Non-Markovian and nonlinear quantum input-output analysis

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Outline

- Introduction
- Gardiner-Collett input-output theory
- Non-Markovian input-output analysis
- Nonlinear IO analysis by Volterra series
- Conclusion

Input-output analysis in solid-state network

Optical input-output network

- Linear input-output response (extremely weak nonlinearity);
- Markovian dynamics;

Solid-state input-output network

- Nonlinear input-output response;
- Non-Markovian dynamics

Input-output formalism should be extended to the non-Markovian & nonlinear cases

Different input-output approaches

Classical version	Quantum version	Advantages/disadvantages (My opinions)
State space description	**Gardiner-Collett IO-theory,** Gardiner & Collett, PRA 31, 3761-3774, 1985 **Hudson-Parthasarathy formalism** Hudson & Parthasarathy, Com. Math. Phys. 93, 301 (1984); Gough & James, IEEE TAC 54, 2530-2544, 2009	1. Valid for nonlinear systems; 2. Invalid for non-conservative components (quantum amplifier); 3. **Invalid for non-Markovian setup;** 4. High computational complexity (interior dynamics considered)
Transfer function, Linear response theory	**Q-transfer function** Yanagisawa & Kimura, IEEE TAC 48, 2107, 2003 **Q-response theory** Clerk el al. RMP 82, 1155, 2010	1. Valid for non-conservative, non-Markovian systems; 2. Low computational complexity (interior dynamics omitted) 3. Invalid for nonlinear systems;
Series approach (Volterra & Fliess	**Quantum version is not shown**	1. Valid for weak-nonlinear, non-conservative systems;

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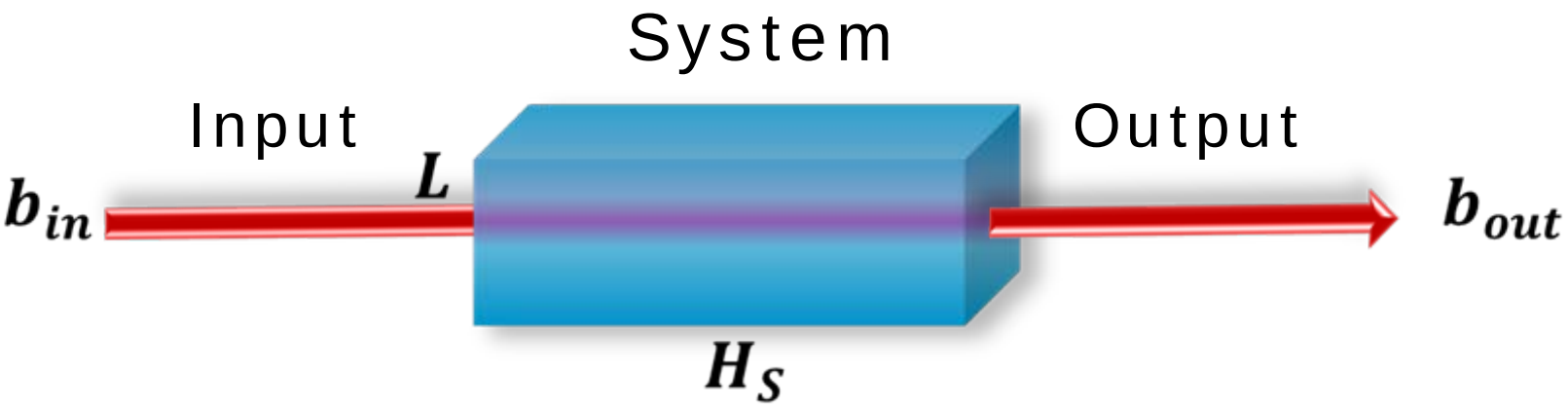
Gardiner-Collett input-output theory

Original model

$$H = H_s + H_b + H_{int}$$

$$H_b = \int_{-\infty}^{\infty} d\omega\, \omega b^{\dagger}(\omega) b(\omega)$$

$$H_{int} = i \int_{-\infty}^{\infty} d\omega \left[\kappa(\omega) b^{\dagger}(\omega) L - \kappa^{*}(\omega) b(\omega) L^{\dagger}\right]$$



Gardiner-Collett input-output theory

Markovian assumption

$$\kappa(\omega) = \sqrt{\frac{\gamma}{2\pi}}$$

Heisenberg equation

$$\dot{X} = -i[X, H_s] + \frac{\gamma}{2}\left\{L^\dagger[X, L] + \left[L^\dagger, X\right]L\right\} + \sqrt{\gamma}\left\{b_{in}(t)\left[L^\dagger, X\right] + [X, L]b_{in}{}^\dagger(t)\right\}$$

