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FACULTY OF
SCIENCE
Institute of Theoretical Physics

A universal set of qubit quantum channels and the Bloch tensor

DANIEL BRAUN

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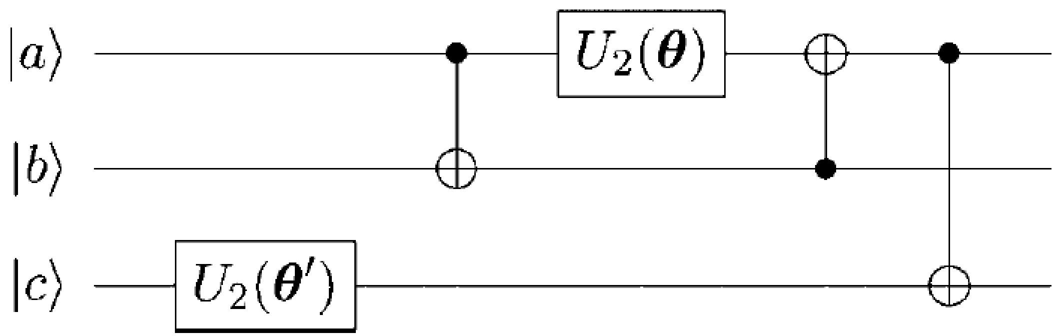
EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Universal gate sets – Unitary operations

- Quantum circuit model
- All unitary operations in Hilbert space of *n* qubits can be obtained by concatenation of
  - arbitrary single qubit unitaries
  - CNOT (or other entangling gate) on any pair of qubits



D. Deutsch et al Proc. Roy. Soc. (London) A '89, '95; T. Sleator and H. Weinfurter, PRL '95; A. Barenco et al PRA '95; A. Barenco Proc. R. Soc. Lond. A '95; D. P. DiVincenzo PRA '95

What about general, non-unitary quantum channels?

Definitions

A channel $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is a *completely positive* and *trace preserving* linear map $\mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$.

Call $\mathcal{C}_d$ the set of channels acting on a d-dimensional quantum system.

A set of quantum channels $\mathcal{F}_d \subset \mathcal{C}_d$ is said to be *universal* for a set of channels $\tilde{\mathcal{C}}_d \subset \mathcal{C}_d$ if for all channels $\Phi \in \tilde{\mathcal{C}}_d$ there exist channels $\Phi_1, \dots, \Phi_n \in \mathcal{F}_d$ such that

$$\Phi = \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_2 \circ \Phi_1.$$

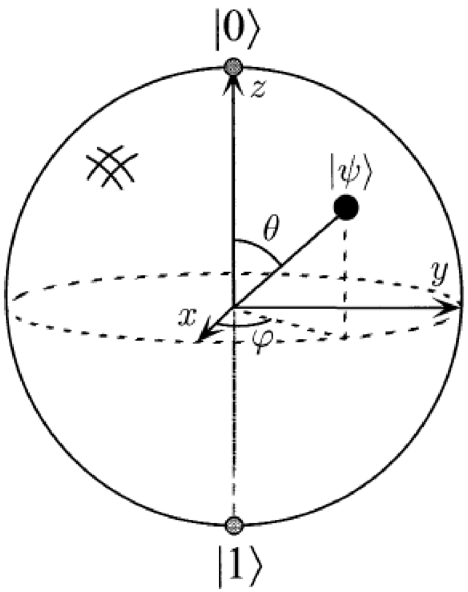
Qubit channels

- Bloch ball representation of density matrix of a single qubit:

Pauli matrices

$$\rho = \frac{1}{2}\sum_{i=0}^{3} r_i \sigma_i$$

r_0=1, $\boldsymbol{r}$=(r_1,r_2,r_3) Bloch vector



Nielsen & Chuang

Most general linear map

$$(1, r_1', r_2', r_3')^t = T_\Phi\,(1, r_1, r_2, r_3)^t$$

$$T_\Phi = \begin{pmatrix} 1 & 0_{1\times 3} \\ \mathbf{t}_\Phi & M_\Phi \end{pmatrix}$$

