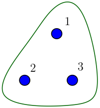
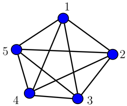
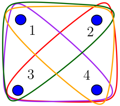
Multiparticle Entanglement in Hypergraph States

Otfried Gühne
D. Bruß, C. Budroni, M. Cuquet, M. Gachechiladze, B. Kraus, C. Macchiavello, T. Moroder, M. Rossi, F. Steinhoff







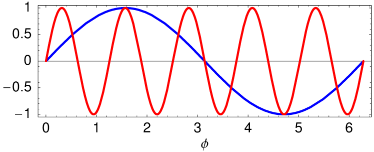
Department Physik, Universität Siegen

Theoretical Quantum Optics

Overview

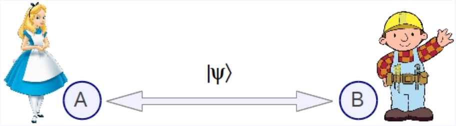
1. Multiparticle entanglement & graph states
2. Hypergraph states: Definitions & classification
3. Hypergraph states: Bell inequalities & applications
4. Conclusion & Outlook

Multiparticle entanglement



Entanglement & separability

Alice and Bob share a state $|\psi\rangle$.



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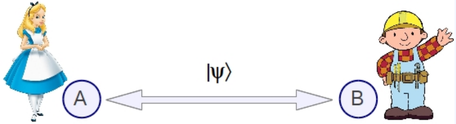

A pure state $|\psi\rangle$ is separable iff it is a product state:

$$|\psi\rangle = |a\rangle_A |b\rangle_B = |a, b\rangle.$$

Otherwise it is entangled.

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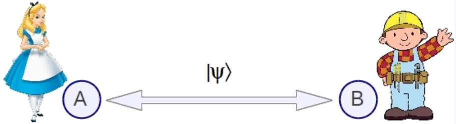

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Otherwise it is entangled.

Mixed states: Consider convex combinations: ϱ is separable, if

$$\varrho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad \text{mit} \quad p_i \geq 0, \ \sum_i p_i = 1.$$

Interpretation: Entanglement cannot be generated by local operations and classical communication.

R. Werner, PRA 40, 4277 (1989).

The separability problem

Question

Given ϱ, is it entangled or separable?

The separability problem

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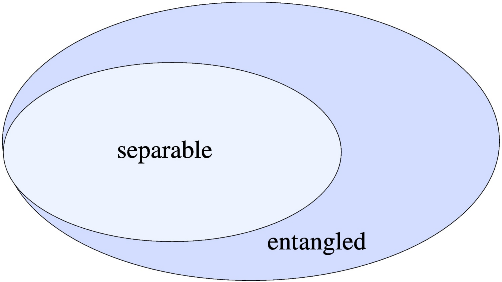
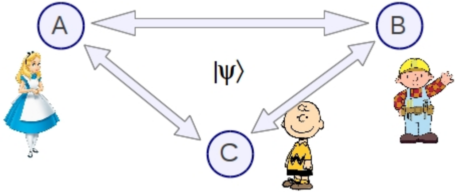
Geometrical interpretation

The set of all separable states is convex.



Multiparticle entanglement



Several possibilities:

- Fully separable:

$$|\psi^{\mathrm{fs}}\rangle = |000\rangle$$

- Biseparable:

$$|\psi^{\mathrm{bs}}\rangle = |0\rangle \otimes (|00\rangle + |11\rangle)$$

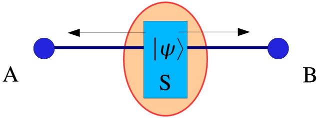
- Genuine multiparticle entangled:

$$|GHZ\rangle = |000\rangle + |111\rangle \quad \text{oder} \quad |W\rangle = |001\rangle + |010\rangle + |100\rangle.$$

- Mixed states: Convex combinations, again.

Why is entanglement interesting?

