Test of Local Realism by Equality (*A hidden-variables version of Gisin's theorem*)

Kazuo Fujikawa

RIKEN Nishina Center, Japan

Plan of the talk:

1. Review of Bell's theorem and Gisin's theorem
2. A simple test of local realism by EQUALITY
3. Numerical illustration
4. Brief sketch of the derivation of new criterion in local hidden-variables models.
5. Conclusion

Ref. K. Fujikawa and K. Umetsu, arXiv:1410.1702.

We start with operator B (Cirel'son)

$$B = \mathbf{a}\cdot\sigma \otimes (\mathbf{b}+\mathbf{b}')\cdot\sigma + \mathbf{a}'\cdot\sigma \otimes (\mathbf{b}-\mathbf{b}')\cdot\sigma,$$

where σ stands for the Pauli matrix and $\mathbf{a},\ \mathbf{a}',\ \mathbf{b},\ \mathbf{b}'$ are 3-dimensional unit vectors. Bounded by

$$||B|| \le |\mathbf{b}+\mathbf{b}'| + |\mathbf{b}-\mathbf{b}'| \le 2\sqrt{2}.$$

Bell's theorem:

Local hidden-variables model in $d=4$ (local realism) gives CHSH inequality

$$|\langle\psi|B|\psi\rangle| \le 2$$

for any $\mathbf{a},\ \mathbf{a}',\ \mathbf{b},\ \mathbf{b}'$.

Gisin's theorem:N. Gisin, PLA**154**, 201 (1991)

If the state $|\psi\rangle$ is an entangled pure quantum state, one can always achieve

$$|\langle\psi|B|\psi\rangle| > 2$$

by suitably choosing $\mathbf{a},\ \mathbf{a}',\ \mathbf{b},\ \mathbf{b}'$.

Contraposition of Gisin's theorem:
If

$$|\langle\psi|B|\psi\rangle| \leq 2$$

for any $\mathbf{a},\ \mathbf{a}',\ \mathbf{b},\ \mathbf{b}'$, the state $|\psi\rangle$ is separable if one considers only pure states.

Combination of Bell's theorem and the contraposition of quantum mechanical Gisin's theorem:

Bell's theorem:
Local realism gives CHSH inequality

$$|\langle\psi|B|\psi\rangle| \leq 2$$

for any $\mathbf{a},\ \mathbf{a}',\ \mathbf{b},\ \mathbf{b}'$, and

Contraposition of Gisin's theorem:
If

$$|\langle\psi|B|\psi\rangle| \leq 2$$

for any $\mathbf{a},\ \mathbf{a}',\ \mathbf{b},\ \mathbf{b}'$, the state $|\psi\rangle$ is separable, if one considers only pure states.

One thus concludes:
Local realism can describe only separable pure quantum states.

The local realism is thus quantified by Equality,

$$
\begin{aligned}
&G(\mathbf{a}, \mathbf{b})/4 \\
&\equiv \langle\psi|P(\mathbf{a}) \otimes P(\mathbf{b})|\psi\rangle \\
&\quad -\langle\psi|P(\mathbf{a}) \otimes \mathbf{1}|\psi\rangle\langle\psi|\mathbf{1} \otimes P(\mathbf{b})|\psi\rangle = 0
\end{aligned}
$$

for any two projection operators $P(\mathbf{a})$ and $P(\mathbf{b})$.

Test of local realism by the deviation of $G(\mathbf{a}, \mathbf{b})$ from $G(\mathbf{a}, \mathbf{b}) = 0$.

Test of Local Realism:

Categorize three different cases:
(i) Quantum mechanics.

$$
\begin{aligned}
&\langle\psi|B|\psi\rangle_{QM}\\
&= \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle + \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b'}\cdot\sigma|\psi\rangle\\
&+\langle\psi|\mathbf{a'}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle - \langle\psi|\mathbf{a'}\cdot\sigma\otimes\mathbf{b'}\cdot\sigma|\psi\rangle
\end{aligned}
$$

which assumes values $|\langle\psi|B|\psi\rangle_{QM}| \leq 2\sqrt{2}$.

