Bell violations for entangled qudit pairs from random mutually unbiased bases

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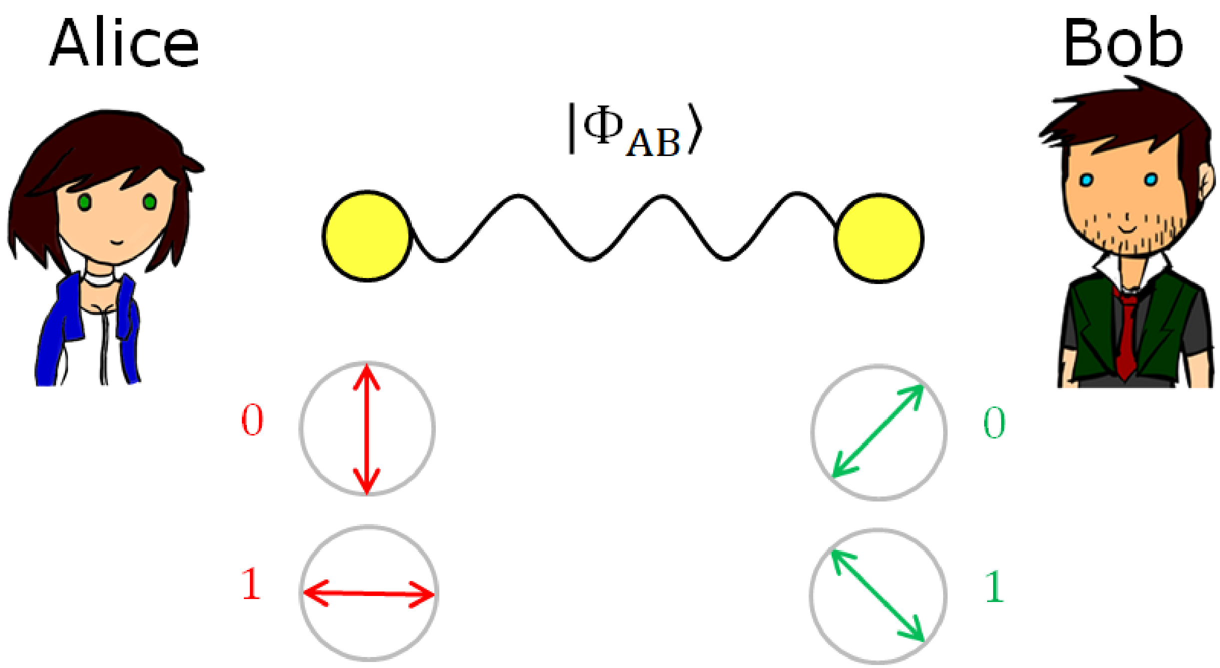
Bell tests

Demonstrating quantum nonlocality from violation of a Bell inequality



CHSH: $\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2$

Shared reference frame

How do Alice and Bob align their frames?

Align z-direction with qubits:

Alice sends $|\vec{z}\rangle = \cos\theta\,|0_B\rangle + e^{i\phi}\sin\theta\,|1_B\rangle$.
Bob measures in $\{|0_B\rangle, |1_B\rangle\}$.

Average fidelity:

$$f_{\vec{z},B} = \frac{1}{4\pi}\int d\vec{z}\left(p_0\,|\langle\vec{z}|0_B\rangle|^2 + p_1\,|\langle\vec{z}|1_B\rangle|^2\right) = \frac{2}{3}.$$

Optimal protocol for n copies $\rightarrow$ $f_{\vec{z},B} = \frac{n+1}{n+2}$

[Massar Popescu PRL 74 (1995) 1259]

Random measurements

Can we obtain Bell violations without a common reference frame?

Yes for CHSH inequality:

Liang, *et al.*: random pair of mutually unbiased bases (MUBs) → probability of violation $\approx 41\%$

Shadbolt, *et al.*: random orthogonal triads (complete set of MUBs) → guaranteed violation

How about in higher dimensions?

