The extent of multipartite (many-body) entanglement II

A pure state $|\psi\rangle$ is k-producible if one can write:[2]

$$|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \cdots \otimes |\varphi_m\rangle$$

where the $|\varphi_i\rangle$ are states at most k-partite.

A mixed state is k-producible if it is a convex mixture of k-producible pure states.

Separable (non-entangled) state $\Leftrightarrow$ 1-producible state.

Entanglement depth: minimum k for which ρ is k-producible.

Genuine k-partite entangled state: quantum state with entanglement depth k.

[2] Gühne, Tóth, and Briegel, New J Phys. **7**, 229 (2005).

Why entanglement depth? Why (non)-k-producibility?

Essential in achieving extreme spin squeezing and high sensitivity in some metrology tasks.[3]

The minimal extent of multipartite (many-body) entanglement needed for quantum information processing tasks.

Need more refined tools to answer this!

Separability test: separable or entangled?

Biseparability test: biseparable or fully-entangled?

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1-producible (separable or not)

(separable)

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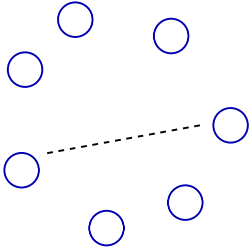

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7-body entangled
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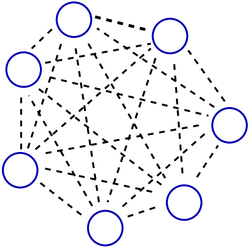

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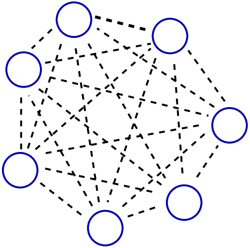

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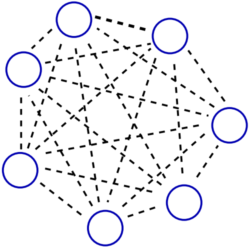

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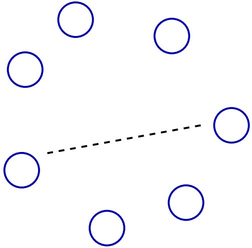
Separability test: separable or entangled?

Biseparability test: biseparable or fully-entangled?

2-body entangled

(biseparable)



(separable or not)

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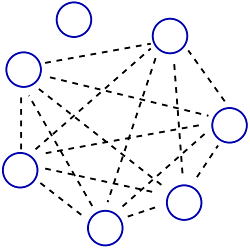
6-body entangled

(biseparable)



(separable or not)

(biseparable or not)

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Correlation in a Nonlocal Game (Bell Experiment)

Consider a nonlocal game btw a Referee and some players

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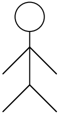
Consider a nonlocal game btw a Referee and some spatially-separated players.

(1) Goal: $a_1 \oplus a_2 \oplus a_3 = x_1 x_2 x_3$; (2) Rule: No communication after game begins

Referee



$P(\vec{a}|\vec{x}) = P(a_1 a_2 a_3 | x_1 x_2 x_3)$

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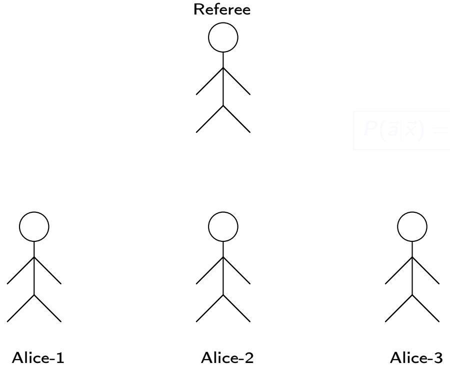
strategy $\Rightarrow$ (linear) constraints on $P(\vec{a}|\vec{x})$

strategy



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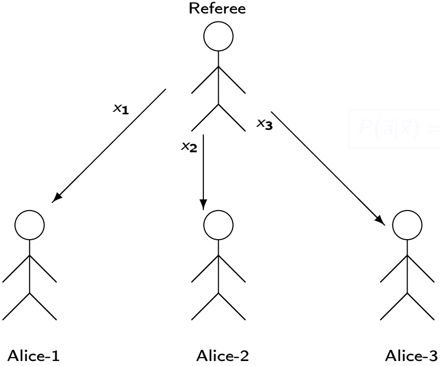