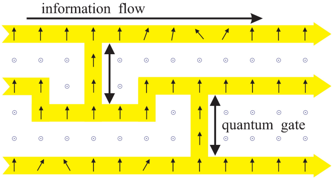
Quantum cryptography



- Source S sends entangled states to A and B.
- From the correlations a key can be generated.
- If the measurement results are compatible with a separable state, then the scheme is not secure.

A.K. Ekert, PRL 67, 661 (1991);
M. Curty et al, PRL 92, 217903 (2004).

One-way quantum computer



- By making local measurement on a cluster state, a quantum computer can be realized.
- Problem: Experimental generation of the cluster state.

R. Raussendorf, H. Briegel, PRL 86, 5188 (2001).

Entanglement and precision measurements

The task

Assume we have a device D inducing the transformation

$$|0\rangle \mapsto |0\rangle, \quad |1\rangle \mapsto e^{i\phi}|1\rangle$$

How can we estimate ϕ?

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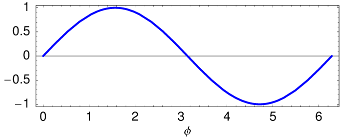

Simple strategy

- Prepare $|\psi\rangle = |0\rangle + |1\rangle$.
- Apply D: $|\psi'\rangle = |0\rangle + e^{i\phi}|1\rangle$.
- Measure $\langle\sigma_x\rangle \sim cos(\phi)$.
- Uncertainty:

$$\Delta\phi = \frac{\Delta\langle\sigma_x\rangle}{|\partial\langle\sigma_x\rangle/\partial\phi|} = 1$$

- Repeat N times:

$$\Delta\phi \geq 1/\sqrt{N}$$

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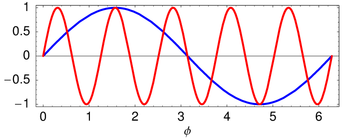

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Using entanglement

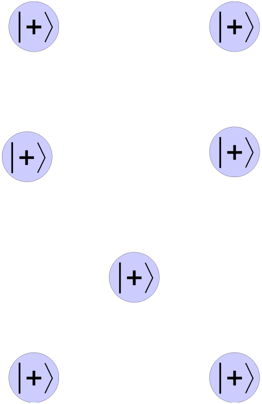
- Prepare N qubit state:

$$|GHZ_N\rangle = |0...0\rangle + |1...1\rangle$$

- Apply D and measure $\langle\sigma_x^{\otimes N}\rangle \sim cos(N\phi)$.
- Uncertainty:

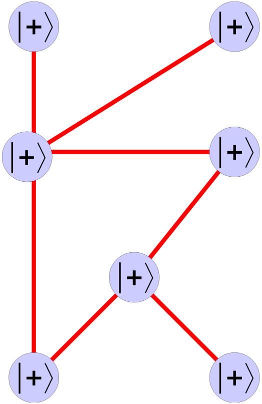
$$\Delta\phi = \frac{1}{N}$$

Review: V. Giovanetti et al, Science 306, 1330 (2004).



1) Start with a product state on N qubits in the state $|+\rangle = |x^+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

Graph states



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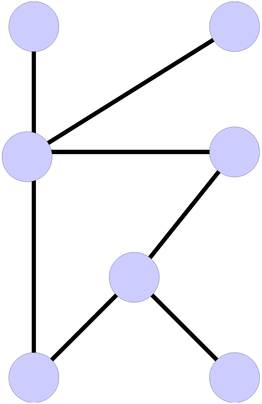
2) Let some of them interact pairwise via some Ising-type interaction:

$$C_{ab} = e^{i\frac{\pi}{4}(\mathbb{1}-\sigma_z^{(a)}-\sigma_z^{(b)}+\sigma_z^{(a)}\sigma_z^{(b)})\}}$$

This corresponds to a phase gate.

Graph states



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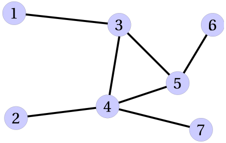
3) Resulting state is the graph state.