Mutually unbiased bases

Orthonormal bases in $\mathbb{C}^d$:

$$B_1 = \{|e_i\rangle\}, \qquad B_2 = \{|f_j\rangle\}$$

B_1 and B_2 are mutually unbiased iff:

$$\left|\langle e_i|f_j\rangle\right|^2 = \frac{1}{d}$$

If we prepare a state from B_1 and measure in B_2, all outcomes are equally likely.

Why we care about MUBs

MUBs exhibit the idea of complementarity

Entropic uncertainty [Maasen Uffink 1988]:

$$H(X) + H(Z) \geq -\log(c)\,, \qquad c = \max_{|x\rangle,|z\rangle} |\langle x|z\rangle|^2$$

When $c = \frac{1}{d} \Leftrightarrow \{|x\rangle\}, \{|z\rangle\}$ are mutually unbiased.

Practical applications:

- Quantum cryptography (BB84)
- Quantum state tomography ($\vec{p} \mapsto \rho$)

Complete set of MUBs

How many MUBs N_d can exist for $\mathbb{C}^d$?

Known results:

$N_d \leq d + 1$ [Delsarte Goethals Seidel 1975]

$N_p = p + 1$ [Ivanovic 1981]

$N_{p^e} = p^e + 1$ [Wootters Fields 1989]

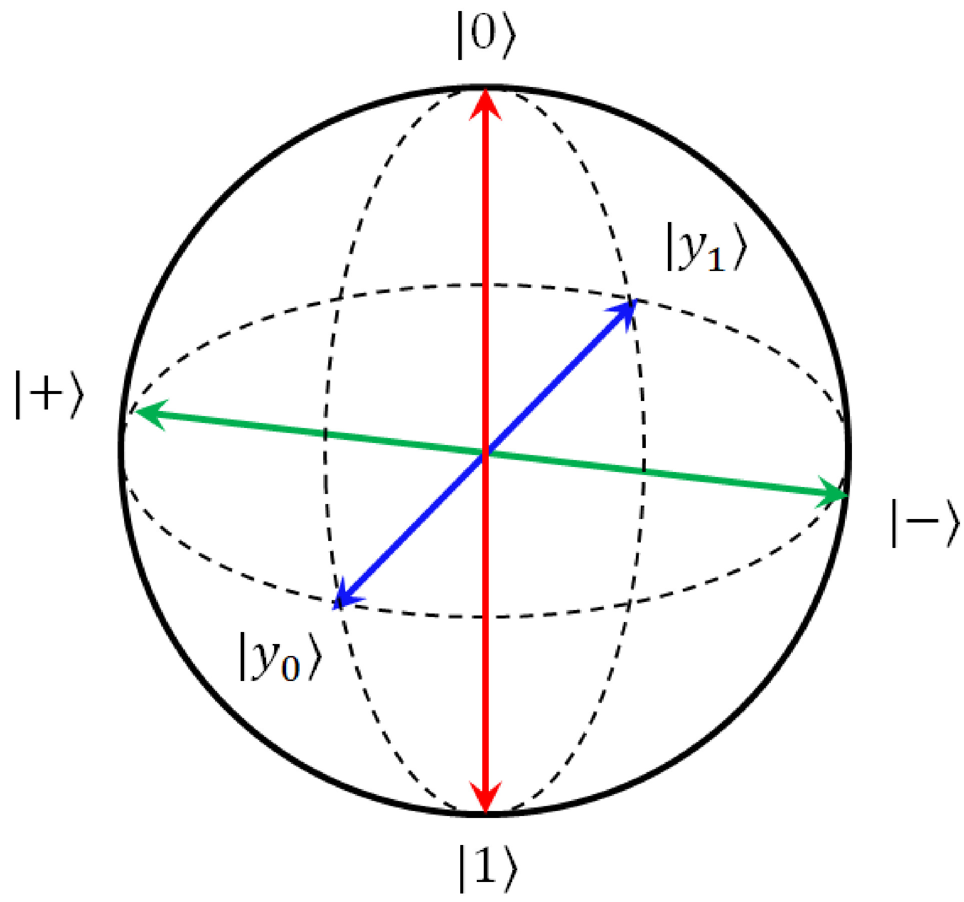
Examples of MUBs

Qubits $(d = 2)$

$$B_1 = \{|0\rangle, |1\rangle\}$$

$$B_2 = \left\{\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}\right\}$$

$$B_3 = \left\{\frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}\right\}$$



