I. Quantum Waveform Detection Theory
II. Quantum Waveform Estimation Theory
III. Quantum Microwave Photonics *

Mankei Tsang
Department of Electrical and Computer Engineering
Department of Physics
National University of Singapore

eletmk@nus.edu.sg
http://mankei.tsang.googlepages.com/

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- Wave:

- Probability (Born's rule $P(x) = |\langle x|\psi\rangle|^2$):

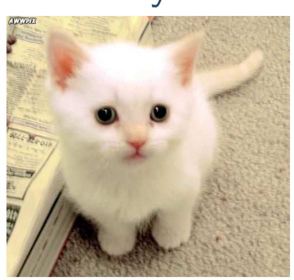
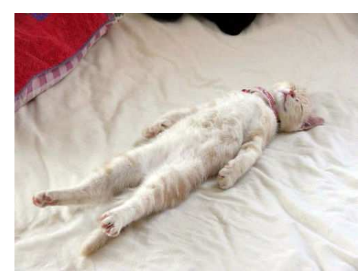
I.2 ON THE QUANTUM MECHANICS OF COLLISIONS

[Preliminary communication]†

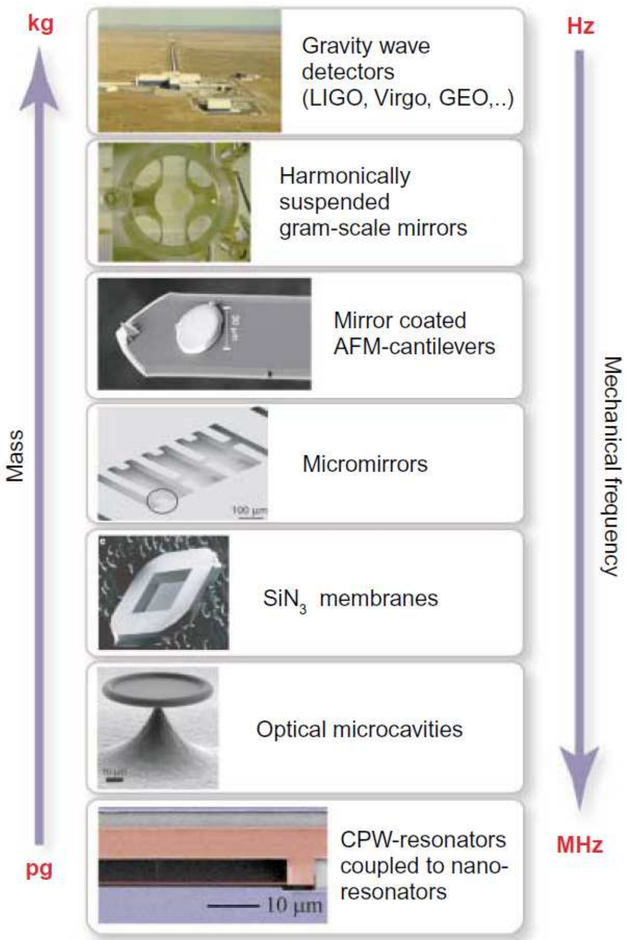
MAX BORN

*** Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n_\tau m}$.**

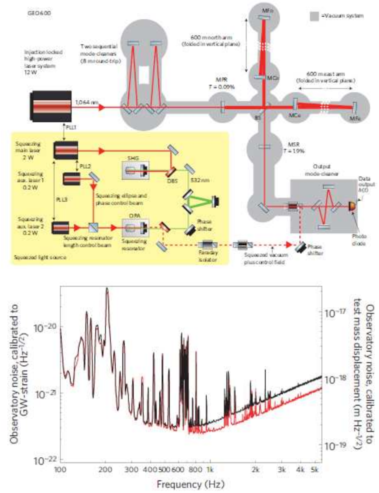
- More than classical probability:

$\Bigg|$  $\Bigg\rangle + e^{i\theta} \Bigg|$  $\Bigg\rangle$

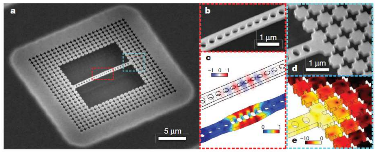
Quantum Probability Experiments



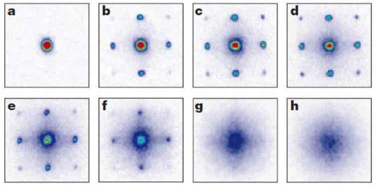
Kippenberg and Vahala, Science **321**, 1172 (2008), and ref. therein.



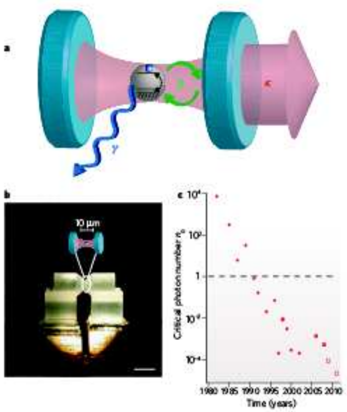
LIGO, Nature Phys. **7**, 962 (2011).



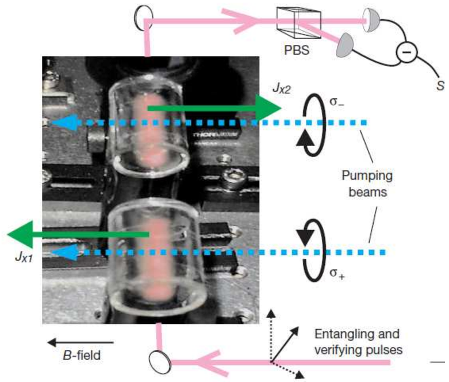
Chan *et al.*, Nature **478**, 89 (2011).



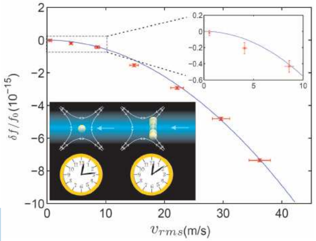
Greiner *et al.*, Nature **415**, 39 (2002).



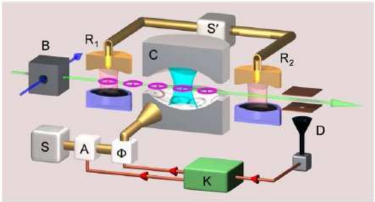
Kimble, Nature **453**, 1023 (2008).



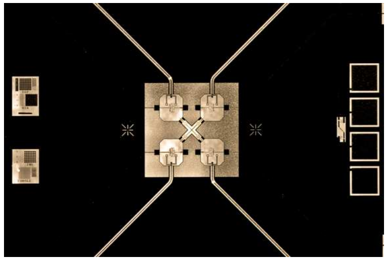
Julsgaard *et al.*, Nature **413**, 400 (2001).



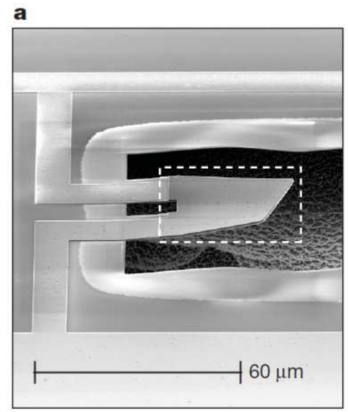
Chou *et al.*, Science **329**, 1630 (2010).



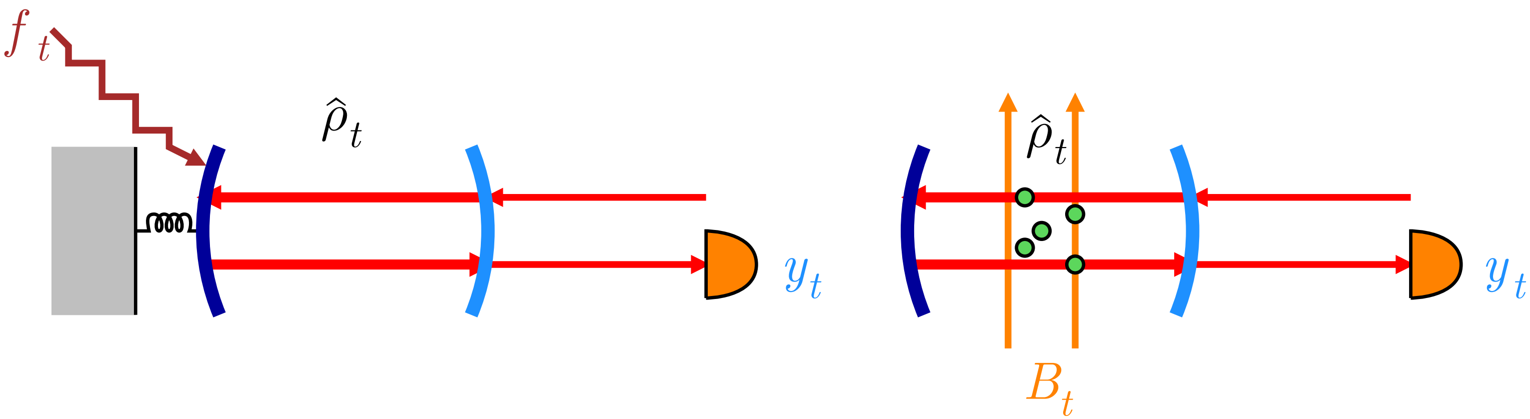
Sayrin *et al.*, Nature **477**, 73 (2011).

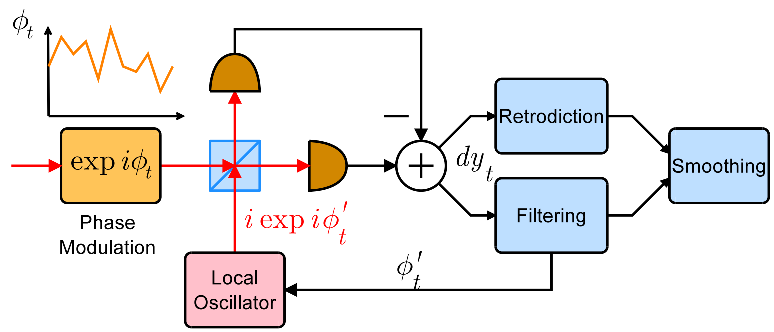


Neeley et al., Nature **467**, 570 (2010)



O'Connell *et al.*, Nature **464**, 697 (2010).

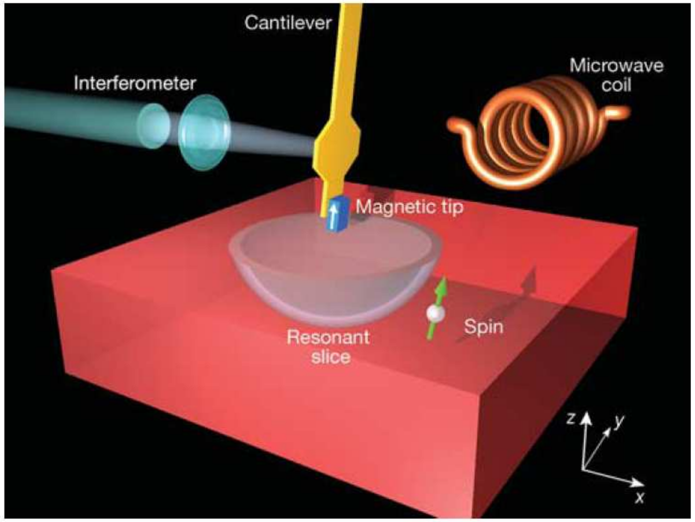




- Fundamental Limits: What is the ultimate sensitivity allowed by quantum mechanics?
- Control: Optimize experiment
- Estimation: Optimize data processing
- Examples: optical interferometry, optical imaging, optomechanical force sensing (gravitational-wave detection), atomic magnetometry, gyroscopes, electrometer, etc.

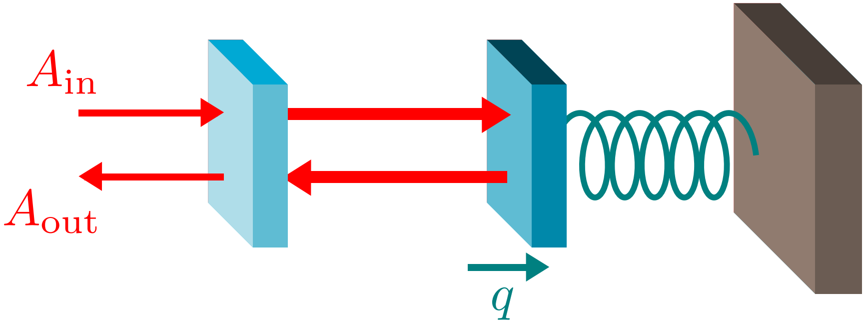
Quantum Optomechanical Force Detection

LIGO, Hanford

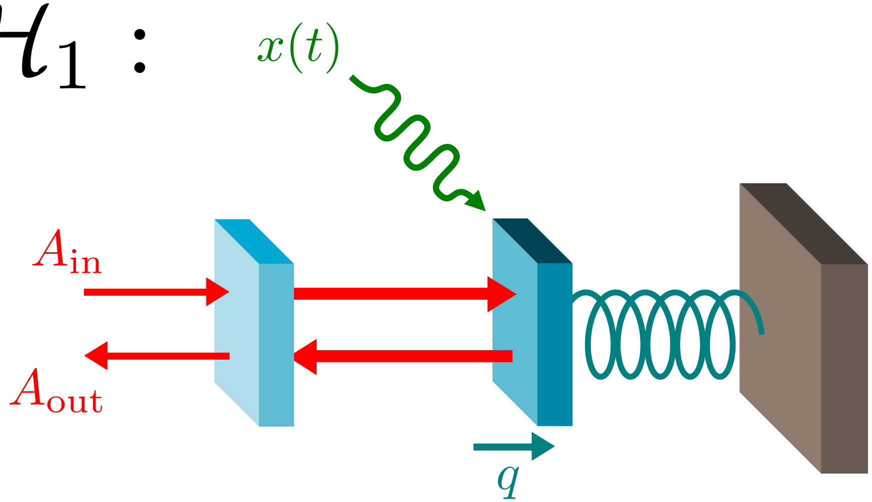


Rugar *et al.*, Nature **430**, 329 (2004).

$\mathcal{H}_0$:



$\mathcal{H}_1$:



- $Y \in \Upsilon$ is an observation.
- Y is noisy: $\Pr(Y|\mathcal{H}_0)$ and $\Pr(Y|\mathcal{H}_1)$
- Given Y, $\Pr(Y|\mathcal{H}_0)$, and $\Pr(Y|\mathcal{H}_1)$, we want to decide which hypothesis is true.
- Decision rule: divide Υ into two regions Υ_0 and Υ_1:
  - If $Y \in \Upsilon_0$, we decide $\mathcal{H}_0$ is true.
  - If $Y \in \Upsilon_1$ we decide $\mathcal{H}_1$ is true.
- Type-I error probability (false-alarm probability):

$$P_{10}(\Upsilon_0, \Upsilon_1) = \sum_{Y \in \Upsilon_1} \Pr(Y|\mathcal{H}_0) \qquad (1)$$

- Type-II error probability (miss probability):

$$P_{01}(\Upsilon_0, \Upsilon_1) = \sum_{Y \in \Upsilon_0} \Pr(Y|\mathcal{H}_1) \qquad (2)$$

- How to choose Υ_0 and Υ_1 in order to minimize errors?

Likelihood-Ratio Test

- Define likelihood ratio:

$$\Lambda \equiv \frac{\Pr(Y|\mathcal{H}_1)}{\Pr(Y|\mathcal{H}_0)} \tag{3}$$

- Likelihood-ratio test given a threshold γ:
  - If $\Lambda \geq \gamma$ decide $\mathcal{H}_1$ is true.
  - If $\Lambda < \gamma$ decide $\mathcal{H}_0$ is true.
- Neyman-Pearson criterion:
  - Constrain $P_{10} \leq \alpha$ and minimize P_{01}
  - set γ such that $P_{10} = \Pr(\Lambda \geq \gamma|\mathcal{H}_0) = \alpha$
- Bayes criterion (given prior probabilities P_0 and P_1):
  - Define the cost of deciding on $\mathcal{H}_j$ given $\mathcal{H}_k$ as C_{jk} (loss function).
  - minimize average cost (Bayes risk):

$$C = \sum_{j,k} P_{jk} P_k C_{jk}. \tag{4}$$

  - e.g., $P_e = P_0 P_{10} + P_1 P_{01}$.
  - set $\gamma = (C_{10} - C_{00}) P_0 / (C_{01} - C_{11}) P_1$.

- H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part I.* (Wiley, New York, 2001).

