# Information-based approach to quantum noise and disturbance: theory and experiment


Francesco Buscemi

Masanao Ozawa

University of Nagoya
Japan



Mark Wilde

Louisana State
University
USA



Yuji Hasegawa

Georg
Sulyok

Bülent
Demirel

Stephan
Sponar

Vienna University
of Technology
Austria

# OUTLINE

- Gain information about observable *A* via some measurement, with some noise or error.

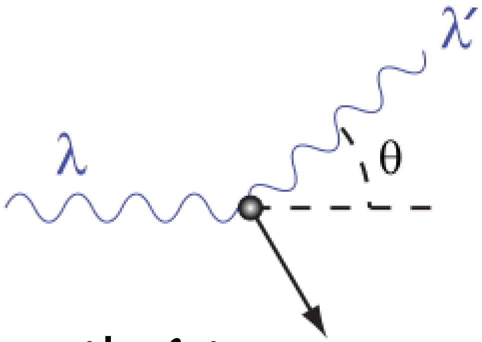
  Q. *What is the cost? – in terms of losing information about observable B, i.e., the disturbance to B?*

- Noise–disturbance operator approach
  - Testable relations
  - But not about "information" (e.g., label dependent)

- Information-theoretic approach
  - All about "information" (and allows for error correction).
  - Recent neutron experiment

# Noise vs disturbance: γ-ray microscope

Heisenberg 1927:




- If use photon of wavelength $\lambda$ to measure the position of an electron, then the error will be on the order of

$$\varepsilon(Q) \sim \lambda.$$

- From the Compton effect, the momentum of the electron will suffer a "discontinuous change" on the order of

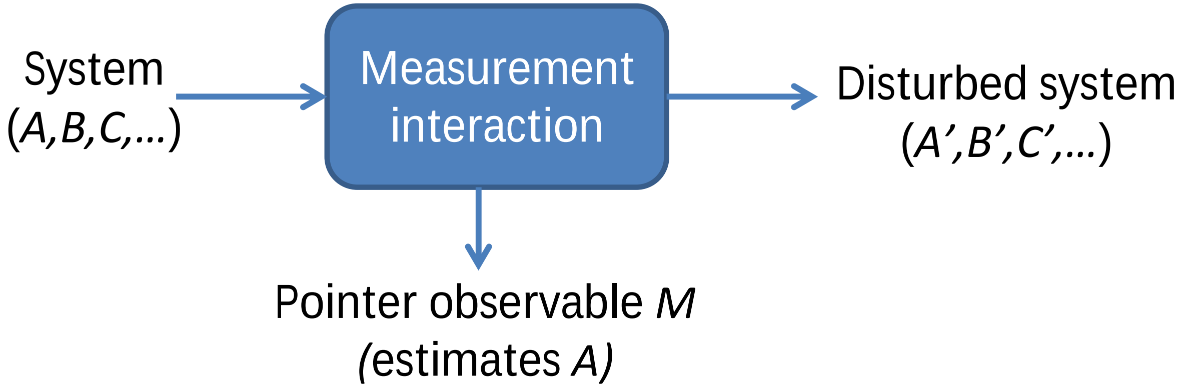
$$\eta(P) \sim p_\gamma = h/\lambda.$$

- Hence,

$$\varepsilon(Q)\ \eta(P) \sim h.$$

- How can this tradeoff be made rigorous – and generalised?

# Noise and disturbance operators




- Noise operator: $M - A$
- RMS noise: $\varepsilon_\rho(A)^2 := \langle (M - A)^2 \rangle_\rho$

- Disturbance operator: $B' - B$
- RMS disturbance: $\eta_\rho(B)^2 := \langle (B' - B)^2 \rangle_\rho$

$$???? \quad \varepsilon_\rho(A)\, \eta_\rho(B) \geq \tfrac{1}{2} \left| \langle [A,B] \rangle_\rho \right| \quad ????$$

# Noise-disturbance-*predictability* relations

- Ozawa (2003):

$$\varepsilon_\rho(A)\,\eta_\rho(B) + \varepsilon_\rho(A)\,\Delta_\rho(B) + \Delta_\rho(A)\,\eta_\rho(B) \geq ½\,|< [A,B] >_\rho|$$

- Generalised and tested extensively

[e.g., Hall 2004, Hasegawa *et al.* 2012&2013; Rozema *et al.* 2012; Weston *et al.* 2013; Branciard 2013&2014; Ringbauer *et al.* 2014; Edamatsu *et al.* 2014; Ozawa 2014; many arXiv eprints]