Orthogonal triad

Examples of MUBs

Qutrits $(d = 3)$: $\omega = e^{2\pi i/3}$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

$$B_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{bmatrix}, \qquad B_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{bmatrix}$$

Examples of MUBs

Ququarts $(d = 4)$: B_1 is standard basis

$$B_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \qquad B_3 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ i & i & -i & i \\ i & i & i & -i \end{bmatrix}$$

$$B_4 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ i & -i & i & i \\ i & i & -i & i \\ 1 & 1 & 1 & -1 \end{bmatrix}, \qquad B_5 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ i & -i & i & i \\ 1 & 1 & -1 & 1 \\ i & i & i & -i \end{bmatrix}$$

CGLMP inequality

2 parties with 2 inputs and d outputs

Observable $\mathcal{O}$ with $\mathrm{eig}(\mathcal{O}) = 0,1, \ldots d-1$:

$$\langle [A_1 - B_1] \rangle + \langle [B_1 - A_2] \rangle + \langle [A_2 - B_2] \rangle + \langle [B_2 - A_1 - 1] \rangle \geq d - 1$$

$$\langle [\mathcal{O}] \rangle = \sum_{i=0}^{d-1} i \cdot \Pr[\mathcal{O} = i \bmod d]$$

D. Collins, N. Gisin, N. Linden, S. Massar, S. Popescu

Numerical simulation

To estimate probability of CGLMP violation from random MUBs:

Alice and Bob each pick k random MUBs and choose pair to measure on $|\Phi\rangle = \sum_{i=0}^{d-1} |i\rangle|i\rangle/\sqrt{d}$.

Look for the best CGLMP value (settings, outcomes, MUB pairs) $\rightarrow$ value of 1 trial/simulation