Input-output relation

$$b_{out}(t) = \sqrt{\gamma}L(t) + b_{in}(t)$$

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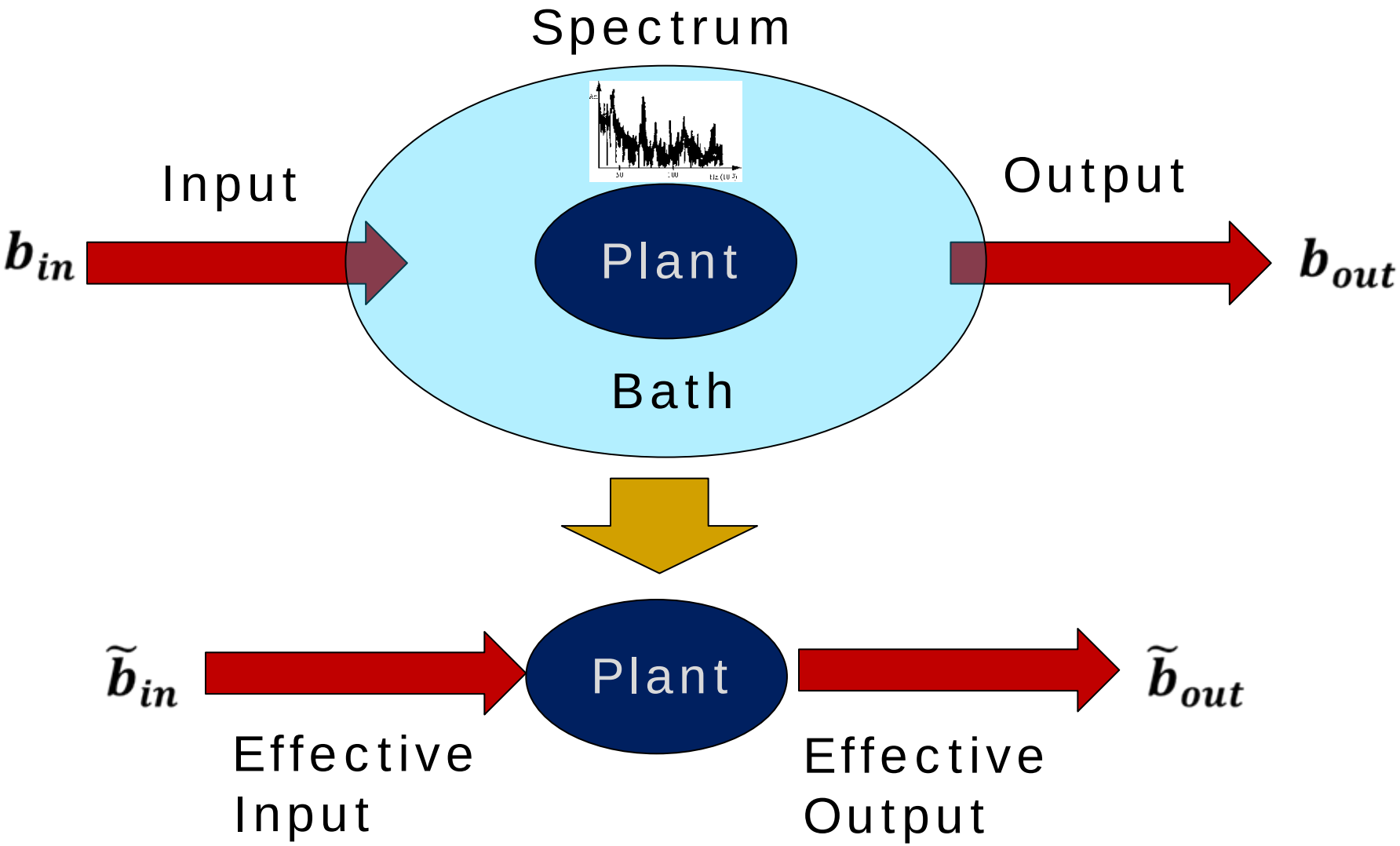
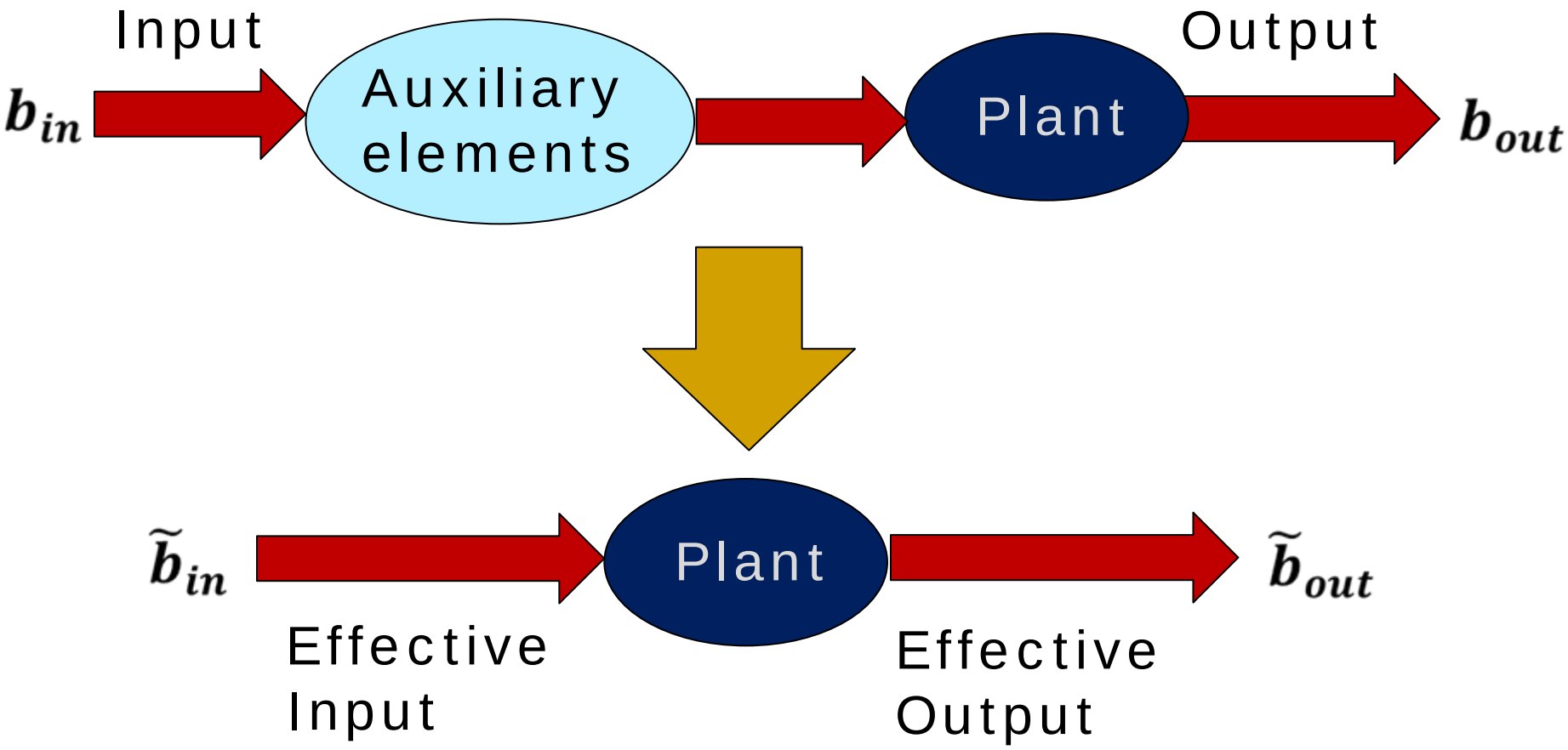
Non-Markovian input-output analysis