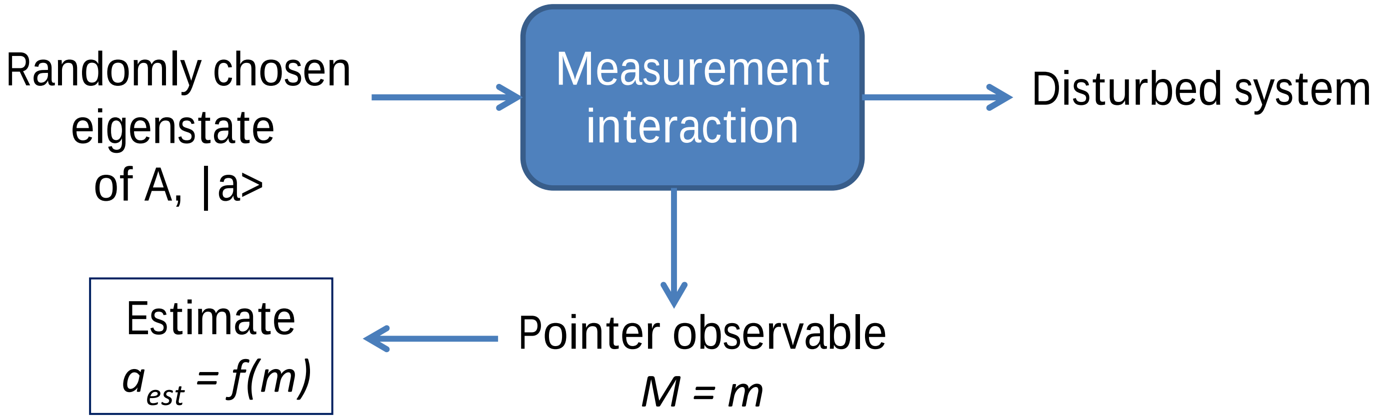
- Not a pure *noise-disturbance* tradeoff, *a la* Heisenberg
- $\varepsilon_\rho(A)$ can underestimate, $\eta_\rho(B)$ can overestimate
- Label-dependent – not about "information" lost or gained

# Some controversy….

- “you really don't want to confuse this freaky maverick paper with Heisenberg's actual discoveries”
- “it is shown that their *alleged* proof includes a loophole”
- “…. has written an attack on our approach. This was worded so aggressively, and was so low on scientific quality, that it would seem to be more of an embarrassment to the author than an argument requiring an answer.”

**Q**. *Is there an alternative approach to characterising a tradeoff between noise and disturbance, for a given apparatus M?*

