Quantum Causal Structures

Christian Majenz
University of Copenhagen
Joint work with Rafael Chaves and David Gross, University of Cologne
(arXiv:1407.3800)

Workshop on Quantum Nonlocality, Causal Stuctures and Device-Independent Quantum Information, NCKU Tainan

10.12.2015

Motivation

Question

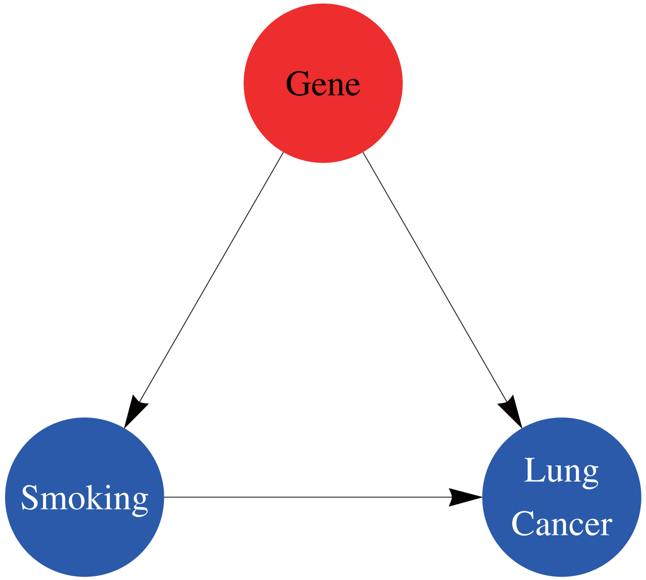
Question

Can we rule out certain causal relationships by just looking at statistical data?

Question

Can we rule out certain causal relationships by just looking at statistical data?

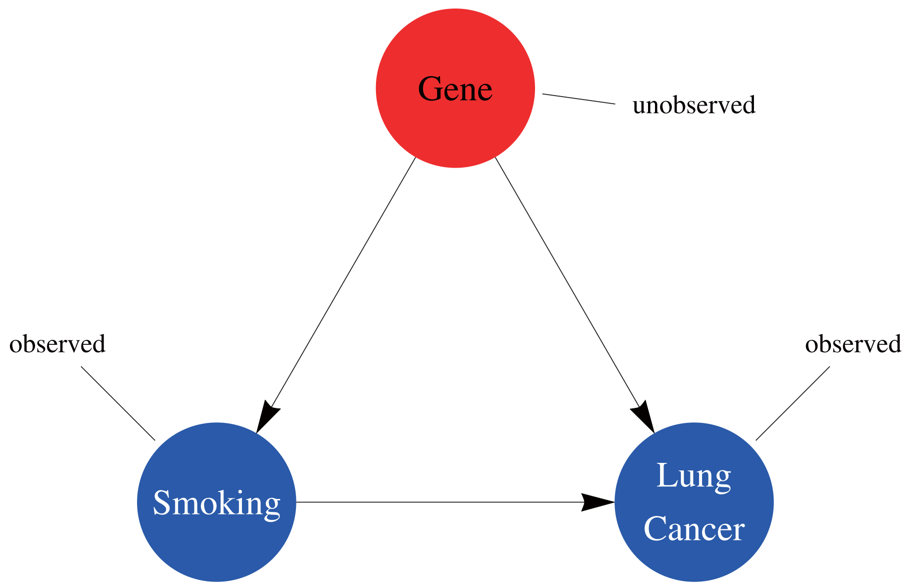
Example:



Question

Can we rule out certain causal relationships by just looking at statistical data?

Example:



Answer: Causal Inference

- Rigorous mathematical way to distinguish different causal relations

Answer: Causal Inference

- Rigorous mathematical way to distinguish different causal relations
- Introduced by Pearl in the 1980s

Answer: Causal Inference

- Rigorous mathematical way to distinguish different causal relations
- Introduced by Pearl in the 1980s
- Necessary conditions for probability distributions to come from more restrictive causal model

Answer: Causal Inference

- Rigorous mathematical way to distinguish different causal relations
- Introduced by Pearl in the 1980s
- Necessary conditions for probability distributions to come from more restrictive causal model
- Tools: Bayesian networks, entropies, convexity

Answer: Causal Inference

- Rigorous mathematical way to distinguish different causal relations
- Introduced by Pearl in the 1980s
- Necessary conditions for probability distributions to come from more restrictive causal model
- Tools: Bayesian networks, entropies, convexity

[?] What if the underlying processes are quantum?

Answer: Causal Inference

- Rigorous mathematical way to distinguish different causal relations
- Introduced by Pearl in the 1980s
- Necessary conditions for probability distributions to come from more restrictive causal model
- Tools: Bayesian networks, entropies, convexity

[?] What if the underlying processes are quantum?

- Bell nonlocality etc. special cases of this

Structure

Motivation

Classical

Bayesian Networks

Quantum

Entropic Description

Quantum Causal Structure

Entropic Description

Application

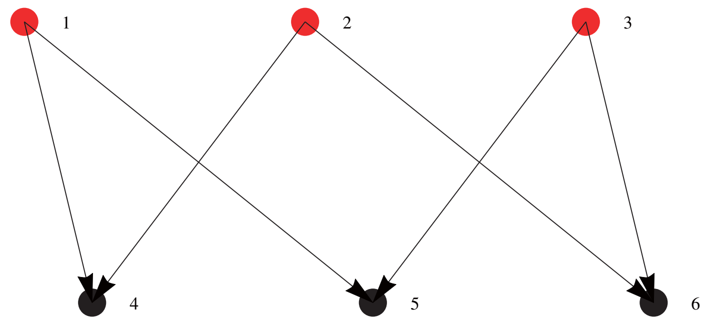
Information Causality

Computational Techniques

Classical

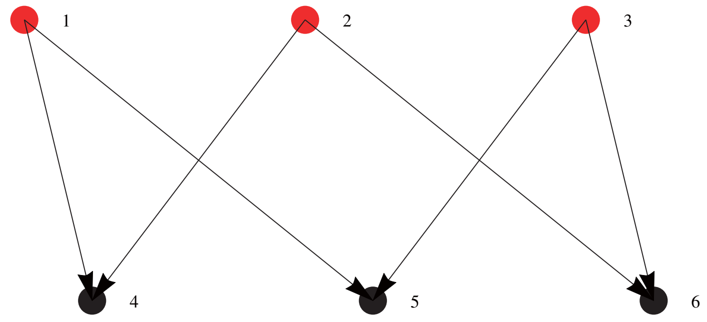
DAGs

- Any directed acyclic graph (DAG) specifies a causal structure:



DAGs

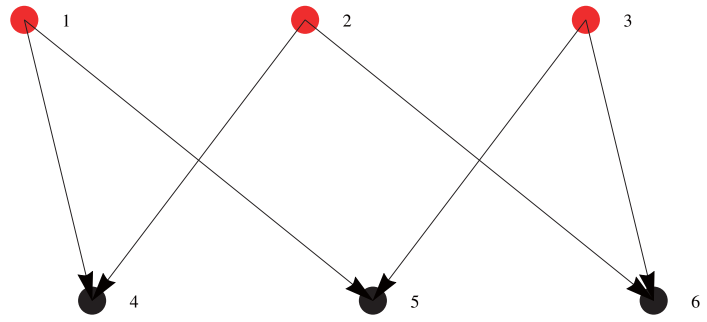
- Any directed acyclic graph (DAG) specifies a causal structure:



- Directed graph: $G = (V, E)$ with $E \subset V \times V$

DAGs

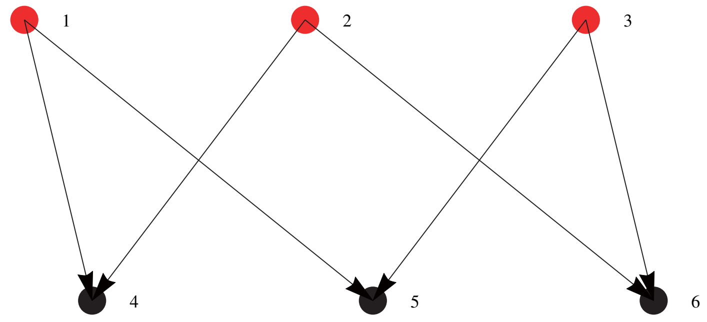
- Any directed acyclic graph (DAG) specifies a causal structure:



- Directed graph: $G = (V, E)$ with $E \subset V \times V$
- Acyclic: no cycle (causality)