Non-Markovian input-output analysis



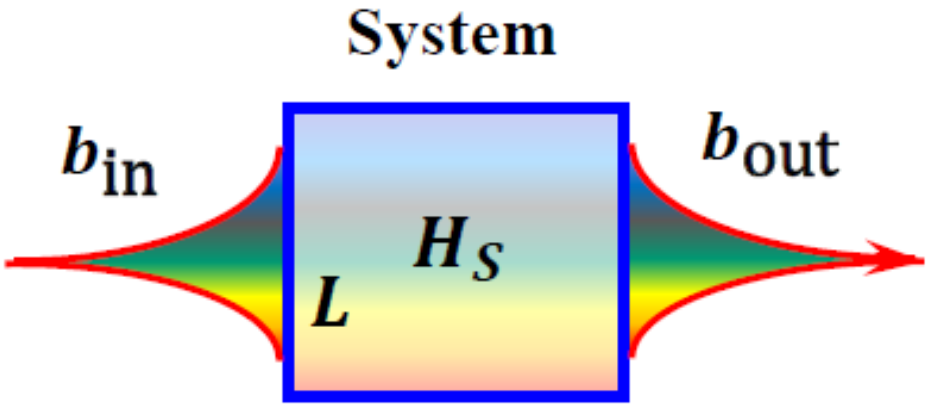
Non-Markovian input-output analysis

Original model

$$H = H_s + H_b + H_{int}$$

$$H_b = \int_{-\infty}^{\infty} d\omega\, \omega b^{\dagger}(\omega) b(\omega)$$

$$H_{int} = i \int^{\infty} d\omega \left[\kappa(\omega) b^{\dagger}(\omega) L - \kappa^{*}(\omega) b(\omega) L^{\dagger}\right]$$



