Uncertainties of Genuinely Incompatible Triple Measurement Based on Statistical Distance

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Abstract

We investigate the measurement uncertainties of a triple of positive operator-valued measures (POVMs) based on statistical distance, and formulate state-independent tight uncertainty inequalities satisfied by the three measurements in terms of triple-wise joint measurability. Particularly, uncertainty inequalities for three unbiased qubit measurements are presented with analytical lower bounds which relates to the necessary and sufficient condition of the triple-wise joint measurability of the given triple. We show that the measurement uncertainties for a triple measurement are essentially different from the ones obtained by pair wise measurement uncertainties by comparing the lower bounds of different measurement uncertainties.[1]

1. Introduction of Backgrounds

The uncertainty principle is arguably one of the most famous features of quantum mechanics The well-known Heisenberg-Robertson uncertainty relation says that for any observables A and B, $\Delta A \Delta B \geq \frac{1}{2}|\langle [A,B]\rangle|$, where $\Delta\Omega = \sqrt{\langle \Omega^2\rangle - \langle \Omega\rangle^2}$ is the standard deviation for observable Ω, $\langle\cdot\rangle$ denotes the expectation of an operator with respect to a given state ρ, and $[A,B] = AB - BA$. Such uncertainty relations based on product form or summation form of deviation have been generalized and studied. There are uncertainty relations, of which the uncertainties are described by approximation error for probabilities of joint measurements [2, 3]:

$$\Delta(M^1, M^2; N^1, N^2) := \max_{\rho} \sum_{i=1}^{2} d_\rho(M^i; N^i). \tag{1}$$

where $d_\rho(M^i;N^i) := \sum_k |p_k^{M^i} - p_k^{N^i}|$, $p_k^{M^i} = Tr(\rho M_k^i)$ ($p_k^{N^i} = Tr(\rho N_k^i)$) and approximation measurement $\{N^i\}$ are jointly measurable.
It has been demonstrated in [4] that three unbiased qubit measurements $\{N_\pm^i = (I \pm \vec{n}_i\cdot\vec{\sigma})/2\}_{i=1}^3$ are triple-wise jointly measurable if and only if

$$\sum_{k=1}^{4} |\vec{p}_k - \vec{p}_F| \leq 4, \tag{2}$$

where $\vec{q}_1 = \vec{n}_{123}$, $\vec{q}_2 = \vec{n}_1 - \vec{n}_{23}$, $\vec{q}_3 = \vec{n}_2 - \vec{n}_{13}$, $\vec{q}_4 = \vec{n}_3 - \vec{n}_{12}$ and $\vec{q}_F$ is the Fermat-Torricelli point of $\{\vec{q}_k\}_{k=1}^4$, ($\vec{m}_{ij} = \vec{m}_i + \vec{m}_j, \vec{m}_{i-j} = \vec{m}_i - \vec{m}_j$). According to these two points, we can also formulate $\sum_{i=1}^3 d_\rho(M^i;N^i)$ under all triple-wise jointly measurable measurements $\{N^i\}_{i=1}^3$ satisfying (2) and maximize $\sum_{i=1}^3 d_\rho(M^i;N^i)$ over all quantum states to obtain $\Delta(M^1,M^2,M^3;N^1,N^2,N^3) := \sum_{i=1}^3 d_\rho(M^i;N^i)$. We show that the approximation error is lower bounded by a quantity which relates to the necessary and sufficient condition of the triple-wise joint measurability of the given triple.

2. Main Results and Outline of Proof

Consider now three unbiased qubit measurements $\{M^i\}_{i=1}^3$ described by positive operator-valued measures

$$M_+^i = \frac{I + \vec{m}_i\cdot\vec{\sigma}}{2}, \quad M_-^i = \frac{I - \vec{m}_i\cdot\vec{\sigma}}{2}, \quad i = 1,2,3,$$

where the three dimensional vectors $\vec{m}_i$ satisfy $|\vec{m}_i| \leq 1$, I is the 2×2 identity matrix, and $\vec{\sigma}$ is the vector with the Pauli matrix σ_i as the i-th entry. Let ρ be a qubit state with Bloch vector representation, $\rho = (I + \vec{r}\cdot\vec{\sigma})/2$ ($|\vec{r}| \leq 1$). Maximizing $\sum_{i=1}^3 d_\rho(M^i;N^i)$ over all ρ, we obtain

$$\Delta(M^1,M^2,M^3;N^1,N^2,N^3) = 2\max_{\vec{r}} \sum_{i=1}^{3} |\vec{r}\cdot(\vec{m}_i - \vec{n}_i)|. \tag{3}$$

Theorem. The approximation error of three unbiased qubit measurements $\{M^i\}_{i=1}^3$ to triple-wise jointly measurable unbiased qubit measurements $\{N^i\}_{i=1}^3$ satisfies the following inequality,

$$\Delta(M^1,M^2,M^3;N^1,N^2,N^3) \geq \frac{1}{2}\Big(\sum_{k=1}^{4} |\vec{p}_F - \vec{p}_k| - 4\Big), \tag{4}$$

where $\vec{p}_1 = \vec{m}_{123}$, $\vec{p}_2 = \vec{m}_1 - \vec{m}_{23}$, $\vec{p}_3 = \vec{m}_2 - \vec{m}_{13}$, $\vec{p}_4 = \vec{m}_3 - \vec{m}_{12}$ and $\vec{p}_F$ is the Fermat-Torricelli point of $\{\vec{p}_k\}_{k=1}^4$