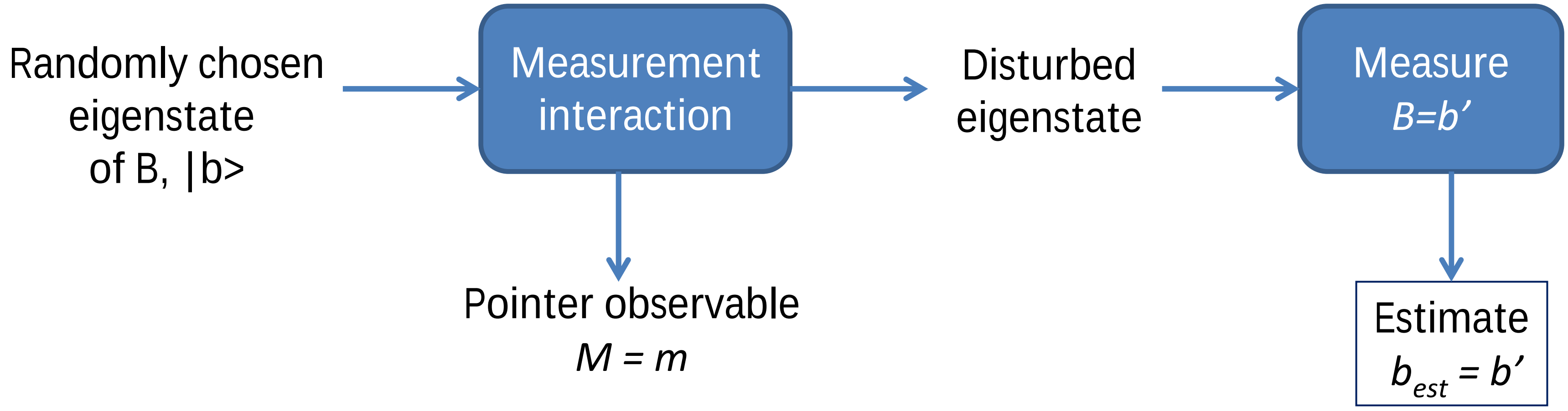
# Information-theoretic noise




- How well can one guess the input, $A=a$, from the output, $M=m$?
- The correlation is described by $p(a,m) = d^{-1} \langle a|M_m|a \rangle$
- The *noise* is defined to be *the amount of information lost via an imperfect correlation*:

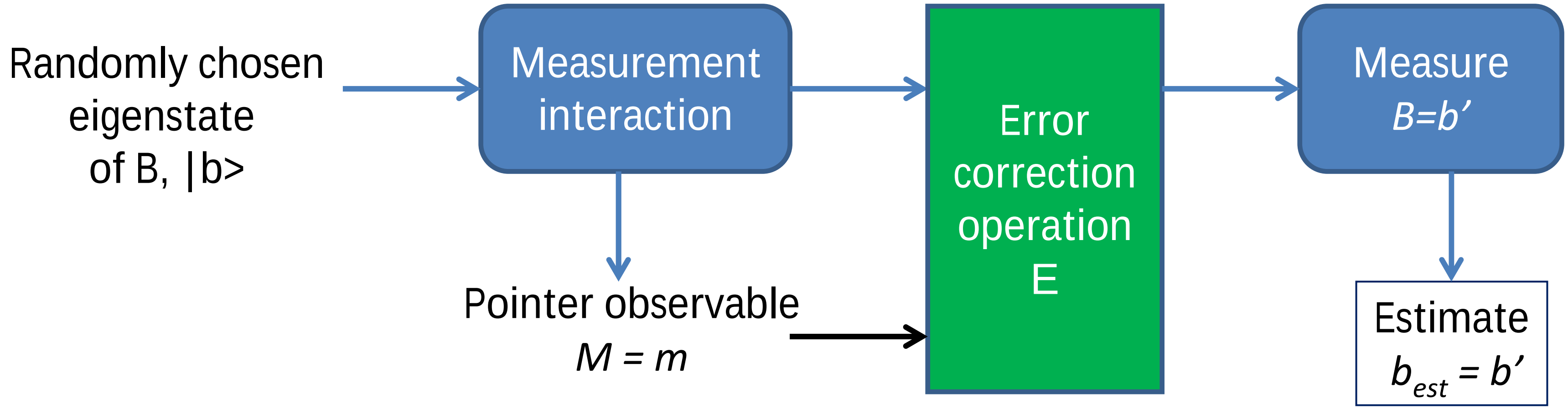
$$N[A,M] := H(A|M) \quad \text{(conditional entropy)}$$

# Information-theoretic disturbance




- How well can one estimate the input, $B=b$, from the output, $B'=b'$?

# Information-theoretic disturbance




- How well can one estimate the input, *B=b*, from the output, *B′=b′*?
- Allow for arbitrary error correction, E.
- The *disturbance* is defined to be the *information irreversibly lost due to imperfect correlation*:

$$D[B,M] := \inf_E \; H_E(B|B')$$

# Noise-disturbance relation

- Can prove that [*Buscemi et al. PRL 112 050410 2014*]:

$$N[A,M] + D[B,M] \geq -\log c$$

- Here $c$ denotes the maximum overlap of the eigenstates of $A$ and $B$, i.e.,

$$c := \max_{a,b} |\langle a|b\rangle|^2 \leq 1.$$

- The relation is label-independent and state-independent.