$x_k \in \{0, 1\}$

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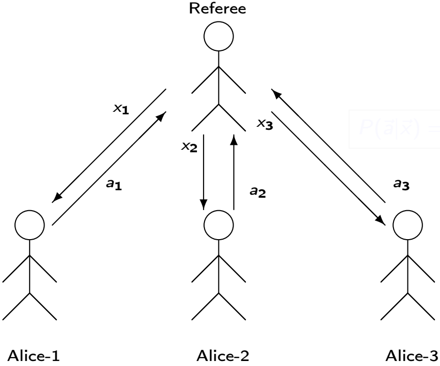

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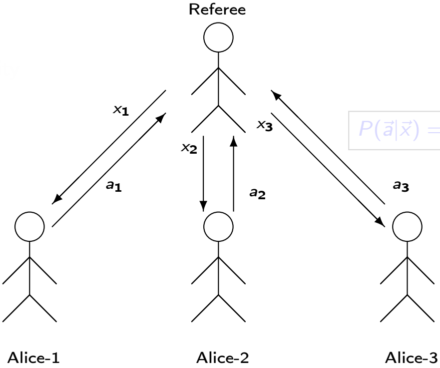
Quantum strategy



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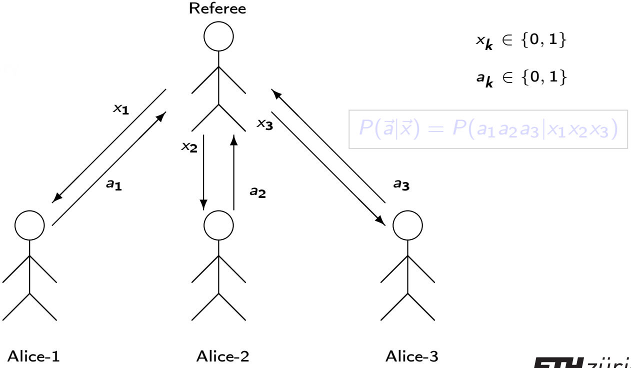

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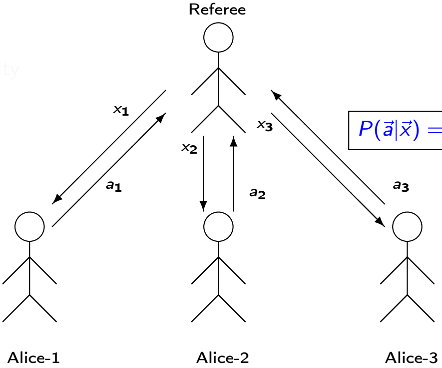
Quantum strategy

$\Downarrow$

violation of Bell inequality

$\Downarrow$

Nonlocal



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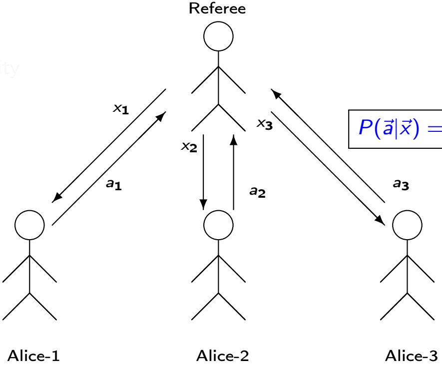

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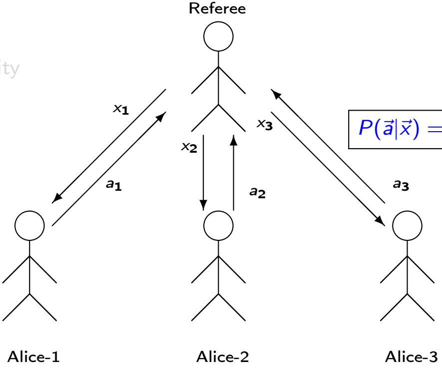

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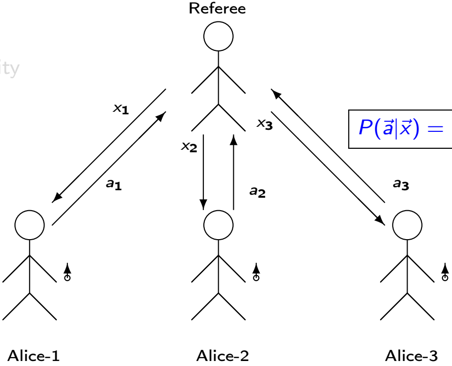