Outline of Proof

By direct calculation we have that state-dependent approximation error $\sum_{i=1}^3 d_\rho(M^i;N^i) = 2\sum_{i=1}^3 |\vec{r}\cdot(\vec{m}_i - \vec{n}_i)| \geq 2\max\{|\vec{g}_1|, |\vec{g}_2|, |\vec{g}_3|, |\vec{g}_4|\}$, where $|\vec{g}_1| = |\vec{m}_{123} - \vec{n}_{123}|$, $|\vec{g}_2| = |\vec{m}_{1-23} - \vec{n}_{1-23}|$, $|\vec{g}_3| = |\vec{m}_{2-13} - \vec{n}_{2-13}|$, $|\vec{g}_4| = |\vec{m}_{3-12} - \vec{n}_{3-12}|$. Maximizing the approximation error $2\sum_{i=1}^3 |\vec{r}\cdot(\vec{m}_i - \vec{n}_i)|$ over all quantum states ρ we could obtain that

$$\max_{\vec{r}} 2\sum_{i=1}^{3} |\vec{r}\cdot(\vec{m}_i - \vec{n}_i)| = 2\max|g_i| := \mathcal{G}. \tag{5}$$

And the optimization is reached by $\vec{r}_0 = \max\{\vec{g}_i/|\vec{g}_i|\}$.
Noting that $\vec{g}_i = \vec{p}_i - \vec{q}_i$ and $\sum_{k=1}^4 |\vec{q}_F - \vec{q}_k| \leq 4$, we have

$$\begin{aligned}
&\Delta(M^1,M^2,M^3;N^1,N^2,N^3) = 2\mathcal{G} \\
&\geq \frac{1}{2}\sum_{k=1}^{4} |\vec{p}_k - \vec{q}_k| = \frac{1}{2}\sum_{k=1}^{4} |\vec{p}_k - \vec{q}_F + \vec{q}_F - \vec{q}_k| \\
&\geq \frac{1}{2}\sum_{k=1}^{4} [|\vec{p}_k - \vec{q}_F| - |\vec{q}_F - \vec{q}_k|] \geq \frac{1}{2}\Big[\sum_{k=1}^{4} |\vec{p}_k - \vec{p}_F| - 4\Big],
\end{aligned} \tag{6}$$

where the second inequality is due to triangle inequality, the third one comes from the definition of the Fermat-Torricelli point of $\{\vec{p}_k\}_{k=1}^4$ and the constraint of the triple wise joint measurability for $\{N^i\}_{i=1}^3$.

Remarks. From the Theorem we could obtain that

- When the lower bound of (4) is zero, then M^1, M^2, M^3 are triple-wise jointly measurable. In this trivial case the uncertainty inequality (4) is tight.
- For sharp unbiased qubit measurements associated with the Pauli operators σ_i, $i = 1,2,3$, we can get that the uncertainty inequality (4) is also tight while

$$\Delta_{lb}(M^1,M^2,M^3) = \frac{1}{2}\Big(\sum_{k=1}^{4} |\vec{p}_k| - 4\Big) = 2\sqrt{3} - 2.$$

- In the above case the degree of the incompatibility of σ_i is $2\sqrt{3} - 2$.

Compared with the the measurement uncertainty relations which are obtained by minimizing $\Delta(M^1,M^2,M^3;N^1,N^2,N^3)$ over pair-wise and triple-wise jointly measurable measurements respectively, we show that (4) captures better incompatible measurement uncertainty of the triple measurements M^1, M^2 and M^3 when one pair of measurements $\{M^i, M^j\}$ are jointly measurable.

The merit of the uncertainty relation (4). The measurement uncertainty relation obtained by pair-wise jointly measurable measurements reads

$$\Delta(M^1,M^2,M^3;N^1,N^2,N^3) \geq \frac{1}{2}\Big[\sum_{i<j}^{3} (|\vec{m}_i + \vec{m}_j| + |\vec{m}_i - \vec{m}_j| - 2)\Big]. \tag{7}$$

When one pair of measurements $\{M^i, M^j\}$ are jointly measurable, comparing (4) and (7) we have

$$\begin{aligned}
2\Delta(M^1,M^2,M^3;N^1,N^2,N^3) &\geq \sum_{k=1}^{4} |\vec{p}_F - \vec{p}_k| - 4 \\
&\geq \max_{i\neq j\neq k\neq l\in\{1,2,3,4\}} (|\vec{p}_i - \vec{p}_j| + |\vec{p}_k - \vec{p}_l|) \\
&\geq 2\max_{i\neq j}(|\vec{m}_i + \vec{m}_j| + |\vec{m}_i - \vec{m}_j| - 2),
\end{aligned} \tag{8}$$

3. Conclusion and Discussion

We found that the n unbiased qubit measurements $\{N^i = (I \pm \vec{n}_i\cdot\vec{\sigma})/2\}_{i=1}^n$ are n-tuple-wise jointly measurable, if

$$\sum_{\mu_i=\pm 1} |\sum_{i=1}^{n} \mu_i \vec{n}_i| \leq 2^n. \tag{9}$$

Then we conjecture that for some special M^i, one may have the following relation from (9)

$$\Delta(M^1,...,M^n;N^1,...,N^n) \geq \Big(\sum_{\mu_i=\pm 1} |\sum_{i=1}^{n} \mu_i \vec{m}_i| - 2^n\Big)/2^{n-2}.$$

By approximating a given triple of unbiased qubit measurements to all possible triple measurements that are triple-wise jointly measurable, we have formulated state-independent tight uncertainty inequalities satisfied by the triple of qubit measurements, with the lower bound giving by the necessary and sufficient condition of the triple-wise joint measurability of the given triple. These uncertainty relations can be experimentally tested, like the case of two qubit measurements [5]. As the measurement uncertainties from a triple of measurements are essentially different from the ones from pair wise measurements, it is of significance to explore the measurement uncertainties for triple or n-tuple measurements by their measurement incompatibilities.

References

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