Error Probabilities

- For a likelihood-ratio test,
  - Error probabilities:

$$P_{10} = \Pr(\Lambda \geq \gamma | \mathcal{H}_0), \qquad P_{01} = \Pr(\Lambda < \gamma | \mathcal{H}_1). \tag{5}$$

  Very hard to calculate, but can be bounded using Chernoff bounds:

$$P_{10} \leq \inf_{0 \leq s \leq 1} \mathbb{E}\left[\Lambda^s | \mathcal{H}_0\right] \gamma^{-s}, \qquad P_{01} \leq \inf_{0 \leq s \leq 1} \mathbb{E}\left[\Lambda^s | \mathcal{H}_0\right] \gamma^{1-s}. \tag{6}$$

  - Lower bounds:

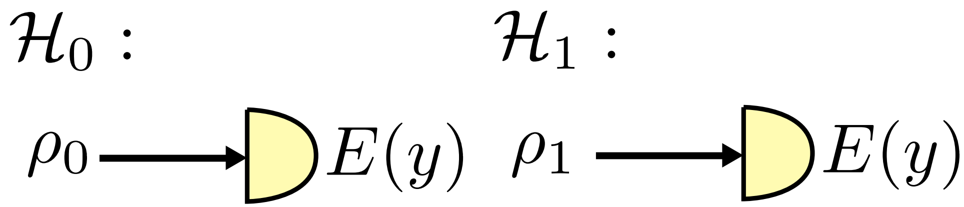
$$\min_{\Upsilon_{0,1}} P_e = \frac{1}{2}\left[1 - ||P_0 \Pr(Y|\mathcal{H}_0) - P_1 \Pr(Y|\mathcal{H}_1)||_1\right] \geq \frac{1}{2}\left(1 - \sqrt{1 - 4P_0 P_1 F}\right), \tag{7}$$

$$||A(Y)||_1 \equiv \sum_Y |A(Y)|, \quad F \equiv \left[\sum_Y \sqrt{\Pr(Y|\mathcal{H}_1)\Pr(Y|\mathcal{H}_0)}\right]^2. \tag{8}$$

  valid for any decision rule.
  - Bayesian posterior probabilities:

$$\Pr(\mathcal{H}_1|Y) = \frac{P_1 \Lambda}{P_1 \Lambda + P_0}, \qquad \Pr(\mathcal{H}_0|Y) = \frac{P_0}{P_1 \Lambda + P_0}. \tag{9}$$

Quantum Hypothesis Testing



- Define density operator as mixture of pure states

$$\rho = \sum_j P_j |\psi_j\rangle\langle\psi_j|, \tag{10}$$

- A generalized measurement (generalized Born's rule) is described by

$$|\Psi_j\rangle = |\psi_j\rangle_A \otimes |\phi\rangle_B, \qquad \Pr(Y) = \sum_j P_j \left|\langle Y|U|\Psi_j\rangle\right|^2 = \mathrm{tr}\left[E(Y)\rho\right], \tag{11}$$

  where $E(Y) = {}_B\langle\phi|U^\dagger|Y\rangle\langle Y|U|\phi\rangle_B$ is called POVM (Positive Operator-Valued Measure).
- Given two density operators ρ_0 and ρ_1,

$$\Pr(Y|\mathcal{H}_0) = \mathrm{tr}[E(Y)\rho_0], \qquad \Pr(Y|\mathcal{H}_1) = \mathrm{tr}[E(Y)\rho_1], \tag{12}$$

  what is the POVM that minimizes the error probabilities?

Quantum Error Bounds

- C. W. Helstrom, Quantum Detection and Estimation Theory, (Academic Press, New York, 1976).
- Given constraint $P_{10} \leq \alpha$,

$$P_{01} \geq \begin{cases} 1 - \left[\sqrt{\alpha F} + \sqrt{(1-\alpha)(1-F)}\right]^2, & \alpha < F, \\ 0, & \alpha \geq F. \end{cases} \tag{13}$$

- Average error probability:

$$\min_{E(Y)} P_e = \frac{1}{2}\left(1 - ||P_1\rho_1 - P_0\rho_0||_1\right) \geq \frac{1}{2}\left(1 - \sqrt{1 - 4P_0P_1F}\right), \tag{14}$$

$$||A||_1 \equiv \operatorname{tr}\sqrt{AA^\dagger}, \quad F \equiv \left(\operatorname{tr}\sqrt{\sqrt{\rho_0}\rho_1\sqrt{\rho_0}}\right)^2. \tag{15}$$

- For pure states,

$$F = |\langle\psi_0|\psi_1\rangle|^2, \tag{16}$$

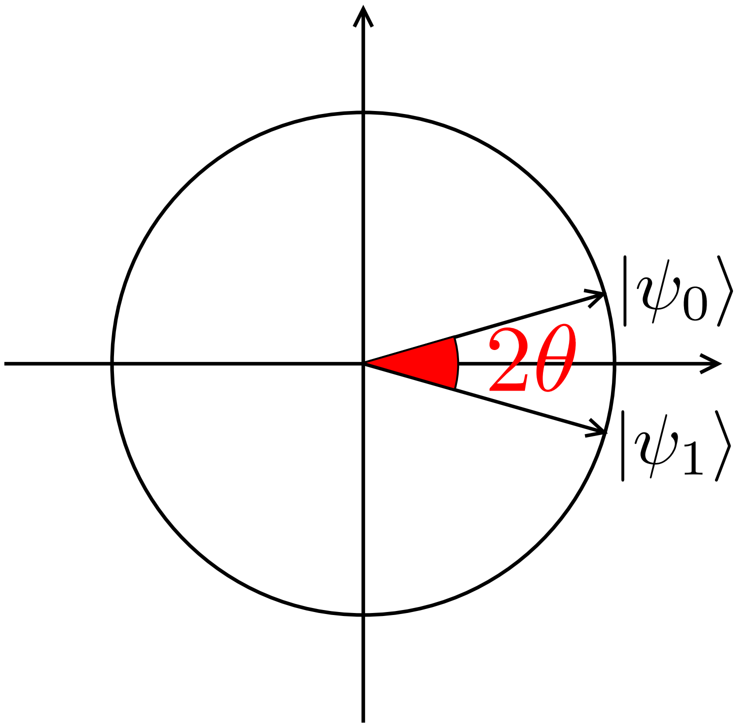
and there exist POVMs such that the fidelity bounds are saturated.

- Distinguishing a photon in two possible polarizations (assume $P_0 = P_1 = 1/2$):

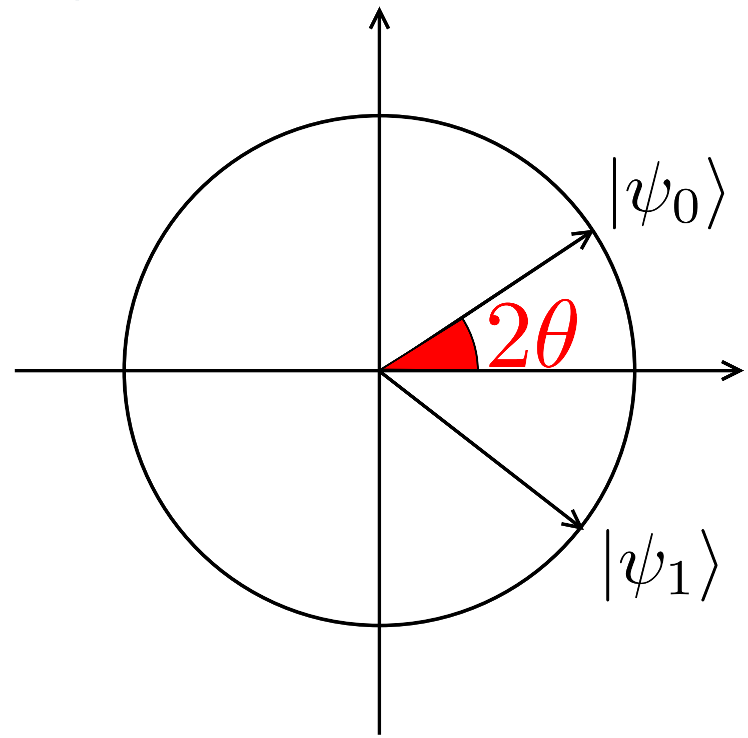
$$|\psi_0\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad |\psi_1\rangle = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \tag{17}$$

$$F = \cos^2 2\theta, \qquad \min_{E(Y)} P_e = \frac{1}{2}\left(1 - \sin 2\theta\right). \tag{18}$$

Two Linear Polarizations:



Bloch Sphere:

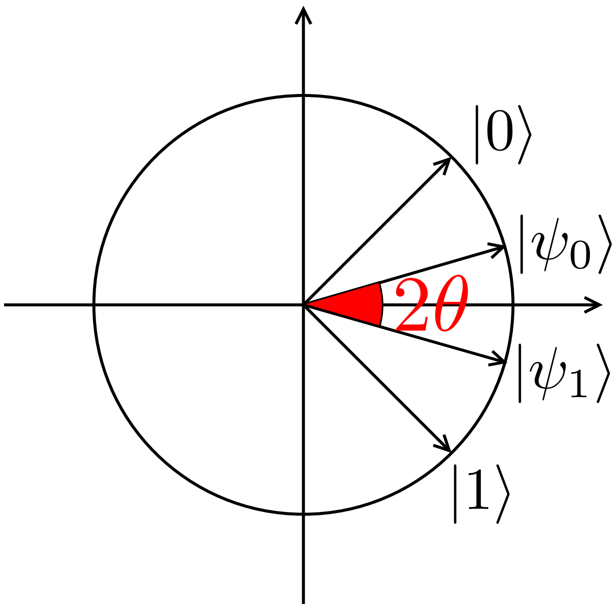


Optimal measurement

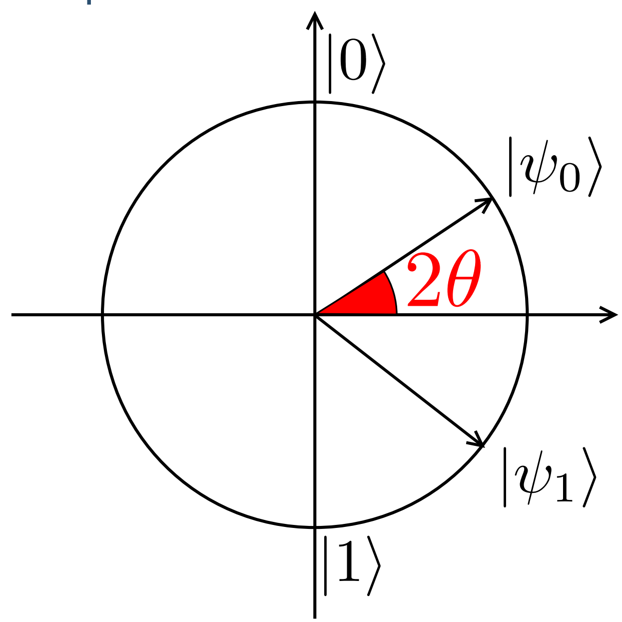
- Consider projective measurement in the following basis:

$$|0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}. \tag{19}$$

Two Linear Polarizations:



Bloch Sphere:



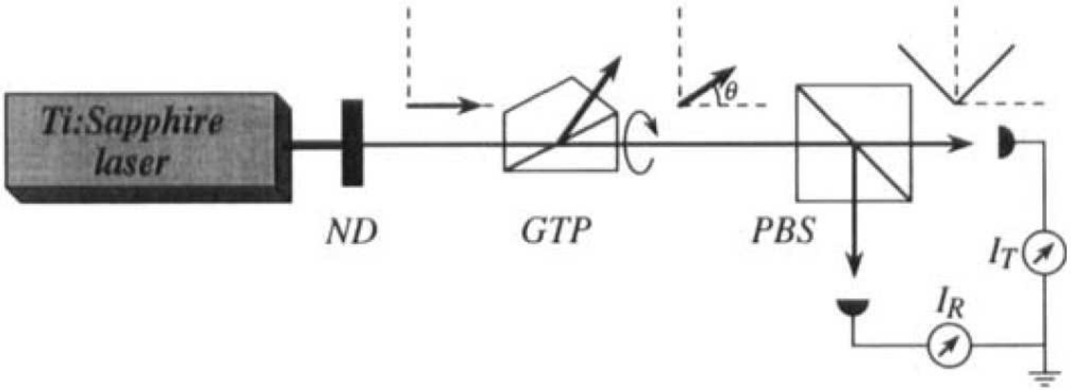
- Error probabilities:

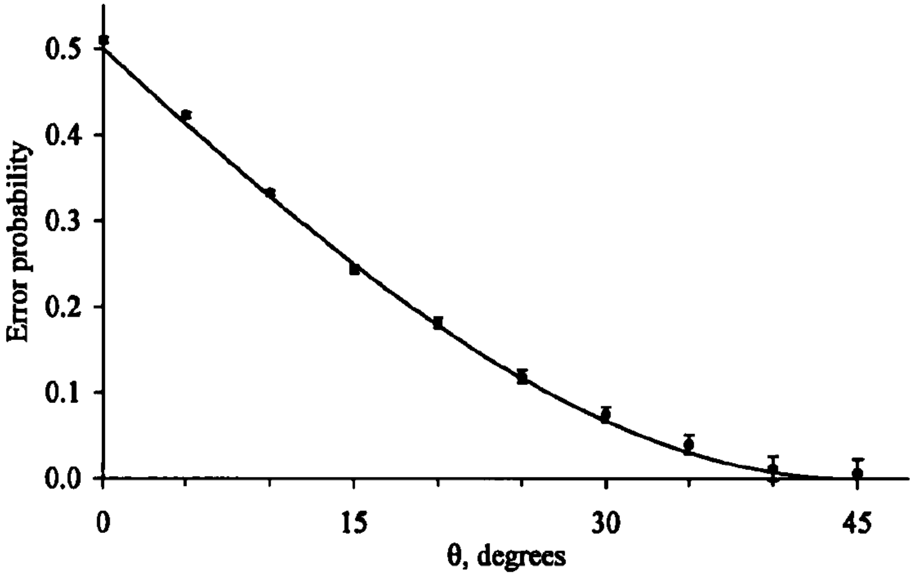
$$P_{10} = |\langle 1|\psi_0\rangle|^2 = \frac{1}{2}(\cos\theta - \sin\theta)^2\,, \qquad P_{01} = |\langle 0|\psi_1\rangle|^2 = \frac{1}{2}(\cos\theta - \sin\theta)^2\,, \tag{20}$$

$$P_e = \frac{1}{2}(1 - \sin 2\theta) = \min_{E(Y)} P_e. \tag{21}$$

This attains Helstrom's bound.

- Barnett and Riis, J. Mod. Opt. **44**, 1061 (1997):

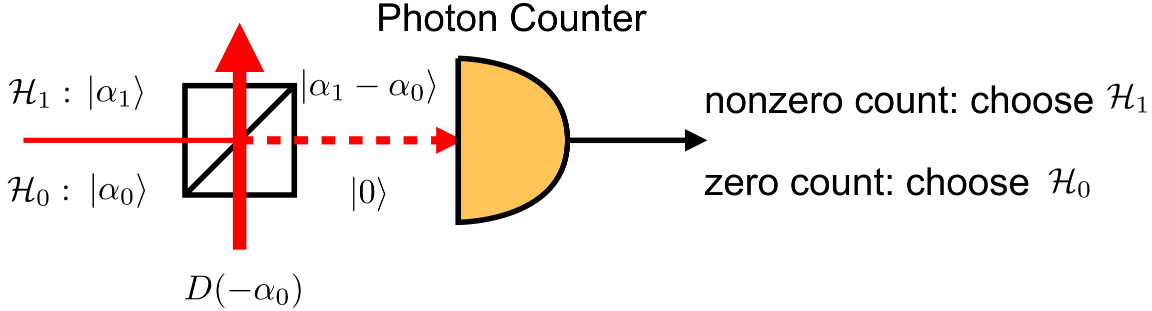




- two coherent states

$$|\alpha_{0,1}\rangle = \exp\left(-\frac{|\alpha_{0,1}|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha_{0,1}^n}{\sqrt{n!}}, \qquad F = \exp(-|\alpha_0-\alpha_1|^2). \tag{22}$$

- Kennedy receiver [Kennedy, MIT RLE Quart. Prog. Rep. **108**, 219-225 (1973)]:

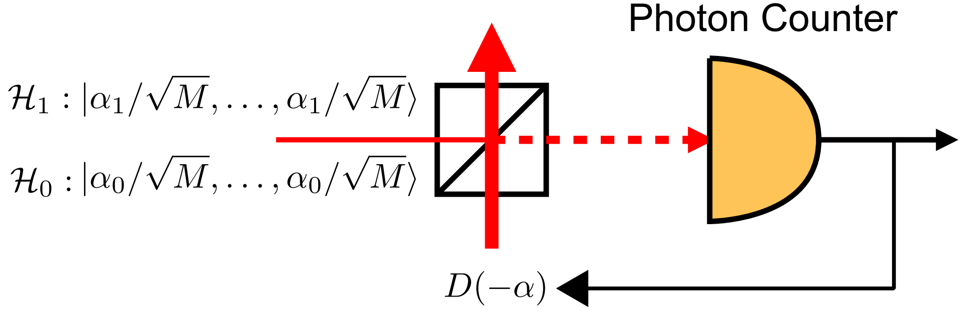


$$P_{10} = 0, \quad P_{01} = \exp(-|\alpha_0-\alpha_1|^2), \tag{23}$$

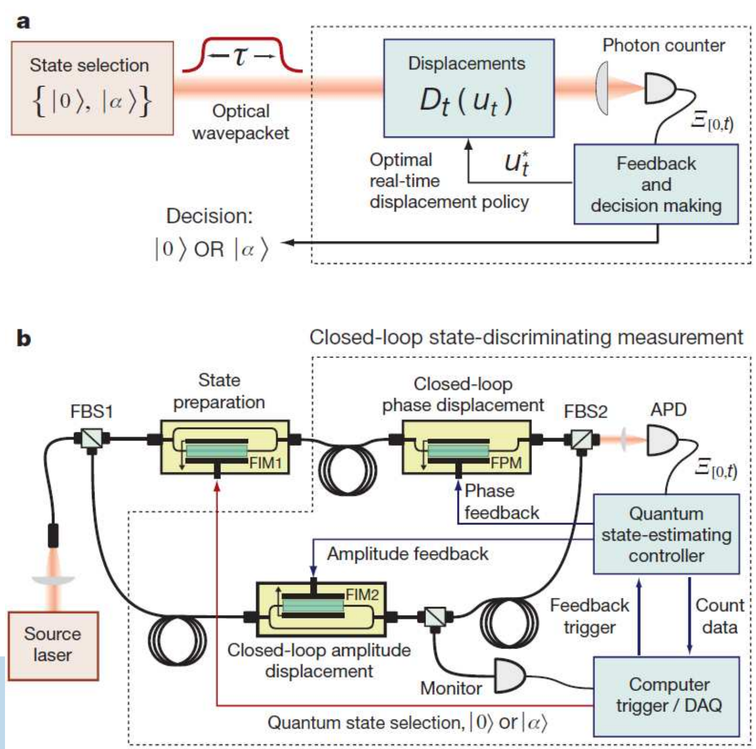
$$P_e = \frac{1}{2}\exp(-|\alpha_0-\alpha_1|^2) = \frac{1}{2}F \approx 2\min_{E(Y)} P_e \text{ if } F \ll 1. \tag{24}$$

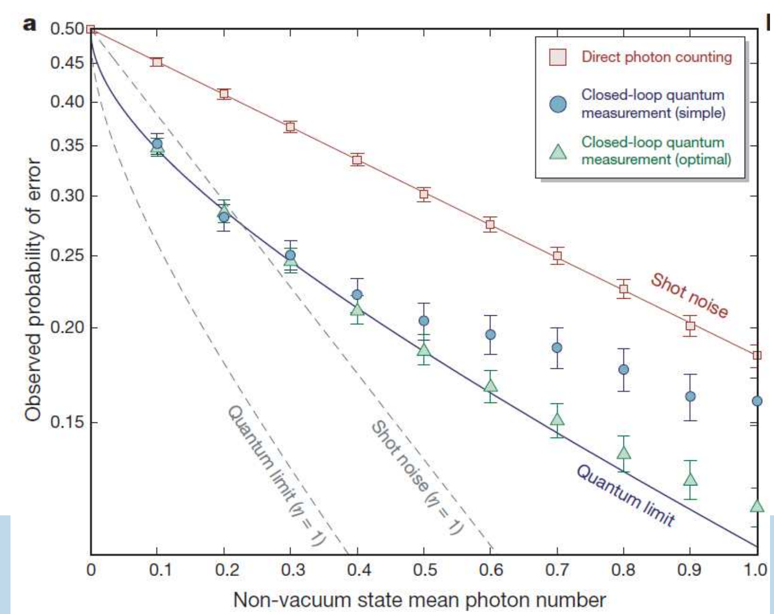
- near-optimal if $|\alpha_0-\alpha_1|^2 \gg 1$.

- Dolinar receiver [Dolinar, MIT RLE Quart. Prog. Rep. **111**, 115-120 (1973)]:



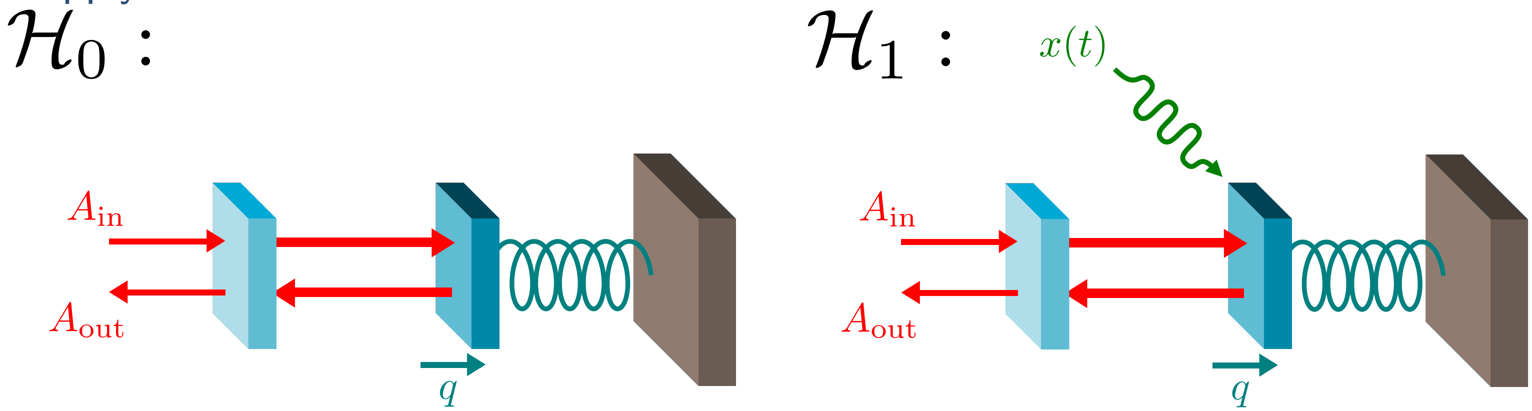
- Cook *et al.*, Nature **446** 774, (2007):





Helstrom bound for Waveform Estimation

- How to apply Helstrom bound to waveform detection?



- Force is time-varying
- Continuous quantum dynamics
- Continuous quantum measurements at the same time as the force perturbation.

- Discretize time, and take continuous limit at the end
- Measurements and dynamics described by a sequence of completely positive maps:

$$\Pr(Y|\mathcal{H}_j) = \operatorname{tr}\left[\mathcal{J}_j(y_M)\mathcal{K}_j \dots \mathcal{J}_j(y_1)\mathcal{K}_j\rho_j(t_0)\right], \tag{25}$$

- In terms of Kraus operators:

$$\mathcal{K}\rho \equiv \sum_z K(z)\rho K^\dagger(z), \qquad \mathcal{J}(y)\rho \equiv \sum_z J(y,z)\rho J^\dagger(y,z). \tag{26}$$

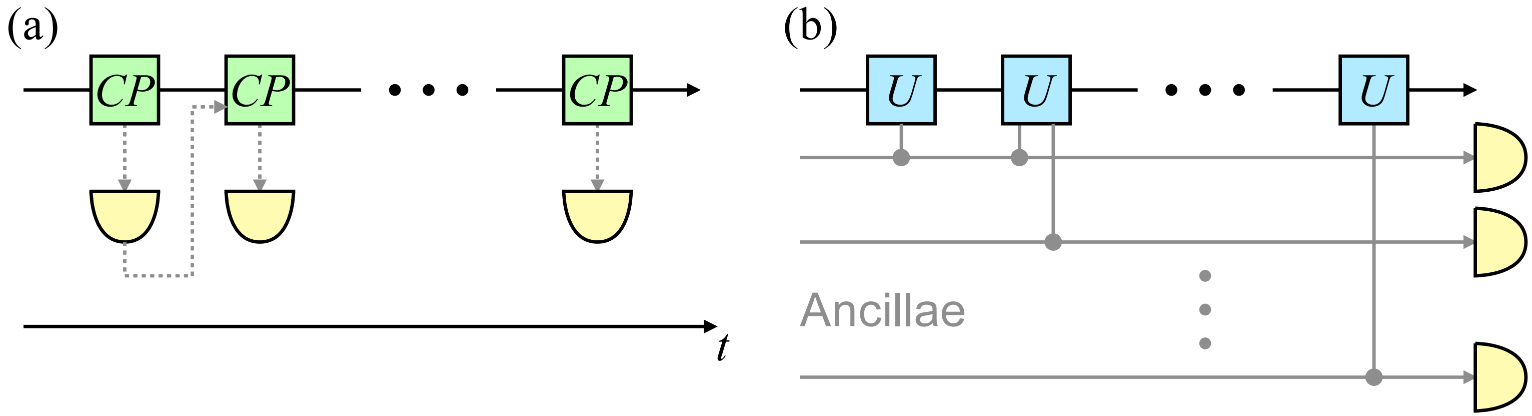
- Purification (Naimark/Kraus): CP maps can always be written as unitary operations with projective measurements in a larger Hilbert space:

$$\mathcal{K}\rho = \operatorname{tr}_B\left[U\left(\rho \otimes |\phi\rangle_B\langle\phi|\right)U^\dagger\right], \qquad \mathcal{J}(y)\rho = {}_B\langle y|U\left(\rho \otimes |\phi\rangle_B\langle\phi|\right)U^\dagger|y\rangle_B. \tag{27}$$

- K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
- M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010).

Church of Larger Hilbert Space

- Principle of deferred measurements:



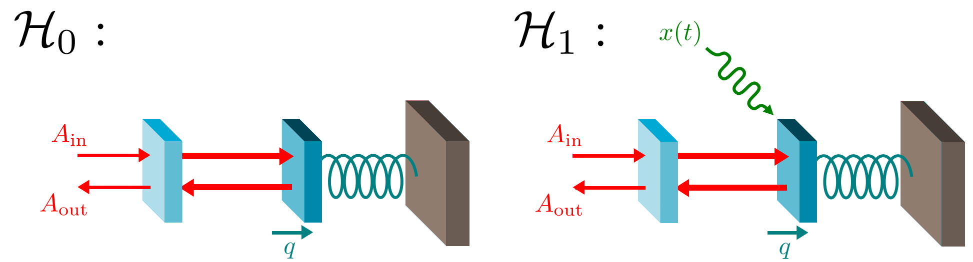
- The waveform detection problem can be written as a quantum-state discrimination problem:

$$\rho_0 = U_0|\psi\rangle\langle\psi|U_0^\dagger, \qquad \rho_1 = U_1|\psi\rangle\langle\psi|U_1^\dagger, \tag{28}$$

$$U_0 = \mathcal{T}\exp\left[\frac{1}{i\hbar}\int dt H_0(t)\right], \qquad U_1 = \mathcal{T}\exp\left[\frac{1}{i\hbar}\int dt H_1(x(t),t)\right], \tag{29}$$

$$F = \left|\langle\psi|U_0^\dagger U_1|\psi\rangle\right|^2. \tag{30}$$

Quantum Bound for Waveform Detection



- Lower bound in terms of fidelity:

$$P_e \geq \frac{1}{2}\left(1 - \sqrt{1 - 4P_0 P_1 F}\right), \tag{31}$$

$$F = \left|\langle\psi|U_0^\dagger U_1|\psi\rangle\right|^2 = \left|\langle\psi|\mathcal{T}\exp\frac{1}{i\hbar}\int dt H_I(t)|\psi\rangle\right|^2, \tag{32}$$

$$H_I(t) \equiv U_0^\dagger(t,t_0)\left[H_1(t) - H_0(t)\right]U_0(t,t_0). \tag{33}$$

- Suppose $H_1 - H_0 = -qx(t)$, $U_0^\dagger q U_0$ is linear with respect to initial positions/momenta, and $|\psi\rangle$ has Gaussian Wigner representation. Then

$$F = \exp\left[-\frac{1}{\hbar^2}\int dt \int dt' x(t)\left\langle : \Delta q_0(t)\Delta q_0(t') : \right\rangle x(t')\right], \tag{34}$$

$$q_0(t) = U_0^\dagger(t,t_0) q U_0(t,t_0), \quad \Delta q_0(t) = q_0(t) - \langle q_0(t)\rangle. \tag{35}$$

- M. Tsang and R. Nair, PRA **86**, 042115 (2012).

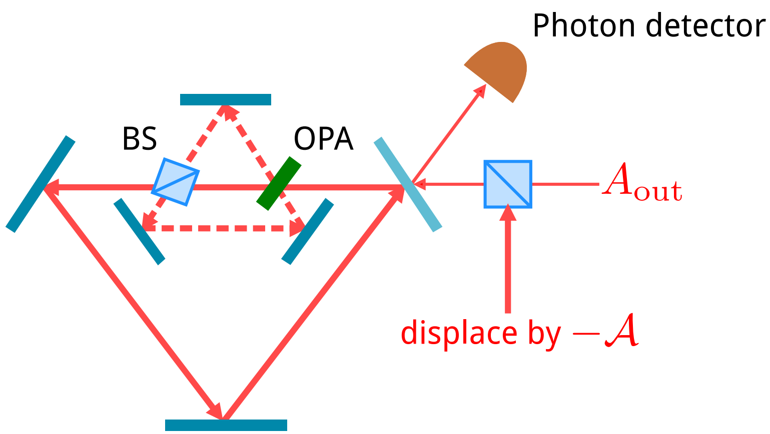
Quantum Receiver Design

- error exponent: measures the decay rate of error:

$$\Gamma \equiv -\lim_{T\to\infty}\frac{1}{T}\ln P_e, \qquad \Gamma^{(\mathrm{Helstrom})} \equiv -\lim_{T\to\infty}\frac{1}{T}\ln \min_{E(Y)} P_e = -\lim_{T\to\infty}\frac{1}{T}\ln F. \tag{36}$$

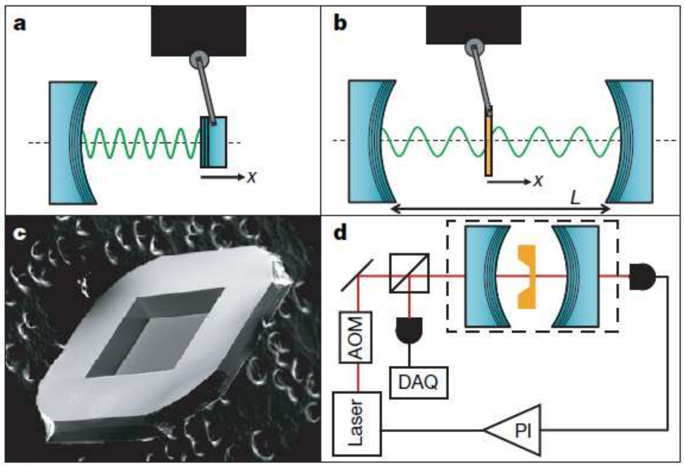
- Homodyne detection of A_{out} is sub-optimal: $\Gamma^{(\mathrm{homodyne})} = \frac{1}{2}\Gamma^{(\mathrm{Helstrom})}$.
- Kennedy receiver is near-optimal if A_{out} is in coherent state:

$$\Gamma^{(\mathrm{Kennedy})} = \Gamma^{(\mathrm{Helstrom})}. \tag{37}$$

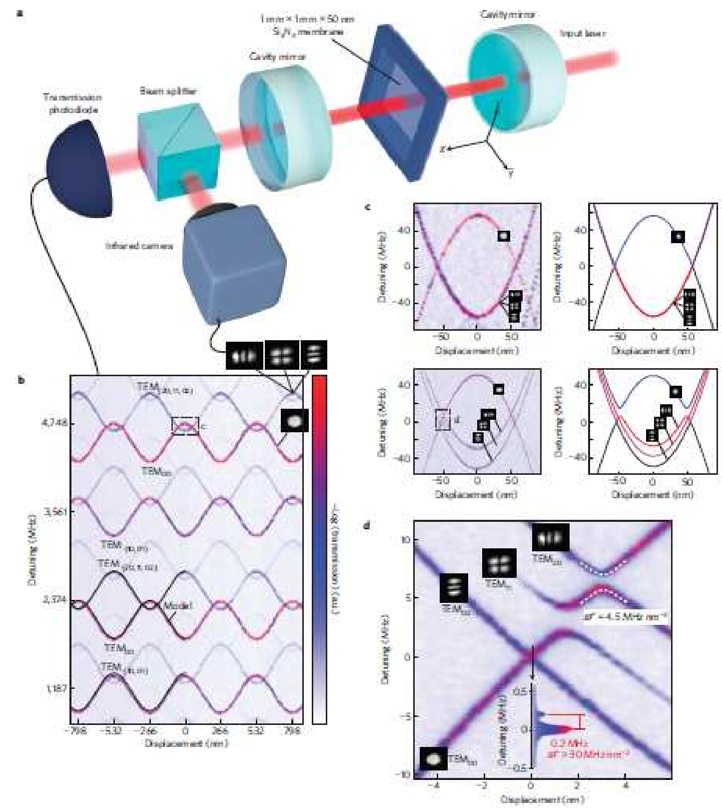


- Require Quantum Noise Cancellation [M. Tsang and C. M. Caves, PRL **105**, 123601 (2010); PRX **2**, 031016 (2012)], a coherent feedforward control technique, if measurement backaction noise is significant.
- Estimation is trivial in theory, but with technical imperfections/homodyne detection, likelihood-ratio test is necessary.

Energy Quantization



Thompson *et al.*, Nature **452**, 72 (2008).



Sankey *et al.*, Nature Phys. **6**, 707 (2010).

- Continuous noisy measurement of mechanical oscillator energy
- Is the energy classical (continuous) or quantum (discrete)?
- Focus on estimation (calculation of likelihood ratio).