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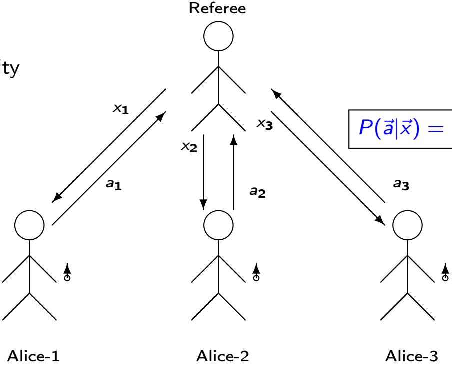

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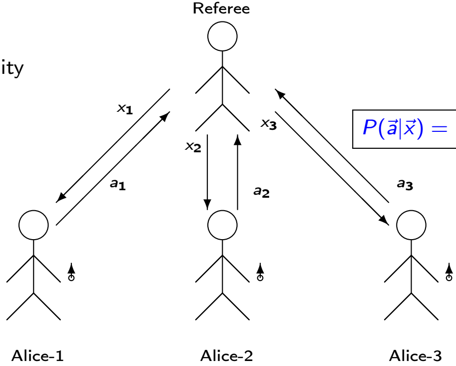

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k-producibility vs nonlocal resource I

Definition: k-producibility of correlations (arXiv:1411.4656)

W.r.t. a given resource $\mathcal{R}$, $\vec{P} = \{P(\vec{a}|\vec{x})\}$ is k-producible using $\mathcal{R}$ if

1. $\vec{P}$ can be decomposed into a convex mixture of products of at most k-partite correlations, and
2. each constituent correlation satisfies the constraints of $\mathcal{R}$.

$\vec{P}$ is 2-producible using a quantum resource if:

$$\begin{aligned} P(\vec{a}|\vec{x}) = &\sum_{\ell_1} p_{\ell_1}^{12|3} P_{\ell_1}^{\mathcal{Q}}(a_1 a_2|x_1 x_2) P_{\ell_1}(a_3|x_3) \\ &+ \sum_{\ell_2} p_{\ell_2}^{13|2} P_{\ell_2}^{\mathcal{Q}}(a_1 a_3|x_1 x_3) P_{\ell_2}(a_2|x_2) \\ &+ \sum_{\ell_3} p_{\ell_3}^{23|1} P_{\ell_3}^{\mathcal{Q}}(a_2 a_3|x_2 x_3) P_{\ell_3}(a_1|x_1). \end{aligned}$$

Quantum resource: $P_{\ell_i}^{\mathcal{Q}}(a_j a_k|x_j x_k) = \mathrm{tr}(\rho_{\ell_i} M_{a_j}^{x_j} \otimes M_{a_k}^{x_k})$

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k-producibility vs nonlocal resource II

For a nonlocal resource $\mathcal{R}$:

$$P(\vec{a}|\vec{x}) \stackrel{?}{=} \sum_{\ell_1} p_{\ell_1}^{12|3} P_{\ell_1}^{\mathcal{Q}}(a_1 a_2|x_1 x_2) P_{\ell_1}(a_3|x_3) + \sum_{\ell_2} p_{\ell_2}^{13|2} P_{\ell_2}^{\mathcal{Q}}(a_1 a_3|x_1 x_3) P_{\ell_2}(a_2|x_2) + \sum_{\ell_3} p_{\ell_3}^{23|1} P_{\ell_3}^{\mathcal{Q}}(a_2 a_3|x_2 x_3) P_{\ell_3}(a_1|x_1).$$

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Non-signaling resource $\mathcal{NS}$:
$\sum_{a_j} P_{\ell_i}^{\mathcal{NS}}(a_j a_k|x_j x_k) = P_{\ell_i}^{\mathcal{NS}}(a_k|x_j x_k) = P_{\ell_i}(a_k|x_k)$,
$\sum_{a_k} P_{\ell_i}^{\mathcal{NS}}(a_j a_k|x_j x_k) = P_{\ell_i}^{\mathcal{NS}}(a_j|x_j x_k) = P_{\ell_i}(a_j|x_j)$.

[Players cheat] Time-ordered 1-way signaling resource $\mathcal{T}$.

[Players cheat] Svetlichny resource $\mathcal{S}$: arbitrary signaling (communication).