King et al. '01,
Ruskai et al. '02,
Bengtsson et al.'06

$$M_\Phi = M_{\Phi_2} M_{\Phi_1}$$

Composition rule

$$\mathbf{t}_\Phi = M_{\Phi_2}\mathbf{t}_{\Phi_1} + \mathbf{t}_{\Phi_2}$$

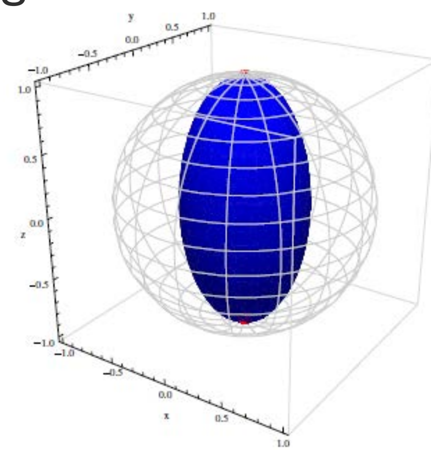
Canonical Form

Concatenate $\mathcal{U} \circ \mathcal{F} \circ \mathcal{V}$ Singular value decomposition of M_Φ up to global sign Up to unitary conjugation:

$$T_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}$$

λ_i are "signed singular values"
 Bloch ball ellipsoid

$\mathbf{t}_\Phi = (t_1, t_2, t_3)$ coordinates (in the Pauli basis) of $\Phi(I_2/2)$
 shift of ellipsoid



Unital channels have $\mathbf{t}_\Phi = 0$

Fujiwara-Algoet conditions for complete positivity of **unital** channel:

$$\begin{cases} 1 + \lambda_3 \geq |\lambda_1 + \lambda_2| \\ 1 - \lambda_3 \geq |\lambda_1 - \lambda_2|. \end{cases}$$

A. Fujiwara & P. Algoet PRA 1999

Equality 4 linear equations faces of a tetrahedron

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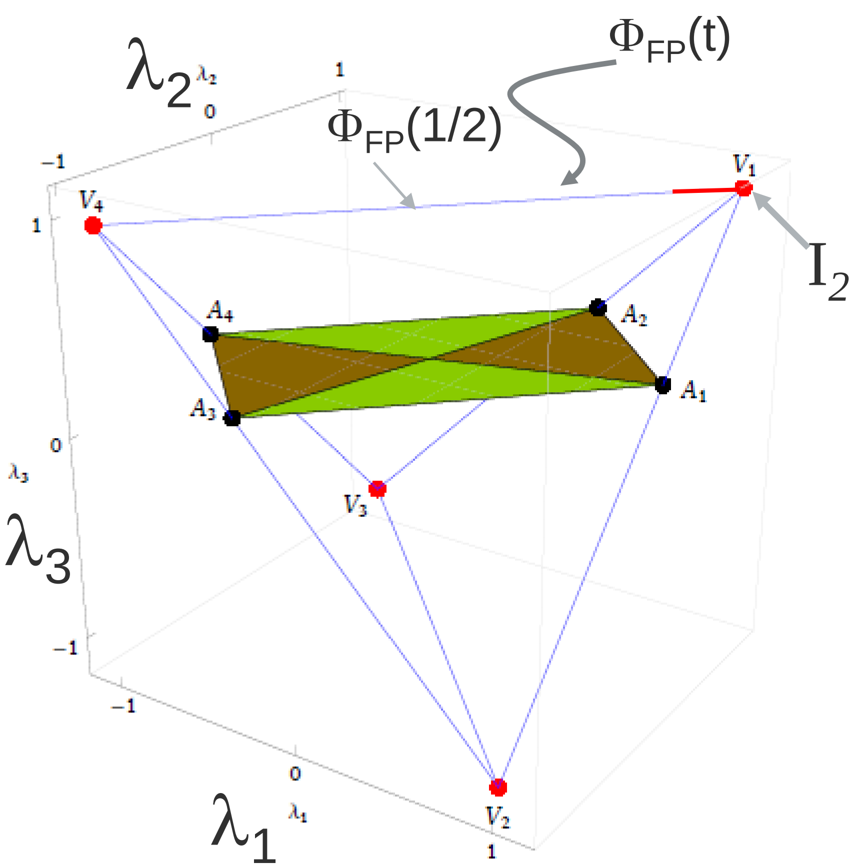
Unital Channels: t=0

$$\rho' = \sum_{k=1}^{r} A_k \rho A_k^\dagger$$



- Vertices: rank r=1 channels (unitaries)

All unitaries needed in universal channel set.