Continuous Quantum Measurements

- Sequential measurements of a quantum system:

$$\Pr(Y|\mathcal{H}_j) = \operatorname{tr}\left[\mathcal{J}_j(y_M)\mathcal{K}_j \dots \mathcal{J}_j(y_1)\mathcal{K}_j\rho_j(t_0)\right], \tag{38}$$

- $\mathcal{K}$ and $\mathcal{J}$ are completely-positive maps. In terms of Kraus operators:

$$\mathcal{K}\rho \equiv \sum_z K(z)\rho K^\dagger(z), \qquad \mathcal{J}(y)\rho \equiv \sum_z J(y,z)\rho J^\dagger(y,z). \tag{39}$$

- Stick with smaller Hilbert space; easier for numerical analysis.
- infinitesimal CP map (Lindblad):

$$\mathcal{K}\rho = \rho + \delta t\mathcal{L}\rho + o(\delta t). \tag{40}$$

- For weak measurements with Gaussian noise,

$$\mathcal{J}(\delta y)\rho = \tilde{P}(\delta y)\left[\rho + \frac{\delta y}{2R}\left(c\rho + \rho c^\dagger\right) + \frac{\delta t}{8Q}\left(2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c\right) + o(\delta t)\right], \tag{41}$$

$$\tilde{P}(\delta y) = \mathcal{N}(0, R\delta t). \tag{42}$$

- H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010).

Likelihood Ratio for Continuous Measurements

- Suppose noise variance R is the same in both hypothesis. Then it can be shown that

$$\Lambda = \frac{\operatorname{tr} f_1(T)}{\operatorname{tr} f_0(T)}, \tag{43}$$

where f_1 and f_0 obey the quantum Duncan-Mortensen-Zakai (DMZ) equation:

$$df_j = dt\mathcal{L}_j f_j + \frac{dy}{2R}\left(c_j f_j + f_j c_j^\dagger\right) + \frac{dt}{8Q_j}\left(2c_j f_j c_j^\dagger - c_j^\dagger c_j f_j - f_j c_j^\dagger c_j\right). \tag{44}$$

- some stochastic calculus:

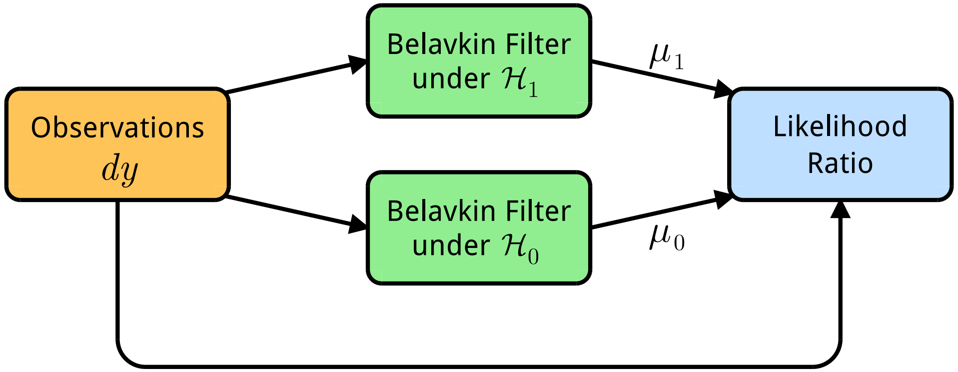
$$d\operatorname{tr} f_j = \operatorname{tr} df_j = \frac{dy}{2R}\operatorname{tr}\left(c_j f_j + f_j c_j^\dagger\right) = \frac{dy}{R}\frac{\operatorname{tr}\left(c_j f_j + f_j c_j^\dagger\right)}{2\operatorname{tr} f_j}\operatorname{tr} f_j, \tag{45}$$

$$\ln\operatorname{tr} f_j(t) = \int_{t_0}^{T}\frac{dy}{R}\mu_j - \int_{t_0}^{T}\frac{dt}{2R}\mu_j^2, \quad \mu_j \equiv \frac{1}{\operatorname{tr} f_j}\operatorname{tr}\left(\frac{c_j + c_j^\dagger}{2} f_j\right). \tag{46}$$

- Final result:

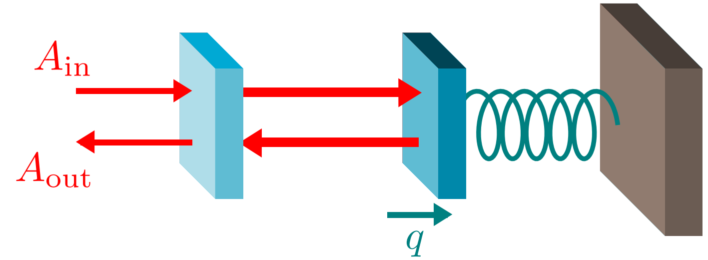
$$\Lambda(T) = \exp\left[\int_{t_0}^{T} \frac{dy}{R}(\mu_1 - \mu_0) - \int_{t_0}^{T} \frac{dt}{2R}\left(\mu_1^2 - \mu_0^2\right)\right]. \tag{47}$$

- μ_j is the expected value of $(c_j + c_j^\dagger)/2$ given the observation record, assuming $\mathcal{H}_j$ is true, can be calculated by quantum filters.

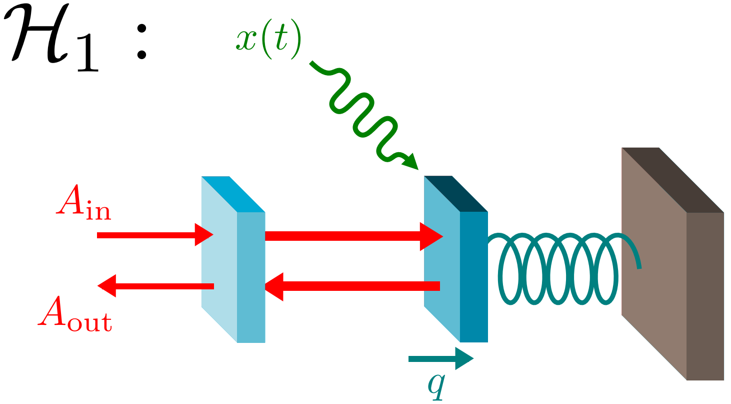


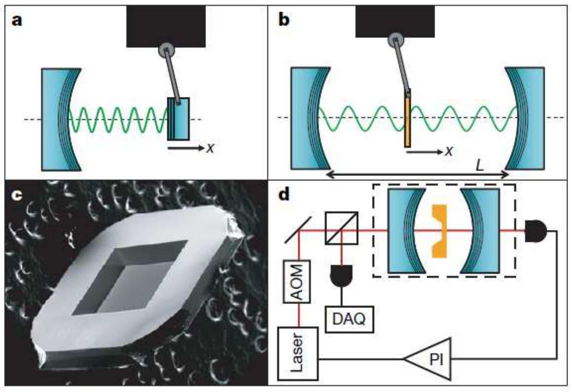
- Similar formula exists for continuous measurements with Poisson noise.
- M. Tsang, PRL **108**, 170502 (2012)
- Quantum generalizations of the Duncan-Kailath estimator-correlator formula [Duncan, Inf. Control **13**, 62 (1968); Kailath, IEEE TIT **15**, 350 (1969)] and Snyder's formula [Snyder, IEEE TIT **18**, 91 (1972)].

$\mathcal{H}_0$:

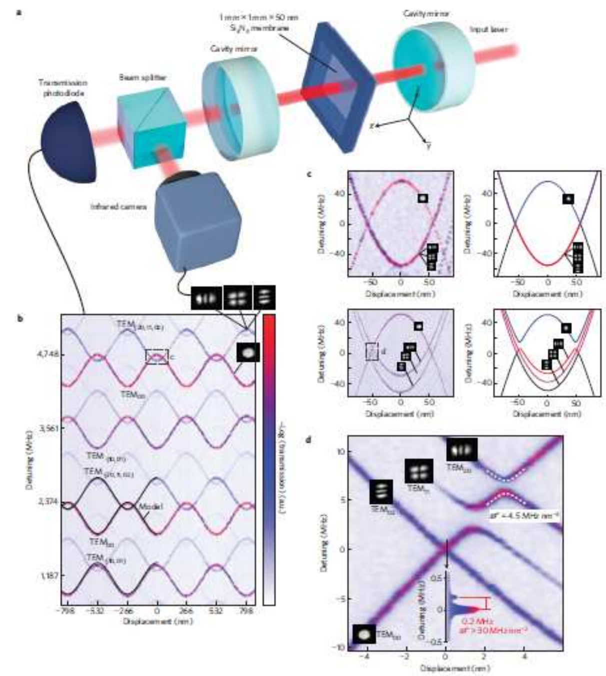


$\mathcal{H}_1$:





Thompson *et al.*, Nature **452**, 72 (2008).



Sankey *et al.*, Nature Phys. **6**, 707 (2010).

Parameter Estimation

- Given y and likelihood function $P(y|x)$, estimate x.
- Let estimate be $\tilde{x}(y)$.
- Mean-square error:

$$\mathbb{E}(\delta x^2) \equiv \int dy \, [\tilde{x}(y) - x]^2 \, P(y|x). \tag{48}$$

- For unbiased estimates, $x = \int dy \tilde{x}(y) P(y|x)$, Cramér-Rao bound:

$$\mathbb{E}\left(\delta x^2\right) \geq J^{-1}, \tag{49}$$

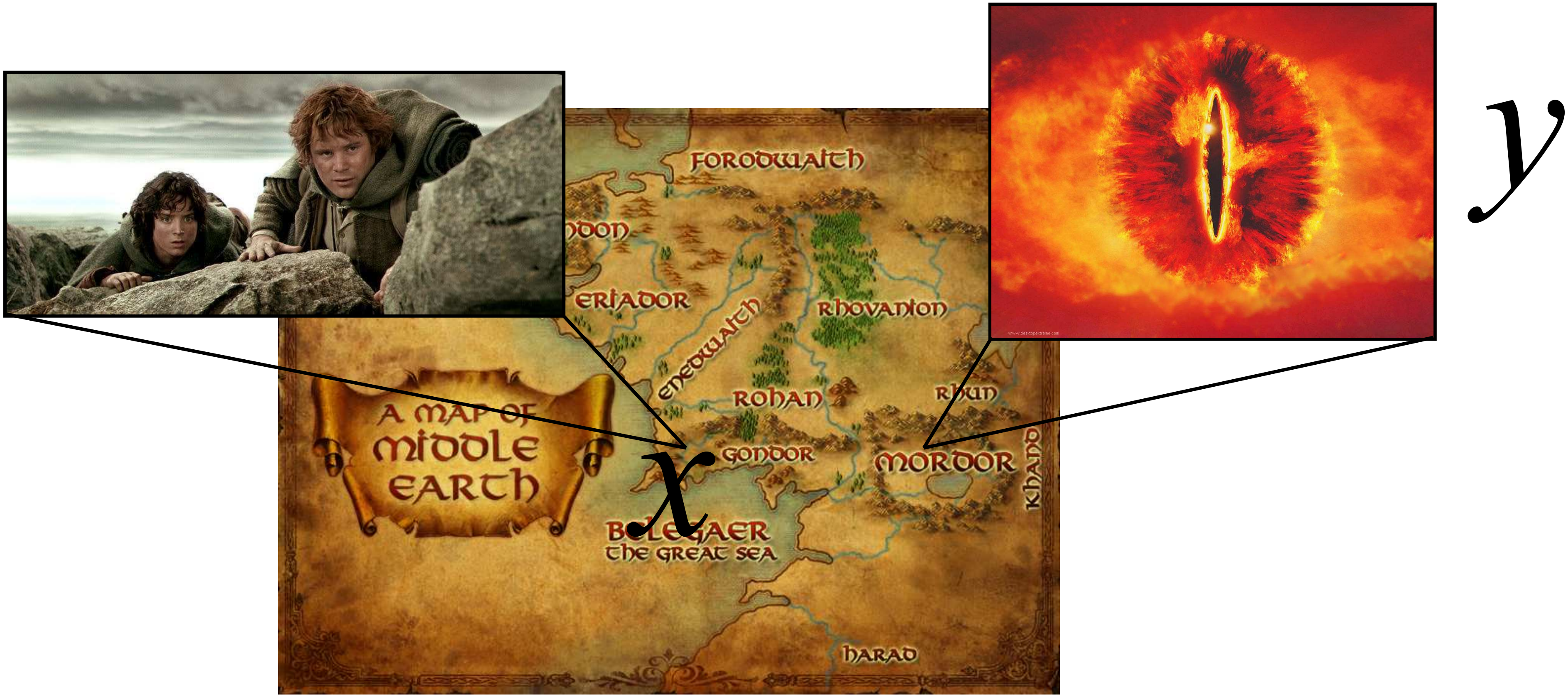
- Fisher information:

$$J \equiv \int dy P(y|x) \left[\frac{\partial \ln P(y|x)}{\partial x}\right]^2. \tag{50}$$

- If $P(y|x)$ is Gaussian:

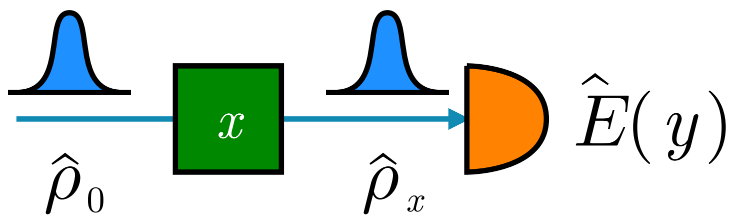
$$P(y|x) = \frac{1}{\sqrt{(2\pi)^K \det R}} \exp\left[-\frac{1}{2}(y - Cx)^\top R^{-1}(y - Cx)\right], \tag{51}$$

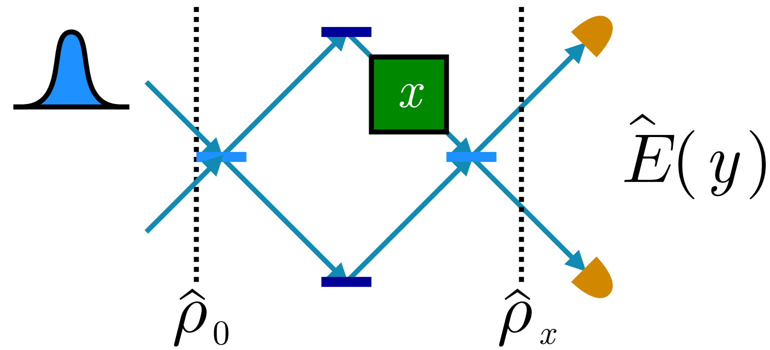
CRB is attainable using maximum-likelihood estimation.


A MAP OF MIDDLE EARTH
FORODWAITH
ERIADOR
ENEDWAITH
RHOVANION
ROHAN
GONDOR
MORDOR
KHAND
THE GREAT SEA
HARAD
x
y

Other Tracking Examples

Quantum Parameter Estimation





- Quantum:

$$P(y|x) = \mathrm{tr}\left[E(y)\rho_x\right]. \tag{52}$$

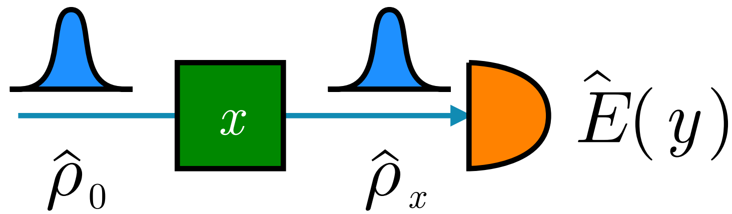
- Quantum Cramér-Rao bound (QCRB) (valid for any POVM but may not be achievable):

$$\mathbb{E}(\delta x^2) \geq 1/J^{(Q)}, \qquad J^{(Q)} \equiv \mathrm{tr}\left(\Delta h^\dagger \Delta h \rho_x\right), \tag{53}$$

$$\frac{\partial \rho_x}{\partial x} = \frac{1}{2}\left(h\rho_x + \rho_x h^\dagger\right), \qquad \Delta h \equiv h - \mathrm{tr}\left(h\rho_x\right). \tag{54}$$

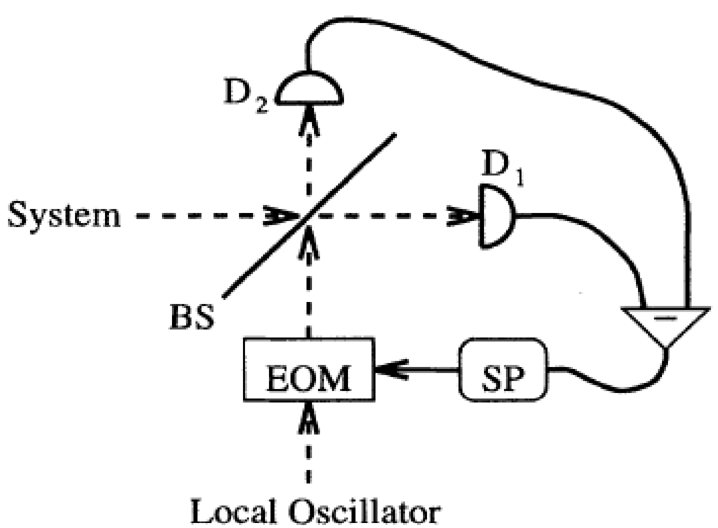
- C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976); V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004); Nature Photon. **5**, 222 (2011).