M. Hein, J. Eisert, H.J. Briegel, PRA 69, 062311 (2004).

Graph states as stabilizer states



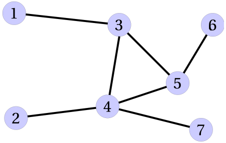
1) For any graph, we define stabilizing operators as ($X_i = \sigma_x^{(i)}$)

$$S_i = X_i \bigotimes_{j \in N(i)} Z_j.$$

2) The graph state $|G\rangle$ is the unique state fulfilling

$$S_i|G\rangle = |G\rangle.$$

Graph states as stabilizer states



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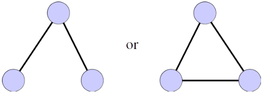
GHZ as example

The GHZ state $|GHZ\rangle = |000\rangle + |111\rangle$ fulfills

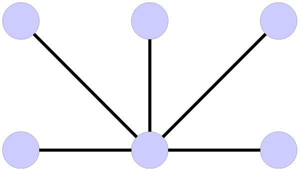
$$\begin{aligned} X_1 X_2 X_3 |GHZ\rangle &= |GHZ\rangle \\ Z_1 Z_2 \mathbb{1} |GHZ\rangle &= |GHZ\rangle \\ \mathbb{1} Z_2 Z_3 |GHZ\rangle &= |GHZ\rangle \end{aligned}$$

and corresponds (up to local rotations) to the graphs

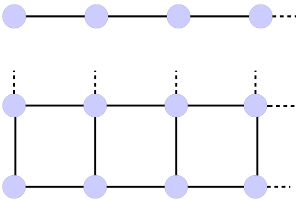


Graph states

Further examples of graph states: General GHZ states:



and cluster states:



Properties of graph states:

- They serve as the central resource in the one-way quantum computer.

  R. Raussendorf, H.J. Briegel, PRL 86, 5188 (2001)

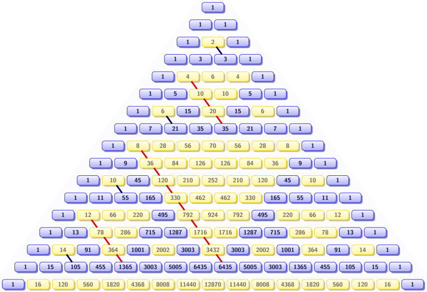
- All code words in quantum error correcting codes correspond to graph states.

  D. Schlingemann and R.F. Werner, PRA 65, 012308 (2002).

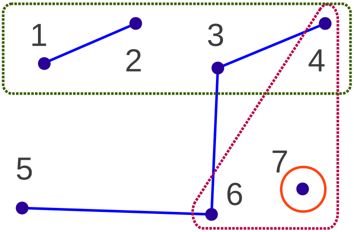
- The violate local realism in an extreme manner.

  O. Gühne et al., PRL 95, 120405 (2005).

Hypergraph states



Basic definitions



In a hypergraph, edges can contain more than two vertices.

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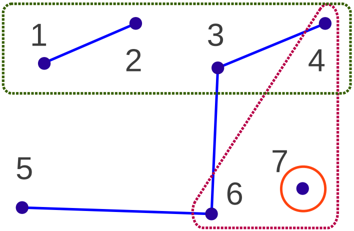

In a hypergraph, edges can contain more than two vertices.
The controlled phase gate on an edge e is given by

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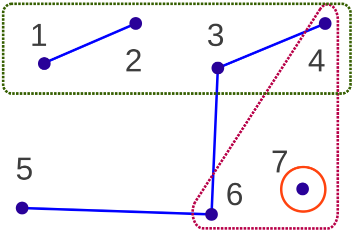

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The hypergraph state is:

$$|H\rangle = \prod_{e\in E} C_e|+\rangle^{\otimes N}$$

C. Kruszynska, B. Kraus. PRA 79, 052304 (2009), M. Rossi, M. Huber, D. Bruß,C. Macchiavello, NJP 15, (2013).

The nonlocal stabilizer

Define for each qubit the operator

$$g_i \equiv \big(\prod_{e\in E} C_e\big) X_i \big(\prod_{e\in E} C_e\big) = X_i \otimes \big(\prod_{e\ni i} C_{e\setminus\{i\}}\big)$$

Then:

$$g_i|H\rangle = |H\rangle \quad \text{for all } i$$

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Then:

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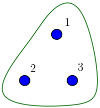
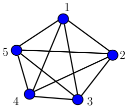
The stabilizing operators g_i:

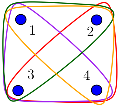
- ... are hermitean, but nonlocal,
- ... commute: $g_i g_j = g_j g_i$,
- ... generate a group with 2^N elements.