DAGs

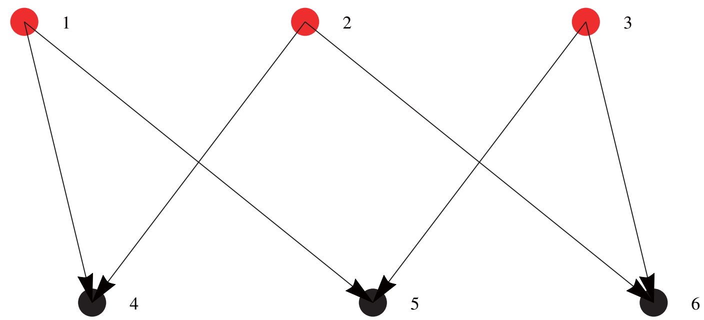
- Any directed acyclic graph (DAG) specifies a causal structure:



- Directed graph: $G = (V, E)$ with $E \subset V \times V$
- Acyclic: no cycle (causality)
- parents of v: $\mathrm{pa}(v) = \{w \in V | (w, v) \in E\}$

DAGs

- Any directed acyclic graph (DAG) specifies a causal structure:



- Directed graph: $G = (V, E)$ with $E \subset V \times V$
- Acyclic: no cycle (causality)
- parents of v: $\mathrm{pa}(v) = \{w \in V | (w, v) \in E\}$
- children, ancestors, descendants, non-descendants etc.

Bayesian network

- Bayesian network: generalization of Markov chain

Bayesian network

- Bayesian network: generalization of Markov chain
- distribution is product of conditional distributions

Bayesian network

- Bayesian network: generalization of Markov chain
- distribution is product of conditional distributions
- characterization: $v \perp \mathrm{nd}(v) | \mathrm{pa}(v)$

Bayesian network

- Bayesian network: generalization of Markov chain
- distribution is product of conditional distributions
- characterization: $v \perp \mathrm{nd}(v) | \mathrm{pa}(v)$
- example: Markov chain,
  $X \to Y \to Z \implies Z \perp X | Y \Leftrightarrow p(x, y, z)p(y) = p(x, y)p(y, z)$

Bayesian network

- Bayesian network: generalization of Markov chain
- distribution is product of conditional distributions
- characterization: $v \perp \mathrm{nd}(v) | \mathrm{pa}(v)$
- example: Markov chain,
  $X \to Y \to Z \implies Z \perp X | Y \Leftrightarrow p(x, y, z)p(y) = p(x, y)p(y, z)$
- formalism requires access to joint probability distribution

Bayesian network

- Bayesian network: generalization of Markov chain
- distribution is product of conditional distributions
- characterization: $v \perp \mathrm{nd}(v) | \mathrm{pa}(v)$
- example: Markov chain,
  $X \to Y \to Z \implies Z \perp X | Y \Leftrightarrow p(x, y, z)p(y) = p(x, y)p(y, z)$
- formalism requires access to joint probability distribution

$\implies$ obstacle for quantum generalization

Bayesian network

- Bayesian network: generalization of Markov chain
- distribution is product of conditional distributions
- characterization: $v \perp \mathrm{nd}(v) | \mathrm{pa}(v)$
- example: Markov chain,
  $X \to Y \to Z \implies Z \perp X | Y \Leftrightarrow p(x, y, z)p(y) = p(x, y)p(y, z)$
- formalism requires access to joint probability distribution

$\implies$ obstacle for quantum generalization

- hard algebraic equations

Bayesian network

- Bayesian network: generalization of Markov chain
- distribution is product of conditional distributions
- characterization: $v \perp \mathrm{nd}(v) | \mathrm{pa}(v)$
- example: Markov chain,
  $X \to Y \to Z \implies Z \perp X | Y \Leftrightarrow p(x, y, z) p(y) = p(x, y) p(y, z)$
- formalism requires access to joint probability distribution

$\implies$ obstacle for quantum generalization

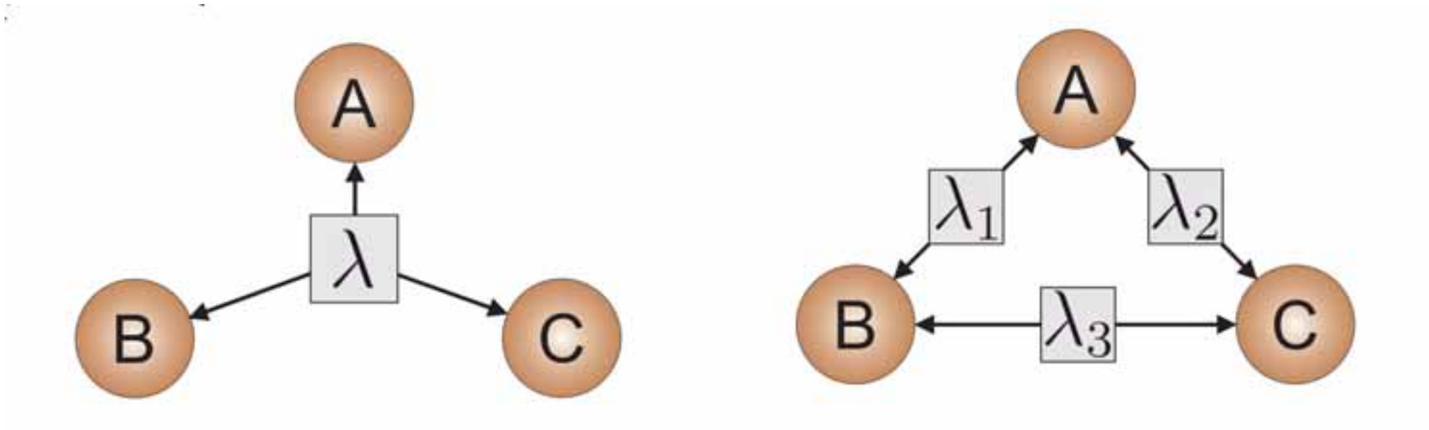
- hard algebraic equations
- easier: look at entropies

Marginal scenario

- Only some RVs are observed

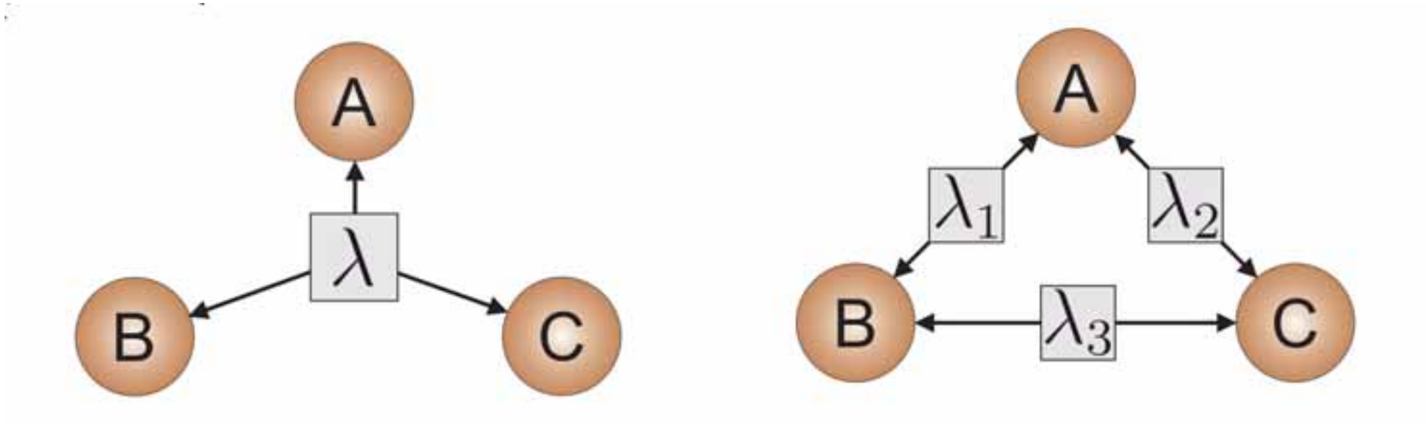
Marginal scenario

- Only some RVs are observed
- Example:



Marginal scenario

- Only some RVs are observed
- Example:



$\rightarrow$ Marginal scenario: $\{A, B, C\}$

Quantum

Interlude: Entropies

Interlude: Entropies

- $\rho \in \mathbb{S}(\mathcal{H})$ mixed quantum state on $\mathcal{H} = \mathbb{C}^d$