- Edges: rank r=2 channels
  - Top edge: phase flip channels
    $\Phi_{\mathrm{PF}}(t) : \rho \mapsto (1-t)\rho + t\sigma_z \rho \sigma_z$
  - All other edges by concatenation with unitaries

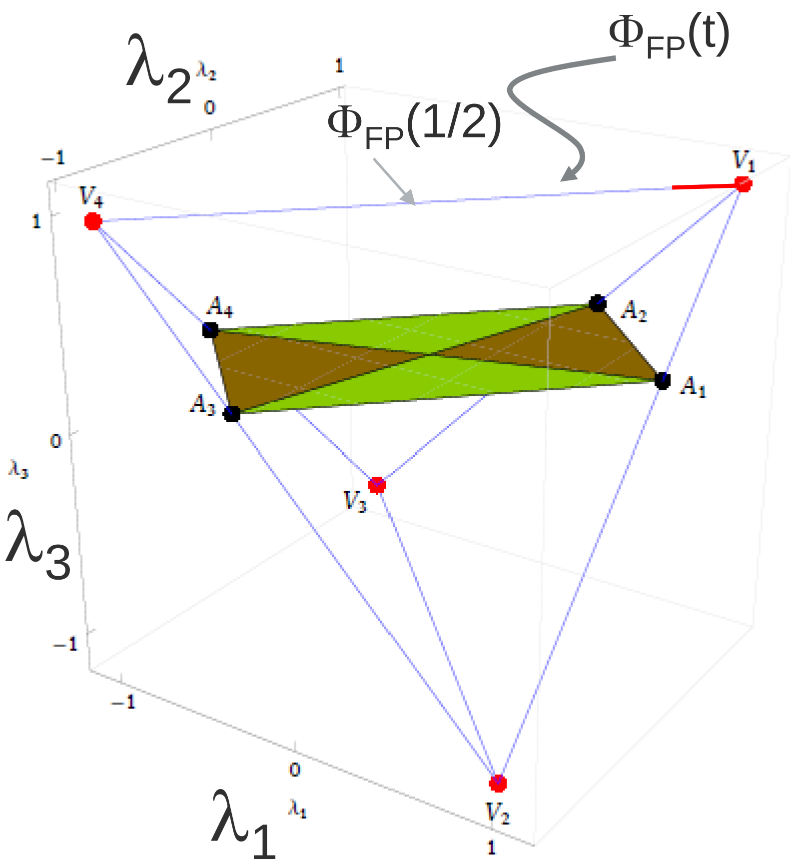
An ε-environment of channels close to V_1 is enough to create all phase flip channels by a finite number of concatenations and conjugation with σ_z, with exception of $\Phi_{FP}(1/2)$.

- Faces: rank r=3 channels
  - Indecomposable

M. Wolf and I. Cirac '08

All channels on faces must therefore be part of the universal channel set.

Unital Channels: t=0



- Inside of tetrahedron: rank 4 channels

Can be obtained as concatenation of phase flip channels (top edge) and rank 3 channels (two faces).

Proof: Concatenations of $\Phi_{FP}(t)$, $0 \le t \le 1$ and channels from A_1A_2 fill brown bow tie. Concatenations of $\Phi_{FP}(t)$, $0 \le t \le 1$ and channels from A_2A_4 fill green bow tie. Varying λ_3 that defines edges A_1A_2 and A_2A_4 therefore fills whole tetrahedron.

Universal channel set for unital channels

Let $\mathcal{I}_2 = \{\Phi \in \mathcal{C}_2 \,:\, \Phi \text{ has Kraus rank } 3\}$ be the set of indivisible qubit channels, $\mathcal{F}_{\mathrm{PF}}(T) = \{\Phi_{\mathrm{PF}}(t)\,,\, t \in [0, T]\}$ a set of phase flip channels, and $\mathcal{U}_2$ the set of unitary qubit channels.

Theorem 1. *For any $\varepsilon > 0$, the set*

$$\mathcal{G}^{\varepsilon} = \mathcal{I}_2 \cup \mathcal{F}_{\mathrm{PF}}(\varepsilon) \cup \{\Phi_{\mathrm{PF}}(1/2)\} \cup \mathcal{U}_2$$

is a universal set *of* unital *qubit channels.*

It is minimal in the sense that all elements of $\mathcal{I}_2$ and $\{\Phi_{\mathrm{PF}}(1/2)\}$ are needed, as well as a channel $\Phi_{\mathrm{PF}}(\varepsilon')$ where $\varepsilon' \leq \varepsilon$.