(ii) Bell-CHSH inequality (local realism).

$$
\begin{aligned}
&|\langle\psi|B|\psi\rangle_{CHSH}| \\
&= |\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle + \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}'\cdot\sigma|\psi\rangle \\
&+\langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle - \langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{b}'\cdot\sigma|\psi\rangle| \\
&\le 2,
\end{aligned}
$$

and this is valid for *any state* $|\psi\rangle$.

(iii) Our proposed *equality*. Local realism describes only separable states and thus

$$
\begin{aligned}
&\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle \\
&\quad = \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes\mathbf{b}\cdot\sigma|\psi\rangle,
\end{aligned}
$$

and valid for *any choice* of $\mathbf{a}$ and $\mathbf{b}$, including

$$\begin{aligned}
&\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}'\cdot\sigma|\psi\rangle \\
&\quad = \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes\mathbf{b}'\cdot\sigma|\psi\rangle, \\
&\langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle \\
&\quad = \langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes\mathbf{b}\cdot\sigma|\psi\rangle, \\
&\langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{b}'\cdot\sigma|\psi\rangle \\
&\quad = \langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes\mathbf{b}'\cdot\sigma|\psi\rangle.
\end{aligned}$$

By asking these relations, one can test if a given state $|\psi\rangle$ is described by local realism.

We here recall the well-known fact in quantum mechanics that

$$-2 \leq \langle\psi|B|\psi\rangle_{QM} \leq 2$$

for any separable state $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle$.

Because of this fact, our criterion automatically implies CHSH, while CHSH for all $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{b}'$ implies our criterion due to Gisin's theorem.

In the analysis of actual experiments, however, we consider only a limited set of parameters $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{b}'$, thus CHSH and our criterion generally give different constraints.

CHSH can test *local realism* by looking at inequality, while our criterion can test local realism by looking at equality.

$|\langle\psi|B|\psi\rangle_{QM}| > 2$ or $G(\mathbf{a},\mathbf{b}) \neq 0$ negates local realism.

Any state

$$\psi = \alpha|+\rangle_1|-\rangle_2 - \beta|-\rangle_1|+\rangle_2,$$

with $\alpha\beta \neq 0$ is excluded by local realism.

Numerical Illustration:

We consider the generic entangled state

$$\psi = \alpha|+\rangle_1|-\rangle_2 - \beta|-\rangle_1|+\rangle_2,$$

where $|\pm\rangle_1$ and $|\pm\rangle_2$ stand for the eigenstates of σ_z^1 and σ_z^2, respectively.

Both α and β are real and positive with $\alpha^2 + \beta^2 = 1$.

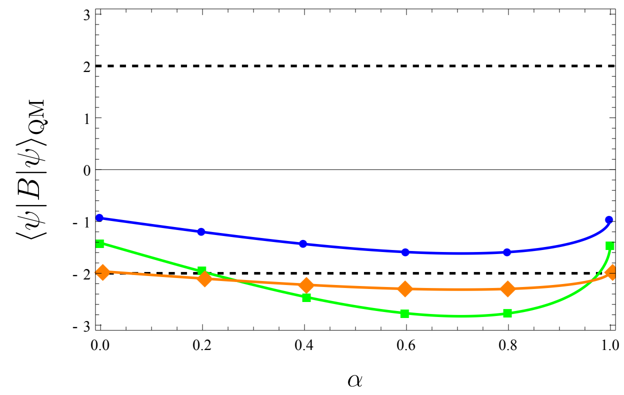

Figure 1: **Inseparable state and CHSH inequality**: Let $\mathbf{a} = (\sin\theta, 0, \cos\theta)$, $\mathbf{b} = (\sin\phi, 0, \cos\phi)$ and similarly for $\mathbf{a}'$ and $\mathbf{b}'$ by choosing y-axis in the direction of two separated parties. We performed numerical tests for three cases, namely, A: $(\theta, \phi, \theta', \phi') = (\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, 0)$, B: $(\frac{\pi}{3}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{6})$, and C: $(\frac{\pi}{6}, \frac{3\pi}{4}, \pi, 0)$. The lines with square, filled circle, and diamond, respectively, indicate $\langle\psi|B|\psi\rangle_{\mathrm{QM}}$ in the case of A, B and C. The dashed lines stand for CHSH inequality, $-2 \leq \langle\psi|B|\psi\rangle_{\mathrm{CHSH}} \leq 2$.