J. Zhang, Y.-X. Liu, R.-B. Wu, K. Jacobs, and F. Nori, Phys. Rev A **87**, 032117 (2013)

Non-Markovian input-output analysis

Quantum Langevin equation with color noises

$$\dot{X} = -i[X, H_s] + \int_0^t \gamma^*(t-\tau) L^\dagger(\tau)\,[X(t), L(t)]\,d\tau + \int_0^t \gamma(t-\tau)\left[L^\dagger(t), X(t)\right] L(\tau)\,d\tau + \left\{\tilde{b}_{in}(t)\left[L^\dagger, X\right] + [X, L]\tilde{b}_{in}{}^\dagger(t)\right\}$$

Non-Markovian effective input field

$$\left[\tilde{b}_{in}(t), \tilde{b}_{in}{}^\dagger(\tau)\right] = \gamma(t-\tau)$$

J. Zhang, Y.-X. Liu, R.-B. Wu, K. Jacobs, and F. Nori, Phys. Rev A **87**, 032117 (2013)

Non-Markovian input-output analysis

Non-Markovian input-output relation

$$b_{out}(t) = b_{in}(t) + \int_0^t \kappa(t-\tau) L(\tau) d\tau$$

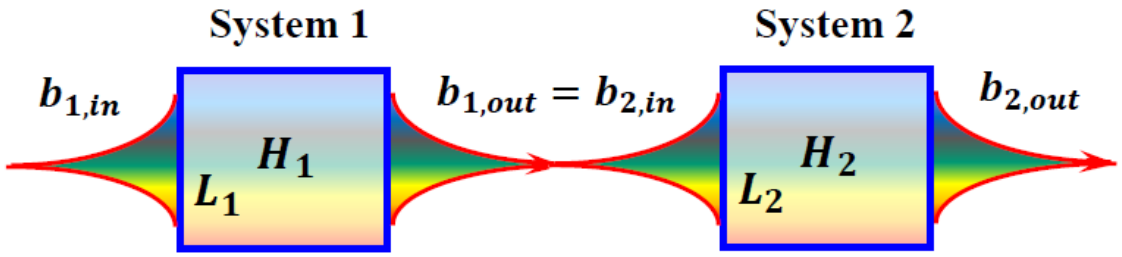
L. Diosi, Phys. Rev. A 85, 034101 (2012)

$$\kappa(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \kappa(\omega)$$

Kernel function

J. Zhang, Y.-X. Liu, R.-B. Wu, K. Jacobs, and F. Nori, Phys. Rev A **87**, 032117 (2013)

Non-Markovian cascade system



Dynamics of Non-Markovian cascade system

$$\dot{\rho} = \sum_{l,r=1}^{2} \int_0^t \{\gamma_{lr}(t-\tau)[L_r\rho(\tau), L_l{}^{\dagger}(\tau-t)] + \text{h.c.}\}d\tau$$
$$-i[H_1 + H_2 + H_{12}, \rho]$$

$$H_{12} = i\int_0^t [\gamma_{lr}(\tau-t)L_r L_l{}^{\dagger}(\tau) - \text{h.c.}]d\tau$$

J. Zhang, Y.-X. Liu, R.-B. Wu, K. Jacobs, and F. Nori, Phys. Rev A **87**, 032117 (2013)

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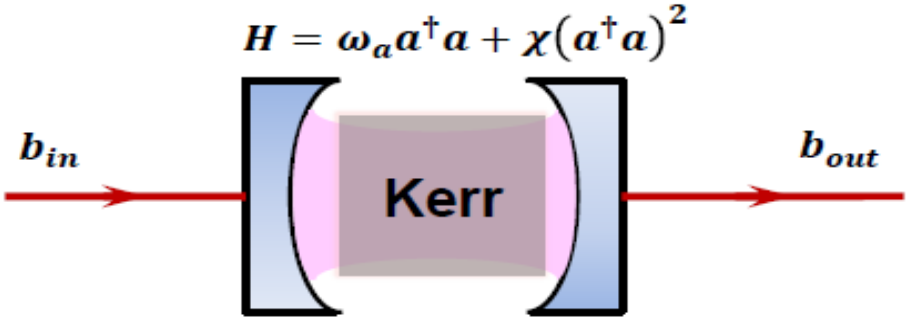
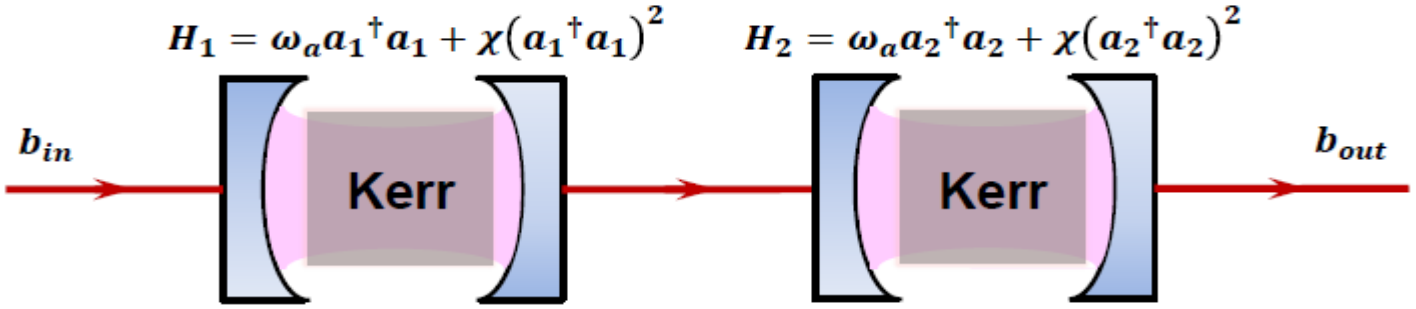
Nonlinear IO analysis