Interlude: Entropies

- $\rho \in \mathbb{S}(\mathcal{H})$ mixed quantum state on $\mathcal{H} = \mathbb{C}^d$
- Von Neumann entropy

$$S(\rho) = -\mathrm{tr}\rho \log \rho$$

Interlude: Entropies

- $\rho \in \mathbb{S}(\mathcal{H})$ mixed quantum state on $\mathcal{H} = \mathbb{C}^d$
- Von Neumann entropy

$$S(\rho) = -\mathrm{tr}\rho \log \rho$$

- Multipartite state $\rho \in \mathbb{S}(\mathcal{H})$ with $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$

Interlude: Entropies

- $\rho \in \mathbb{S}(\mathcal{H})$ mixed quantum state on $\mathcal{H} = \mathbb{C}^d$
- Von Neumann entropy

$$S(\rho) = -\mathrm{tr}\rho \log \rho$$

- Multipartite state $\rho \in \mathbb{S}(\mathcal{H})$ with $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$
- $\rho_I = \mathrm{tr}_{I^c} \rho$

Interlude: Entropies

- $\rho \in \mathbb{S}(\mathcal{H})$ mixed quantum state on $\mathcal{H} = \mathbb{C}^d$
- Von Neumann entropy

$$S(\rho) = -\mathrm{tr}\rho \log \rho$$

- Multipartite state $\rho \in \mathbb{S}(\mathcal{H})$ with $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$
- $\rho_I = \mathrm{tr}_{I^c} \rho$
- Von Neumann entropy vector: $s(\rho) = (S(\rho_I))_{I \subset \{1,\ldots,n\}} \in \mathbb{R}^{2^n}$

Convex cones

Convex cones

What is a convex cone?

Convex cones

What is a convex cone?

Convex cones

What is a convex cone?

Entropy Cone

Set of all quantum entropy vectors:

Entropy Cone

Set of all quantum entropy vectors:

$$\Gamma_n = \{s(\rho) | \rho \text{ n-party quantum state}\}$$

Entropy Cone

Set of all quantum entropy vectors:

$$\Gamma_n = \{s(\rho) | \rho \text{ n-party quantum state}\}$$

$\overline{\Gamma}_n$ is a convex cone (Pippenger, 2003)

Entropy Cone

Set of all quantum entropy vectors:

$$\Gamma_n = \{s(\rho) | \rho \text{ n-party quantum state}\}$$

$\overline{\Gamma}_n$ is a convex cone (Pippenger, 2003)

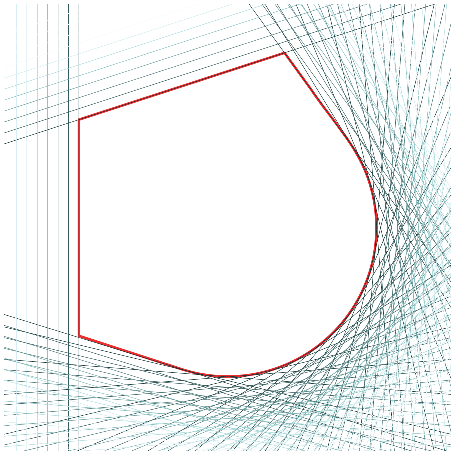
Classical entropy cone: $\overline{\Sigma}_n$

Information Inequalities

Describe convex cone via linear inequalities

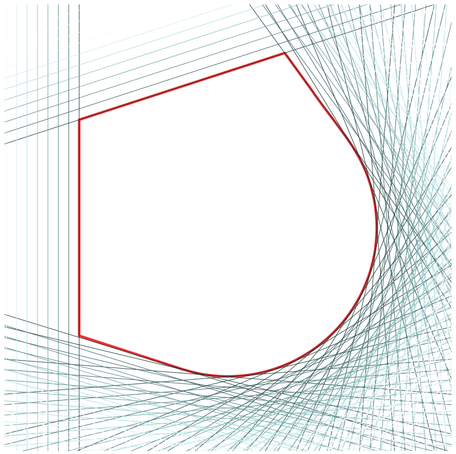
Information Inequalities

Describe convex cone via linear inequalities



Information Inequalities

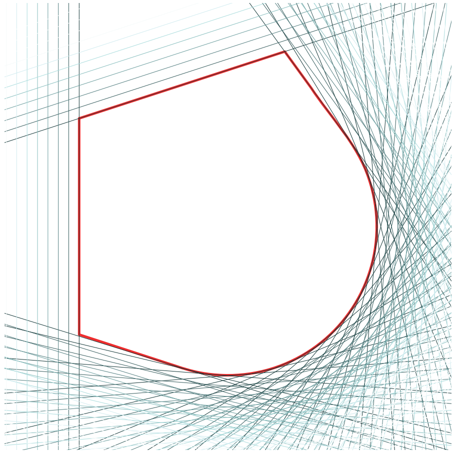
Describe convex cone via linear inequalities



Inequalities for Σ_n: *information inequalities*

Information Inequalities

Describe convex cone via linear inequalities

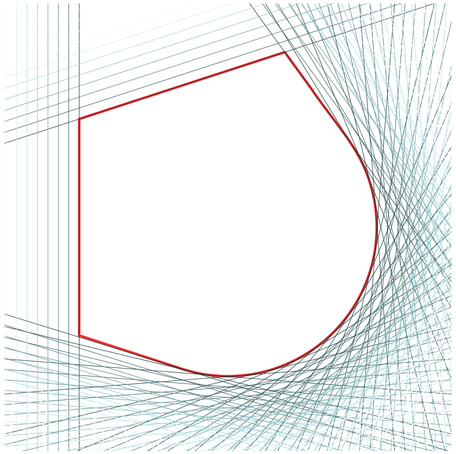


Inequalities for Σ_n: *information inequalities*

Example: monotonicity $H(X_1 X_2) - H(X_2) \geq 0$

Information Inequalities

Describe convex cone via linear inequalities



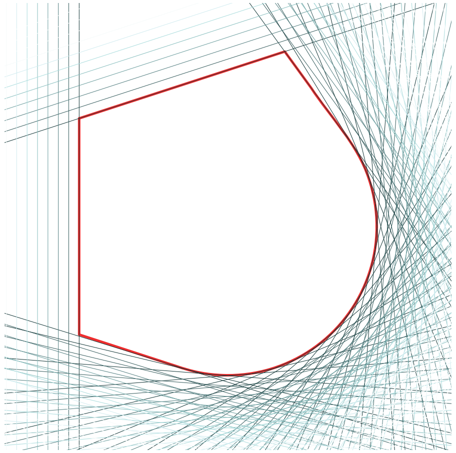
Inequalities for Σ_n: *information inequalities*

Example: monotonicity $H(X_1 X_2) - H(X_2) \geq 0$

Inequalities for Γ_n: *quantum information inequalities*

Information Inequalities

Describe convex cone via linear inequalities



Inequalities for Σ_n: *information inequalities*

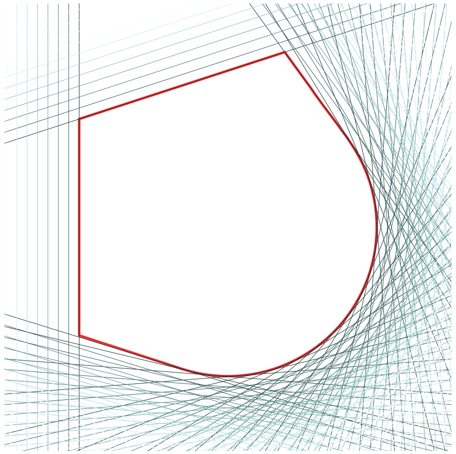
Example: monotonicity $H(X_1X_2) - H(X_2) \geq 0$

Inequalities for Γ_n: *quantum information inequalities*

Example: weak monotonicity $S(\rho_{12}) + S(\rho_{13}) - S(\rho_2) - S(\rho_3) \geq 0$

Information Inequalities

Describe convex cone via linear inequalities



Inequalities for Σ_n: *information inequalities*

Example: monotonicity $H(X_1X_2) - H(X_2) \geq 0$

Inequalities for Γ_n: *quantum information inequalities*

Example: weak monotonicity $S(\rho_{12}) + S(\rho_{13}) - S(\rho_2) - S(\rho_3) \geq 0$

known (quantum) information inequalities provide outer approximation of entropy cones