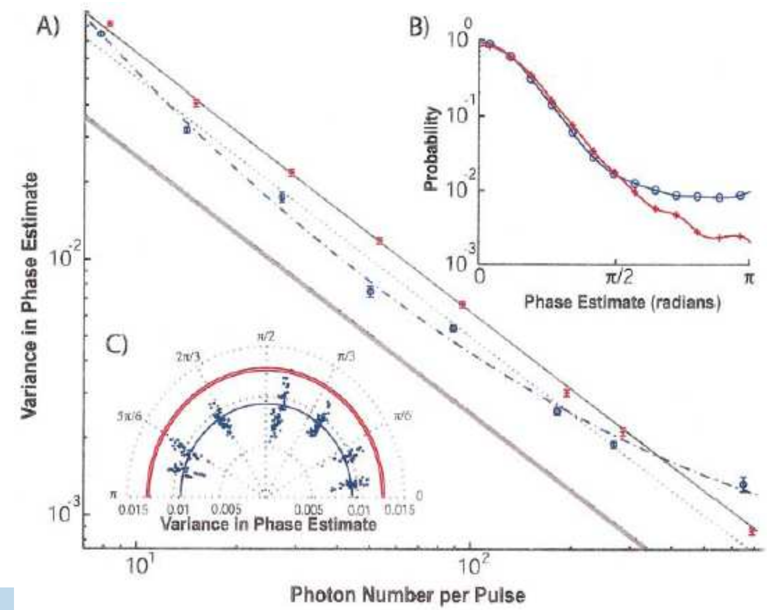
- Optical phase modulation:



$$\rho_x = \exp(i\hat{n}x)\rho_0 \exp(-i\hat{n}x), \qquad \frac{\partial \rho_x}{\partial x} = i\,[\hat{n}, \rho_x]\,, \qquad \hat{h} = 2i\hat{n}, \qquad \mathbb{E}(\delta x^2) \ge \frac{1}{4\,\langle \Delta n^2 \rangle}. \qquad (55)$$

- Optimal measurement for coherent states: adaptive homodyne [H. M. Wiseman, PRL **75**, 4587 (1995)]; Experiment: Armen *et al.*, PRL **89**, 133602 (2002).





Generalizations of Classical Cramér-Rao Bound

- Error covariance matrix for multiple parameters with prior distribution $P(x)$:

$$\Sigma \equiv \mathbb{E}\left[(\tilde{x}-x)(\tilde{x}-x)^\top\right] \equiv \int dydx P(y|x)P(x)(\tilde{x}-x)(\tilde{x}-x)^\top . \quad (56)$$

- Define loss function in terms of positive-semidefinite matrix Λ as

$$C(\tilde{x},x) = (\tilde{x}-x)^\top \Lambda (\tilde{x}-x) . \quad (57)$$

- Average cost/Bayes risk

$$C \equiv \mathbb{E}\left[C(\tilde{x},x)\right] = \mathrm{tr}(\Lambda\Sigma) \geq 0. \quad (58)$$

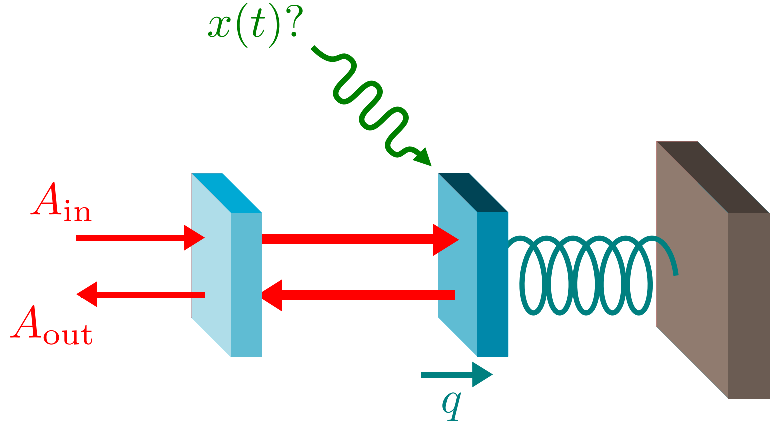
- Bayesian Cramér-Rao bound (Van Trees) for any Λ:

$$C \geq \mathrm{tr}\left(\Lambda J^{-1}\right), \qquad J = J^{(Y)} + J^{(X)}, \quad (59)$$

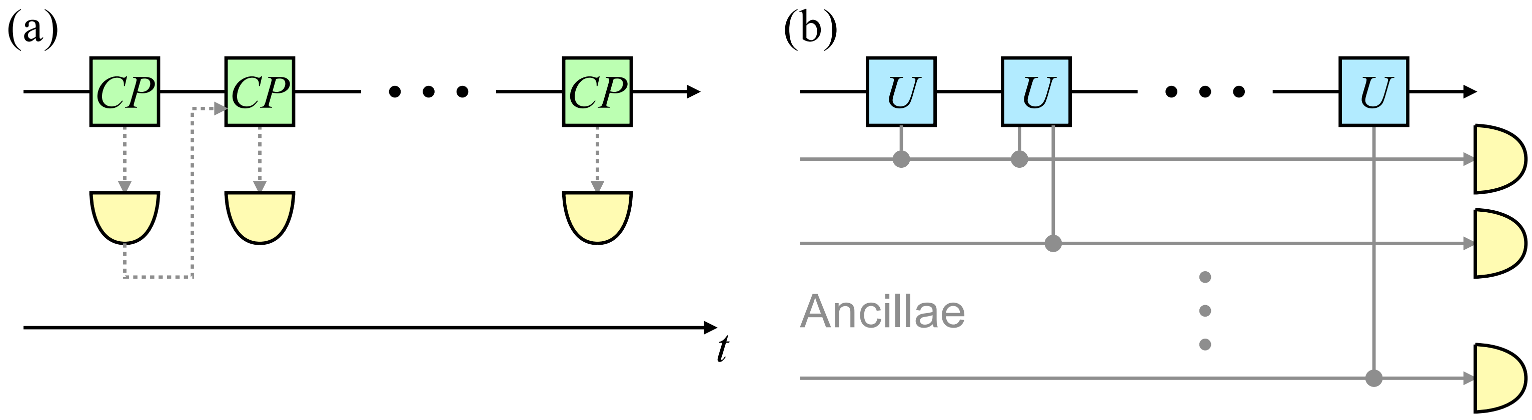
$$J^{(Y)}_{jk} = \mathbb{E}\left[\frac{\partial \ln P(y|x)}{\partial x_j}\frac{\partial \ln P(y|x)}{\partial x_k}\right], \qquad J^{(X)}_{jk} = \mathbb{E}\left[\frac{\partial \ln P(x)}{\partial x_j}\frac{\partial \ln P(x)}{\partial x_k}\right]. \quad (60)$$

- More compact way: $\Sigma \geq J^{-1}$; i.e., $\Sigma - J^{-1}$ is positive-semidefinite.

- How to deal with continuous sensing?



- Discretize time, purification in larger Hilbert space, principle of deferred measurements:



- Fisher information matrix $J(t,t')$

$$\int dtdt'\Lambda(t,t')\left\{\mathbb{E}\left[\delta x(t)\delta x(t')\right]-J^{-1}(t,t')\right\}\geq 0, \tag{61}$$

$J^{-1}(t,t')$ is defined by $\int dt' J(t,t')J^{-1}(t',\tau)=\delta(t-\tau)$.

- Two components:

$$J(t,t')=J^{(Q)}(t,t')+J^{(X)}(t,t'). \tag{62}$$

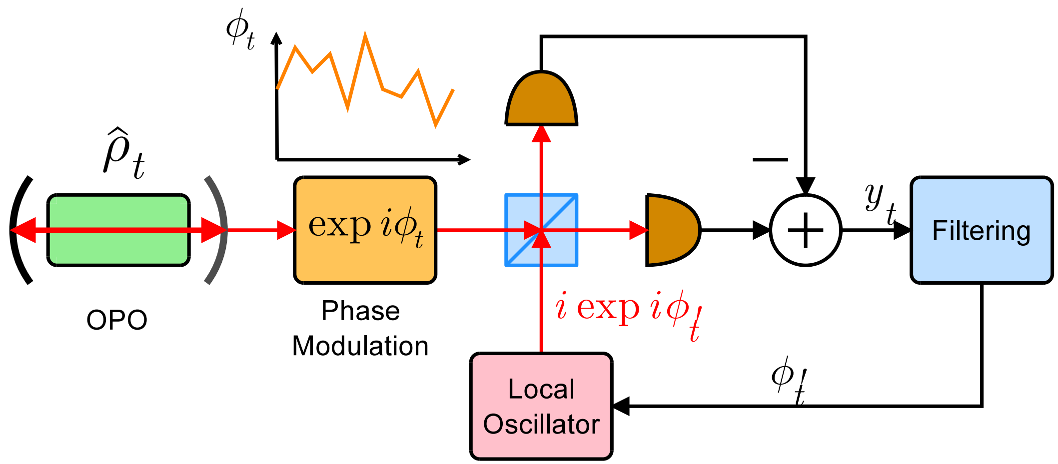
- $J^{(Q)}$ is a two-time quantum covariance function:

$$J^{(Q)}(t,t')=\frac{4}{\hbar^2}\mathbb{E}\Big\{\operatorname{tr}\left[:\Delta\hat{h}(t)\Delta\hat{h}(t'):\rho_0\right]\Big\},\quad \hat{h}(t)\equiv\hat{U}^\dagger(t,t_0)\frac{\partial\hat{H}(t)}{\partial x(t)}\hat{U}(t,t_0).$$

- $J^{(X)}$ incorporates *a priori* waveform information

$$J^{(X)}(t,t')=\mathbb{E}\left\{\frac{\delta\ln P[x]}{\delta x(t)}\frac{\delta\ln P[x]}{\delta x(t')}\right\}. \tag{63}$$

- M. Tsang, H. M. Wiseman, and C. M. Caves, PRL **106**, 090401 (2011).



- M. Tsang, arXiv:1301.5733v3 (2013):

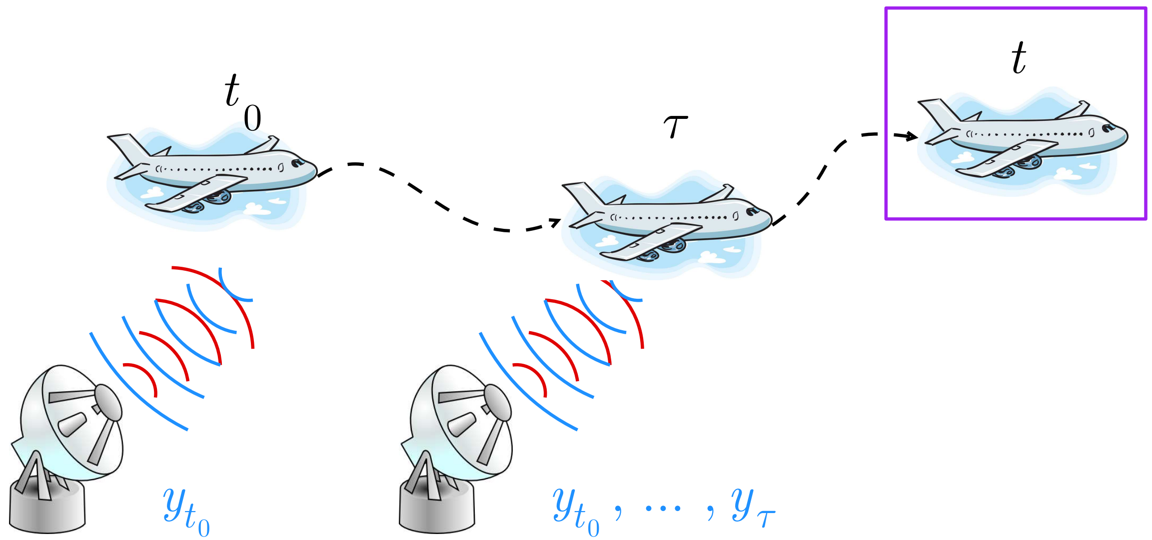
$$\phi(t) = \int_{-\infty}^{\infty} d\tau g(t-\tau)x(\tau), \tag{64}$$

$$\mathbb{E}[\delta x^2(t)] \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{4|g(\omega)|^2 S_{\Delta I}(\omega) + 1/S_x(\omega)}, \tag{65}$$

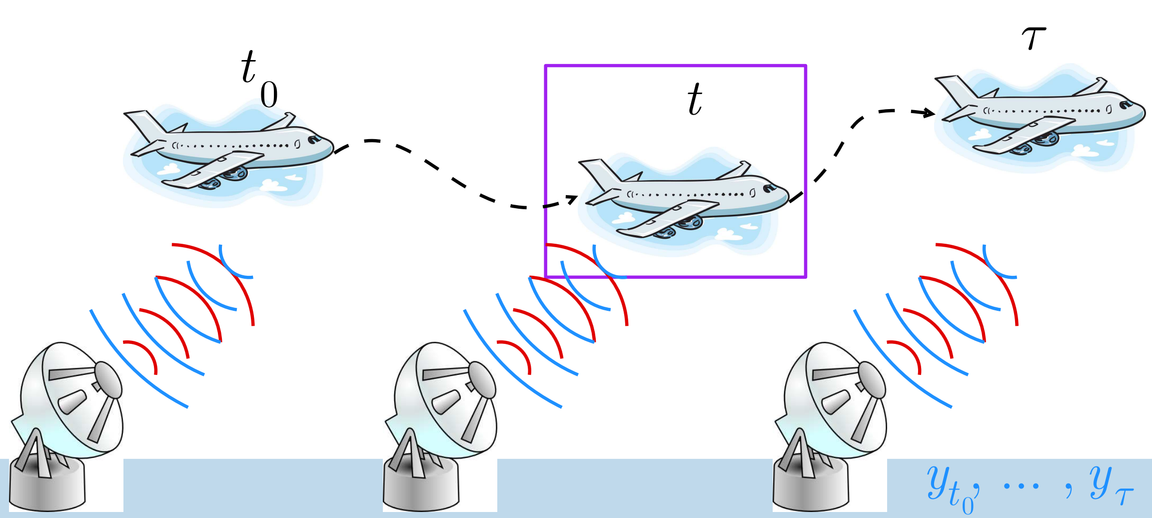
$$\text{e.g. } S_{\Delta I}^{\text{coh}}(\omega) = \frac{\bar{P}}{\hbar\omega_0}, \quad S_x^{\text{OU}}(\omega) = \frac{\kappa}{\omega^2 + \epsilon^2}. \tag{66}$$

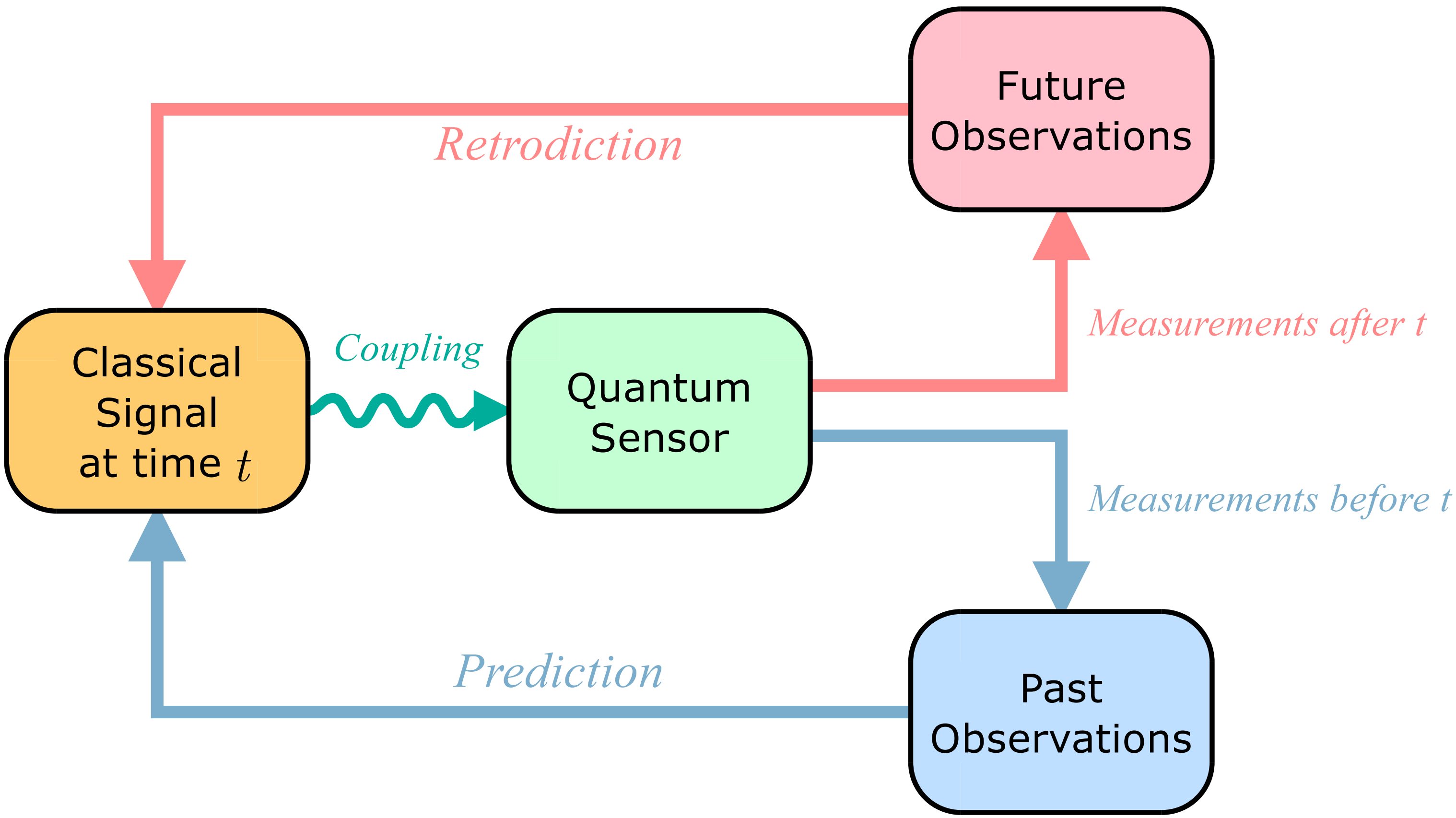
- Filtering: Berry and Wiseman, PRA **65**, 043803 (2002); **73**, 063824 (2006).
- Filtering does not saturate QCRB!