Interior degrees of freedom up to third order terms

$$\vec{X} = (I \quad a \quad a^\dagger \quad a^\dagger a \quad a^{\dagger 2} \quad a^2 \quad a^\dagger a^2 \quad a a^{\dagger 2})^T$$

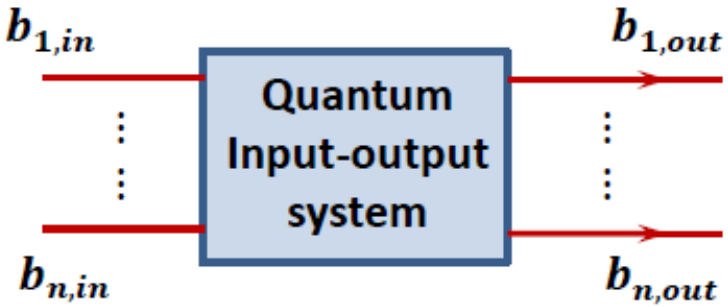
8 dimension !



Interior degrees of freedom up to third order terms **8×8 = 64 dimension !**

Is it possible to reduce the interior degrees of freedom?

Quantum Volterra series



Linear input-output systems

Linear response coefficient or transfer function

$$b_{j,out}(\omega) = \chi_j{}^i(\omega) b_{i,in}(\omega)$$

Nonlinear input-output systems

$$b_{j,out}(\omega) = \sum_{n=1}^{+\infty} \underbrace{\int \cdots \int}_{\omega_1+\cdots+\omega_n=\omega} b^{\pm}_{j_1,in}(\omega_1) \cdots b^{\pm}_{j_n,in}(\omega_n)$$
$$\chi^{j_1 \cdots j_n}_{j,\pm \cdots \pm}(\omega_1, \cdots, \omega_n) d\omega_1 \cdots d\omega_n$$

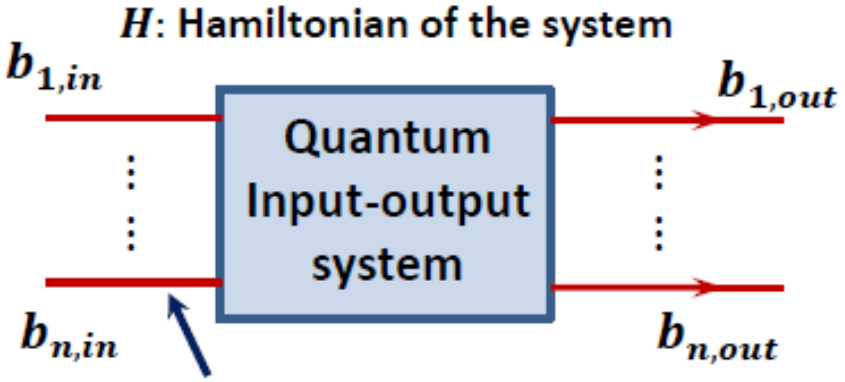
Volterra series

Averaging out the interior degrees of freedom !

Susceptibility coefficient

J. Zhang et. al., arXiv:1407.8108 [quant-ph]

Truncation for weak-nonlinear systems



$\mu \ll 1$

Weak nonlinearity !

$$H = H_{quadratic} + \mu H_{nonlinear}, \qquad L = L_{linear} + \mu L_{nonlinear}$$

Truncate up to lower-order nonlinear terms !