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Quantum resource $\mathcal{Q}$: $P_{\ell_i}^{\mathcal{Q}}(a_j a_k|x_j x_k) = \mathrm{tr}(\rho_{\ell_i} M_{a_j}^{x_j} \otimes M_{a_k}^{x_k})$.

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[Players cheat] Time-ordered 1-way signaling resource $\mathcal{T}$.

[Players cheat] Svetlichny resource $\mathcal{S}$: arbitrary signaling (communication).

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Goal: determine the smallest k for which $\vec{P}$ is k-producible — "Minimal Group Size" (MGS) — the extent to which nonlocal resource $\mathcal{R}$ needs to be shared among players to produce $\vec{P}$.

If $\vec{P}$ has MGS $= k$,

- w.r.t $\mathcal{R} = \mathcal{Q}$, generation of $\vec{P}$ quantum-mechanically requires at least genuinely k-partite entanglement.
  $\because$ $(k-1)$-producible $\Longrightarrow$ $(k-1)$-producible correlation.
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GHZ state $|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$ and the local measurement of σ_x and σ_y.

For odd $n \geq 3$, the resulting correlation $P^n_{\mathrm{GHZ}}(\vec{a}|\vec{x})$[4]

1. is not $(n-1)$-producible by *any* quantum resource;
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If only quantum resource is allowed, nonlocal collaboration of all n parties (genuine n-partite entangled state) is needed.

If signaling resource is available, nonlocal collaboration of only $\lceil \frac{n}{2} \rceil$ parties is needed.

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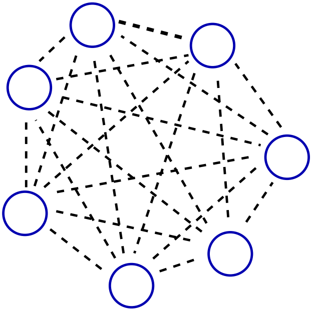

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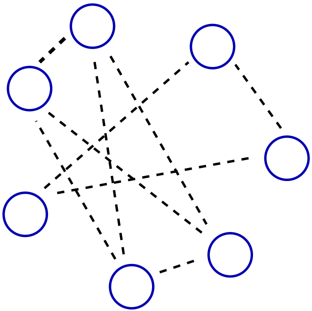
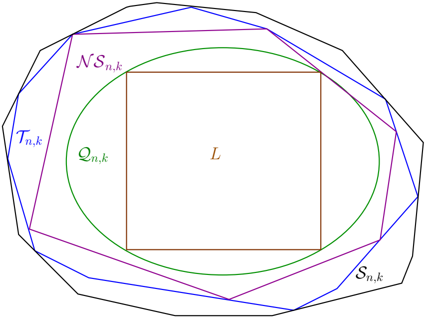


Characterization of k-producible correlations

For general non-signaling resource, or classical but signaling resources, $\mathcal{R}_{n,k}$ is a convex polytope:[6] linear programming.

Assuming quantum resource: solving a hierarchy of semidefinite programs (SDP)s.[7]



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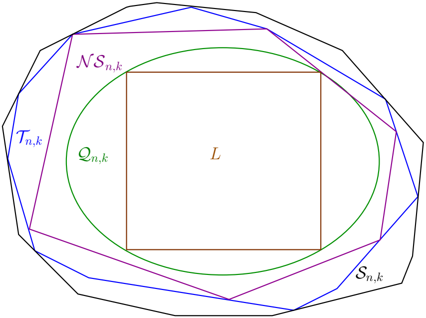

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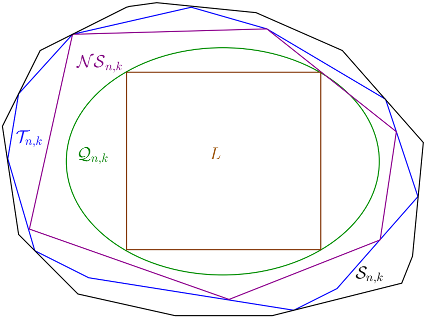

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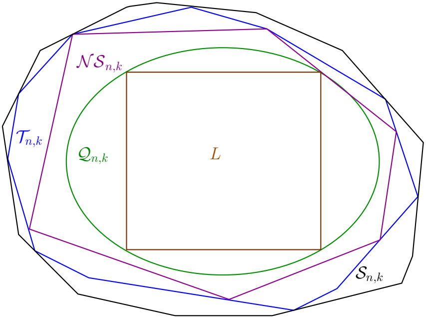

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Convexity: Bell-like inequalities to witness non-k-producibility:

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Witnessing entanglement depth using Bell-like inequalities I

If $\vec{P}'$ has MGS $= k$ w.r.t. $\mathcal{R} = \mathcal{Q} \Rightarrow \vec{P}'$ originates from genuinely k-partite entangled state.