Examples

The three-qubit HG state







For the simplest nontrivial HG we have

$$|H_3\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle)$$

after a Hadarmard transformation on the third qubit:

$$|H_3\rangle = \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |111\rangle).$$

This state was also called “logical AND state”.

S. Abramsky, C. Costantin, arXiv:1412.5213

Pauli operations

Three possibilities

How does $|HG\rangle \mapsto \sigma_?^{(i)}|HG\rangle$ change the hypergraph?

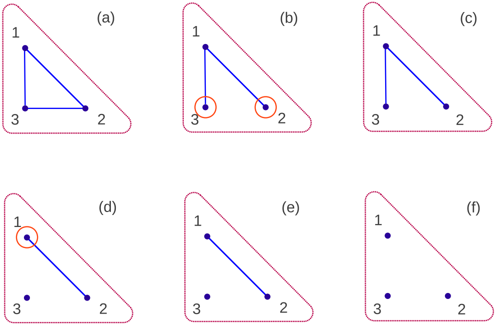
- Z-tranformation: Add / remove the edge $e = \{i\}$, since $C_{\{i\}} = \sigma_z^{(i)}$.
- X-transformation: Determine the set

  $$\mathcal{E}^{(k)} = \{e \setminus \{k\} | e \in E(k)\}.$$

  by taking all edges e which contain k and then removing k out of all these edges.
  Then, remove or add the edges from $\mathcal{E}$ to the HG, depending on whether they exist already in the HG or not.
- Y-transformation: Combined X- and Z- transformation.

R. Qu, J. Wang, Z. Li, Y. Bao, PRA 87, 022311 (2013).

Example



$$X_1 \mapsto Z_2, Z_3 \mapsto X_2 \mapsto Z_1 \mapsto X_3$$

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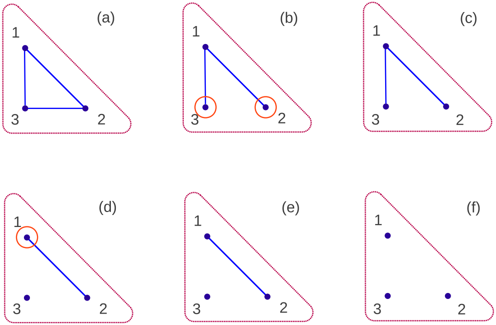


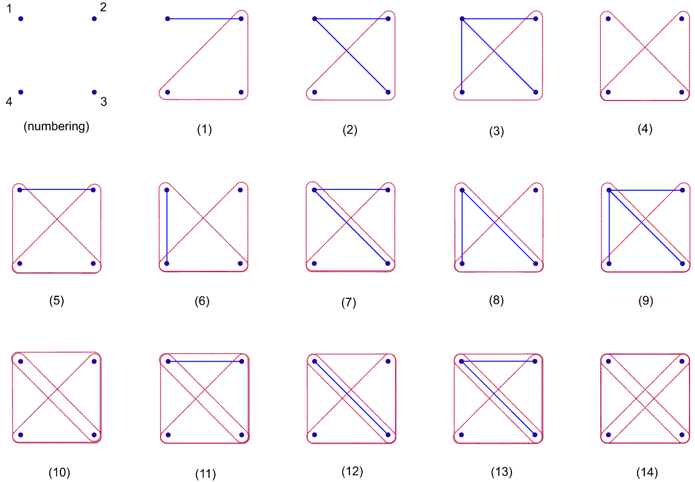
$$X_1 \mapsto Z_2, Z_3 \mapsto X_2 \mapsto Z_1 \mapsto X_3$$

Consequence

There is only one HG state for three qubits.