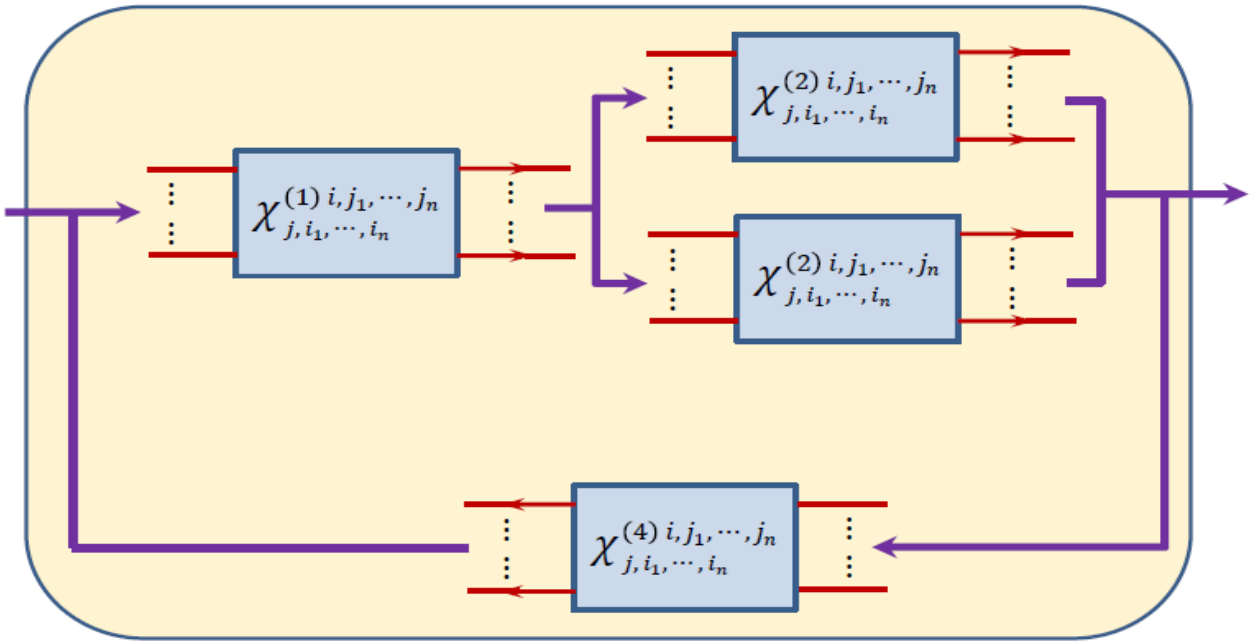
$$b_{j,out}(\omega) = \chi_j{}^{i}(\omega) b_{i,in}(\omega)$$ **Linear terms**

$$+\mu \int \cdots \int \chi^{j_1 j_2}_{j,\pm\pm}(\omega_1,\omega_2)\, b^{\pm}_{j_1,in}(\omega_1)\, b^{\pm}_{j_2,in}(\omega_2)\, d\omega_1\, d\omega_2 + o(\mu)$$

$\omega_1+\omega_2=\omega$ **Quadratic terms**

J. Zhang et. al., arXiv:1407.8108 [quant-ph]

Nonlinear quantum network



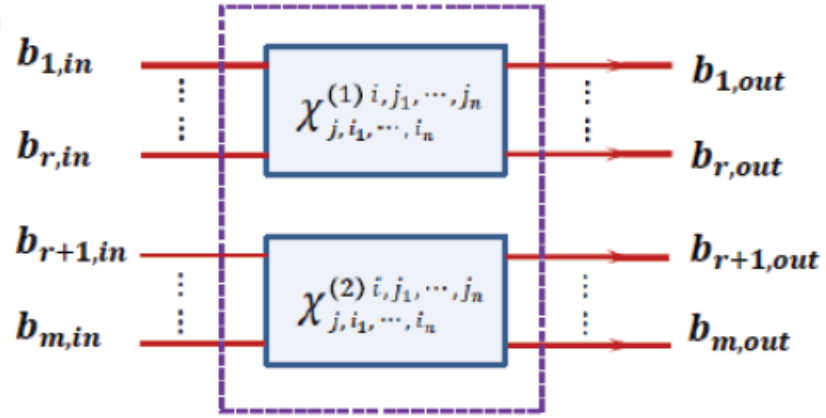
$$\chi^{i,j_1,\cdots,j_n}_{j,i_1,\cdots,i_n}$$

Main result

The n-th order susceptibility coefficients $\chi^{i,j_1\cdots j_n}_{j,i_1\cdots i_n}$ of a multi-component quantum network can be expressed as the polynomials of lower-order susceptibility coefficients of each component $\left\{\chi^{(\beta)\,i,j_1\cdots j_n}_{j,i_1\cdots i_n}\right\}$

