} \rangle_{\text{noise}} = 0$ and write the Fourier transform

$$A_\nu(t) = \int_{-\infty}^{\infty} f_\nu(\omega) e^{i\omega t} \frac{d\omega}{\sqrt{2\pi}} \tag{6}$$

where ν is a Cartesian coordinate. We will assume that independent, phase-random noise occurs at each frequency for each direction, i.e.,

$$\langle f_\nu(\omega) f_\mu(\omega') \rangle_{\text{noise}} = 2\pi S_\nu(\omega) \delta_{\nu\mu} \delta(\omega + \omega'). \tag{7}$$

Furthermore, this noise is Gaussian, that is, all the moments of $f_\nu(\omega)$ are Gaussian distributed.

Plugging back in for $A(t)$, we can then evaluate the correlation function,

$$\begin{aligned}
\langle A_\nu(t+\tau) A_\mu(t) \rangle_{\text{noise}} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f(\omega) f(\omega') \rangle_{\text{noise}} e^{i\omega(t+\tau) + i\omega' t} \frac{d\omega \; d\omega'}{2\pi} &(8)\\
&= \delta_{\nu\mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\pi S_\nu(\omega) \delta(\omega + \omega') e^{i\omega(t+\tau) + i\omega' t} \frac{d\omega \; d\omega'}{2\pi} &(9)\\
&= \delta_{\nu\mu} \int_{-\infty}^{\infty} S_\nu(\omega) e^{i\omega(t+\tau) - i\omega t} d\omega &(10)\\
&= \delta_{\nu\mu} \int_{-\infty}^{\infty} S_\nu(\omega) e^{i\omega\tau} d\omega &(11)
\end{aligned}$$

If we assume that A_x and A_y have no high frequency components (components near ω_0, similar to the standard approximations used in the rotating wave approximation), the effects of A_x and A_y may be neglected: the noise cannot inject enough energy to satisfy energy conservation sufficient to induce a spin flip. We will revisit this assumption at the end. Thus, we write, in the rotating wave approximation and rotating frame,

$$H_S + H_{SB} = \hbar A_z(t) \sigma_z / 2 \tag{12}$$

Consider the effects of this hamiltonian on an initial coherent superposition of the qubit:

$$|+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \tag{13}$$

We can write the evolution operator associated with H directly:

$$\begin{aligned}
U(t_f, t_i) &= \hat{T} \exp[-i \int_{t_i}^{t_f} A_z(t) dt \sigma_z / 2] &(14)\\
&= \exp[-i \int_{t_i}^{t_f} A_z(t) dt \sigma_z / 2] &(15)\\
&= \cos[\int_{t_i}^{t_f} A_z(t) dt/2] - i\sigma_z \sin[\int_{t_i}^{t_f} A_z(t) dt/2] &(16)
\end{aligned}$$

where $\hat{T}$, the time-ordering operator, is irrelevant as H commutes with itself at different times. Let us set the angle of rotation: $\theta = \int_{t_i}^{t_f} A_z(t) dt$. This is a random variable, which we will have to average over when finding some observable. One observable of interest is the probability of projecting back onto the initial state: $P = |+\rangle\langle+|$.

$$\begin{aligned}
\langle P \rangle &= \langle | \langle + | U(t, 0) | + \rangle |^2 \rangle_{\text{noise}} &(17)\\
&= \langle \cos^2(\theta/2) \rangle &(18)\\
&= \langle (e^{i\theta} + e^{-i\theta} + 2) \rangle / 4 &(19)\\
&= \frac{1}{2} \left[1 + \langle e^{i\theta} + e^{-i\theta} \rangle / 2 \right] &(20)\\
&= \frac{1}{2} \left[1 + \exp(-\langle \theta^2 \rangle) \right] &(21)
\end{aligned}$$

where the final step relies upon the Gaussian noise properties of θ. This result indicates that as $\langle\theta^2\rangle$ grows, the probability that the initial coherence is maintained decreases. This corresponds to a *pure dephasing* process. For example, an initial state along the Bloch sphere's z-axis has no decay: $|\langle\uparrow|U(t,0)|\uparrow\rangle|^2 = 1$.