LU classes for four qubits

One finds 27 LP equivalence classes, which turn out to be LU inequivalent



O. Gühne et al., J. Phys. A: Math. Theor. 47, 335303 (2014)

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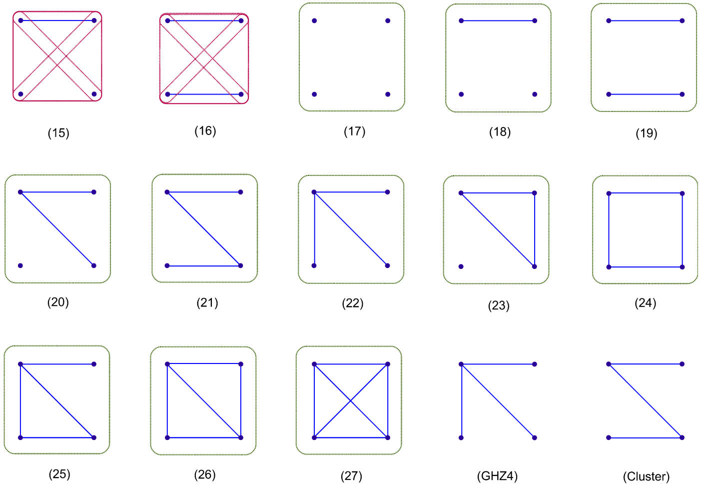

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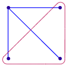
Some interesting states

States with maximally mixed single-qubit marginals are:

- No. 3:
$$|V_3\rangle = \sqrt{\frac{3}{4}}|D_4\rangle + \frac{1}{2}|GHZ_4^-\rangle.$$

- No. 9: With $|\gamma\rangle = (|00\rangle + |01\rangle - |10\rangle + |11\rangle)/2$ one has:
$$|V_9\rangle = \frac{1}{\sqrt{2}}|GHZ_4^-\rangle + \frac{1}{2}|01\rangle|\gamma\rangle + \frac{1}{2}|10\rangle|\overline{\gamma}\rangle,$$

- No. 14:
$$|V_{14}\rangle = \sqrt{\frac{3}{4}}|D_4\rangle + \frac{1}{2}|\overline{GHZ_4^-}\rangle,$$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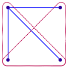
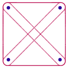

(9)


(14)

Discussion

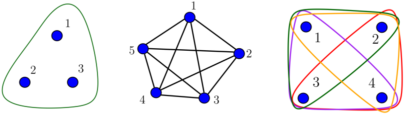
LU-LP Problem

- Is LU equivalence always equivalent to LP equivalence?
- For many cases yes, but in general ... ?
- Counterexamples would be useful.

Questions

- Is there a general rule to identify maximally entangled HG states?
- What are the applications of these states?

Bell inequalities for HG states



The first idea

First Problem

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- The state $|H_3\rangle$ is a $+1$ eigenstate of

$$g_1 = X_1 \otimes C_{23} = X_1 \otimes (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|)$$

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- So we have

$$P(+--|XZZ) = 0.$$

- Furthermore:

$$P(-++|XZZ) + P(-+-|XZZ) + P(--+|XZZ) = 0,$$

$\Rightarrow$ The non-local stabilizer predicts some local perfect correlations!

Hardy argument

If a LHV model satisfies the conditions from zero correlations from the state $|H_3\rangle$ then it must fulfill

$$P(+--|XXX) + P(-+-|XXX) + P(--+|XXX) = 0.$$

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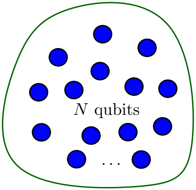
$$P(+--|XXX) + P(-+-|XXX) + P(--+|XXX) = 0.$$

In contrast, for $|H_3\rangle$ we have

$$P(+--|XXX) = \frac{1}{16}$$

This argument can be generalized to N qubits.