# Qubit example

- For spin-1/2 observables $A=\sigma_z$ and $B=\sigma_y$:

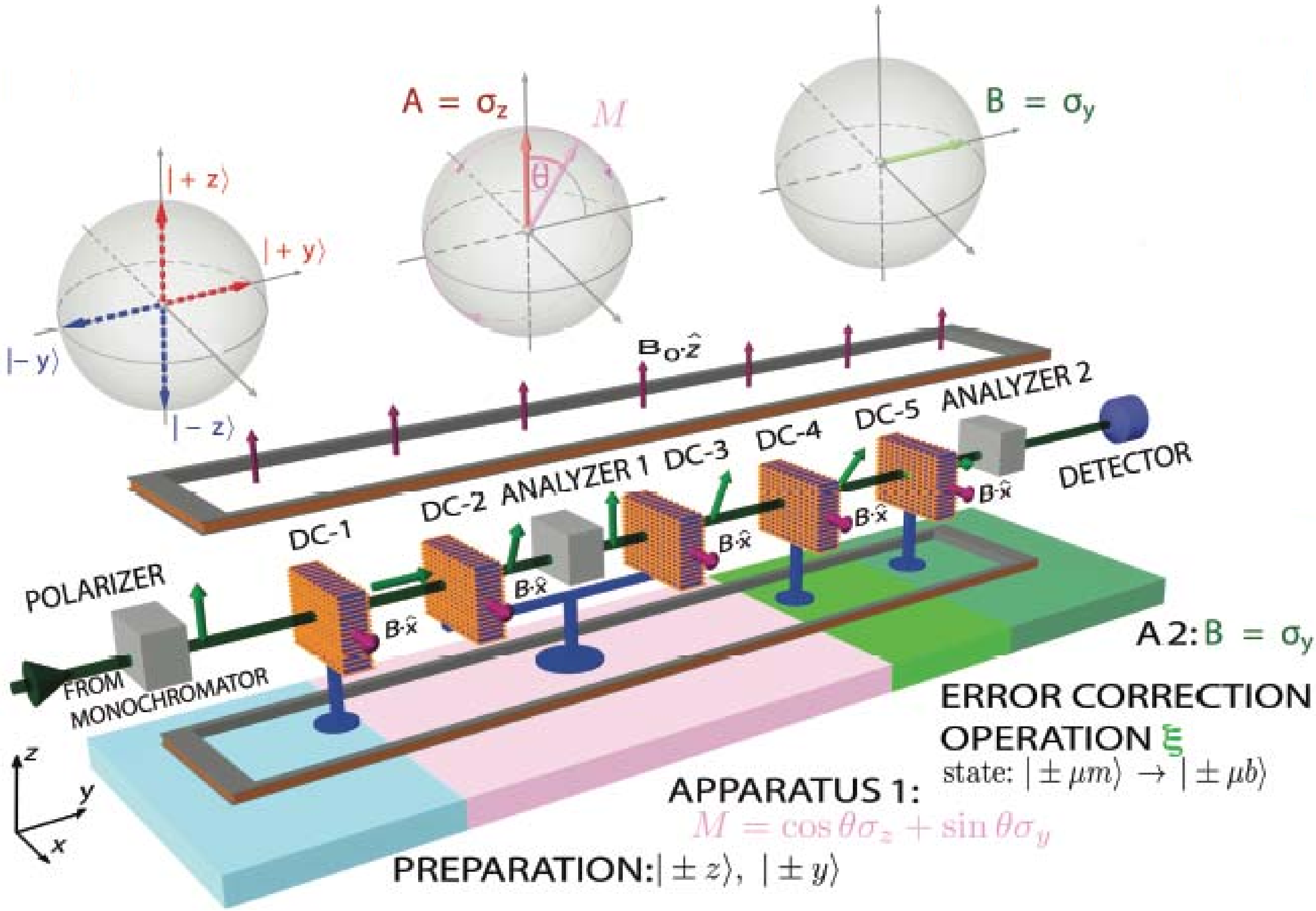
$$N[A,M] + D[B,M] \geq \log 2.$$

- Thus, if a measurement device gives perfect information about $\sigma_z$ (no noise) , then it destroys all information about $\sigma_y$ (maximum disturbance).

- Neutron experiment has been recently performed in Vienna, exploring the qubit case