$\Leftrightarrow \exists\ \beta_{\vec{a}}^{\vec{x}}$ and $B_{n,k-1}^{\mathcal{Q}}$ s.t.

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Conclusion drawn from observed correlation — no assumption about Hilbert space dimension nor local measurements performed

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Witnessing multipartite entanglement using Bell-like inequalities II

The n-partite, 2-setting, 2-outcome Bell inequalities of Mermin-Ardehali-Belinskiĭ-Klyshko (MABK)

$$\mathcal{M}_n = 2^{\frac{1-n}{2}} \left| \sum_{\vec{x}\in\{0,1\}^n} \cos\left\{\frac{\pi}{4}\left[1+(n-2\mathrm{x})\right]\right\} E_n(\vec{x}) \right| \overset{\text{LHV}}{\leq} 1,$$

full correlator $E_n(\vec{x}) = \sum_{\vec{a}}(-1)^{\sum_i a_i} P(\vec{a}|\vec{x})$.

From Nagata *et al.*[8] & Yu *et al.*:[9] maximal quantum violation by k-producible quantum state is:

$$\max_{k-prod} \mathcal{M}_n = \begin{cases} 2^{\frac{k-1}{2}} : n \leq 5, & 2 \leq k \leq n-1 \\ 2^{\frac{n-3}{2}} : n > 5, & \lceil \frac{n}{2} \rceil \leq k < n-1 \end{cases}$$

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A family of Bell inequalities

A family of n-partite, 2-setting, 2-outcome Bell inequalities:[10]

$$\mathcal{I}_n : \mathcal{S}_n = 2^{1-n} \sum_{\vec{x}\in\{0,1\}^n} E_n(\vec{x}) - E_n(\vec{1}_n) \overset{\text{LHV}}{\leq} 1$$

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A facet-defining inequality of the local full-correlation polytope.

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$\mathcal{I}_n$ can be violated by entangled quantum state.

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k	2	3	4	5	6	7	8	∞
$\mathcal{S}_k^{\mathcal{Q},*}$	$\sqrt{2}$	$\frac{5}{3}$	1.8428	1.9746	2.0777	2.1610	2.2299	3

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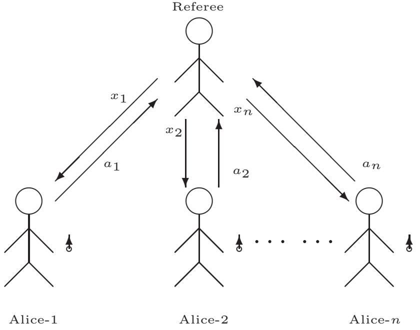

$x_k \in \{0, 1\}$

$a_k \in \{0, 1\}$

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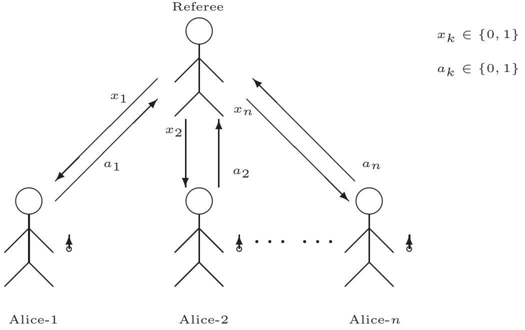

Conclusion

Summary

- Notion of k-producibility for correlations — resource dependent.
- Examples where the difference in k-producibility between a quantum resource and a signaling resource gets arbitrarily large with number of parties.
- Techniques to determine k-producibility.
- A family of multipartite Bell inequalities which (together with its quantum violation) serve as a natural family of device-independent witnesses for entanglement depth.
- A recipe to construct witnesses for non-k-producibility for non-signaling resources based on lifting of Bell-like inequalities.

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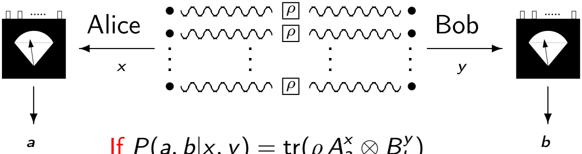
Characterizing quantum resource in a black-box scenario

With local measurements, entanglement is necessary to produce Bell inequality violation.