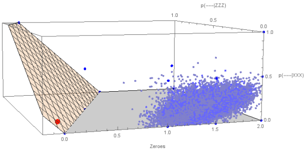


Taking the zero terms and $P(---|XXX)$ and $P(---|ZZZ)$ one has a Bell-Svetlichny inequality for genuine multiparticle nonlocality,

$$\begin{aligned}\langle\mathcal{B}_3^{(2)}\rangle = \big[&P(+--|XZZ) + P(-++|XZZ)\\ &+P(-+-|XZZ) + P(--+|XZZ) + \text{ permutat.}\big]\\ &+P(---|XXX) - P(---|ZZZ) \geq 0,\end{aligned}$$

which is violated by $|H_3\rangle$ with $\langle\mathcal{B}_3^{(2)}\rangle = -1\backslash 16$.

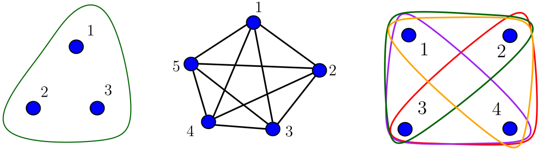
This inequality is a facet of the classical polytope.



Scaling

Question & Answer

- Does the violation of local realism increase with the number of particles? What are the interesting many-qubit HG states?
- Take three- and four-uniform fully connected HG states. They can be seen as generalizations of GHZ states.



Scaling

- For three-uniform HG states and for even m with $1 < m < N$:

$$\langle \underbrace{X \dots X}_{m} Z \dots Z \rangle = \begin{cases} +\frac{1}{2} & \text{if } m = 2 \bmod 4, \\ -\frac{1}{2} & \text{if } m = 0 \bmod 4. \end{cases}$$

- A similar result holds for four-uniform HG states
- This can be combined with the Mermin-type Bell operator:

$$\mathcal{B}_N = -\left[AAA \dots AA\right] + \left[BBA \dots A + \text{ permutat.}\right] - \\ -\left[BBBBA \dots A + \text{ permutat.}\right] + \left[\dots\right] - \dots$$

Results

- For three-uniform HG states the violation of Bell inequalities scales exponentially with the number of particles:

$$\frac{\langle \mathcal{B}_N \rangle_Q}{\langle \mathcal{B}_N \rangle_C} \overset{N\to\infty}{\sim} \sqrt{2}^N$$

- For four-uniform HG states the scaling is:

$$\frac{\langle \mathcal{B}_N \rangle_Q}{\langle \mathcal{B}_N \rangle_C} \overset{N\to\infty}{\sim} 1.20711^N$$

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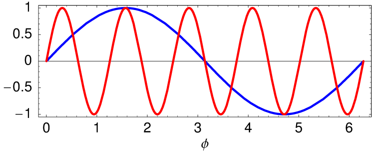
- For four-uniform HG states also the state after loosing one qubit violates Bell inequalities *with the same scaling.*
- For three-uniform HG states the reduced state is still highly entangled.

M. Gachechiladze, C. Budroni, O. Gühne, arXiv:1507.03570

Applications

- HG states are useful in the standard scheme of metrology



Reason: The visibility of the $\cos(N\phi)$ component is related to the violation of the Mermin inequality.

W.B. Gao et al., Nat. Phys. 6, 331 (2010)

- HG states are useful in some schemes of measurement based quantum computation.

M. Gachechiladze, C. Budroni, O. Gühne, arXiv:1507.03570

- Open Question: HG states & topological models?

B. Yoshida, arXiv:1508.03468, J. Miller, A Miyake, arXiv:1508.02695.

Conclusion

- HG states are a generalization of graph states
- They can be described by a non-local stabilizer formalism
- They violate Bell inequalities in many ways and are robust against particle loss
- The can be useful in metrology & quantum computation

Literature

- O. Gühne, M. Cuquet, F.E.S. Steinhoff, T. Moroder, M. Rossi, D. Bruß, B. Kraus, C. Macchiavello, J. Phys. A: Math. Theor. 47, 335303 (2014)
- M. Gachechiladze, C. Budroni, O. Gühne, arXiv:1507.03570

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