We now evaluate $\langle\theta^2\rangle$:

$$\begin{aligned}
\langle\theta^2\rangle &= \int_0^t\int_0^t \langle A_z(t')A_z(t'')\rangle dt''dt' && (22)\\
&= \int_0^t\int_0^t\int_{-\infty}^{\infty} e^{i\omega(t'-t'')}S(\omega)d\omega dt''dt' && (23)\\
&= \int_{-\infty}^{\infty}\int_0^t\int_0^t e^{i\omega(t'-t'')}dt''dt'S(\omega)d\omega && (24)\\
&= \int_{-\infty}^{\infty}\frac{\sin^2(t\omega/2)}{(\omega/2)^2}S(\omega)d\omega && (25)
\end{aligned}$$

B. Evaluation of T_2^* noise

Consider one scenario, where

$$S(\omega) = (1/T_2^*)^2(2\Gamma)^{-1} \qquad (26)$$

from $-\Gamma$ to Γ and zero otherwise. For $t \ll 1/\Gamma$, we can expand the sin, and get

$$\begin{aligned}
\langle\theta^2\rangle &\approx t^2\int_{-\Gamma}^{\Gamma}\frac{1}{(T_2^*)^2 2\Gamma}d\omega && (27)\\
&= (t/T_2^*)^2 && (28)
\end{aligned}$$

This leads to decay of the coherence going as $\exp[-(t/T_2^*)^2]$. We call this "time-ensemble-averaged" dephasing, because it arises from averaging over classical noise which occurs on possibly long time scales.

Another scenario would be Lorenztian decay (exponential decay of the correlation function); this gives

$$S(\omega) = \frac{\Gamma/\pi}{(T_2^*)^2(\omega^2+\Gamma^2)}\ . \qquad (29)$$

Then

$$\int_{-\infty}^{\infty}\frac{\sin^2(t\omega/2)}{(\omega/2)^2}\frac{\Gamma/\pi}{(T_2^*)^2(\omega^2+\Gamma^2)}d\omega = \frac{t\Gamma-(1-e^{-t\Gamma})}{(T_2^*\Gamma)^2/2} \qquad (30)$$

For $t\Gamma \ll 1$, this behaves the same as the flat spectrum, above, yielding initially quadratic decay. However, for short-correlation times ($\Gamma T_2^* \gg 1$), it instead is dominated by the term proportional to t, giving decay as

$$\exp\left(-\frac{t}{T_2^*(T_2^*\Gamma)}\right) \qquad (31)$$

That is, the effective decay becomes exponential (rather initially quadratic) but the effective decay time is increased from the "bare" T_2^* by a factor $T_2^*\Gamma \gg 1$. This corresponds to so-called motional averaging.

III. SPIN ECHO DECAY

In the cases considered above, spin-echo can greatly change the final dynamics. Recall the Hahn echo sequence: an initial coherence (i.e., a state $|+\rangle$) is prepared by a $\pi/2$ x rotation. After waiting a time $\tau/2$, a π (x) rotation flips 0 and 1. After another waiting time $\tau/2$, the the probability the system still is in the state $|+\rangle$ is greatly enhanced when compared to the Ramsey sequence in the first section.

A. Echo sequence

In particular, the sequence $(\tau/2 - \pi - \tau/2)$, or:

$$U_{se} = U(\tau, \tau/2) X U(\tau/2, 0) \tag{32}$$

For the Gaussian noise $A(t)$ we can evaluate the effects exactly:

$$U_{se} = X X^\dagger e^{-i \int_{\tau/2}^{\tau} A(t) dt \sigma_z/2} X e^{-i \int_0^{\tau/2} A(t) dt \sigma_z/2} \tag{33}$$

$$= X e^{i(\int_{\tau/2}^{\tau} A(t) dt - \int_0^{\tau/2} A(t) dt)\sigma_z/2} \tag{34}$$