Non-unital qubit channels

Theorem 2 (Generalized Fujiwara-Algoet conditions). *Let $\Phi : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ be a non-unital linear map in canonical form.*

Let $t = \|\mathbf{t}\|$ and $\mathbf{u} = \mathbf{t}/t$ the corresponding unit vector. Then the map Φ is a quantum channel if and only if $q_i \geq 0, \quad i = 0, 1, 2, 3$ and $t^2 \leq r - \sqrt{r^2 - q}$.

$$q_0 = (1 + \lambda_1 + \lambda_2 + \lambda_3)/4$$
$$q_1 = (1 + \lambda_1 - \lambda_2 - \lambda_3)/4$$
$$q_2 = (1 - \lambda_1 + \lambda_2 - \lambda_3)/4$$
$$q_3 = (1 - \lambda_1 - \lambda_2 + \lambda_3)/4\,.$$

$$r = 1 - \sum_i \lambda_i^2 + 2 \sum_i \lambda_i^2 u_i^2,$$
$$q = 256 \prod_{i=0}^{3} q_i.$$

Remarks:

1. *Positivity of the q_i* is equivalent to the complete positivity of the corresponding *unital* channel (with $\mathbf{t}$=0), i.e. Fujiwara Algoet conditions.
2. Second inequality restricts how far the ellipsoid to which the Bloch ball is mapped can be shifted inside the Bloch sphere. No shift possible for channels on surface of tetrahedron. Maximum shift depends on *signs* of λ_i.

Idea of proof of generalized Fujiwara-Algoet condition

- Positivity of Choi matrix C_Φ is equivalent to complete positivity of channel
- Analyze positivity of Choi matrix using Descartes' rule of signs for characteristic polynomial

$$C_\Phi = \begin{pmatrix} \frac{1}{2}(1+\lambda_3+t_3) & 0 & \frac{1}{2}(t_1+it_2) & \frac{\lambda_1+\lambda_2}{2} \\ 0 & \frac{1}{2}(1-\lambda_3+t_3) & \frac{\lambda_1-\lambda_2}{2} & \frac{1}{2}(t_1+it_2) \\ \frac{1}{2}(t_1-it_2) & \frac{\lambda_1-\lambda_2}{2} & \frac{1}{2}(1-\lambda_3-t_3) & 0 \\ \frac{\lambda_1+\lambda_2}{2} & \frac{1}{2}(t_1-it_2) & 0 & \frac{1}{2}(1+\lambda_3-t_3) \end{pmatrix}$$

