General state transitions with exact resource morphisms:a unified resource-theoretic approach

-quantifying resources without using distillation or dilution-

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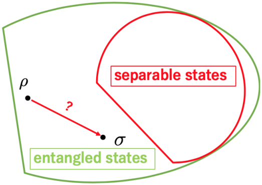
Outline

Motivation

- Given two quantum states, is it possible to transform one into another under a restricted quantum channel? (ex: LOCC)



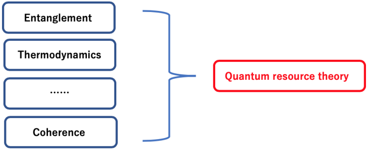
- In 99' M. A. Nielsen showed sufficient and necessary condition for one pure state transformed into another pure state under LOCC.
- Or, does there exist a state that can be transformed into any other states under the restricted channel?
- In bipartite entanglement theory, there exists maximally entangled state that can be transformed into any other states under LOCC.

Motivation

- What drives the transformation between states?
- Might be the amount of entanglement (ebit), as you see the example of maximally entangled state.
- What is special about LOCC here?
- LOCC does not create entanglement at no cost.
- Wait...Is it similar to another familiar theory?
- In the theory of thermodynamics, whether a macrostate can be transformed to another macrostate by **isothermal processes without external work** (does not increase free energy) is determined by the order of the free energy of the two macrostates.
- ... the list goes on as you look into more theories, coherence, asymmetric distinguishability, measurement incompatibility, etc.

Motivation

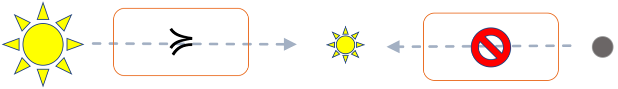
- What is the unified theory behind since we study unification in physics and a unified theory always makes things easier to understand?



- Quantum resource theory is a unified mathematical framework that deals with quantification and manipulation of different (possible all) statistical properties of quantum system.

What are the ingredients of quantum resource theory?

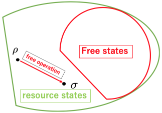
- The framework of quantum resource theory:



(more)Resource Allowed process (less)Resource Free Resource

EX) Entanglement theory with **operationally constrained** process:
• Entangled state (Separable state); • LOCC process; • Entanglement measure

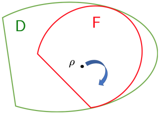
- Ingredients: resource state, allowed transformation (can't generate resource at no cost, or, **free operation**), free state.



- **Note**: conventionally, free state is special case of free operation with trivial input.
- This platform provides us recipes to study quantification of the resources as well as the possibility of resource transformations.

Free operations: resource morphisms (exact)

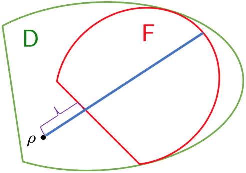
- In our work, we focus on **geometric approach**, where we define free state first, then free operation is defined as free-state-preserving map.
- Def: **resource morphisms** are defined as completely positive, trace-preserving (CPTP) linear maps $\mathcal{E} : \mathrm{D}(\mathbb{C}^m) \to \mathrm{D}(\mathbb{C}^m)$ such that $\mathcal{E}(\mathrm{F}) \subseteq \mathrm{F}$.



- We don't (operationally) give any constrains to free operations, so this structure-preserving map is considered as the largest set of free operations: maximal resource theory.
- Resourcefulness preorder: $\rho \succ_\epsilon \sigma$ whenever there exists a resource morphism $\mathcal{E}$ such that $\|\sigma - \mathcal{E}(\rho)\|_1 \leqslant 2\epsilon$.
- Maximally resourceful element: an element $\alpha \in \mathrm{D}(\mathbb{C}^m)$ is said to be *maximally resourceful* if $\alpha \succ \sigma$ for any $\sigma \in \mathrm{D}(\mathbb{C}^m)$.

Quantifying resource with entropies

- Constructions of monotones with smoothing.
  1. Hypothesis testing relative entropy:
     $\mathfrak{D}_h^\epsilon(\rho) := -\log \max_{\omega\in \mathsf{F}} \min_{P\in \mathsf{P}^\epsilon(\rho)} \mathrm{Tr}\{P\ \omega\}$;
  2. Max-divergence: $\mathfrak{D}_{\max}^\epsilon(\rho) := \inf_{\omega\in \mathsf{F}} D_{\max}^\epsilon(\rho\|\omega)$;
  3. Max-divergence of resourcefulness: $\mathfrak{D}_{\max,\mathsf{F}}^\epsilon(\rho) := \inf_{\omega\in \mathsf{F}} D_{\max,\mathsf{F}}^\epsilon(\rho\|\omega)$,
     where $D_{\max,\mathsf{F}}(\rho\|\sigma) := \log\inf\left\{\lambda\in\mathbb{R} : \frac{\lambda\sigma-\rho}{\lambda-1}\in \mathsf{F}\right\}$.