We plot

$$\langle\psi|B|\psi\rangle_{QM}$$
$$= \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle + \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}'\cdot\sigma|\psi\rangle$$
$$+\langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle - \langle\psi|\mathbf{a}'\cdot\sigma\otimes\mathbf{b}'\cdot\sigma|\psi\rangle.$$

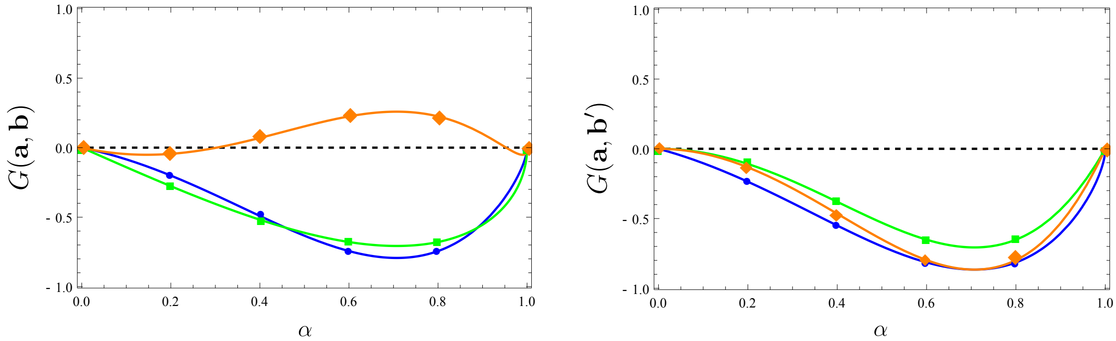

Figure 2: **Inseparable state and our proposed criterion**: We use the same set of parameters as in Fig. 1. The lines with square, filled circle, and diamond indicate $G(\mathbf{a},\mathbf{b})$ and $G(\mathbf{a},\mathbf{b}')$ for the case of A, B and C in Fig. 1, respectively. The dashed lines stand for the prediction of local realism.

We plot

$$G(\mathbf{a},\mathbf{b}) \equiv \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle - \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes\mathbf{b}\cdot\sigma|\psi\rangle,$$

for the same set of parameters as in Fig.1.

The experimental setups close to the test of our criterion

$$
\begin{aligned}
&G(\mathbf{a},\mathbf{b})/4 \\
&\equiv \langle\psi|P(\mathbf{a})\otimes P(\mathbf{b})|\psi\rangle \\
&\qquad -\langle\psi|P(\mathbf{a})\otimes \mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes P(\mathbf{b})|\psi\rangle \\
&= 0
\end{aligned}
$$

have in fact been used in the past by Freedman and Clauser in 1972, and by Aspect, Grangier and Roger in 1981. Those experiments are based on the measurement of the transverse linear polarization of the photon.

In terms of measured quantities of Aspect, $G(\varphi) = G(\mathbf{a}, \mathbf{b})$ is written as

$$
\begin{aligned}
G(\varphi) &= 4[\frac{R(\varphi)}{R_0} - \frac{R_1 R_2}{R_0^2}] \\
&= (0.971 - 0.029)(0.968 - 0.028)0.984 \cos 2\varphi
\end{aligned}
$$

where φ stands for the angle between $\mathbf{a}$ and $\mathbf{b}$. The quantities $R(\varphi),\ R_1,\ R_2$ and R_0 are defined in eq.(2) of Aspect, and the numerical factors in front of $\cos 2\varphi$ are also given there.

A. Aspect et al., PRL **47**, 460 (1981).(before better known exp. in '82)

Note that for the maximally entangled state in quantum mechanics, we have

$$G(\mathbf{a}, \mathbf{b}) = -\cos\varphi$$

for the spin, while we have

$$G(\mathbf{a}, \mathbf{b}) = \cos 2\varphi$$

for the **photon**. For the ideal measurement, the coefficient of $\cos 2\varphi$ is unity.