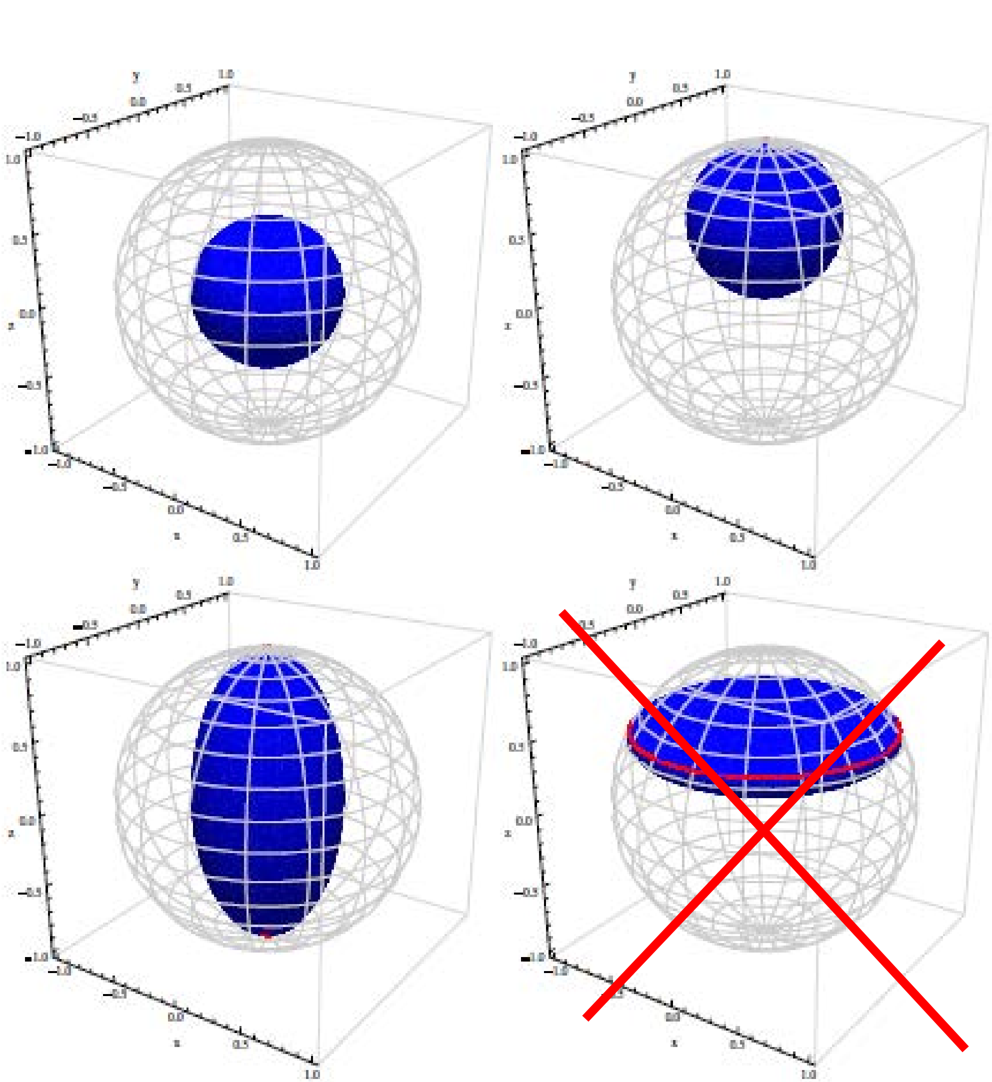
Pure output of qubit channels

Definition .1. *The* pure output *(PO) of a quantum channel* Φ *is the set of pure states in the image of* Φ: $\mathrm{PO}(\Phi) = \Phi(\mathcal{D}_d) \cap \mathcal{P}_d$.

Theorem 3. *Let* $\Phi \in \mathcal{C}_2$ *be a qubit channel. One of the following holds:*

1. $\mathrm{PO}(\Phi) = \emptyset$, *the channel has no pure output, all output states are mixed;*
2. $\mathrm{PO}(\Phi) = \{\xi\}$, $\xi \in \mathcal{P}_2$, *the channel has a unique pure output* ξ;
3. $\mathrm{PO}(\Phi) = \{\xi, \zeta\}$, $\xi, \zeta \in \mathcal{P}_2$, *the channel has exactly two pure outputs* ξ, ζ;
4. $\mathrm{PO}(\Phi) = \mathcal{P}_2$, *all pure states are outputs of* Φ. *In this case,* Φ *is a unitary conjugation* $\Phi(X) = UXU^\dagger$, *for some unitary matrix* U .

Idea of proof: Geometry + generalized Fujiwara Algoet conditions



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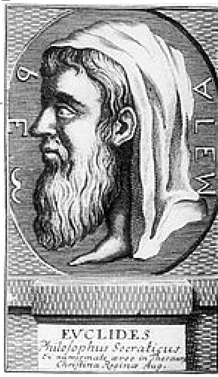
T. Olivier, Géométrie descriptive, Impriméric de l'Université Royale de France (1845)



Uni Jena

Pure output in form of a circle cannot be produced by a qubit channel, as it violates complete positivity

Remarks:

1. This generalizes the "no-pancake" theorem: the image of the Bloch sphere cannot be a disk that touches the sphere. Blume et al. PRA '10
2. The result is implicit in Ruskai et al. '02: a quasi-extreme qubit channel can have at most 2 pure outputs.

M. B. Ruskai, S. Szarek, E. Werner, Lin. Alg. Appl. 347, 159 (2002)

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Extremal non-unital channels

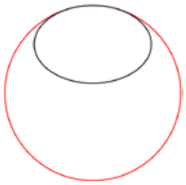
- Extremal channel: not a convex combination of other channels
- Parametrization by Ruskai et al.:

M. B. Ruskai, S. Szarek, E. Werner, Lin. Alg. Appl. 347, 159 (2002)

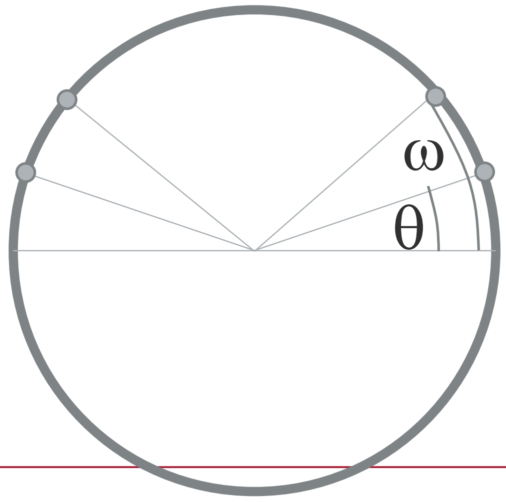
- 1 pure output (PO) extremal channel:

$$\Phi_{1\mathrm{PO}}(\lambda) \text{ is a channel with } \mathbf{t}_\Phi = (0, 0, 1-\lambda^2) \text{ and } \boldsymbol{\lambda}_\Phi = (\lambda, \lambda, \lambda^2).$$