- Filtering, Prediction: real-time or advanced estimation



- Smoothing: delayed estimation, more accurate when $x(t)$ is a stochastic process





- M. Tsang, PRL **102**, 250403 (2009).

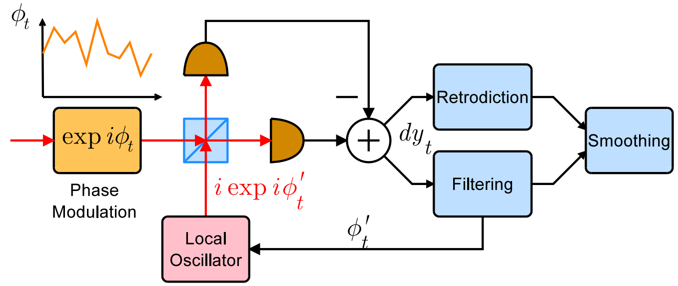
$$df = dt\mathcal{L}(x)f + \frac{dt}{8}\left(2C^\top R^{-1} f C^\dagger - C^{\dagger\top} R^{-1} C f - f C^{\dagger\top} R^{-1} C\right) + \frac{1}{2} dy^\top R^{-1}\left(Cf + fC^\dagger\right)$$

$$-dg = dt\mathcal{L}^*(x)g + \frac{dt}{8}\left(2C^{\dagger\top} R^{-1} g C - C^{\dagger\top} R^{-1} C g - g C^{\dagger\top} R^{-1} C\right) + \frac{1}{2} dy^\top R^{-1}\left(C^\dagger g + gC\right)$$

$$h(x,t) = P(x_t = x | Y_{\text{past}}, Y_{\text{future}}) = \frac{\operatorname{tr}\left[g(x,t) f(x,t)\right]}{\int dx\,(\text{numerator})}$$

Experimental Demonstration

- Optical phase-locked loop with smoothing: M. Tsang, J. H. Shapiro, and S. Lloyd, PRA **78**, 053820 (2008); **79**, 053843 (2009).



- Wheatley *et al.*, PRL **104**, 093601 (2010): very close to QCRB for coherent state:

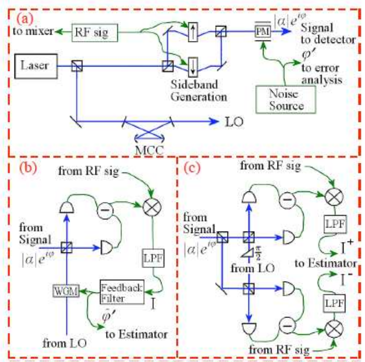


FIG. 1: Schematic diagrams showing: (a) source and local oscillator generation; (b) adaptive phase estimation; (c) dual homodyne phase estimation. Although illustrated as a single device, both AOMs as drawn are actually a pair of AOMs which shift by 110 MHz and 105 MHz in opposite directions to achieve a 5 MHz frequency shift. LO = local oscillator; RF sig = radio-frequency signal; PM = phase modulator; WGM = waveguide modulator; LPF = low-pass filter; MCC = mode-cleaning cavity; AOM = acousto-optic modulator.

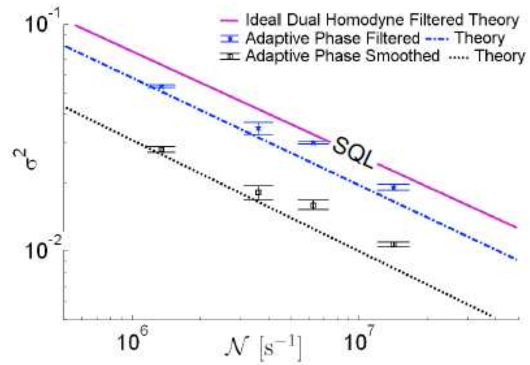


FIG. 3: The variance σ^2 of the adaptive phase estimation for quantum filtering and smoothing as a function of photon number $\mathcal{N}$, compared to the relevant theoretical predictions, and the theoretical predictions for nonadaptive measurements.

- QCRB can be lowered by squeezing.

- Yonezawa *et al.*, Science **337**, 1514 (2012)

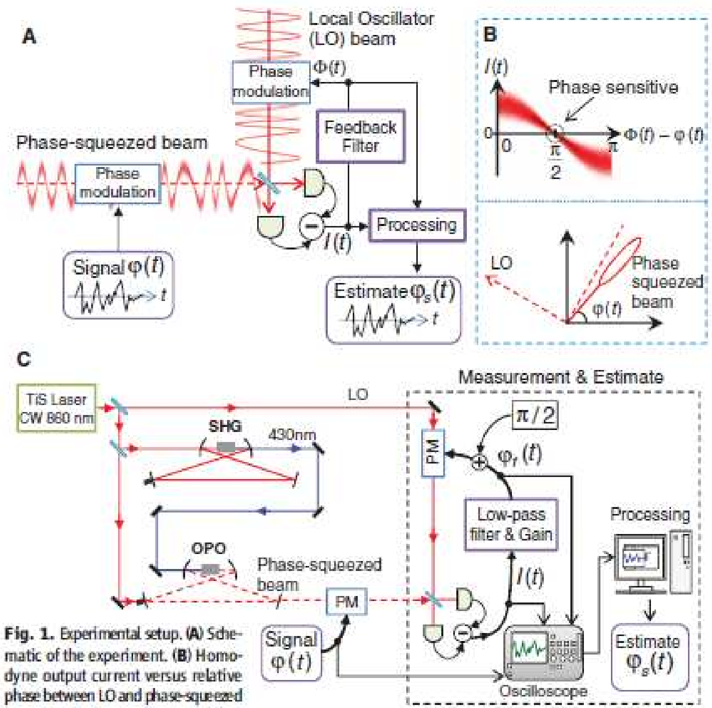


Fig. 1. Experimental setup. **(A)** Schematic of the experiment. **(B)** Homodyne output current versus relative phase between LO and phase-squeezed beam and phasor diagram for a slightly nonoptimal relative phase (as will occur in the phase-tracking problem). **(C)** Detail of the experimental setup. The abbreviations are TiS, titanium sapphire; CW, continuous-wave; PM, phase modulator; and SHG, second-harmonic-generation.

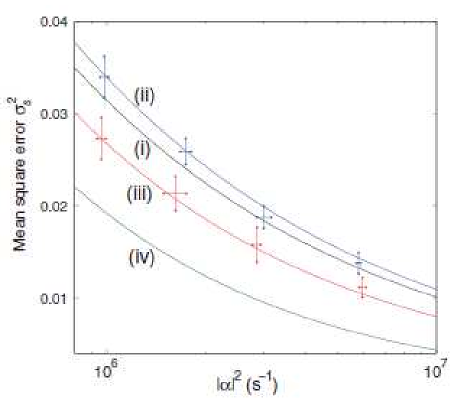


Fig. 2. Time domain results of phase estimate. **(A)** Signal phase to be estimated $\varphi(t)$. **(B)** Homodyne output current $I(t)$. **(C)** Filtered estimate $\varphi_f(t)$. **(D)** Smoothed estimate $\varphi_s(t)$.

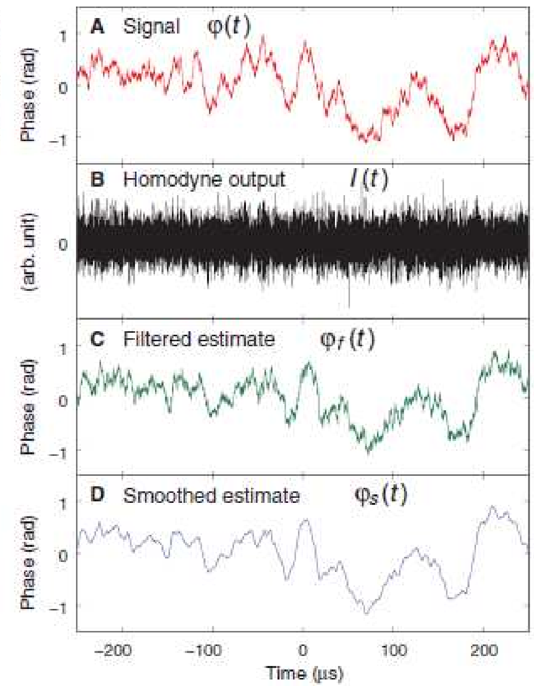
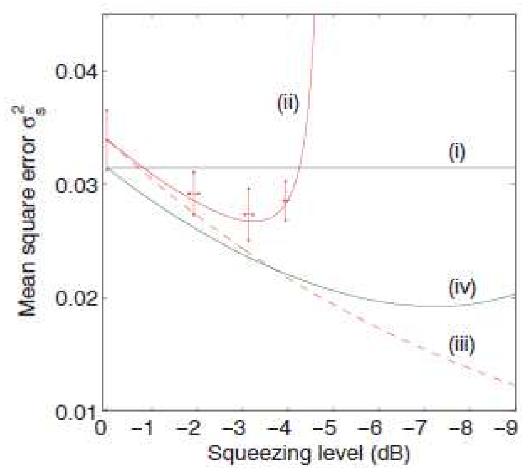
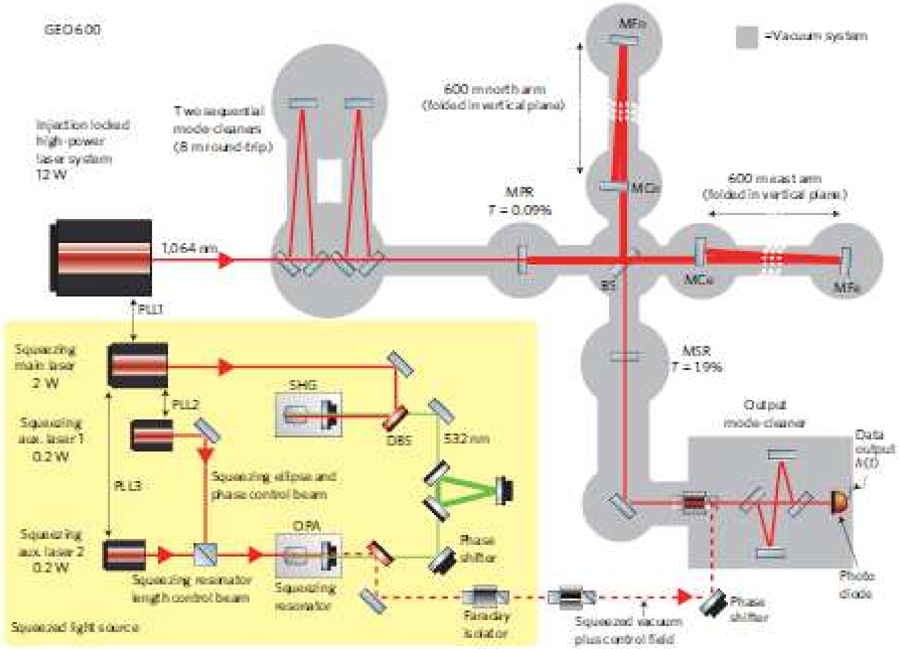
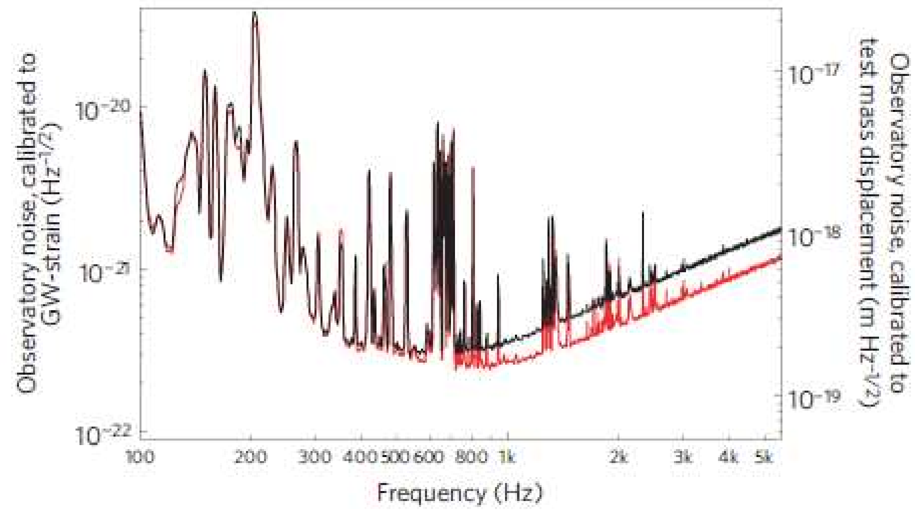


Fig. 3. Smoothed MSE σ_s^2 versus squeezing level. Red crosses represent experimental data. Trace (i) is the coherent-state limit, which is reachable with a coherent beam only if we have unit-detection efficiency $\eta = 1$. Trace (ii) is the theoretical curve from Eq. 3. Trace (iii) is the theoretical curve based on approximating the homodyne output current $I(t)$ to only first order in $[\varphi(t) - \varphi_f(t)]$ so that $\bar{R}_{sq} = e^{-2r_m}$. Trace (iv) is the theoretical curve from Eq. 3 for pure squeezed beams (i.e., without loss).

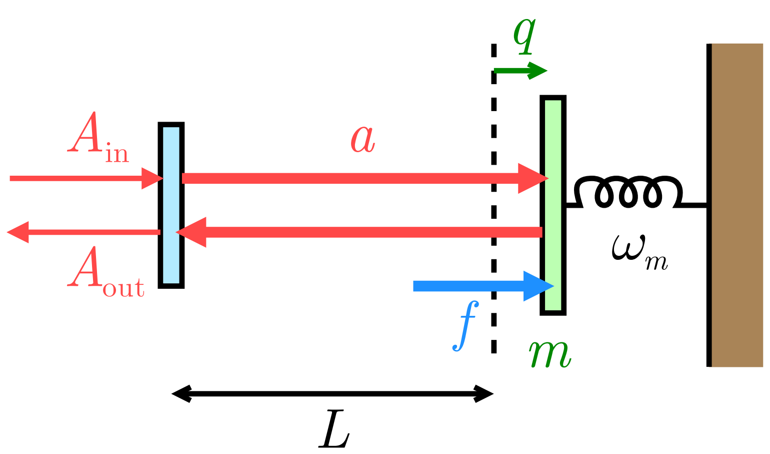


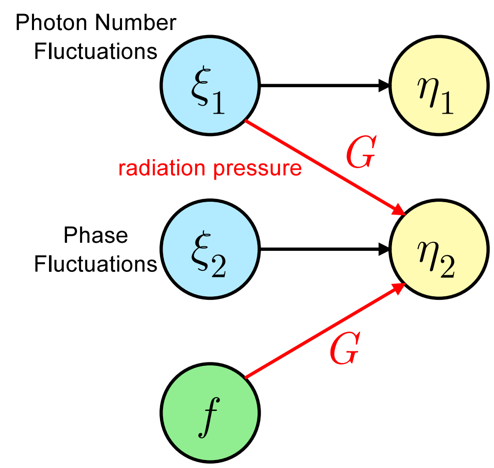
- LIGO, Nature Phys. **7**, 962 (2011).