We show this quantum mechanical prediction

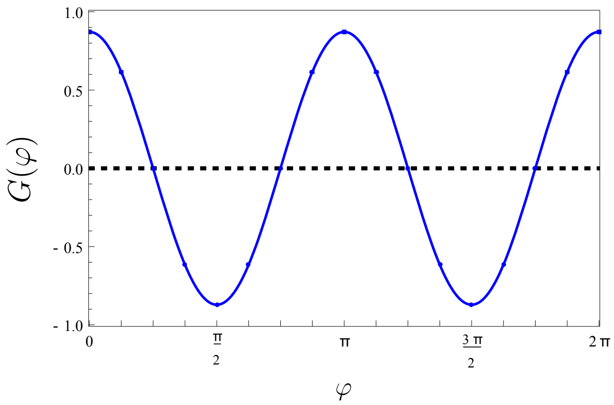

Figure 3: **Aspect's experiment and our proposed criterion**: The solid line represents the quantum mechanical prediction of $G(\varphi)$ corresponding to Aspect's experimental setup in 1981. The dashed line represents the prediction of local realism.

Fig.3 shows that our criterion is very effective and provides a decisive test of the deviation of local realism from quantum mechanics.

It should be emphasized that Aspect et al. have not discussed figure 4 in their paper, which corresponds to Fig.3 in the present case, as a test of local realism; instead they discussed only the conventional CHSH inequality as a test of local realism. Also, Bell himself was not aware of "Gisin's theorem" around 1990.

This shows that our criterion is not universally recognized as a decisive prediction of local realism.

Recently, more and more sophisticated tests of Bell-CHSH inequalities have been performed. It should be interesting to look at those experiments from the point of view of our criterion illustrated in Fig.3; simply stated, any entangled state negates local realism, which is the original idea of Einstein.

Since our criterion is mathematically much simpler, it is expected that it may avoid some of the technical complications (experimental loopholes) involved in the tests of Bell-CHSH inequalities.

Sketch of Direct Derivation of our Criterion:

$$\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle$$
$$= \int \rho(\lambda_1, \lambda_2)d\lambda_1 d\lambda_2 a(\psi, \lambda_1)b(\psi, \lambda_2)$$

with dichotomic variables $a(\psi, \lambda_1)$ and $b(\psi, \lambda_2)$, and we recover the more common hidden-variables model if one sets

$$\rho(\lambda_1, \lambda_2) = \rho(\lambda_1)\delta(\lambda_1 - \lambda_2).$$

We have quantum mechanical relations

$$
\begin{aligned}
&\langle B\rangle \\
&= \langle \mathbf{a}\cdot\sigma\otimes(\mathbf{b}+\mathbf{b}')\cdot\sigma\rangle + \langle \mathbf{a}'\cdot\sigma\otimes(\mathbf{b}-\mathbf{b}')\cdot\sigma\rangle \\
&= \langle \mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma\rangle + \langle \mathbf{a}\cdot\sigma\otimes\mathbf{b}'\cdot\sigma\rangle \\
&+\langle \mathbf{a}'\cdot\sigma\otimes\mathbf{b}\cdot\sigma\rangle - \langle \mathbf{a}'\cdot\sigma\otimes\mathbf{b}'\cdot\sigma\rangle.
\end{aligned}
$$

If one moves to the hidden-variables representation from the last expression, one obtains the standard $|\langle B\rangle| \le 2$ by noting,

$$
\begin{aligned}
&a(\psi,\lambda_1)b(\psi,\lambda_2) + a(\psi,\lambda_1)b'(\psi,\lambda_2) \\
&+a'(\psi,\lambda_1)b(\psi,\lambda_2) - a'(\psi,\lambda_1)b'(\psi,\lambda_2) = \pm 2
\end{aligned}
$$

On the other hand, if one moves from the first relation

$$|\mathbf{b}+\mathbf{b}'|\int \rho(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 a(\psi,\lambda_1)\tilde{b}(\psi,\lambda_2)$$

$$+\ |\mathbf{b}-\mathbf{b}'|\int \rho(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 a(\psi,\lambda_1)\tilde{b}'(\psi,\lambda_2),$$

one *cannot prove* $|\langle B\rangle| \leq 2$ in general.