  - Degenerate (det M=0) iff λ=0



- 2 PO extremal channel:
  - Parametrize in terms of latitudes θ,ω of pure inputs (PI) and PO on Bloch sphere
  - Degenerate (det M=0) iff θ=0



Universal q-channel set of extremal qubit channels

Theorem 4. *For any* $\varepsilon > 0$*, the set* $\mathcal{X}(\varepsilon) = \mathcal{U}_2 \cup \mathcal{X}_{1\mathrm{PO}}(\varepsilon) \cup \mathcal{X}_{2\mathrm{PO}}(\varepsilon)$ *is a universal set for* extremal qubit channels.

$$\mathcal{X}_{1\mathrm{PO}}(\varepsilon) = \{\Phi_{1\mathrm{PO}}(\lambda), \lambda \in (1-\varepsilon, 1) \cup \{0\}\}$$

$\Phi_{1\mathrm{PO}}(\lambda)$ is a channel with $\mathbf{t}_\Phi = (0, 0, 1-\lambda^2)$ and $\boldsymbol{\lambda}_\Phi = (\lambda, \lambda, \lambda^2)$.

$$\mathcal{X}_{2\mathrm{PO}}(\varepsilon) = \{\Phi_{2\mathrm{PO}}(\omega, 0), \omega \in (0, \varepsilon)\} \cup \{\Phi_{2\mathrm{PO}}(\omega, \theta), \omega - \theta \in (0, \varepsilon)\},$$

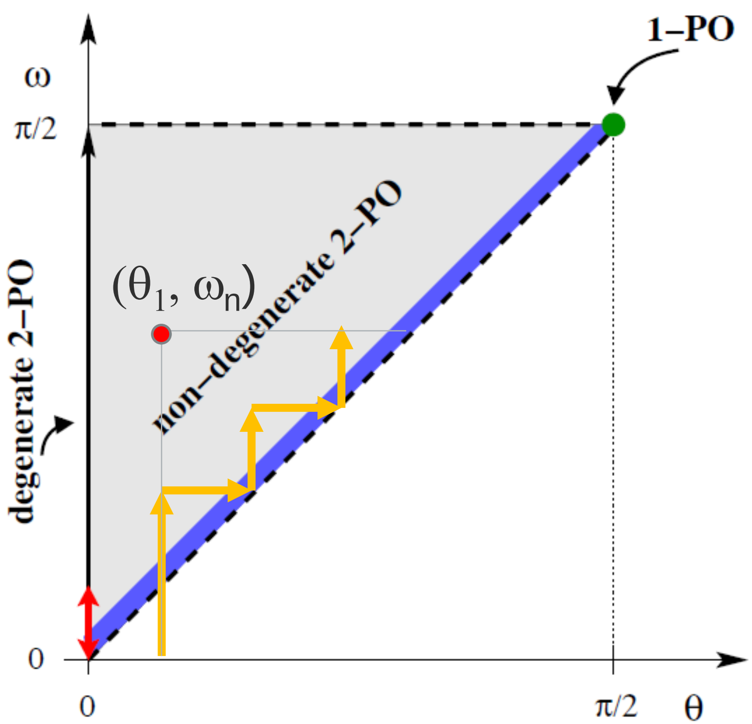
$\Phi_{2\mathrm{PO}}(\omega, \theta)$ is an extremal channel mapping pure input states $(\pm\cos\theta, 0, \sin\theta)$ to pure output states $(\pm\cos\omega, 0, \sin\omega)$.

Remark: these sets of channels are also *necessary* in any universal set of qubit channels.

Idea of proof

- **1 PO degenerate**: a single channel (up to U(2)). Is needed.
- **1 PO non-degenerate**: $\Phi_{1\mathrm{PO}}(\lambda\mu) = \Phi_{1\mathrm{PO}}(\lambda) \circ \Phi_{1\mathrm{PO}}(\mu),\ \lambda, \mu \in (0,1)$

This allows one to show that one gets all non-degenerate 1PO channels by concatenation from small initial ε-environment of λ=1.



2-PO degenerate:

For any $\varepsilon > 0$, the set of all 2PO degenerate extremal channels $\mathcal{X}^{\deg}_{2\mathrm{PO}}$ can be obtained by concatenation of channels from the set $\mathcal{X}^{\deg}_{2\mathrm{PO}}(\varepsilon) = \{\Phi_{2\mathrm{PO}}(\omega, 0), \omega \in (0, \varepsilon)\}$, and channels from the set $\mathcal{X}^{\mathrm{nd}}_{2\mathrm{PO}}$.