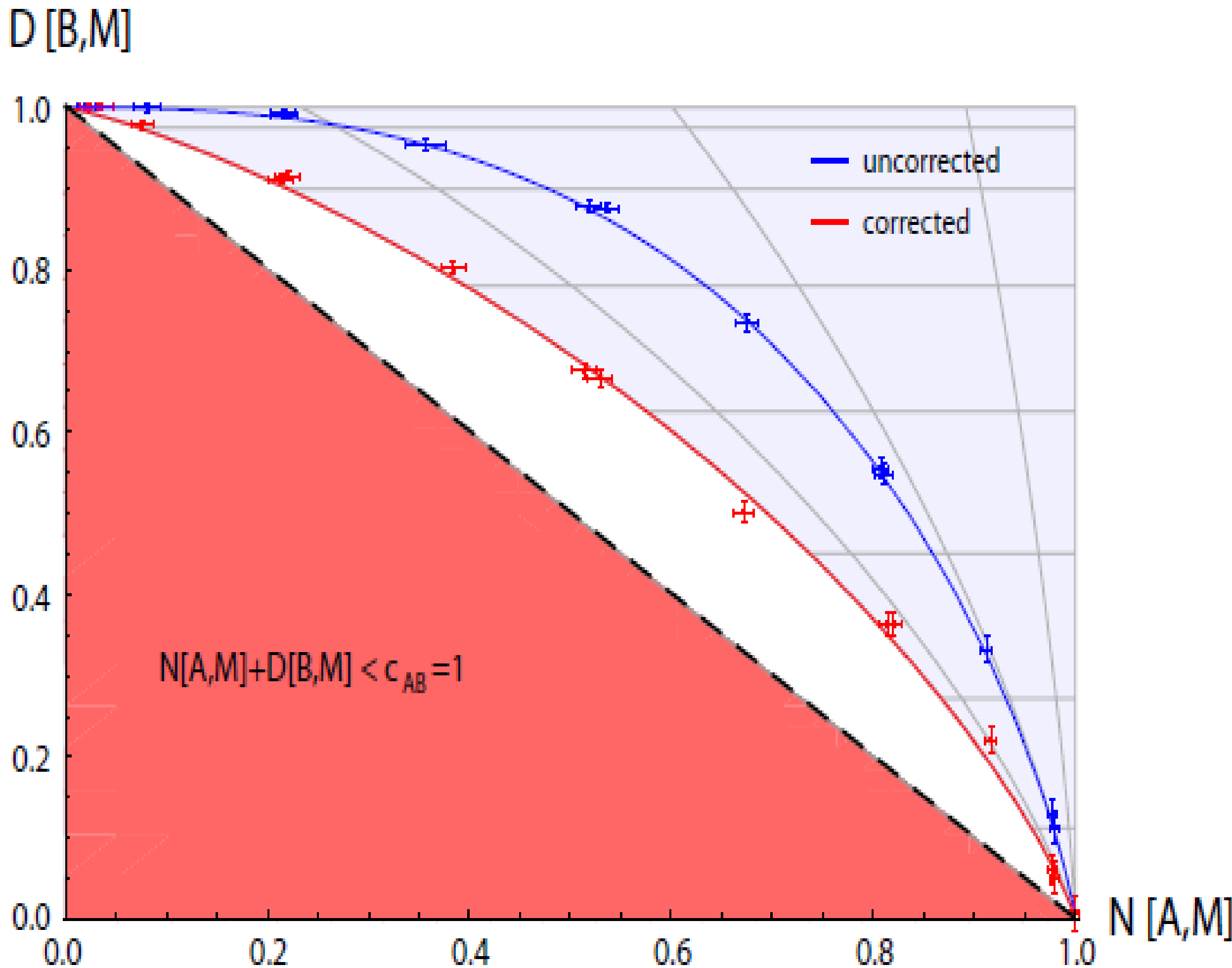
# Noise and disturbance with neutrons

- Prepare eigenstates of $A=\sigma_z$ and $B=\sigma_y$
- Measure $M=\sigma.m$
- Make error correction, based on outcome $\mu=\pm 1$
- Measure *B*
- Calculate *N[A,M* and *D[B,M]*, as the measurement M is varied



# Experimental results

- Blue curve: no error correction
- Red curve: optimal error correction
- Red curve gives ultimate noise-disturbance tradeoff, for any *M*




$\{ (N,D) = ( h(\sin\phi), h(\cos\phi) ) : \phi \in [0,\pi/2] \}$, $h(x)$ := entropy of $\{(1\pm x)/2\}$

# Joint measurement relation

- For any measurement device which gives joint estimates of observables *A* and *B*, one has the noise-tradeoff relation [*Buscemi et al. PRL 112 050410 2014*]:

$$N[A,M] + N[B,M] \geq -\log c\,.$$

- Thus, *no measurement device can gain perfect information about both observables*, if $c<1$.

- This result implies the noise-disturbance relation

# Heisenberg-type relations?

- One can compare any estimate $a_{est}=f(m)$ with the input value $A=a$, in the "noise" experiment.