If $P(a,b|x,y) = \mathrm{tr}(\rho\, A_a^x \otimes B_b^y)$

$$\neq \int q_\lambda\, P(a|x,\lambda)P(b|y,\lambda)\,\mathrm{d}\lambda \Rightarrow \rho \text{ is entangled!}$$

Conclusion drawn directly from measurement statistics, independent of dimension of ρ nor any assumption/ knowledge of the device implementing A_a^x, B_b^y!

Examples

$\lvert\psi\rangle$	Witness(es)	**2**	**3**	**4**	**5**	**6**	**7**
$\lvert\mathrm{GHZ}_n\rangle$	$\mathcal{I}_n^k$, $\mathcal{M}_n^k$, QFI	2^*	3^*	4^*	5^*	6^*	7^*
$\lvert\mathrm{W}_n\rangle$	$\mathcal{M}_n^k$, QFI	2^*	3^*	3	3	3	3
$\lvert\mathrm{W}_n\rangle$	$\mathcal{I}_n^k$	2^*	3^*	2	2	2	2
$\lvert C_n^-\rangle$	QFI	2^*	3^*	2	3	2	3
$\lvert C_n^-\rangle$	$\mathcal{I}_n^k$, $\mathcal{M}_n^k$	2^*	3^*	2	2	2	2
$\lvert C_n^o\rangle$	$\mathcal{I}_n^k$	2^*	3^*	2	2	2	2
$\lvert C_n^o\rangle$	$\mathcal{M}_n^k$	2^*	3^*	2	1	1	1
$\lvert C_n^o\rangle$	QFI	2^*	3^*	1	1	1	1

Table : Lower bound on entanglement depth based on violation of device-independent witness for entanglement depth [cf. lower bound from Quantum Fisher Information (QFI) is due to Florian Froewis].

Lifting:[11] A recipe to construct non-k-producible witnesses

Theorem

If I_n is a a Bell-like inequality satisfied by all correlations in $\mathcal{R}_{n,k}$ and $\mathcal{R}$ is a non-signaling resource, then

$$I_n = \sum_{\vec{x},\vec{a}} \beta_{\vec{x}}^{\vec{a}} P(\vec{a}|\vec{x}) \overset{\mathcal{R}_{n,k}}{\leq} 0 \Rightarrow I_{n+h} = \sum_{\vec{x},\vec{a}} \beta_{\vec{x}}^{\vec{a}} P(\vec{a},\vec{o}|\vec{x},\vec{s}) \overset{\mathcal{R}_{n+h,k}}{\leq} 0,$$

where $\vec{o}$ and $\vec{s}$ refer, respectively, to arbitrary but fixed combination of measurement outcomes and measurement settings for the h additional parties.

[11] Pironio, J. Math. Phys. (2005).

Lifting: An example

Consider $\rho = v|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3|\otimes|0\rangle\langle 0| + (1-v)\frac{\mathbb{1}}{2^3}\otimes|1\rangle\langle 1|$.

ρ is 3-producible. Is ρ 2-producible for some $v > 0$?

For $v \leq \frac{1}{5}$, all tripartite reduced density matrices are separable.

With appropriate measurements, $|\mathrm{GHZ}_3\rangle$ violates

$$I_{S,3} = \sum_{\vec{x},\vec{a}} (-1)^{\sum_i a_i + \left\lfloor \frac{\sum_i x_i - 1}{2} \right\rfloor} P(a_1 a_2 a_3|x_1 x_2 x_3) - 4 \overset{\mathcal{NS}_{3,2}}{\leq} 0,$$

Lifting $I_{S,3}$ to a fourth-party

$$\sum_{\vec{x},\vec{a}} \beta_{\vec{x}}^{\vec{a}} P(a_1 a_2 a_3, a_4 = 0|x_1 x_2 x_3, x_4 = 0) - 4P(a_4 = 0|x_4 = 0) \overset{\mathcal{NS}_{4,2}}{\leq} 0$$

4th party measures σ_z, conditions on getting outcome $+$ $(a_4 = 0)$, ρ violates lifted inequality for all $0 < v \leq 1$ $\Rightarrow \rho$ is genuinely 3-partite entangled for all $v > 0$!

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