Quantum Causal Structures

- DAG $G = (V, E)$

Quantum Causal Structures

- DAG $G = (V, E)$
- Sink nodes $s \in V$: Hilbert space $\mathcal{H}_s$

Quantum Causal Structures

- DAG $G = (V, E)$
- Sink nodes $s \in V$: Hilbert space $\mathcal{H}_s$
- Nodes $v \in V$ with children: Hilbert space

$$\mathcal{H}_v = \bigotimes_{\substack{w \in V \\ (v,w) \in E}} \mathcal{H}_{v,w}$$

Quantum Causal Structures

- DAG $G = (V, E)$
- Sink nodes $s \in V$: Hilbert space $\mathcal{H}_s$
- Nodes $v \in V$ with children: Hilbert space

$$\mathcal{H}_v = \bigotimes_{\substack{w \in V \\ (v,w) \in E}} \mathcal{H}_{v,w}$$

-

$$\mathcal{H} = \bigotimes_{v \in V} \mathcal{H}_v$$

Quantum Causal Structures

- DAG $G = (V, E)$
- Sink nodes $s \in V$: Hilbert space $\mathcal{H}_s$
- Nodes $v \in V$ with children: Hilbert space

$$\mathcal{H}_v = \bigotimes_{\substack{w \in V \\ (v,w) \in E}} \mathcal{H}_{v,w}$$

-

$$\mathcal{H} = \bigotimes_{v \in V} \mathcal{H}_v$$

- Parent Hilbert space

$$H_{\mathrm{pa}(v)} = \bigotimes_{\substack{w \in V \\ (w,v) \in E}} \mathcal{H}_{w,v}$$

Quantum Causal Structures

- Initial product state $\rho_0 = \bigotimes_q \rho_q$ on Hilbert spaces of source nodes q

Quantum Causal Structures

- Initial product state $\rho_0 = \bigotimes_q \rho_q$ on Hilbert spaces of source nodes q
- CPTP maps Φ_v for each non-source node:

$$\Phi_v : L\left(\mathcal{H}_{\mathrm{pa}(v)}\right) \to L(\mathcal{H}_v)$$

Quantum Causal Structures

- Initial product state $\rho_0 = \bigotimes_q \rho_q$ on Hilbert spaces of source nodes q
- CPTP maps Φ_v for each non-source node:

$$\Phi_v : L\left(\mathcal{H}_{\mathrm{pa}(v)}\right) \to L(\mathcal{H}_v)$$

-

$$\rho_v = \Phi_v \rho_{\mathrm{pa}(v)}$$

Quantum Causal Structures

- Initial product state $\rho_0 = \bigotimes_q \rho_q$ on Hilbert spaces of source nodes q
- CPTP maps Φ_v for each non-source node:

$$\Phi_v : L\left(\mathcal{H}_{\mathrm{pa}(v)}\right) \to L(\mathcal{H}_v)$$

- $$\rho_v = \Phi_v \rho_{\mathrm{pa}(v)}$$

[!] no global state

Quantum Causal Structures

- Initial product state $\rho_0 = \bigotimes_q \rho_q$ on Hilbert spaces of source nodes q
- CPTP maps Φ_v for each non-source node:

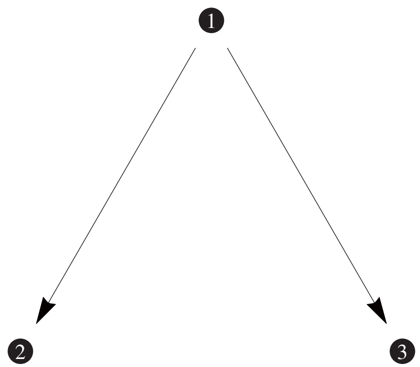
$$\Phi_v : L\left(\mathcal{H}_{\mathrm{pa}(v)}\right) \to L(\mathcal{H}_v)$$

- $$\rho_v = \Phi_v \rho_{\mathrm{pa}(v)}$$

[!] no global state

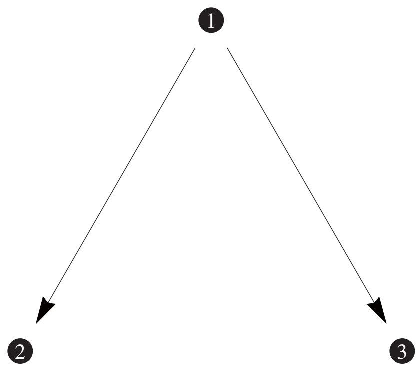
- want classical nodes: pick the right Φ_v

Example



- $\mathcal{H} = \mathcal{H}_{1,2} \otimes \mathcal{H}_{1,3} \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$

Example



- $\mathcal{H} = \mathcal{H}_{1,2} \otimes \mathcal{H}_{1,3} \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$
- States on the coexisting subsets of systems:

$$\begin{aligned}
\rho_{(1,2),(1,3)} &= \rho_0 \\
\rho_{(1,3),2} &= \left(\Phi_2 \otimes \mathbb{1}\right) \rho_0 \\
\rho_{(1,2),3} &= \left(\mathbb{1} \otimes \Phi_3\right) \rho_0 \\
\rho_{2,3} &= \left(\Phi_2 \otimes \Phi_3\right) \rho_0
\end{aligned}$$

The entropy cone

- Look at entropy cone of states constructed like this

The entropy cone

- Look at entropy cone of states constructed like this
- $\mathcal{H}$ is a tensor product of $n = |E| + |V_s|$ hilbert spaces, V_s: Set of sinks

The entropy cone

- Look at entropy cone of states constructed like this
- $\mathcal{H}$ is a tensor product of $n = |E| + |V_s|$ hilbert spaces, V_s: Set of sinks
- Formally: Entropy vector $v \in \mathbb{R}^{2^n}$

The entropy cone

- Look at entropy cone of states constructed like this
- $\mathcal{H}$ is a tensor product of $n = |E| + |V_s|$ hilbert spaces, V_s: Set of sinks
- Formally: Entropy vector $v \in \mathbb{R}^{2^n}$
- $v_I = S(\rho_I)$ if the state ρ_I exists

The entropy cone

- Look at entropy cone of states constructed like this
- $\mathcal{H}$ is a tensor product of $n = |E| + |V_s|$ hilbert spaces, V_s: Set of sinks
- Formally: Entropy vector $v \in \mathbb{R}^{2^n}$
- $v_I = S(\rho_I)$ if the state ρ_I exists
- v_I arbitrary if ρ_I doesn't exist

The entropy cone

- Look at entropy cone of states constructed like this
- $\mathcal{H}$ is a tensor product of $n = |E| + |V_s|$ hilbert spaces, V_s: Set of sinks
- Formally: Entropy vector $v \in \mathbb{R}^{2^n}$
- $v_I = S(\rho_I)$ if the state ρ_I exists
- v_I arbitrary if ρ_I doesn't exist
- for each $J \subset E \cup V_s$ of coexisting systems: quantum entropy cone

The entropy cone

- Look at entropy cone of states constructed like this
- $\mathcal{H}$ is a tensor product of $n = |E| + |V_s|$ hilbert spaces, V_s: Set of sinks
- Formally: Entropy vector $v \in \mathbb{R}^{2^n}$
- $v_I = S(\rho_I)$ if the state ρ_I exists
- v_I arbitrary if ρ_I doesn't exist
- for each $J \subset E \cup V_s$ of coexisting systems: quantum entropy cone
- extra monotonicities for classical systems

Data processing inequality

- replace conditional independence relations?

Data processing inequality

- replace conditional independence relations?

$\rightarrow$ Data processing inequality: For a CPTP map $\Phi : \mathcal{H}_A \rightarrow \mathcal{H}_B$

$$I(A : C|D) \geq I(B : C|D)$$

for any other systems C and D.

Data processing inequality

- replace conditional independence relations?

$\rightarrow$ Data processing inequality: For a CPTP map $\Phi : \mathcal{H}_A \rightarrow \mathcal{H}_B$

$$I(A : C|D) \geq I(B : C|D)$$

for any other systems C and D.