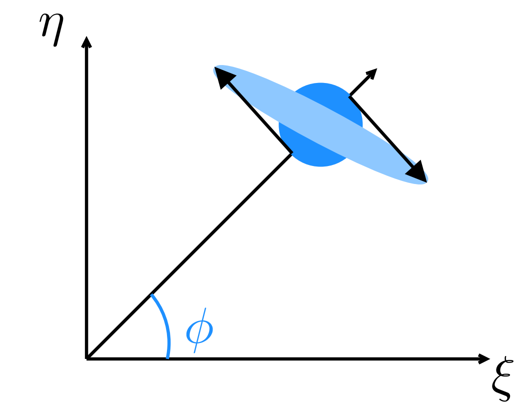




Example 2: Optomechanical Force Estimation



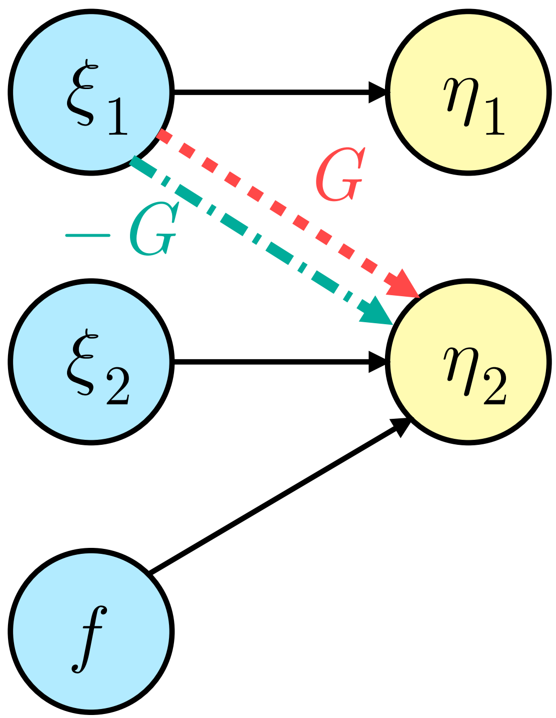


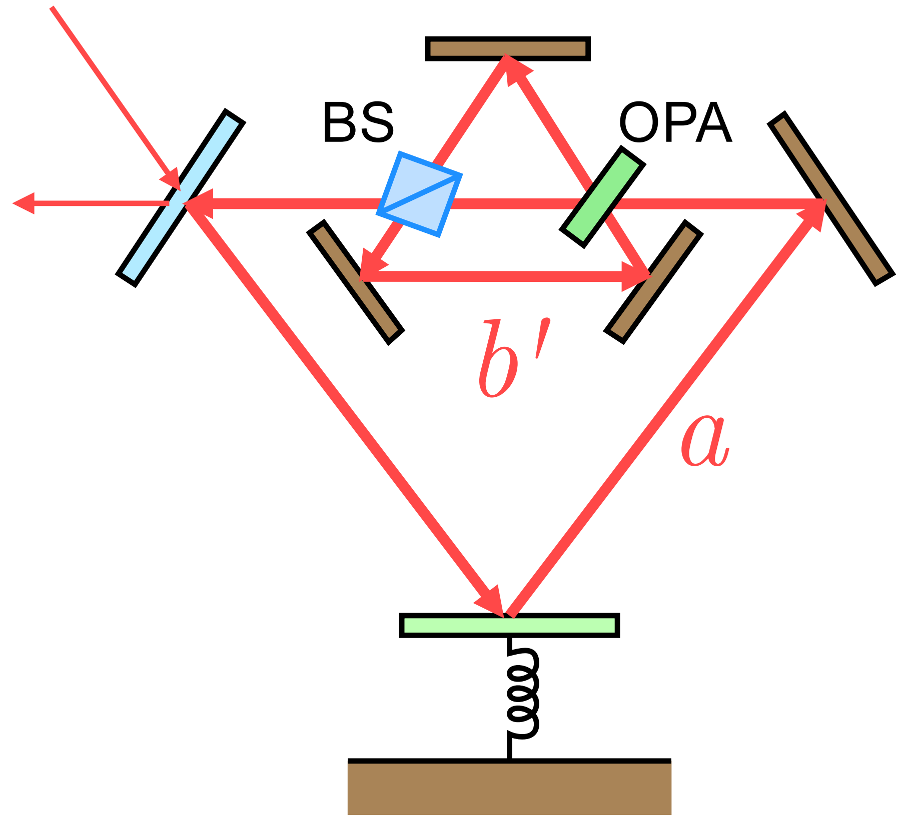


- $\hat{H}_I = -\hat{q} f$,

$$\mathbb{E}[\delta f^2(t)] \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(4/\hbar^2) S_{\Delta q}(\omega) + 1/S_f(\omega)}. \quad (67)$$

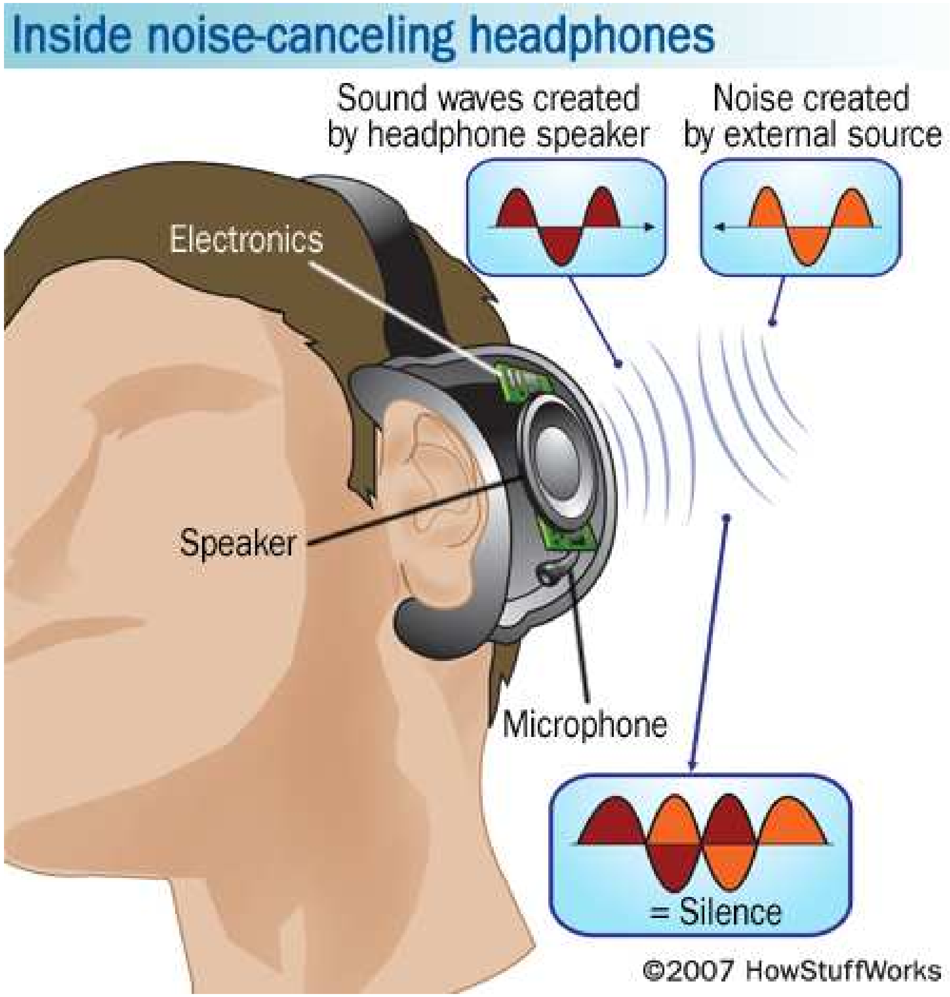
- Smoothing can't saturate QCRB due to the presence of measurement backaction noise
- Standard Quantum Limit (Braginsky, Caves *et al.*): backaction noise cannot be removed.





- Coherent Feedforward Quantum Control
- M. Tsang and C. M. Caves, PRL **105**, 123601 (2010).
- See also Kimble *et al.*, PRD **65**, 022002 (2001).

Noise Cancellation



Quantum Non-Demolition (QND) Observables

- Consider Heisenberg picture:

$$\frac{dq(t)}{dt} = \frac{p(t)}{m}, \qquad \frac{dp(t)}{dt} = -m\omega^2 q(t). \tag{68}$$

Since $[q, p] \neq 0$,

$$\left[q(t), q(t')\right] \neq 0 \text{ for } t \neq t'. \tag{69}$$

$q(t)$ and $q(t')$ are incompatible observables. uncertainty principle (roughly) says measurement of one will disturb the other.

- Pair the harmonic oscillator with another with negative mass:

$$\frac{dq'(t)}{dt} = -\frac{p'(t)}{m}, \qquad \frac{dp'(t)}{dt} = m\omega^2 q'(t), \tag{70}$$

$$\frac{d[q(t) + q'(t)]}{dt} = \frac{[p(t) - p'(t)]}{m}, \qquad \frac{d[p(t) - p'(t)]}{dt} = -m\omega^2 [q(t) + q'(t)]. \tag{71}$$

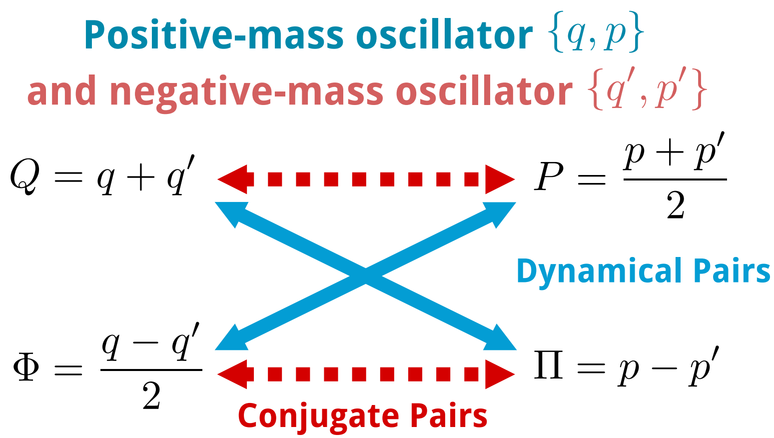
- Since $[q + q', p - p'] = 0$,

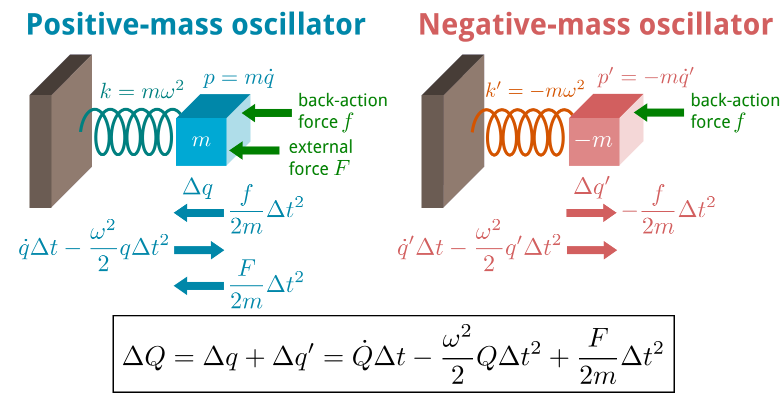
$$\left[q(t) + q'(t), q(t') + q'(t')\right] = 0, \qquad \left[p(t) - p'(t), p(t') - p'(t')\right] = 0, \tag{72}$$

$$\left[q(t) + q'(t), p(t') - p'(t')\right] = 0. \tag{73}$$

These observables that commute with each other in Heisenberg picture are called QND observables.

Quantum-Mechanics-Free Subsystems



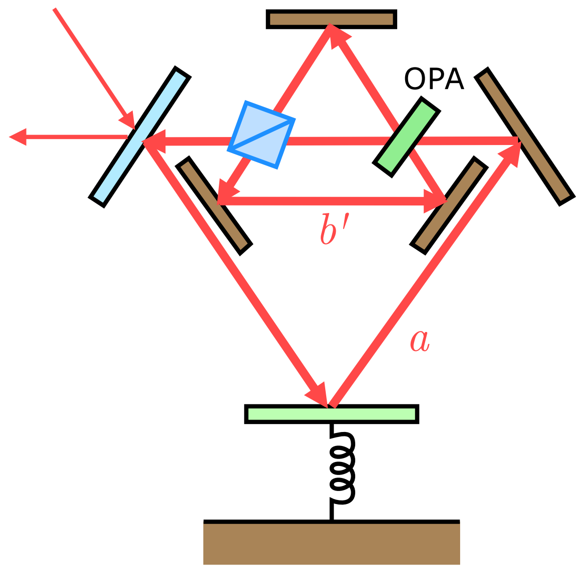


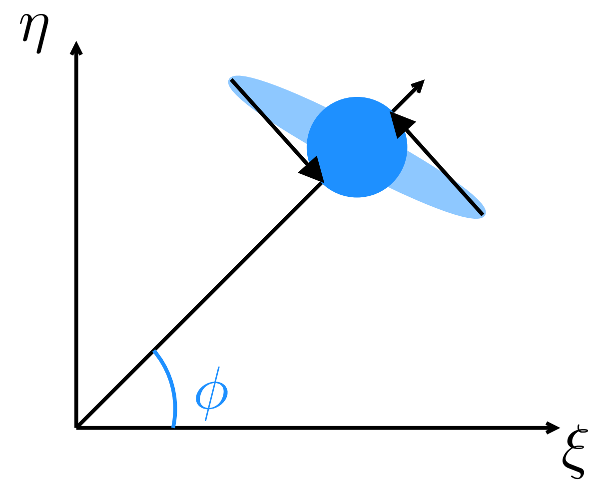
- QND observables: commute with each other at different times in the Heisenberg picture,

$$\left[X_j(t), X_k(t')\right] = 0. \tag{74}$$

  No backaction noise if measurements are made at the times at which they commute.
- Measurements of QND observables are also called backaction-evading measurements.
  - Caves *et al.*, Rev. Mod. Phys. **52**, 341 (1980).
- QND observables are equivalent to classical stochastic processes according to spectral theorem; i.e., they can be measured to any arbitrary accuracy.
- By choosing an appropriate Hamiltonian, it is possible to make a subsystem of observables
  - QND
  - obey any linear/nonlinear classical dynamics.
  - M. Tsang and C. M. Caves, PRX **2**, 031016 (2012).
  - Ars Technica review article: http://arstechnica.com/science/2012/09/demolishing-heisenberg-with-clever-math-and-experiments/

Optimal Force Estimation

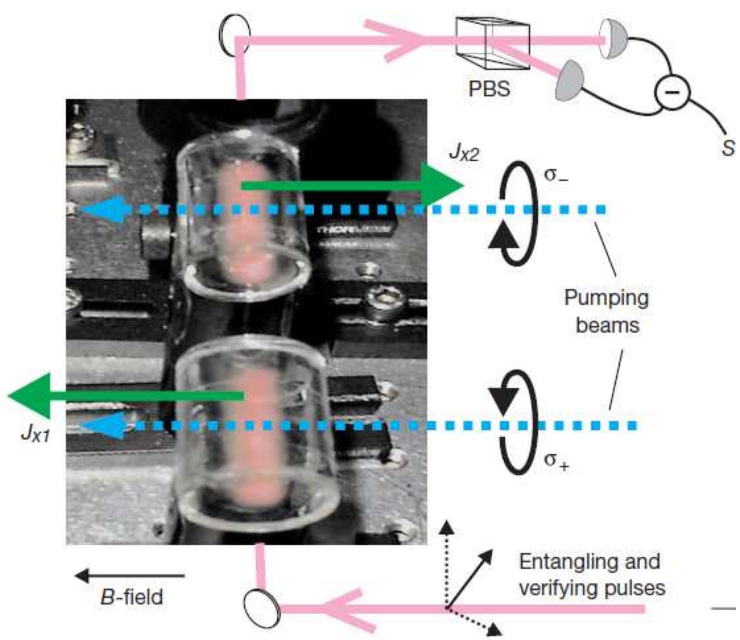


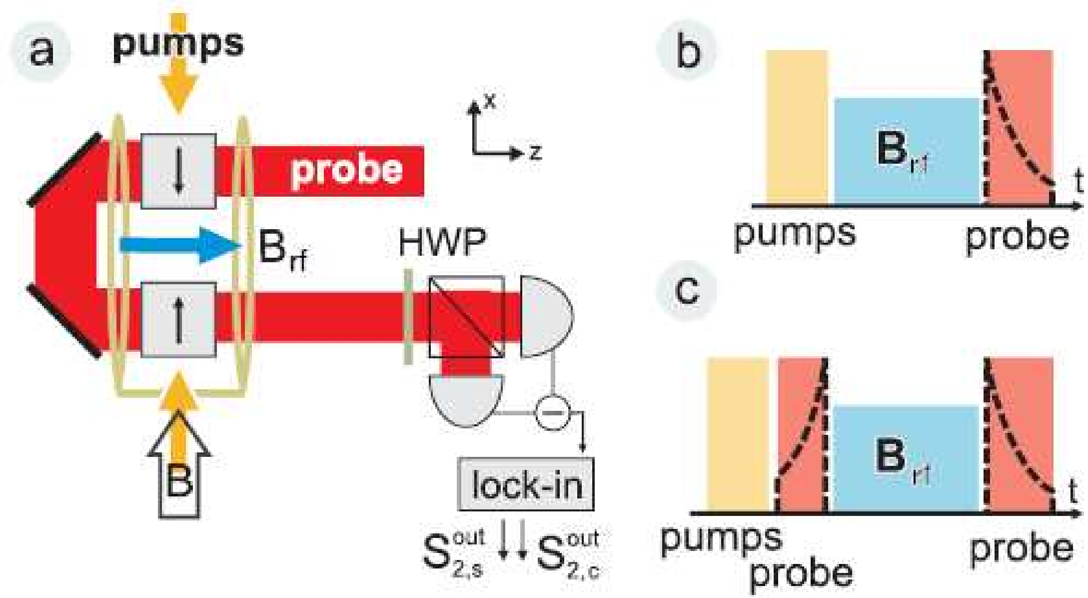


- QNC + Smoothing saturate QCRB for coherent state.
- Optical squeezing of input light can lower the QCRB.