J. Zhang et. al., arXiv:1407.8108 [quant-ph]

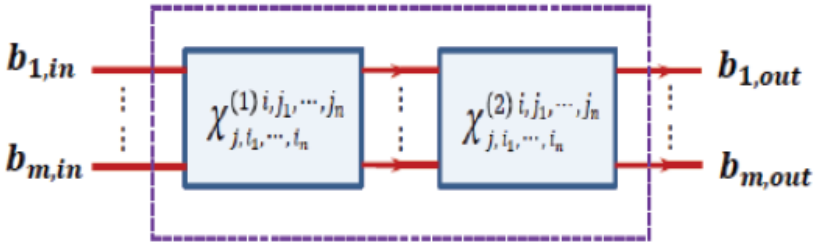
Nonlinear quantum network (II)



$$\chi^{i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n)$$

$$= \begin{cases} \chi^{(1)\,i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n), & i_k,j_k \in \{1,\cdots,r\} \\ \chi^{(2)\,i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n), & i_k,j_k \in \{r+1,\cdots,m\} \\ 0, & Otherwise \end{cases}$$

Concatenation product

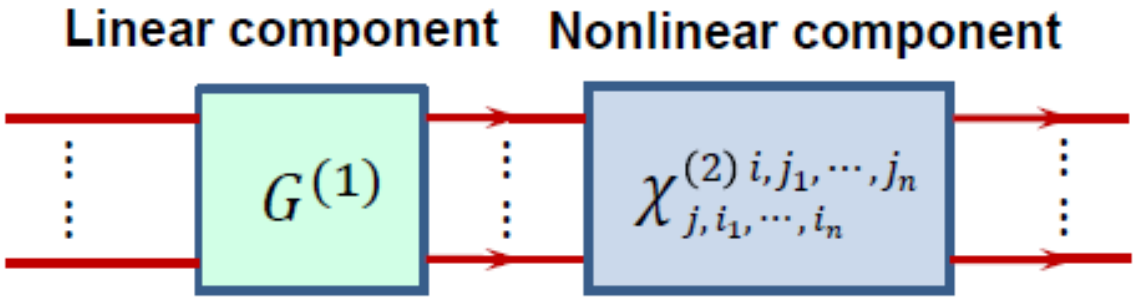


$$\chi^{i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n)$$

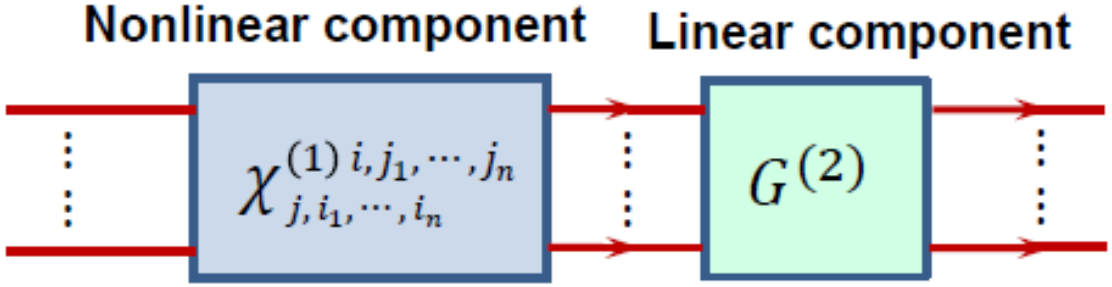
$$= \sum_{\substack{r=1,\cdots,n \\ \alpha_1+\cdots\alpha_r=n}} \chi^{(2)\,i,l_1\cdots l_r}_{j,k_1\cdots k_r}(\omega_1 + \cdots \omega_{\alpha_1},\cdots)\,\chi^{(1)\,k_1,j_1\cdots j_{\alpha_1}}_{l_1,i_1\cdots i_{\alpha_1}}(\omega_1,\cdots,\omega_{\alpha_1})\cdots$$