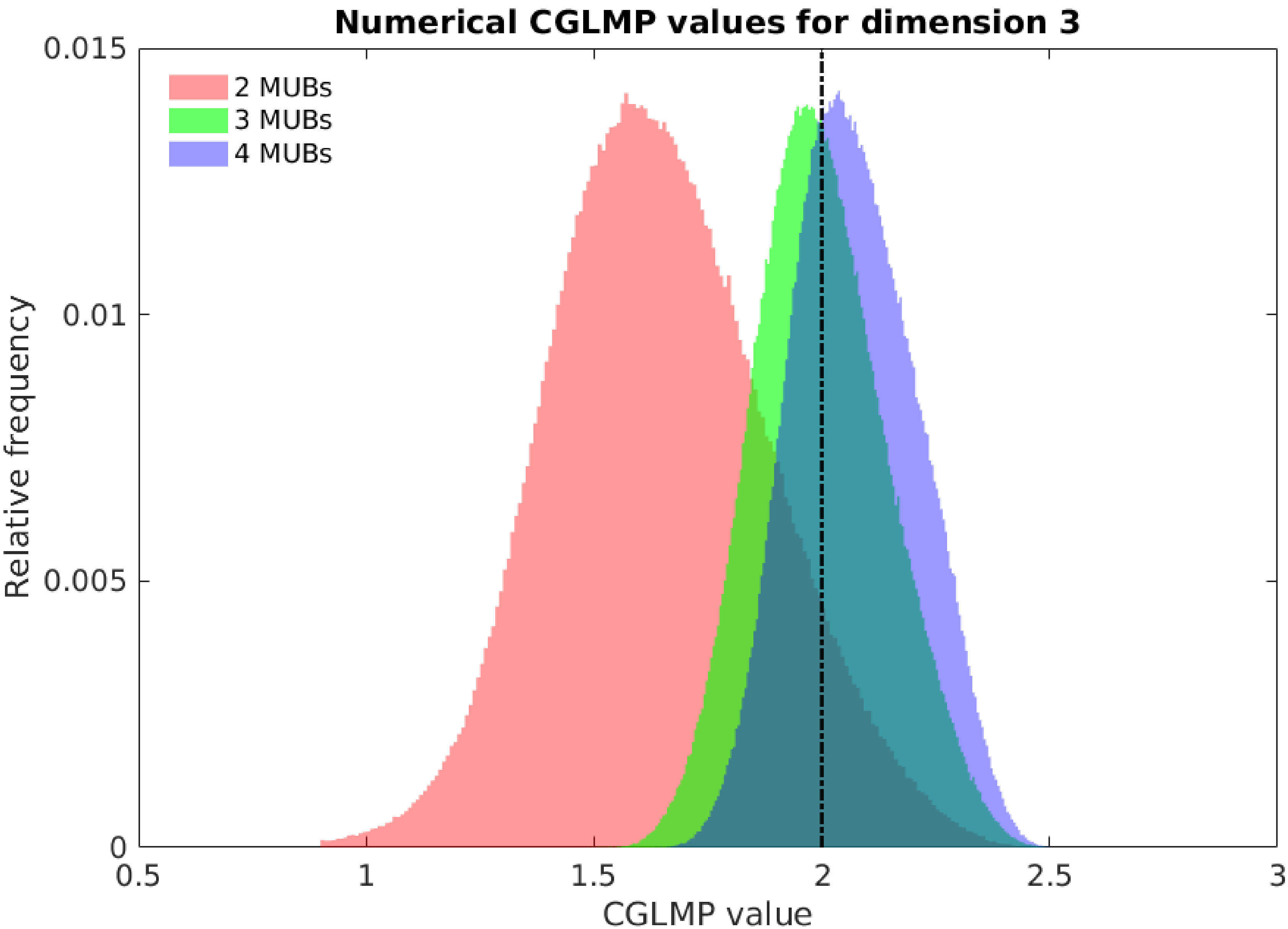
Fraction of trials that violate local bound give an estimate of probability of violation