- One can compare the measurement result $B'=b'$, with the input value $B=b$, in the "disturbance" experiment.

- Define corresponding *mean square errors* for the noise and disturbance:

$$V_N(A) := \Sigma_{a,m}\, p(a,m)\, [a_{est} - a]^2$$

$$V_D(B) := \Sigma_{b,b'}\, p(b,b')\, [b' - b]^2.$$

(NB: Busch, Lahti and Werner [PRL 111 160405 2013] replace averages over $a$ and $b$ by maximums over $a$ and $b$)

# Example: position and momentum

- For $A=Q$ and $B=P$ (in the limit where position and momentum eigenstates are input) [*Buscemi et al. PRL 112 050410*]:

$$V_N(Q)\, V_D(P) \geq \hbar^2/4.$$

- Stronger than a recent result by Busch, Lahti and Werner [*PRL 111 (2013)160405 (2013)*] (which uses maximum rather than mean-square errors).

- Looks nice, but is only of theoretical interest: $V_N(Q) \to \infty$ for any position-measurement device having a finite measurement range (similarly for BLW's "maximum error").

# Conclusions

- *Information-theoretic* form of Heisenberg's noise-disturbance concept:

  $$N[A,M] + D[B,M] \geq -\log c.$$

- A measurement device can gain information about an observable *A* only by destroying information about an incompatible observable *B.*

- Independent of how eigenvalues and measurement outcomes are *labelled* – only information matters.

- Does not depend on the *predictability* of outcomes – e.g, on $\Delta_\rho(A)$ or $\Delta_\rho(B)$.

- Depends only on the measurement device, not on a particular state (is there a state-dependent form?)

# State-dependent information-theoretic joint-measurement uncertainty relation

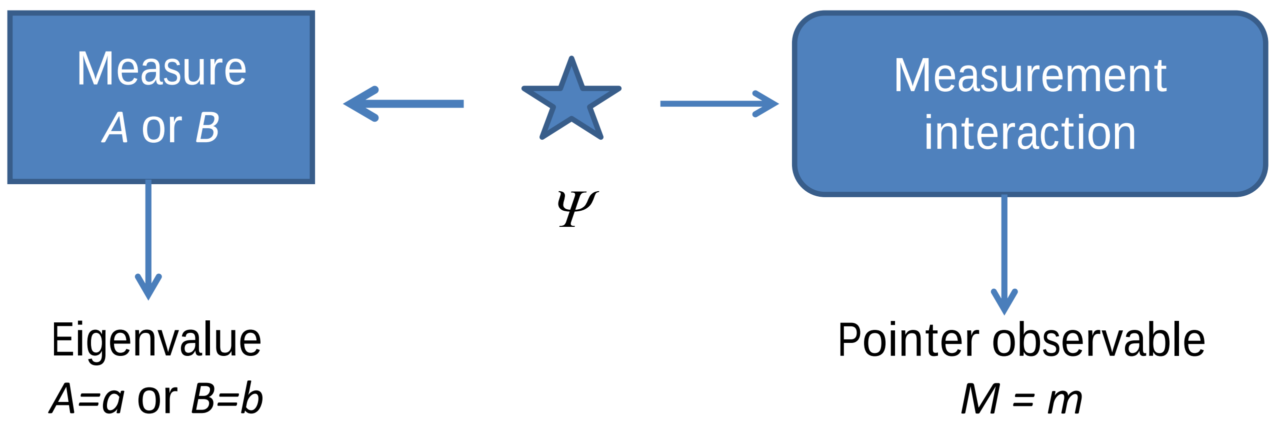
- If a system and probe interact, to give some joint state $\rho$, then for any system observables $A$ and $B$, and probe observable $M$, one has the *information exclusion relation* [Hall, PRA 55, 100, 1997]

$$H_\rho(A:M) + H_\rho(B:M) \leq \log d^2 c$$

- For position and momentum, one obtains

$$H_\rho(Q:M) + H_\rho(P:M) \leq \log [\Delta_\rho Q \, \Delta_\rho P/(\hbar/2)].$$

# Idea behind proof of N(A)+N(B)≥-log c




- Measure *A* or *B* precisely, on one component of a suitable maximally-entangled state
- Carry out the measurement interaction of interest on the other component
- Estimate $A=a_{est}$ and $B=b_{est}$ from $M=m$
- Use suitable entropic uncertainty relation for

$$H_\Psi(A|M) + H_\Psi(B|M)$$