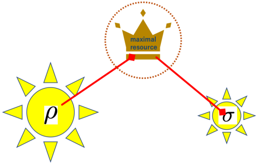


- Why do we consider different monotones? ...resourcefulness is a partial order, inducing incomparable states.

Question

Q: What drives the transformation between resources?

- The amount of resource? ...there are many (infinite) resource monotones that are inequivalent to each other, there is no reason one is better than another.
- Possible answer: refering to maximal resource, compare distillable resource and resource cost. ex) bipartite entanglement manipulation.



Example of “distill-and-dilute” transition

- Resource distillation and dilution are important operational tasks both theoretically and practically.
- In resource theory of bipartite entanglement (2011',Brandao-Datta), the zero-error one-shot SEPP-distillable entanglement $E^{(1)}_{D,\mathrm{SEPP}}(\rho)$ and the zero-error one-shot SEPP-entanglement cost $E^{(1)}_{C,\mathrm{SEPP}}(\sigma)$ satisfy,

$$E^{(1)}_{D,\mathrm{SEPP}}(\rho) \geqslant \lfloor \mathfrak{D}_h(\rho) \rfloor \text{ and } E^{(1)}_{C,\mathrm{SEPP}}(\sigma) \leqslant \mathfrak{D}_{\max,\mathrm{F}}(\sigma) + 1 \,,$$

- Lemma: $\lfloor \mathfrak{D}_h(\rho) \rfloor \geqslant \mathfrak{D}_{\max,\mathrm{F}}(\sigma) + 1 \implies \rho \succ \sigma$.
- When maximally resourceful state doesn't exist, such as in resource theory of multipartite entanglement with LOCC. Better answer?

Main theorem 1:

Theorem

Given $\rho, \sigma \in \mathrm{D}(\mathbb{C}^m)$ *and* $\epsilon_1, \epsilon_2 \in [0, 1]$,

$$\mathfrak{D}_h^{\epsilon_1}(\rho) \geqslant \mathfrak{D}_{\mathrm{max,F}}^{\epsilon_2}(\sigma) \implies \rho \succ_{(\epsilon_1+\epsilon_2)} \sigma.$$

- No requirements for the transformation to pass through the maximally resourceful state.
- Hence, less restrictive condition than previous one that guaranteed transformation between entangled states. $\lfloor \mathfrak{D}_h(\rho) \rfloor \geqslant \mathfrak{D}_{\mathrm{max,F}}(\sigma) + 1$ is relaxed to $\mathfrak{D}_h(\rho) \geqslant \mathfrak{D}_{\mathrm{max,F}}(\sigma)$.
- Choosing different monotones for comparing quantum resource is essentially crucial due to the possible infinite number of inequivalent monotones.
- **You can quantify resources without distillation to or dilution from the maximal resource.**(No currency is needed!!)

Main theorem 2:

Theorem

Given a state $\rho \in \mathrm{D}(\mathbb{C}^m)$, *suppose that* $\max_{\omega \in \mathrm{F}} D_h^{\epsilon_1}(\rho \| \omega) = \min_{\omega \in \mathrm{F}} D_h^{\epsilon_1}(\rho \| \omega)$. *Then, for any* σ,

$$\mathfrak{D}_h^{\epsilon_1}(\rho) \geqslant \mathfrak{D}_{\max}^{\epsilon_2}(\sigma) \qquad \Longrightarrow \qquad \rho \succ_{(\epsilon_1+\epsilon_2)} \sigma.$$

- **Warning**: this is a special case derived from main theorems that are not all listed here.
- Whole picture: we search for conditions that are expressed in terms of resource monotones, computed separately for the initial state and the target state.

Application in thermodynamics

- Consider an input system, with initial state $\rho \in \mathrm{D}(\mathbb{C}^m)$ and free singleton $\mathsf{F} = \{\gamma\}$, and an output system, with target state $\sigma \in \mathrm{D}(\mathbb{C}^n)$ and free singleton $\mathsf{F}' = \{\gamma'\}$.
- Then, $D_h^{\epsilon_1}(\rho\|\gamma) \geqslant D_{\max}^{\epsilon_2}(\sigma\|\gamma') \implies \rho \succ_{(\epsilon_1+\epsilon_2)} \sigma$.
- With physics interest, we can view the singleton as Gibbs state (thermal equilibrium state, free state) and the free operation as Gibbs-preserving operation.