Thus, we are in the same scenario as before, but with a slightly different angle, $\tilde{\theta}$, and a known single-particle unitary, X (which we will proceed to ignore).

$$\tilde{\theta} = \int_{\tau/2}^{\tau} A(t) dt - \int_0^{\tau/2} A(t) dt \tag{35}$$

As before, we need to evaluate

$$\langle \tilde{\theta}^2 \rangle = \int_{\tau/2}^{\tau} \int_{\tau/2}^{\tau} \langle A(t') A(t) \rangle dt' dt - \int_0^{\tau/2} \int_{\tau/2}^{\tau} \langle \{A(t') A(t)\}_+ \rangle dt' dt + \tag{36}$$

$$\int_0^{\tau/2} \int_0^{\tau/2} \langle A(t') A(t) \rangle dt' dt \tag{37}$$

$$= 2\langle \theta(\tau/2)^2 \rangle - \int_{-\infty}^{\infty} S(\omega) \int_0^{\tau/2} \int_{\tau/2}^{\tau} [e^{-i\omega(t-t')} + e^{i\omega(t-t')}] dt' dt d\omega \tag{38}$$

$$= 2 \int_{-\infty}^{\infty} S(\omega) \frac{\sin^2(\omega\tau/4)}{(\omega/2)^2} \left[1 - \cos(\omega\tau/2)\right] d\omega \tag{39}$$

$$= \int_{-\infty}^{\infty} S(\omega) \frac{\sin^4(\omega\tau/4)}{(\omega/2)^2} d\omega \tag{40}$$

B. Spin echo for various scenarios

Again we can consider the case of low frequency white noise, Eq. 26. Assuming $\tau\Gamma \ll 1$, we an do the small angle expansion of the sin term. This yeilds

$$\langle \tilde{\theta}^2 \rangle \approx \frac{\Gamma^2 \tau^4}{3 \times 2^6 (T_2^*)^2} \tag{41}$$

Thus, the time scale for spin-echo decay is given by $\sqrt{T_2^*/\Gamma}$, i.e., the geometric mean of the "bare" dephasing time and the correlation time of the environment.

In the case of the Lorentzian spectrum (Eq. 29), the $1/\omega^2$ high frequency tail plays an important role. We will approximate the integral by splitting it into a low frequency part and a high frequency part,

$$\int_0^{\infty} \rightarrow \int_0^{2/\tau} + \int_{2/\tau}^{\infty} \tag{42}$$

(As the spectrum and prefactors are even, we can multiple by two to include the negative frequency components.) The point of the split, $2/\tau$, is the point where the small angle parameter used in Taylor expansion of the sin becomes of order unity.

The low frequency part gives:

$$\int_0^{2/\tau} \frac{(\omega\tau/4)^4 \Gamma/\pi}{(T_2^*)^2 (\omega^2 + \Gamma^2)(\omega/2)^2} d\omega = \frac{\Gamma \tau^3}{32\pi (T_2^*)^2} \tag{43}$$

where we have again expanded the sin into a small angle approximation.

For the high frequency part, we note that the $\sin^4$ function is oscillating quickly, and it becomes convenient to replace it by its average of a small region of frequency: $\sin^4(\tau\omega/4) \rightarrow 3/8$. Then

$$\int_{2/\tau}^{\infty} \frac{\Gamma/\pi}{(T_2^*\omega)^2} \times \frac{3/8}{\omega^2/4} d\omega = \frac{\Gamma\tau^3}{16\pi(T_2^*)^2} \tag{44}$$

Combining these two terms, we find the decay has a time scale on the order of $(T_2^*)^{2/3}\Gamma^{1/3}$; the numerical pre-factor depends upon the choice of split point, but is irrelevant for understanding to overall timescale. This illustrates the role high frequency noise can play in reducing spin-echo lifetimes. Generally, for a spectrum decaying as $1/\omega^3$ or faster the low frequency white noise result is a good approximation. Otherwise, one must be careful.