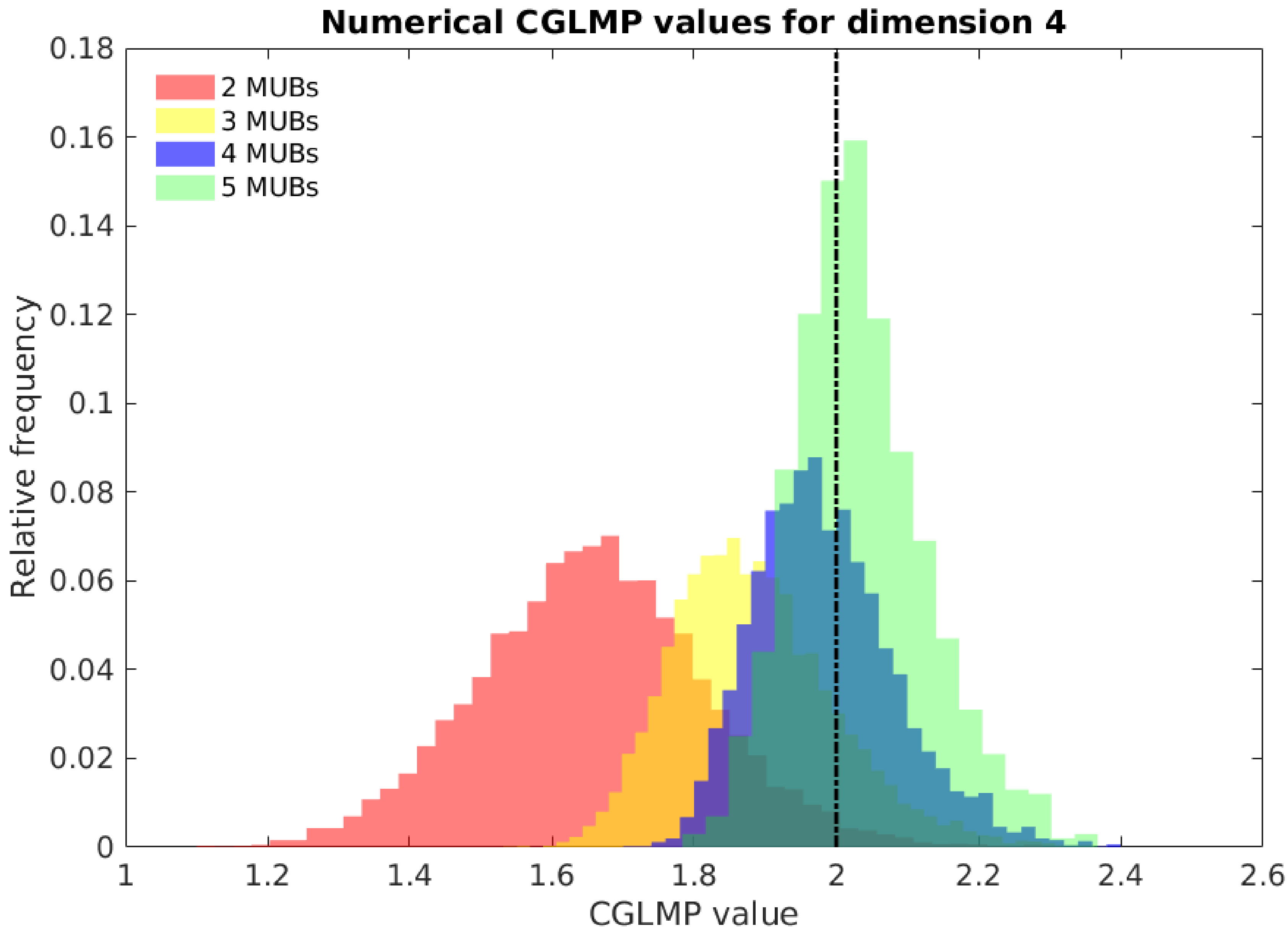
General Bell violation

We also estimate the probability of violation without using a specific Bell inequality