- relates entropies of noncoexisting systems

Marginal Scenario

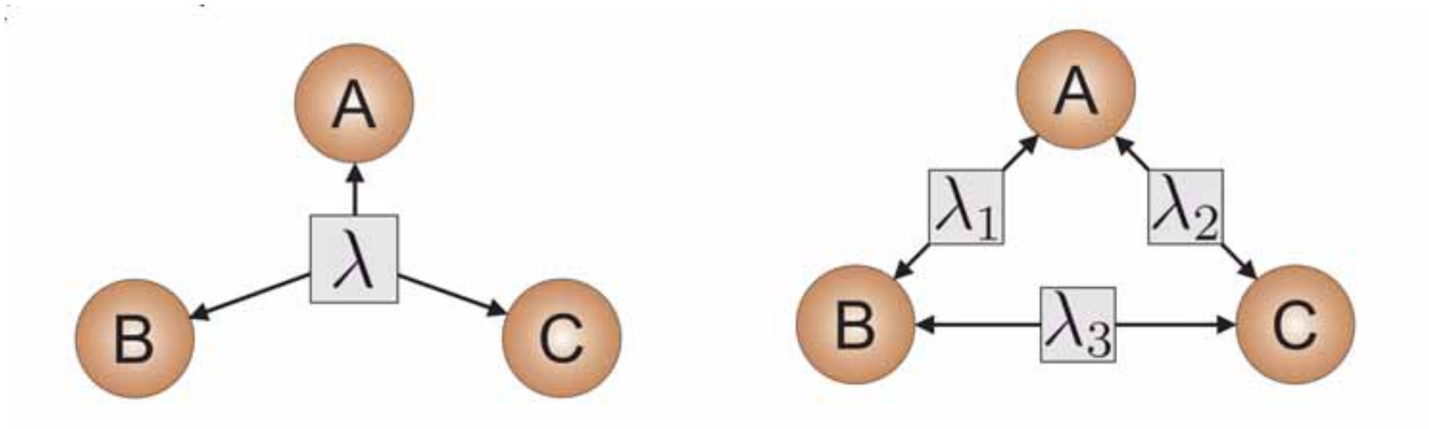
- Again, only some systems are observed

Marginal Scenario

- Again, only some systems are observed
- Example:

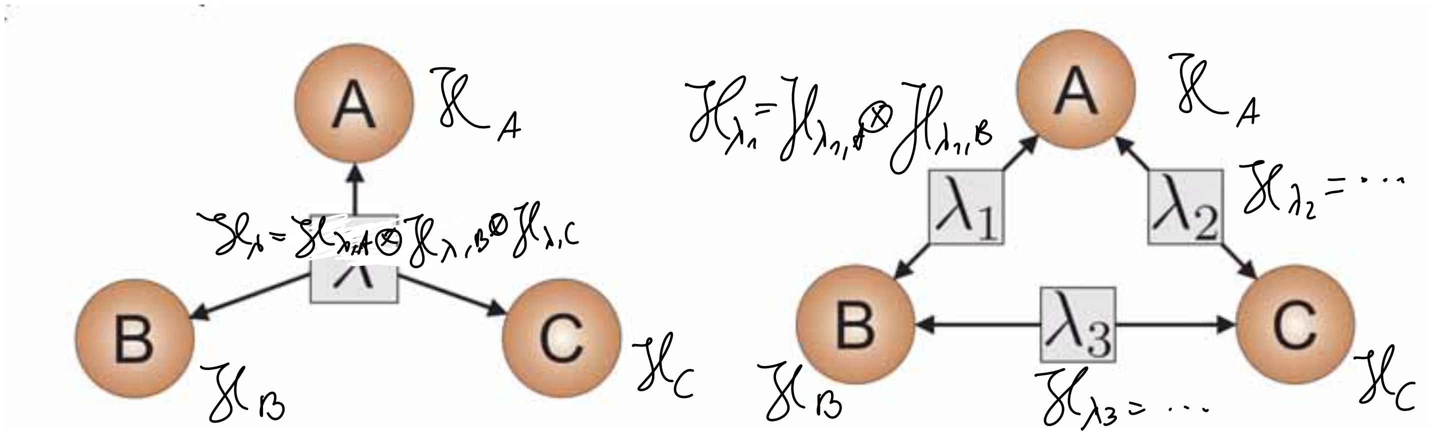
Marginal Scenario

- Again, only some systems are observed
- Example:



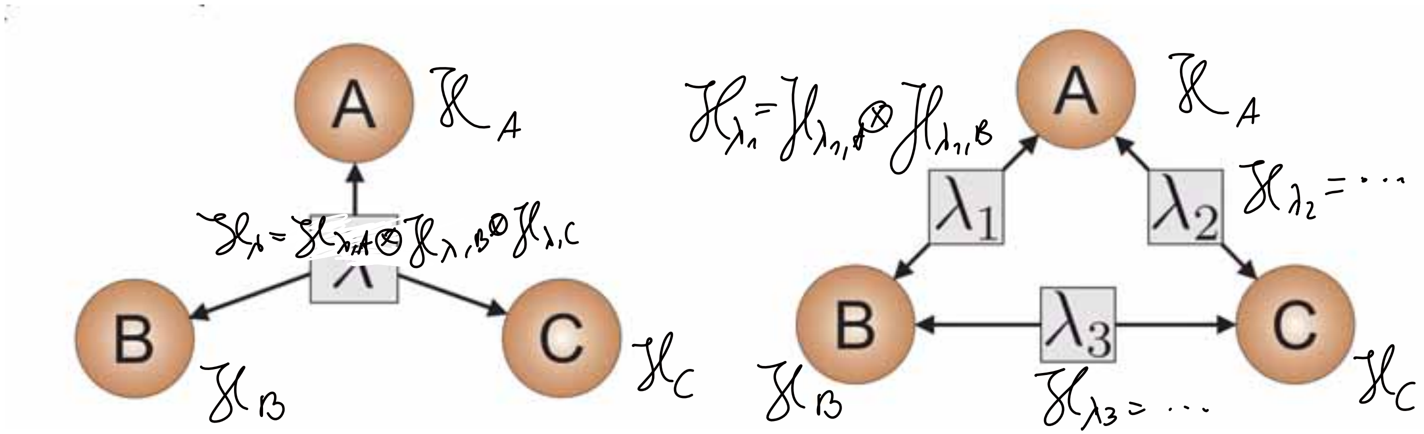
Marginal Scenario

- Again, only some systems are observed
- Example:



Marginal Scenario

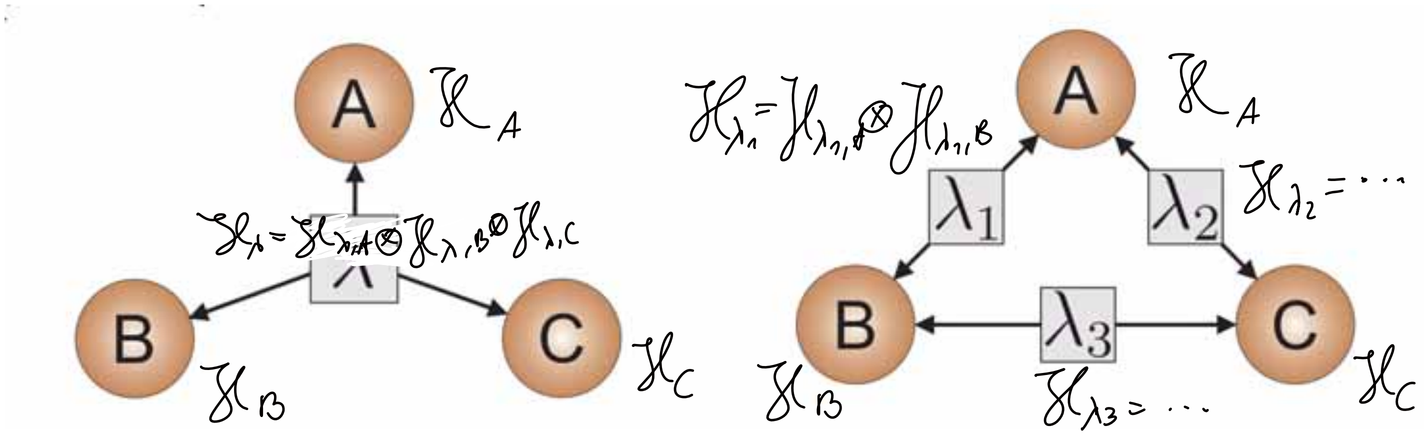
- Again, only some systems are observed
- Example:



- most interesting marginal scenario: $\{A, B, C\}$ & assume these nodes classical