IV. GAP PROTECTION AND CONTINUOUS SPIN ECHO

We would like to have a general framework within which to understand spin-echo; after all, it is rare that you can always reduce environmental couplings to a σ_z (only) coupling. Towards this end, there are some general concepts that are useful, and a mathematical formalism (the Magnus expansion) which helps make rigorous the general ideas.

Consider when the spin-echo sequence is repeated many times. One can think of the long time dynamics of the system arising from the *time averaged* version of the Hamiltonian: the repeated spin flips at periodic intervals tend to average (to zero) the effects of an unknown term coupling along z or y. The general approach to this case, of periodically modulated Hamiltonians, is to use a Magnus expansion.

A. Magnus expansion

Let us start with an evolution operator (U) which is periodic in time ($U(t+\tau) = U(t)$). This could come about from spin echo, or it could be a Hamiltonian in the interaction picture, oscillating with the frequency ω_0. Regardless, we write the Magnus expansion describing $U(\tau)$, i.e., the propagation over the period, as:

$$U(\tau) = \hat{T}\exp\left(-i\int_0^\tau H(t')dt'\right) = \exp[-i\tau(H_0 + H_1 + \ldots)] \tag{45}$$

where

$$H_0 = \frac{1}{\tau}\int_0^\tau H(t)dt \tag{46}$$

is the first term, corresponding to the *time averaged* Hamiltonian. One example is spin echo, as described above. Then,

$$H_0 = \frac{1}{\tau}\left(\int_0^{\tau/2} A(t)dt - \int_{\tau/2}^{\tau} A(t)dt\right)\sigma_z/2 \tag{47}$$

In principle H_0 itself depends upon time—the Hamiltonian is only quasi-periodic in this case due to the time dependence of $A(t)$. However, if we have a high frequency cutoff in $S(\omega)$ occurring at a lower frequency than $1/\tau$ this approach will still be appropriate. In general, we would like to work in an adiabatic limit, where H_0's time dependence is slow with respect to τ.

This formalism lets us determine the first correction to the rotating wave approximation used in the beginning of these notes. Over a time $2\pi/\omega_0$, we consider $A_\nu(t)$ to be fixed; as ω_0 is a (very large) frequency, this may be a reasonable approximation. Then, $H_0 = A_z(t)\sigma_z/2$, as first written in the beginning. The first correction, H_1, comes from the time-ordering operator, i.e., that

$$\int_0^\tau \int_0^{t_1} H(t_1)H(t_2)dt_2dt_1 \neq \frac{1}{2}\int_0^\tau \int_0^\tau H(t_1)H(t_2)dt_2dt_1 \ . \tag{48}$$

In terms of the propagator,

$$\begin{aligned} \hat{T}\exp[-i\int_0^\tau H(t')dt'] &= 1 - i\tau H_0 - \int_0^\tau \int_0^{t_1} H(t_1)H(t_2)dt_2dt_1 + \ldots & (49)\\ &= 1 - i\tau H_0 - \frac{\tau^2}{2}H_0^2 - \frac{1}{2}\int_0^\tau \int_0^{t_1}[H(t_1), H(t_2)]dt_2dt_1 + \ldots & (50)\\ &= \exp[-i\tau(H_0 + H_1 + \ldots)] & (51) \end{aligned}$$

where

$$H_1 = \frac{1}{2i\tau}\int_0^\tau \int_0^{t_1}[H(t_1), H(t_2)]dt_2dt_1 \tag{52}$$

As an aside, there are a variety of approaches to improving the effects of spin-echo and related sequences, and Eq. 52 gives a hint on how to construct these sequences. In particular, you can confirm that for time-symmetric Hamiltonians ($H(t) = H(\tau - t)$) the correction term H_1 vanishes. This leaves only the next term to correct. The associated spin-echo sequence ($\tau/4 - \pi - \tau/2 - \pi - \tau/4$) is called a Carr-Purcell sequence, and can improve protection against low frequency noise.