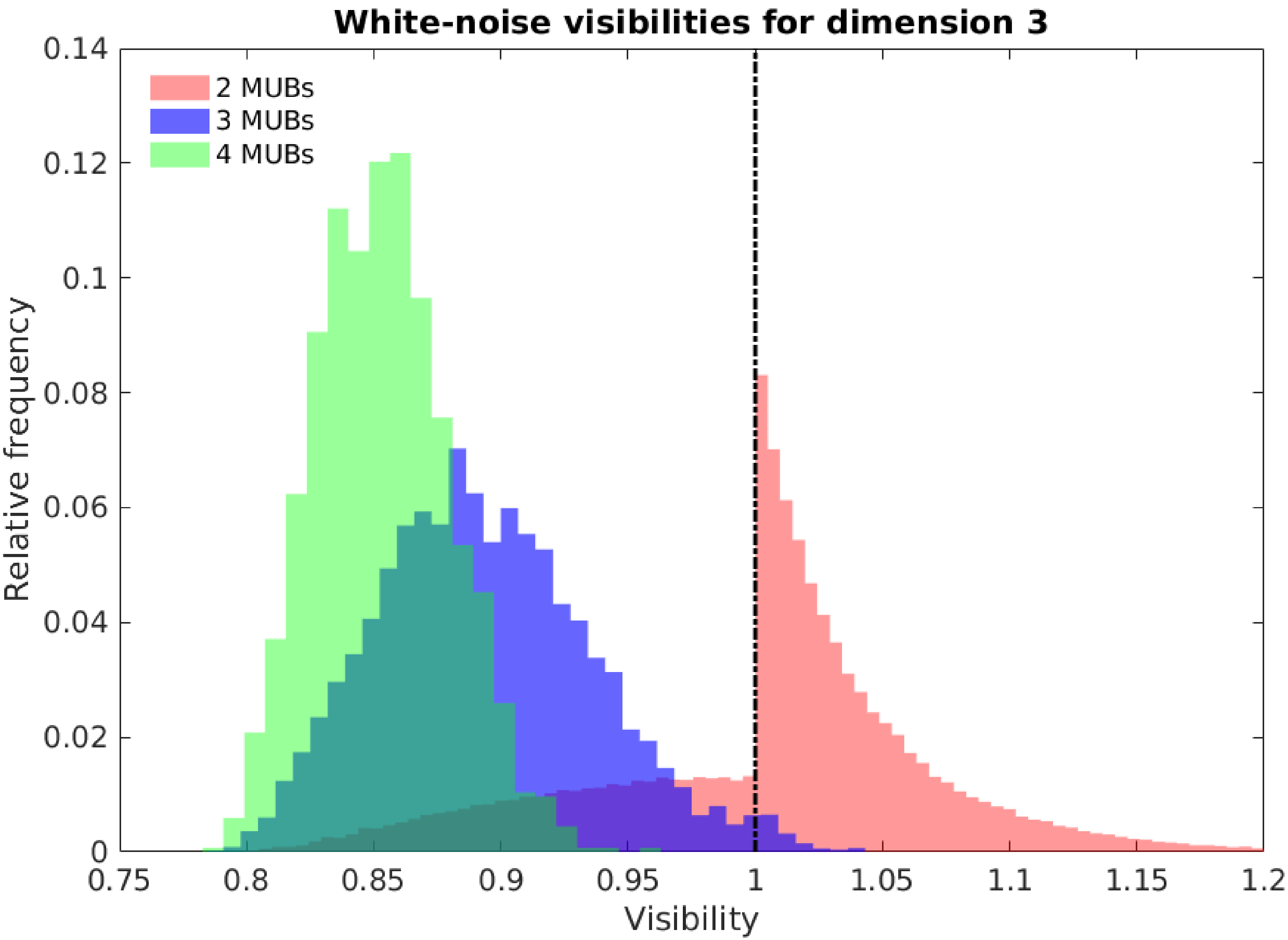
We run a linear program that tests whether a correlation lies within the local polytope