Series product

Two examples

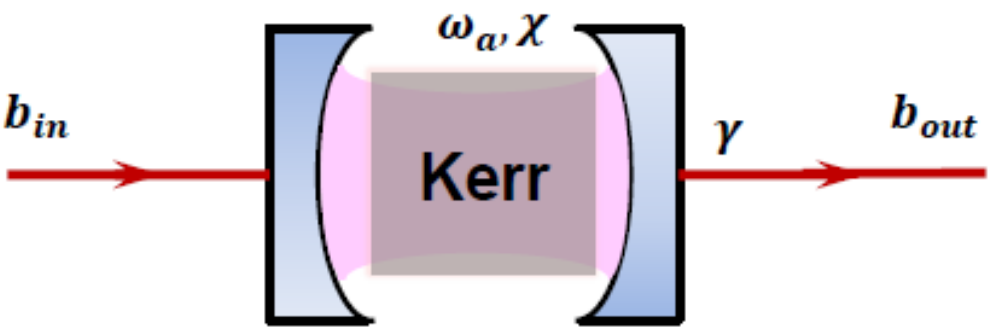


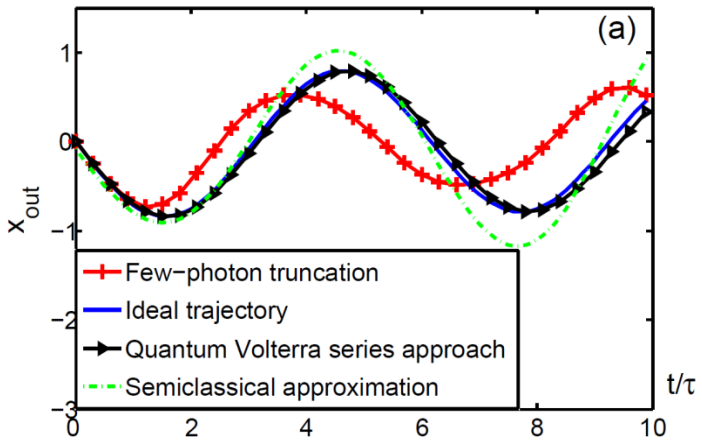
$$\chi^{i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n)=\chi^{(2)\,i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n)\prod_{k=1}^{n}G^{(1)}(i\omega_k)$$



$$\chi^{i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n)=G^{(1)}\left(i\sum_{k=1}^{n}\omega_k\right)\chi^{(1)\,i,j_1\cdots j_n}_{j,i_1\cdots i_n}(\omega_1,\cdots,\omega_n)$$

Application I: Kerr cavity

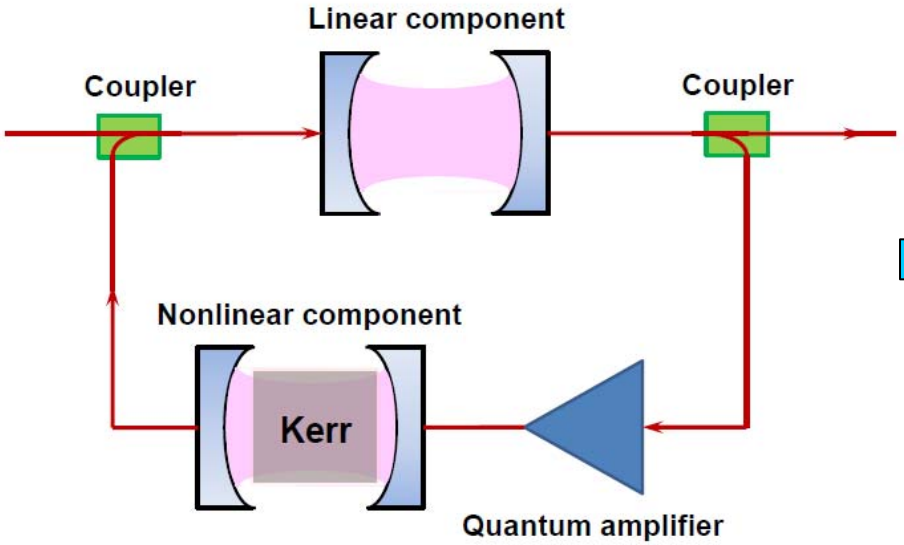


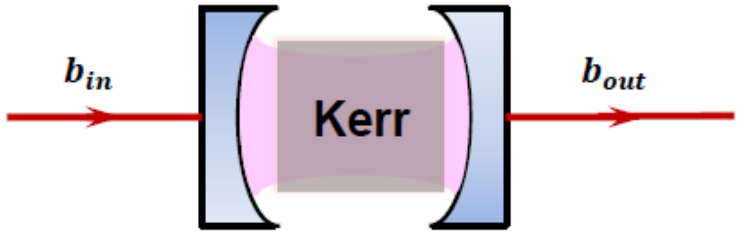


Voltera series approach performs better than few photon truncation and semiclassical approximation for weak-nonlinear systems !

J. Zhang et. al., arXiv:1407.8108 [quant-ph]

Application II: Enhancing Kerr nonlinearity





$\tilde{\chi}_a$: Effective Kerr coefficient

χ_b: Kerr nonlinear coefficient

G: Gain of the quantum amplifier

$$\tilde{\chi}_a \approx G\chi_b$$

J. Zhang et. al., arXiv:1407.8108 [quant-ph]

Conclusions

Main results

1) Extend the input-output theory to non-Markovian case and use it to model non-Markovian quantum input-output network

2) Introduce Volterra series approach to model weak-nonlinear quantum input-output network

3) Applications to various system: Kerr cavity, Optomechanical transducer, nonlinear coherent feedback systems

More...

Please see on arxiv a review paper on quantum feedback.

Comments are welcome !

Jing Zhang, Yu-xi Liu, Re-Bing Wu, Kurt Jacobs, and Franco Nori, Quantum feedback: theory, experiments, and applications, arXiv:1407.8536 [quant-ph]

Thank you!