B. Gap protection

We now return to the interaction $H_{SB} = \vec{A}\cdot\vec{\sigma}/2$, used in the first section. In the interaction picture,

$$H(t) = A_z(t)\sigma_z/2 + A_+\sigma_- e^{i\omega_0 t}/2 + A_-\sigma_+ e^{-i\omega_0 t}/2 \tag{53}$$

That is, terms that flip the spin are rotating at an angular frequency ω_0. These terms correspond to the part of V that does not commute with the system Hamiltonian, $H_S = \hbar\omega_0\sigma_z/2$. Averaging over one cycle, we find $H_0 = A_z(t)\sigma_z/2$. The other terms enter with the first correction.

We now can evaluate the first correction, H_1. The only terms which will be relevant arise from $[\sigma_-, \sigma_+]e^{i\omega_0(t-tp)}$, which can have a finite average over $2\pi/\omega_0$. We find:

$$H_1 = -\frac{\{A_+, A_-\}}{8\omega_0}\sigma_z + \frac{[A_-, A_z]}{4\omega_0}\sigma_+ + \text{H.c.} \tag{54}$$

In addition, if there are terms of $\vec{A}$ which evolve on time scales close to a harmonic of ω_0, they will contribute to this correction term.

We remark that $\langle A_-A_+\rangle \neq 0$ in general, even for a quantum field at zero temperature. Thus, there is an overall frequency shift induced on the qubit due to the mere existence of the environment. For example, this leads to the Lamb shift in quantum optics. Furthermore, all the terms in the correction enter with a factor of $1/\omega_0$, i.e., are suppressed in magnitude by the gap (in energy) between the two states of the qubit.

There are two consequences of this gap dependence. If we know the form of the interaction H_{SB}, then we can reduce or remove its effects on the qubit through either a pulse sequence (like spin-echo) leading to an average Hamiltonian $H_0 = 0$. Or, we can seek a qubit system that has a natural gap with respect to the dominant noise sources, i.e., a qubit Hamiltonian H_S such that no part of H_{SB} commutes with H_S. For a single qubit, this reduces to the requirement that H_{SB} consist only of terms in the $x - y$ plane, i.e., perpendicular to the qubit's intrinsic axis.

C. More complex pulse sequences

As a final example of the first case, let us consider the pulse sequence:

$$\sigma_z U \sigma_z \sigma_y U \sigma_y \sigma_x U \sigma_x U \tag{55}$$

where U is the propagator over a time $\tau/4$. Let us consider what happens to a Hamiltonian (without gap) of the form

$$H = \vec{A} \cdot \vec{\sigma}/2 \tag{56}$$

For the first fourth of the propagation, evolution is governed by H. For the second fourth, we can think of evolution occurring the unitary transformed $H_X = \sigma_x H \sigma_x$. The same holds true for the third (H_Y) and fourth (H_Z) parts of the evolution. The average Hamiltonian is

$$H_0 = H + H_X + H_Y + H_Z \tag{57}$$

$$= \left[\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} + \begin{pmatrix} A_x \\ -A_y \\ -A_z \end{pmatrix} + \begin{pmatrix} -A_x \\ A_y \\ -A_z \end{pmatrix} + \begin{pmatrix} -A_x \\ -A_y \\ A_z \end{pmatrix} \right] \cdot \vec{\sigma}/2 \tag{58}$$

$$= 0 \tag{59}$$

APPENDIX A: GAUSSIAN NOISE

All higher moments of A_ν are Gaussian distributed with $\langle A_\nu \rangle = 0$, which in essence means:

$$\langle \exp(-i \int_0^t A_\nu(t')dt') \rangle = \exp(-1/2 \int_0^t \int_0^t \langle A_\nu(t') A_\nu(t'') \rangle dt' dt'') \tag{A1}$$

I.e., higher order correlation functions $\langle ABCD \rangle$ can be written as

$$\langle ABCD \rangle = \langle AB \rangle \langle CD \rangle + \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle \tag{A2}$$

The zero mean implies all odd moments are zero. For a Gaussian process, these statements are exact, and generalize to all higher orders.