2-PO non-degenerate:

$\theta_1 \rightarrow \omega_1 = \theta_2 \rightarrow \omega_2 = \theta_2 \rightarrow \ldots \omega_n$

Allows to create any pair $(\theta_{1,} \omega_n)$

Conclusion part I

- Unital qubit channels:
  - Found minimal universal channel set

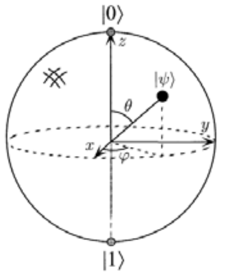
- Non-unital qubit channels:
  - Generalization of Fujiwara-Algoet conditions for complete positivity of qubit channel to non-unital case: geometrical interpretation in terms of maximum shift of image of Bloch sphere
  - Classification of qubit channels in terms of pure output
  - Found minimal universal channel set for extremal qubit channels

WITH
OLIVIER GIRAUD, ION NECHITA,
CLÉMENT PELLEGRINI, MARKO
ZNIDARIC

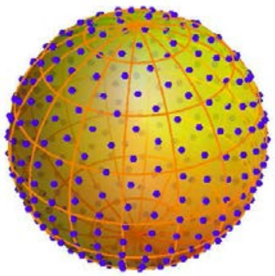
arXiv:1311.7571 - J. Phys. A 47, 135 202 (2014)

Geometrical descriptions of spin-j states

Nielsen & Chuang



- J=1/2:
  - Bloch sphere picture: Bloch vector determines spin state
  - Mixed or pure
- Any J:
  - Majorana representation (roots of overlap with SU(2) coherent state)
  - Pure states only



- Convenient properties:
  - Rigid rotation under SU(2) unitaries
  - For SU(2) coherent state, all Majorana points are anti-podal to coherent state
- Generalization to mixed states of any spin J?

Weinberg matrices $S_{\mu_1\mu_2\ldots\mu_{2j}}$

- Overcomplete set of matrices: Expand (square of) Lorentz boost operator in powers of **q** and identify terms $q_{\mu 1}q_{\mu 2}\ldots q_{\mu N}$

S. Weinberg, PR 1964

$$\Pi^{(j)}(q) \equiv (q_0^2 - |\mathbf{q}|^2)^j \, e^{-2\eta_q \, \hat{\mathbf{q}}\cdot\mathbf{J}}$$

$$\Pi^{(j)}(q) = (-1)^{2j} q_{\mu_1} q_{\mu_2} \ldots q_{\mu_{2j}} S_{\mu_1\mu_2\ldots\mu_{2j}}$$

w/ Einstein summation Convention, μ_i=0,1,2,3 j=N/2

- E.g. spin-1/2 (N=1):

$$\Pi^{(1/2)}(q) = -q_0 - 2\mathbf{q}\cdot\mathbf{J}$$

So $S_0 = \sigma_0$ and $S_a = 2J_a = \sigma_a$

$$\rho = \frac{1}{2} x_{\mu_1} S_{\mu_1}$$

Bloch sphere picture!

- Spin-1 (N=2):

$$\Pi^{(1)}(q) = (q_0^2 - \mathbf{q}^2) + 2\mathbf{q}\cdot\mathbf{J}\,(\mathbf{q}\cdot\mathbf{J} + q_0) = q_{\mu_1} q_{\mu_2} S_{\mu_1\mu_2}.$$

$$S_{00} = J_0,\; S_{a0} = J_a \text{ and } S_{ab} = J_a J_b + J_b J_a - \delta_{ab} J_0$$

$$\rho = \frac{1}{4} x_{\mu_1\mu_2} S_{\mu_1\mu_2}$$

Properties of Weinberg matrices

- 4^N Hermitian matrices (overcomplete set!)
- Traceless in the relativistic sense $\quad g_{\mu_1\mu_2} S_{\mu_1\mu_2\ldots\mu_{2j}} = 0, \quad g \equiv \mathrm{diag}(-,+,+,+)$

Theorem: The Weinberg matrices $S_{\mu1\mu2\ldots\mu N}$ are given by the projection of tensor products of Pauli matrices into the subspace $\mathcal{H}_S$ of states that are invariant under permutation of particles.

$$\langle D_N^{(k)} | S_{\mu_1\mu_2\ldots\mu_N} | D_N^{(\ell)} \rangle = \langle D_N^{(k)} | \boldsymbol{\sigma}_{\mu_1} \otimes \boldsymbol{\sigma}_{\mu_2} \otimes \ldots \otimes \boldsymbol{\sigma}_{\mu_N} | D_N^{(\ell)} \rangle$$

$$|D_N^{(k)}\rangle = \mathcal{N} \sum_{\pi} |\underbrace{0\ldots0}_{N-k}\underbrace{1\ldots1}_{k}\rangle, \quad k = 0,\ldots,N$$

Symmetric Dicke states of *N* two level systems with *k* excitations

Proof: use SU(2) disentangling theorem and SU(2) coherent state representation

Corrollary 1: The Weinberg matrices form a 2^N – **tight frame**.