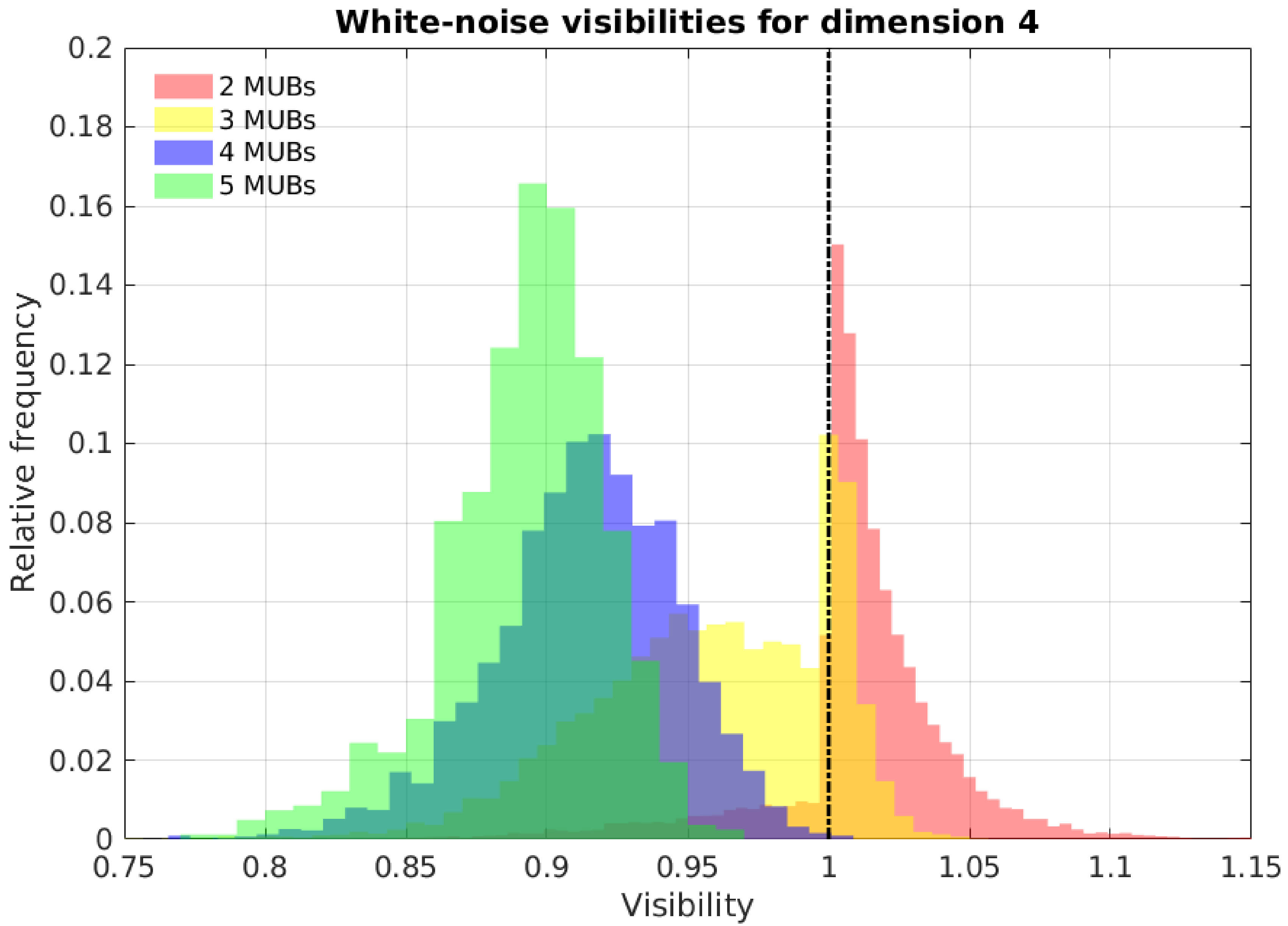
For each correlation P, find largest visibility v
s.t. $M = vP + (1 - v)U$ stays in local polytope.
- U is the uniform distribution.
- If $v < 1$, P is Bell nonlocal.


Numerical CGLMP values for dimension 3
2 MUBs
3 MUBs
4 MUBs
Relative frequency
0.015
0.01
0.005
0
0.5
1
1.5
2
2.5
3
CGLMP value



Numerical CGLMP values for dimension 4
2 MUBs
3 MUBs
4 MUBs
5 MUBs
Relative frequency
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
CGLMP value


White-noise visibilities for dimension 3
2 MUBs
3 MUBs
4 MUBs
Relative frequency
Visibility
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2


White-noise visibilities for dimension 4
2 MUBs
3 MUBs
4 MUBs
5 MUBs
Relative frequency
Visibility
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15

Dim	#MUBs	CGLMP%	Visibility%	#CGLMP trials	#Visibility trials
3	2	7.67	31.14	10^6	10^5
	3	47.01	98.36	10^6	10^4
	4	68.24	100	10^6	10^3
4	2	1.87	14.48	10^4	$2x10^4$
	3	14.97	77.75	10^4	10^4
	4	40.0	99.87	$5x10^3$	$3x10^3$
	5	61.1	100	10^3	10^3

Conclusions

In contrast with qubit case, CGLMP violation is not guaranteed even a complete set of MUBs.

However, for general Bell violations, our results show guaranteed violation from MUBs in the qutrit and ququart cases.

We conjecture that this behavior will persist in higher dimensions.

Constructing MUBs

For prime d:

$$X = \sum_j |j \oplus 1\rangle\langle j| , \qquad Z = \sum_k e^{\frac{2\pi i}{d}} |k\rangle\langle k|$$

Take eigenbases of $Z, X, XZ, \ldots, XZ^{d-1}$