Here, unit vectors $\tilde{\mathbf{b}} \equiv (\mathbf{b}+\mathbf{b}')/|\mathbf{b}+\mathbf{b}'|$ and $\tilde{\mathbf{b}}' \equiv (\mathbf{b}-\mathbf{b}')/|\mathbf{b}-\mathbf{b}'|$ for non-collinear $\mathbf{b}$ and $\mathbf{b}'$.

Note that this operation is consistent with assumed *locality*.

To achieve the conventional CHSH $|\langle B\rangle| \leq 2$ uniquely, one needs to satisfy the linearity of the probability measure

$$\langle \mathbf{a}\cdot\sigma \otimes (\mathbf{b}+\mathbf{b}')\cdot\sigma\rangle$$
$$= \langle \mathbf{a}\cdot\sigma \otimes \mathbf{b}\cdot\sigma\rangle + \langle \mathbf{a}\cdot\sigma \otimes \mathbf{b}'\cdot\sigma\rangle.$$

We then derive

$$\langle\psi|\mathbf{a}\cdot\sigma \otimes \mathbf{b}\cdot\sigma|\psi\rangle$$
$$= \int \rho_1(\lambda_1)d\lambda_1 a(\psi,\lambda_1) \int \rho_2(\lambda_2)d\lambda_2 b(\psi,\lambda_2).$$

We then have

$$
\begin{aligned}
&G(\mathbf{a},\mathbf{b}) \\
&\equiv \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle \\
&\quad -\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes\mathbf{b}\cdot\sigma|\psi\rangle \\
&= \int \rho_1(\lambda_1)d\lambda_1 a(\psi,\lambda_1)\int \rho_2(\lambda_2)d\lambda_2 b(\psi,\lambda_2) \\
&\quad - \int \rho_1(\lambda_1)d\lambda_1 a(\psi,\lambda_1)\int \rho_2(\lambda_2)d\lambda_2 b(\psi,\lambda_2) \\
&= 0,
\end{aligned}
$$

for any **a** and **b**.

End

References:

K. Fujikawa, Prog. Theor. Phys. **127**, 975 (2012).

K. Fujikawa, CH. Oh and Chengjie Zhang, Int. J. Mod. Phys. A**29**, 1450017 (2014).

End

References

[1] J. S. Bell, Physics **1**, 195 (1965).

[2] See N. D. Mermin, Rev. Mod. Phys. **65**, 803 (1993), for a classification of hidden-variables models.

[3] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[4] A. Aspect, P. Grangier, G. Roger, Phys. Rev. Lett. **47**, 460 (1981).

[5] R.F. Werner, Phys. Rev. A**40**, 4277 (1989).

[6] N. Gisin, Phys. Lett. A**154**, 201 (1991).

[7] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).

[8] E. G. Beltrametti and G. Gassinelli, *The Logic of Quantum Mechanics*, (Addison-Wesley Pub., Reading, 1981).

[9] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).

[10] K. Fujikawa, Prog. Theor. Phys. **127**, 975 (2012).

[11] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, 1955).

[12] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).

[13] K. Fujikawa, CH. Oh and Chengjie Zhang, Int. J. Mod. Phys. A**29**, 1450017 (2014).

[14] S.J. Freedman and J.F. Clauser,Phys. Rev. Lett. **28**, 938 (1972).

[15] A. Aspect, J. Dalibard and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).

[16] W. Tittel, J. Brendel, H. Zbinden, N. Gisin, Phys. Rev. Lett. **81**, 3563 (1998).

[17] G. Weihs, et al., Phys. Rev. Lett. **81**, 5039 (1998).

[18] Jian-Wei Pan, et al., Nature **403** 515 (2000).

[19] M.A. Rowe, et al., Nature **409** 791 (2001).

[20] S. Groblacher, et al., Nature **446**, 871 (2006).

[21] D. Salart, et al., Phys. Rev. Lett. **100**, 220404 (2008).

[22] Markus Ansmann, et al., Nature **461**, 504 (2009).

[23] Marissa Giustina, et al., Nature **497**, 227 (2013).