Application: conditions for existence of maximally resourceful state.

Proposition

$\alpha \in \mathrm{D}(\mathbb{C}^d)$ *is maximally resourceful if either of the following conditions holds:*

1. $\mathfrak{D}_h(\alpha) = \max_{\rho \in \mathrm{D}(\mathbb{C}^d)} \mathfrak{D}_{\max}(\rho)$, *and* $\mathrm{Tr}\{\omega\, \Pi_\alpha\} = \text{constant}$, *for any* $\omega \in \mathrm{F}$,
2. $\mathfrak{D}_h(\alpha) = \max_{\rho \in \mathrm{D}(\mathbb{C}^d)} \mathfrak{D}_{\max,\mathrm{F}}(\rho)$.

- the generalized GHZ state is therefore maximally resourceful under resource morphisms (resource theory of genuine multipartite entanglement [19'RBTL]).
- The first condition coincides with resource theory of coherence, and the second one coincides with resource theory of entanglement.

Application: bounds for dilution and distillation

Proposition (sufficient and necessary condition)

When dealing with transitions from an input system $(\mathbb{C}^m, \mathsf{F})$ *to an output system* $(\mathbb{C}^n, \mathsf{F}')$, *suppose that* $\alpha \in \mathsf{D}(\mathbb{C}^m)$ *satisfies* $\mathfrak{D}_h(\alpha) = \mathfrak{D}_{\max,\mathsf{F}}(\alpha)$,*then*

1. *for any* $\sigma \in \mathsf{D}(\mathbb{C}^n)$,
$$\alpha \succ \sigma \qquad \Longleftrightarrow \qquad \mathfrak{D}_h(\alpha) \geqslant \mathfrak{D}_{\max,\mathsf{F}'}(\sigma)\ .$$
2. *for any* $\rho \in \mathsf{D}(\mathbb{C}^m)$,
$$\rho \succ \alpha \qquad \Longleftrightarrow \qquad \mathfrak{D}_h(\rho) \geqslant \mathfrak{D}_{\max,\mathsf{F}'}(\alpha)\ .$$

- A special case is when α is the maximal resource state, then we found optimal conditions for resource manipulation.

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III. MAIN RESULTS

In this section, we state and prove the main results of this paper. Firstly we derive, for any finite-dimensional resource theory in which the set of free states is non-empty closed and convex, sufficient conditions for the existence of a resource morphism between any two states, given in terms of resource monotones. Such conditions are formulated so to allow, in general, non-zero errors in the state transition, while the operation implementing the transition is an exact resource morphism.

Theorem 1. *Let us arbitrarily fix two states, $\rho, \sigma \in \mathsf{D}(\mathbb{C}^m)$, and two values $\epsilon_1, \epsilon_2 \in [0,1]$. Let us moreover choose $\tilde{\sigma} \in \mathsf{B}^{\epsilon_2}(\sigma)$ and $\tilde{\sigma}_+ \in \mathsf{F}$ so that $D_{\max}(\tilde{\sigma}\|\tilde{\sigma}_+) = \mathfrak{D}^{\epsilon_2}_{\max}(\sigma)$.*

(i) If $\mathfrak{D}^{\epsilon_1}_h(\rho) = +\infty$, then $\rho \succ_{\epsilon_1} \sigma$.

(ii) If $\mathfrak{D}^{\epsilon_2}_{\max}(\sigma) = 0$, then $\rho \succ_{\epsilon_2} \sigma$.

(iii) If $\mathfrak{D}^{\epsilon_1}_h(\rho) < +\infty$ and $\mathfrak{D}^{\epsilon_2}_{\max}(\sigma) > 0$, then

(a) either $D_{\max,\mathsf{F}}(\tilde{\sigma}\|\tilde{\sigma}_+) < +\infty$; in such a case, $\rho \succ_{\epsilon_1+\epsilon_2} \sigma$ if the following two conditions simultaneously hold:

$$\mathfrak{D}^{\epsilon_1}_h(\rho) \geqslant \mathfrak{D}^{\epsilon_2}_{\max}(\sigma) \tag{15}$$

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Summary

- We derived a unified resource theory that deals with resource state transformations.
- We showed various applications in physics that could be derived directly from main theorems.
- **You can quantify resources without using distillation or dilution.**
- **Open question**: Asymptotic case of this work?