Marginal Scenario

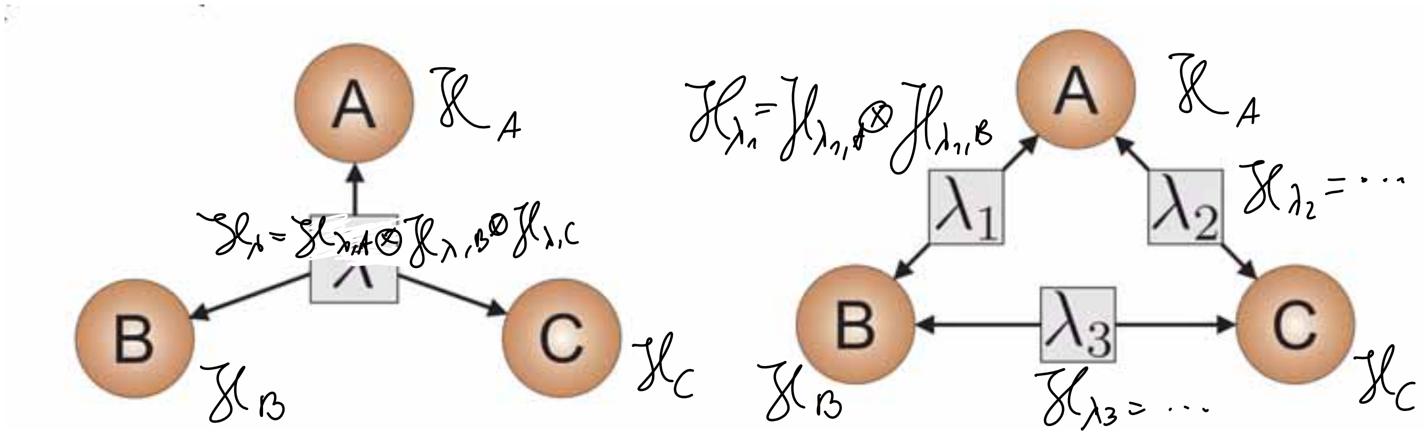
- Again, only some systems are observed
- Example:



- most interesting marginal scenario: $\{A, B, C\}$ & assume these nodes classical
- entropy cone: project onto observable entropies

Marginal Scenario

- Again, only some systems are observed
- Example:



- most interesting marginal scenario: $\{A, B, C\}$ & assume these nodes classical
- entropy cone: project onto observable entropies

$\Longleftrightarrow$ take all implied inequalities for obs. entropies

Application

Information Causality

The IC principle is defined by a game:

Information Causality

The IC principle is defined by a game:

Information Causality

The IC principle is defined by a game:

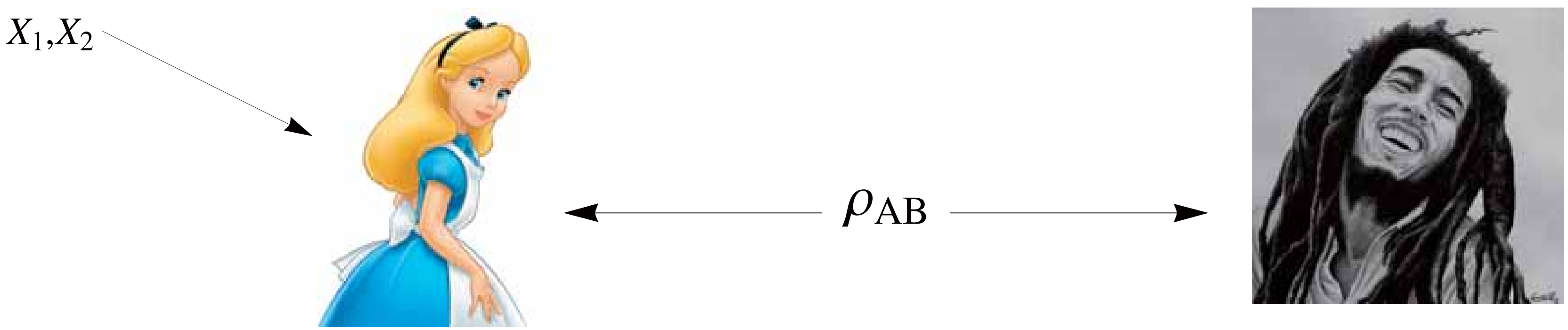
Information Causality

The IC principle is defined by a game:

$$\longleftarrow \rho_{\mathrm{AB}} \longrightarrow$$

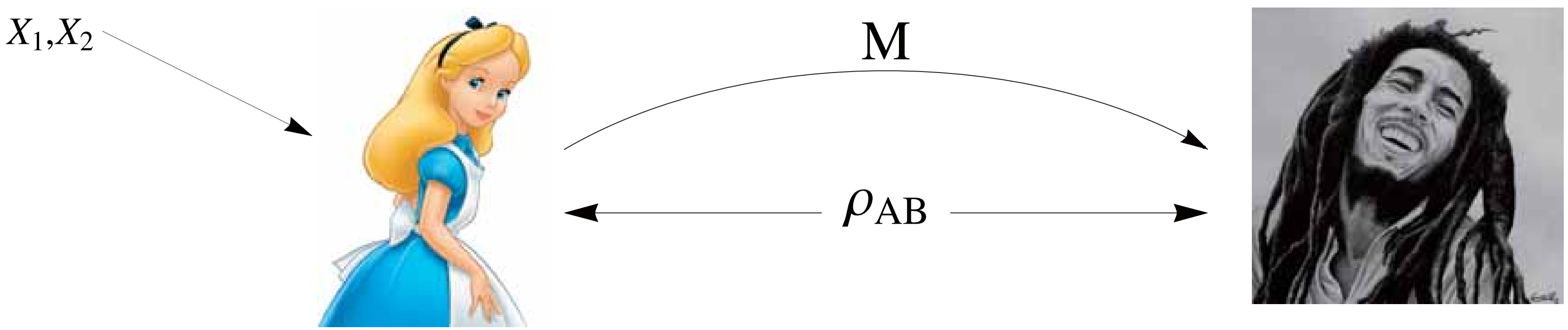
Information Causality

The IC principle is defined by a game:



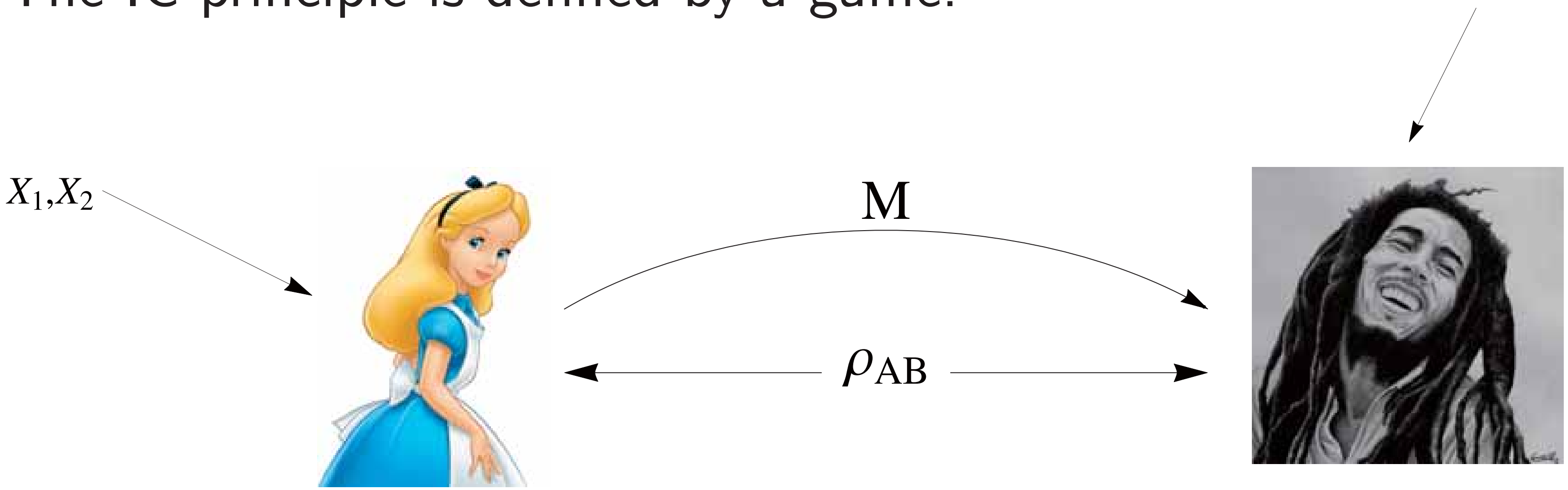
Information Causality

The IC principle is defined by a game:



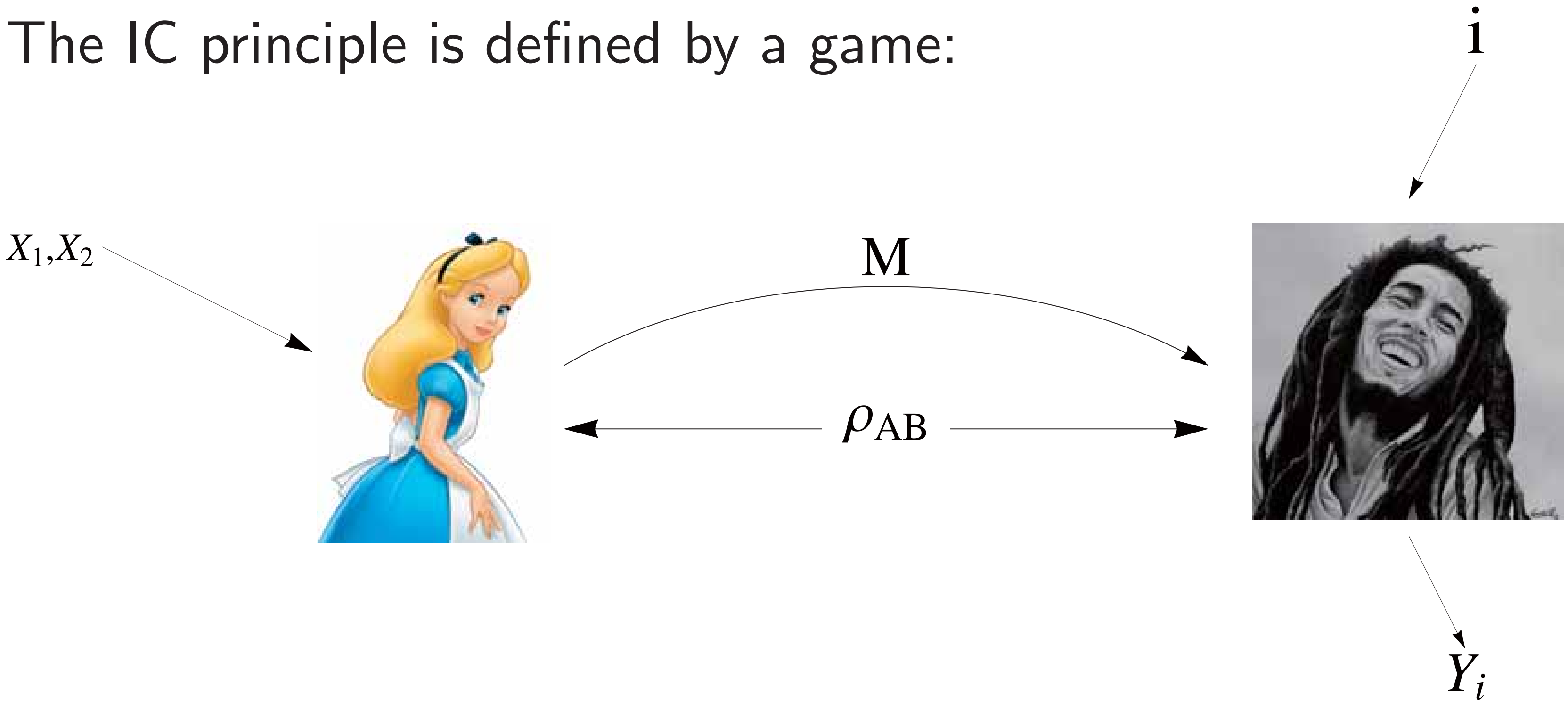
Information Causality

The IC principle is defined by a game:



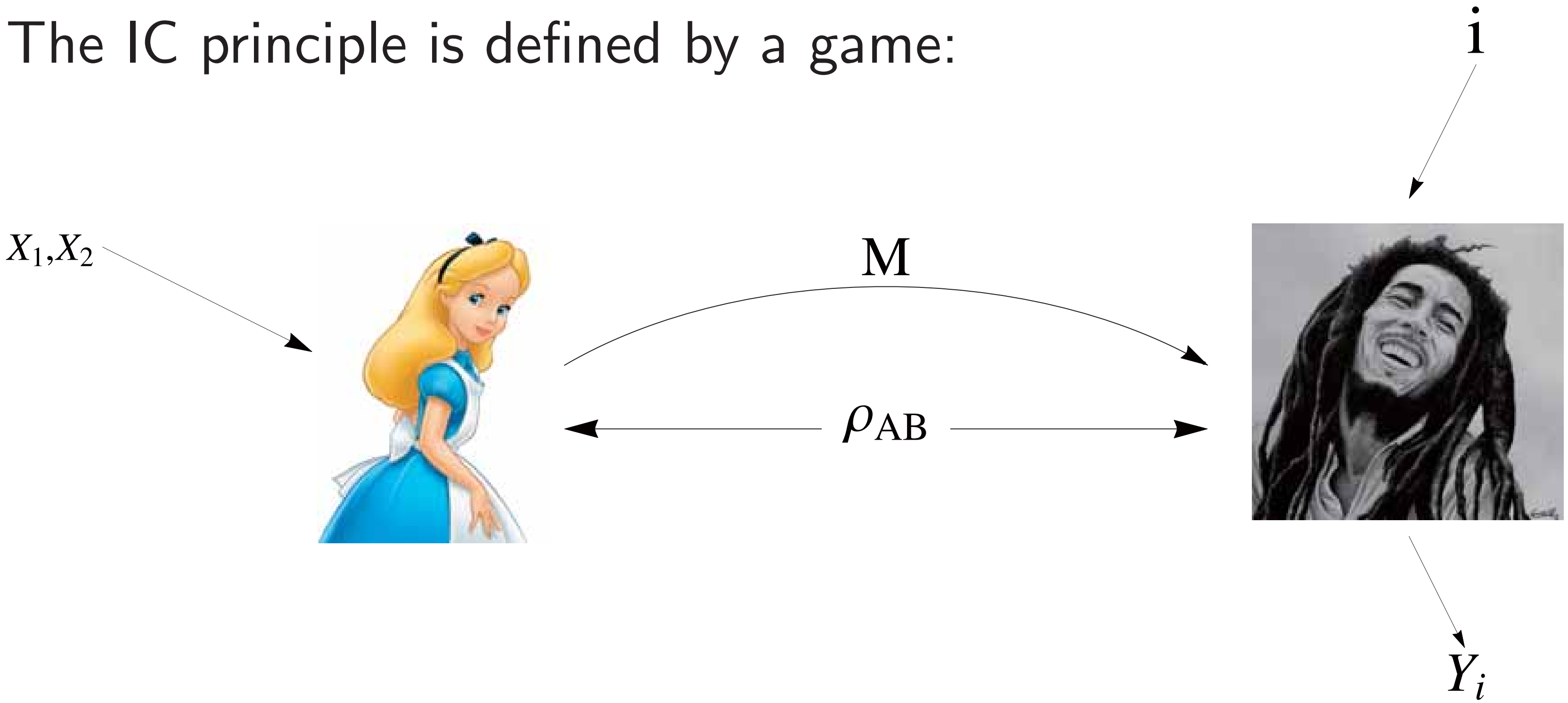
Information Causality

The IC principle is defined by a game:



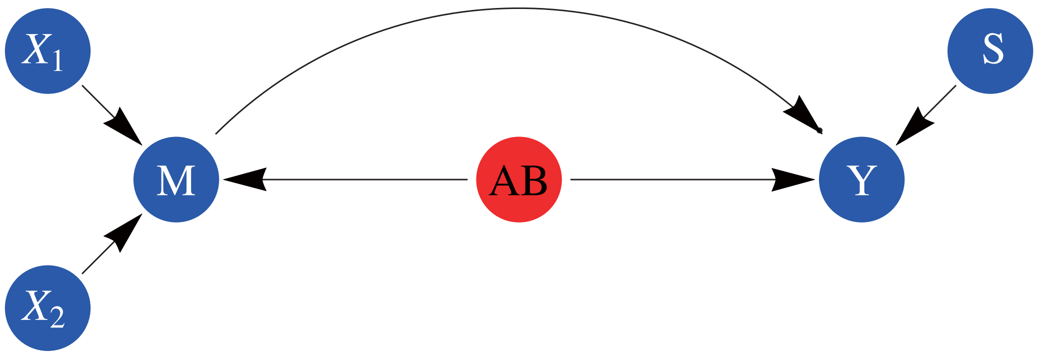
Information Causality

The IC principle is defined by a game:



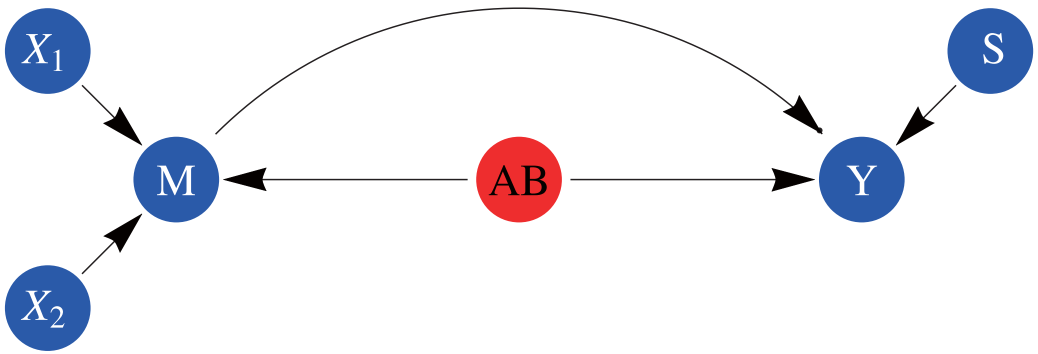
Information Causality

- Corresponding DAG:



Information Causality

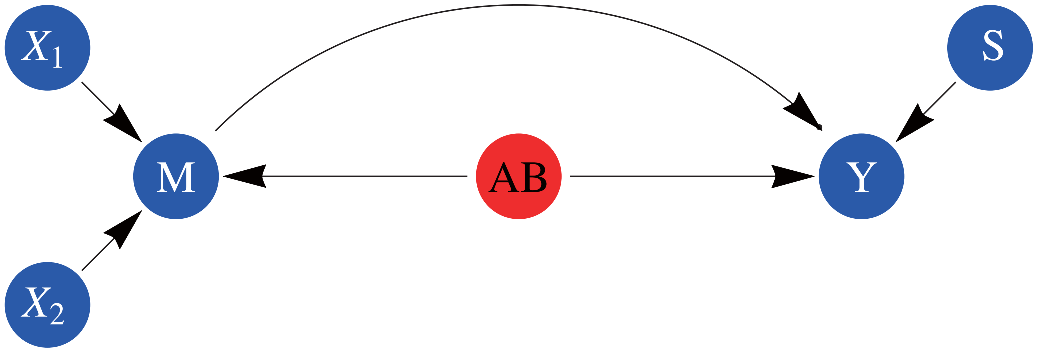
- Corresponding DAG:



- Nodes X_i, Y, M, S are classical, only the AB node is quantum

Information Causality

- Corresponding DAG:



- Nodes X_i, Y, M, S are classical, only the AB node is quantum
- Counterfactual reformulation: $Y_i = (Y|S = i)$

Information Causality

- Marginal scenario $\{X_1, Y_1\}$, $\{X_2, Y_2\}$, $\{M\}$ yields original information causality inequality

$$I(X_1 : Y_1) + I(X_2 : Y_2) \leq H(M)$$

Information Causality

- Marginal scenario $\{X_1, Y_1\}$, $\{X_2, Y_2\}$, $\{M\}$ yields original information causality inequality

$$I(X_1 : Y_1) + I(X_2 : Y_2) \leq H(M)$$

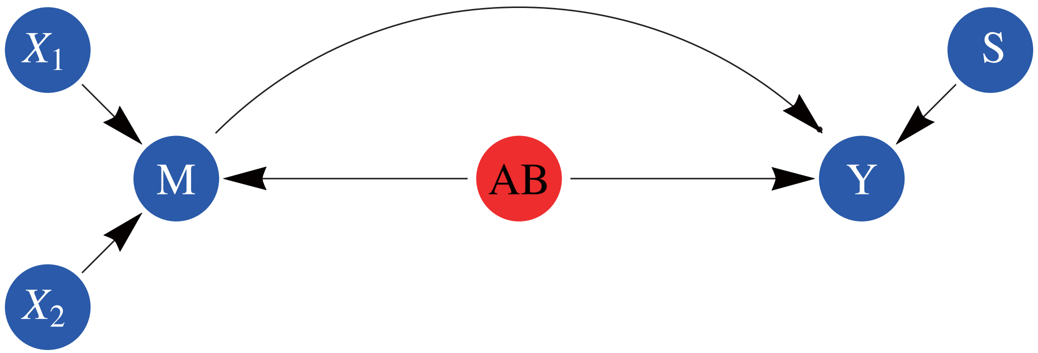
$\rightarrow$ More general marginal scenario $\{X_1, X_2, Y_1, M\}$, $\{X_1, X_2, Y_2, M\}$:

Information Causality

- Marginal scenario $\{X_1, Y_1\}$, $\{X_2, Y_2\}$, $\{M\}$ yields original information causality inequality

$$I(X_1 : Y_1) + I(X_2 : Y_2) \leq H(M)$$

$\rightarrow$ More general marginal scenario $\{X_1, X_2, Y_1, M\}$, $\{X_1, X_2, Y_2, M\}$:



Information Causality

- Marginal scenario $\{X_1, Y_1\}$, $\{X_2, Y_2\}$, $\{M\}$ yields original information causality inequality

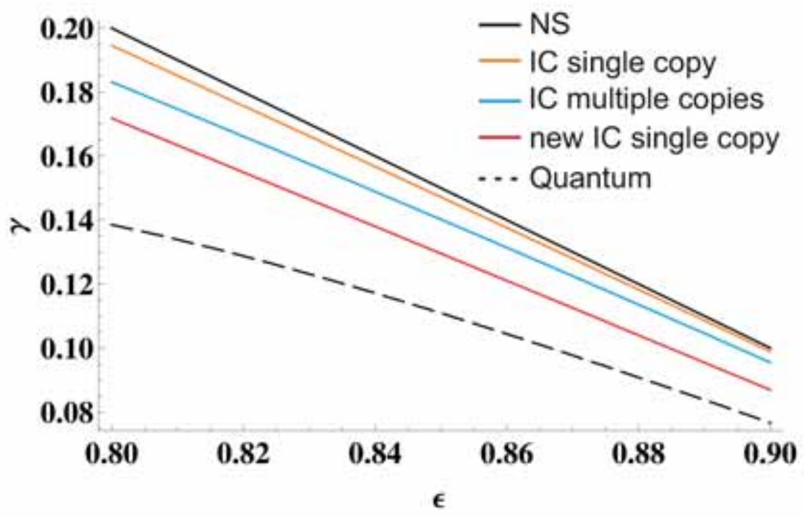
$$I(X_1 : Y_1) + I(X_2 : Y_2) \leq H(M)$$

$\rightarrow$ More general marginal scenario $\{X_1, X_2, Y_1, M\}$, $\{X_1, X_2, Y_2, M\}$:

Strengthened information causality inequality

$$I(X_1 : Y_1, M) + I(X_2 : Y_2, M) + I(X_1 : X_2 | M) \leq H(M) + I(X_1 : X_2)$$

Information Causality



Computational Techniques

Computational Techniques

- Fourier Motzkin elimination: remove variables from inequalities

Computational Techniques

- Fourier Motzkin elimination: remove variables from inequalities
- Problem: double exponential in *number of variables*...

Computational Techniques

- Fourier Motzkin elimination: remove variables from inequalities
- Problem: double exponential in *number of variables*...

... i.e. triple exponential in number of nodes

Computational Techniques

- Fourier Motzkin elimination: remove variables from inequalities
- Problem: double exponential in *number of variables*...

... i.e. triple exponential in number of nodes

- works for small instances (triangle, IC)

Computational Techniques

- Fourier Motzkin elimination: remove variables from inequalities
- Problem: double exponential in *number of variables*...

... i.e. triple exponential in number of nodes

- works for small instances (triangle, IC)
- Easier: check candidate inequality

Computational Techniques

- Fourier Motzkin elimination: remove variables from inequalities
- Problem: double exponential in *number of variables*...

... i.e. triple exponential in number of nodes

- works for small instances (triangle, IC)
- Easier: check candidate inequality
- formulation as LP:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{I \subset \{1,\ldots,n\}} \alpha_I v_I \\ \text{subject to} & \mathrm{SSA}, \mathrm{WM}, \mathrm{DPs} \end{array}$$

Summary

- Defined quantum causal structures

Summary

- Defined quantum causal structures
- Algorithm for characterization

Summary

- Defined quantum causal structures
- Algorithm for characterization
- Applications: strengthening of information causality, quantum networks

PhD positions available: Application deadline Jan. 3rd 2016
check out our website (QMath Copenhagen)

Thank you for your attention!