Corrollary 2:

$$\rho = \frac{1}{2^N} x_{\mu_1\mu_2\ldots\mu_N} S_{\mu_1\mu_2\ldots\mu_N}$$

$$x_{\mu_1\mu_2\ldots\mu_N} = \mathrm{tr}(\rho\, S_{\mu_1\mu_2\ldots\mu_N})$$

Bloch **tensor** picture for a spin-j, j=N/2

Tight frames

A family of vectors $|\phi_i\rangle$, $i \in \{1, \ldots, M\}$, is called a frame for a Hilbert space $\mathcal{H}$ with bounds $A, B \in]0, \infty[$, if

$$A||\psi||^2 \leqslant \sum_{i=1}^{M} |\langle\psi|\phi_i\rangle|^2 \leqslant B||\psi||^2, \quad \forall\ |\psi\rangle \in \mathcal{H}.$$

If $A = B$, then the frame is called an A-tight frame.

Properties of Bloch tensor

- Rotation under SU(2) transformation:

$$x_{\mu_1\ldots\mu_N} \to R_{\mu_1\nu_1}\ldots R_{\mu_N\nu_N} x_{\nu_1\ldots\nu_N}$$

generalizes rotation of Bloch vector: $x_a \to R_{ab}x_b$

- Coordinates of SU(2) coherent state pointing in direction **n**:

$$x_{\mu_1\mu_2\ldots\mu_N} = n_{\mu_1}n_{\mu_2}\ldots n_{\mu_N}$$

- Spin-k reduced density matrix for symmetric state of a multi-qubit system:

$$x_{\mu_1\ldots\mu_{2k}} = x_{\mu_1\ldots\mu_{2k}0\ldots0}$$

- Scalar product

$$\mathrm{tr}(\rho\rho') = \frac{1}{2^N} x_{\mu_1\mu_2\ldots\mu_N} x'_{\mu_1\mu_2\ldots\mu_N},$$

Applications

- A new characterization of anti-coherent spin states:
  A spin state is said to be anti-coherent to order t, if $\langle (\mathbf{n}\cdot\mathbf{J})^k \rangle$ is independent of **n** for all *k*=1,…,*t*. Application e.g. for unpolarized states of light.

- **Theorem**: A spin-j state is anti-coherent to order *t* iff its spin-(*t*/2) reduced density matrix is the maximally mixed state.
- **Corrollary**: A spin-j state is anti-coherent to order t iff in the SU(2) irreducible tensor operator expansion,

$$\rho = \sum_{k=0}^{2j} \sum_{q=-k}^{k} \rho_{kq} T_{kq}^{(j)} \qquad T_{kq}^{(j)} = \sqrt{\frac{2k+1}{2j+1}} \sum_{m,m'=-j}^{j} C_{jm,kq}^{jm'} |j,m'\rangle\langle j,m|,$$

  one has $\rho_{kq} = 0, \ \forall\, k \leqslant t \ (-k \leqslant q \leqslant k)$

- Proof: based on Bloch tensor and expansion in spherical harmonics

Conclusion part II

- Tensor representation of spin-j states with nice geometrical properties
  - Generalizes Bloch vector for spin-1/2
  - Based on Weinberg's covariant matrices
- Proofed that these matrices form a tight frame, and found a simple way to calculate them
- First applications to anti-coherent spin states

With
Olivier Giraud, John Martin,
Thierry Bastin, Dorian
Baguette

arXiv:1409.1106

Pure output theorem

- Idea of proof:

1. Purely geometrical considerations of an ellipsoid touching a sphere from the inside lead to the above four cases + the possibility of the ellipsoid touching the sphere in a circle.
2. Using the generalized Fujiwara-Algoet conditions, one shows that the pure output in form of a circle does not correspond to a completely positive map.

- Remarks:

1. This generalizes the “no-pancake” theorem: the image of the Bloch sphere cannot be a disk that touches the sphere. Blume et al. PRA ‘10
2. The result is implicit in Ruskai et al. ’02: a quasi-extreme qubit channel can have at most 2 pure outputs.

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