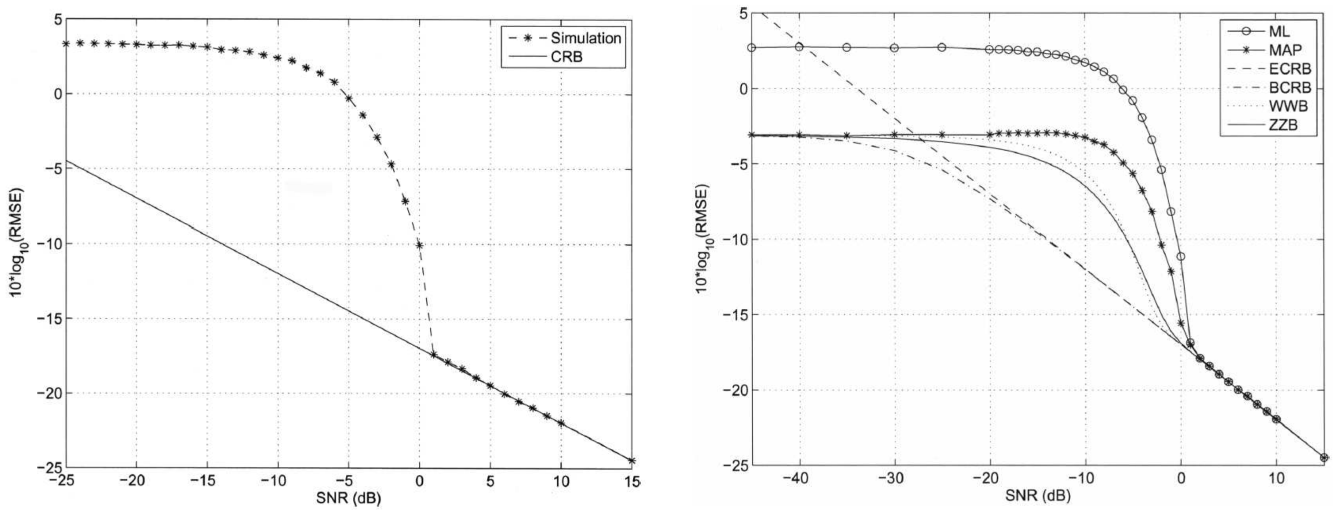
Example 3: Magnetometry

- Linear Gaussian smoother for magnetometry: Petersen and Mølmer, PRA **74**, 043802 (2006).
- QNC:
  - Julsgaard, Kozhekin, and Polzik, Nature **413**, 400 (2001).
  - Wasilewski *et al.*, PRL **104**, 133601 (2010).





- If $P(y|x)$ is not a Gaussian, CRB often grossly underestimate the achievable estimation error.
- Consider M repeated measurements, such that $P(y_1, \ldots, y_M|x) = \prod_{m=1}^{M} P(y_m|x)$.
  - Σ v.s. M (log-log plot) for a classical phase estimation problem:



Van Trees and Bell, Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking (Wiley, Hoboken, 2007)

- Ziv-Zakai bound:

$$\mathbb{E}(\delta x^2) \geq \frac{1}{2}\int_0^{\infty} d\tau\tau \int_{-\infty}^{\infty} dx 2 \min\left[P_X(x), P_X(x+\tau)\right] P_e(x, x+\tau), \quad (75)$$

$P_X(x)$ is prior, $P_e(x, x+\tau)$ is the error probability of a binary hypothesis testing problem with $P(y|\mathcal{H}_0) = P(y|x)$, $P(y|\mathcal{H}_1) = P(y|x+\tau)$, and $P_0 = P_1 = 1/2$.

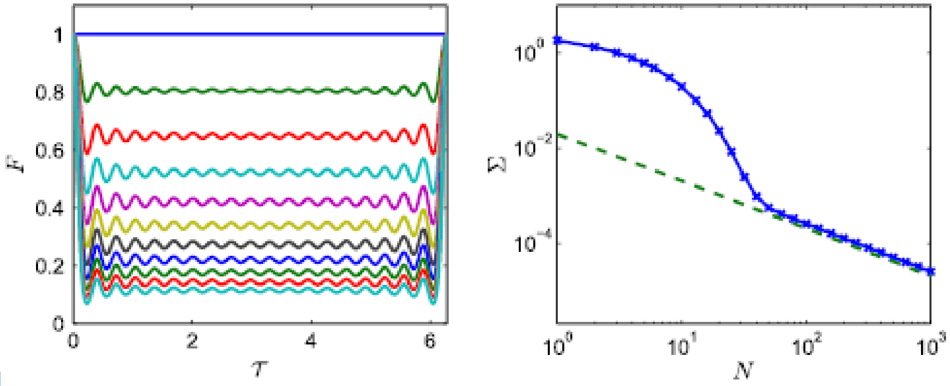
- If $P(y|x) = \mathrm{tr}\,[E(y)\rho_x]$, for any POVM $E(y)$,

$$P_e(x, x+\tau) \geq \frac{1}{2}\left[1 - \frac{1}{2}||\rho_x - \rho_{x+\tau}||_1\right] \geq \frac{1}{2}\left[1 - \sqrt{1 - F(\rho_x, \rho_{x+\tau})}\right]. \tag{76}$$

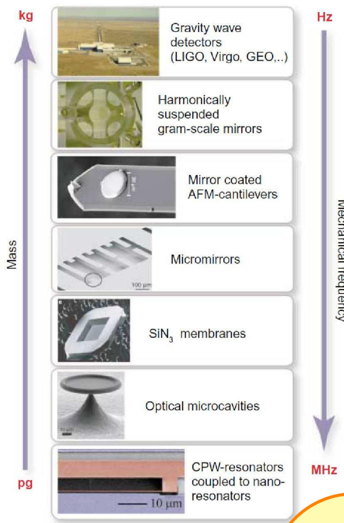
  We immediately obtain quantum Ziv-Zakai bounds on $\mathbb{E}(\delta x^2)$.
- M. Tsang, PRL **108**, 230401 (2012).
- Can be used to prove Heisenberg limit:

$$\mathbb{E}(\delta x^2) \geq \frac{C}{\langle n\rangle^2}. \tag{77}$$

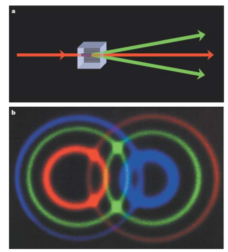
  see also Giovannetti *et al.*, PRL **108**, 260405 (2012); Hall *et al.*, PRA **85**, 041802(R) (2012).
- Compared with QCRB $\mathbb{E}(\delta x^2) \geq 1/4\,\langle \Delta n^2\rangle$, Heisenberg limit is much higher than QCRB when $\langle \Delta n^2\rangle \gg \langle n\rangle^2$
- QZZB vs QCRB for a state proposed by Rivas and Luis in New J. Phys. **14**, 093052 (2012):



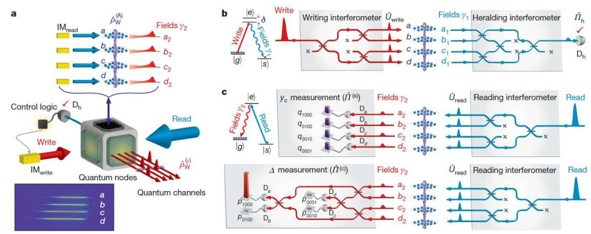
Hybrid Quantum Systems

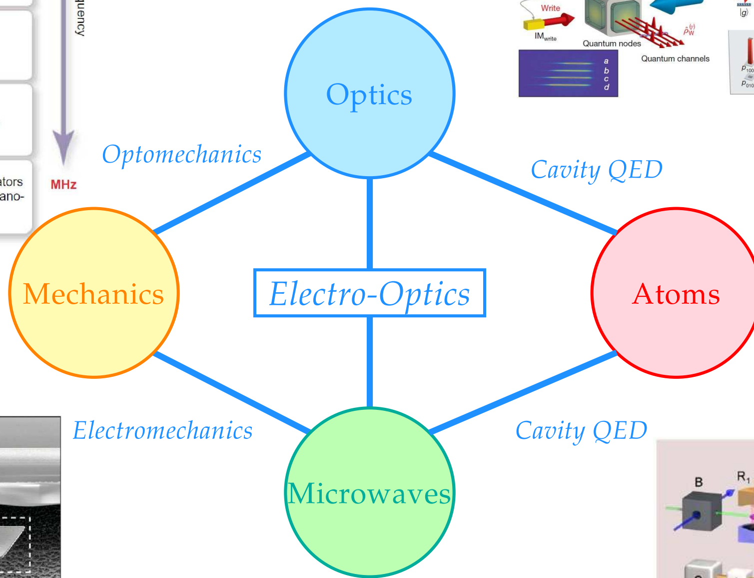


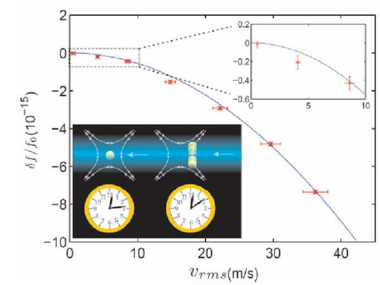
Kippenberg and Vahala,
Science 321, 1172 (2008)



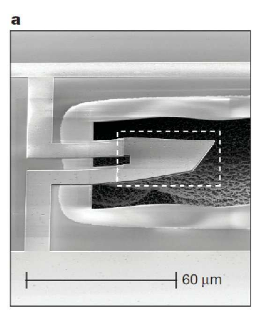
Choi et al.,
Nature 468, 412 (2010)



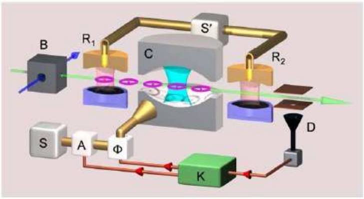




Chou et al., Science 329, 1630 (2010).

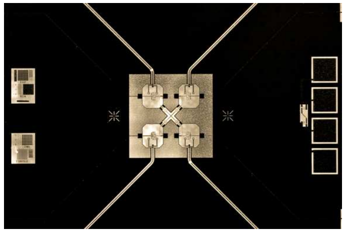


O'Connell et al.,
Nature 464, 697 (2010)



Sayrin et al.,
Nature 477, 73 (2011)

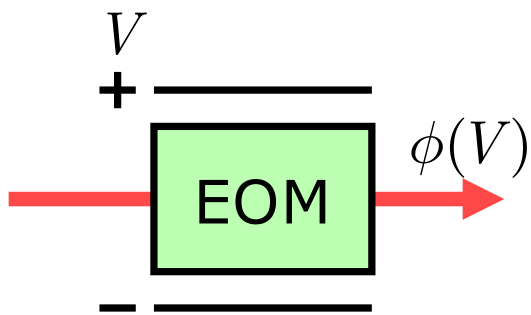
Haroche and Raimond,
Exploring the Quantum



Neeley et al., Nature 467, 570 (2010)

Electro-Optic Modulation

jiadong.com

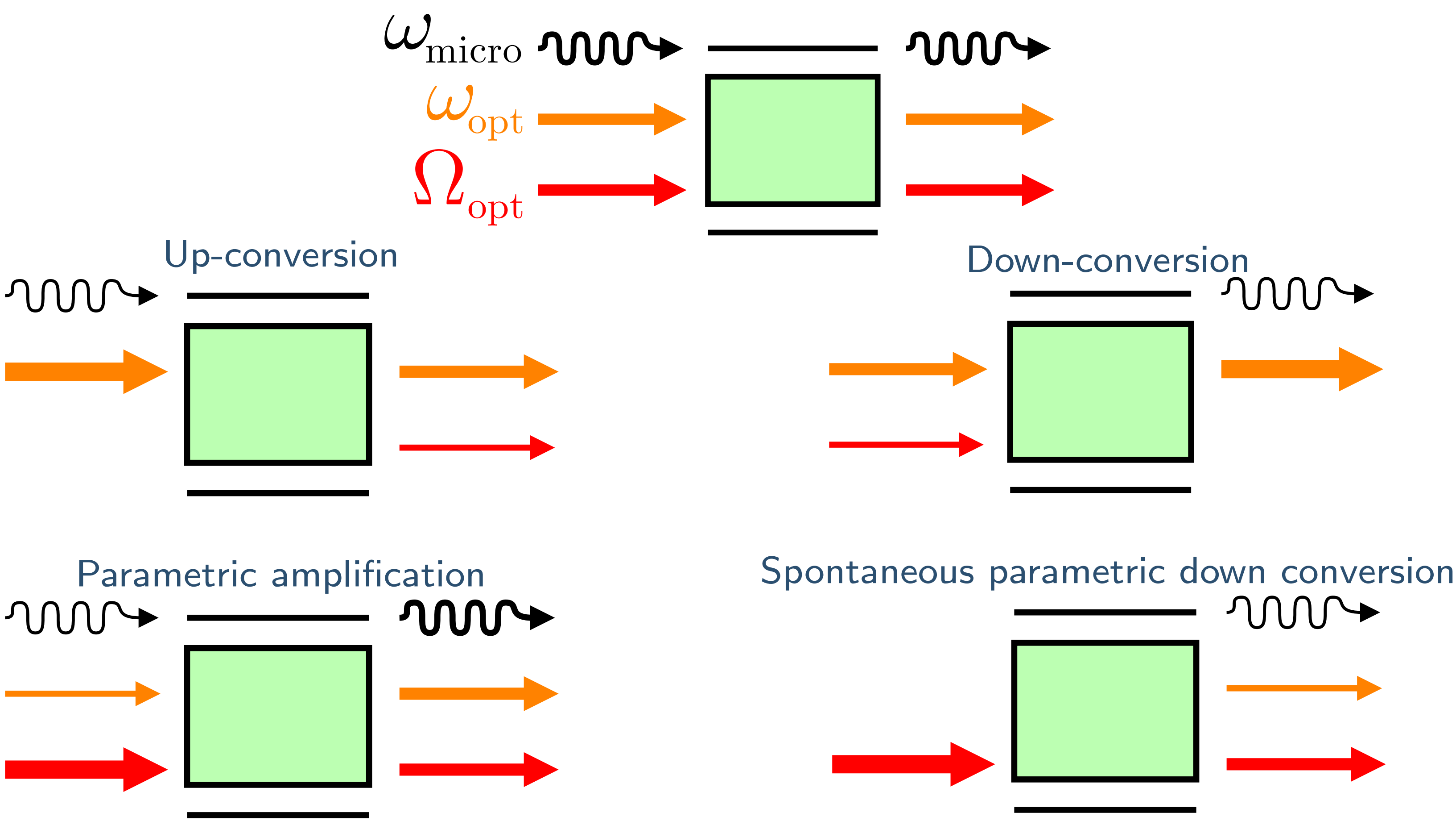


thorlabs.com

- $\boldsymbol{\epsilon} = \epsilon_0 \left(1 + \boldsymbol{\chi}^{(1)} + \boldsymbol{\chi}^{(2)}\boldsymbol{E} + \boldsymbol{\chi}^{(3)} : \boldsymbol{E}\boldsymbol{E} + \ldots\right)$
- $\boldsymbol{\chi}^{(2)}$ (Pockels): $\Delta\phi(V) \propto V$: e.g., Lithium Niobate ($LiNbO_3$)
- Optical:
  - transparent between 350nm-5μm
  - intrinsic $Q \sim 10^6$ resonator at 1.55μm [Ilchenko *et al.*, JOSAB 20, 333 (2003)]
  - 10dB squeezing [Vahlbruch *et al.* PRL 100, 033602 (2008)]
- Microwave:
  - intrinsic $\epsilon_l \approx 28$, $\epsilon_t \approx 45$, $Q \approx 2.3 \times 10^3$ at 9GHz, 300K [Bourreau *et al.*, EL 22, 399 (1986)], loss should decrease with temp.
  - Cu half-wave resonator: $Q \approx 100$ at 9GHz, 300K [Ilchenko *et al.*]
  - 26.5GHz EOM with Nb electrode on $LiNbO_3$ at 4.2K [Yoshida *et al.*, IEEE TMTT 47, 1201 (1999)]

Three-Wave Mixing

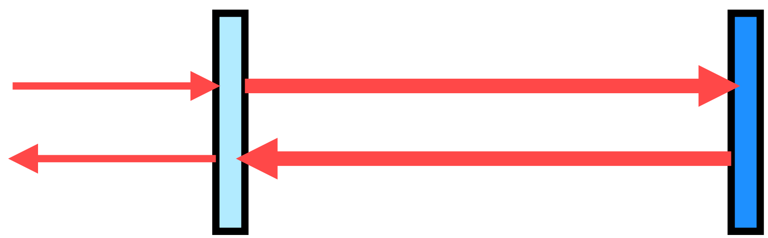
$$\omega_{\text{micro}} + \omega_{\text{opt}} = \Omega_{\text{opt}} \tag{78}$$

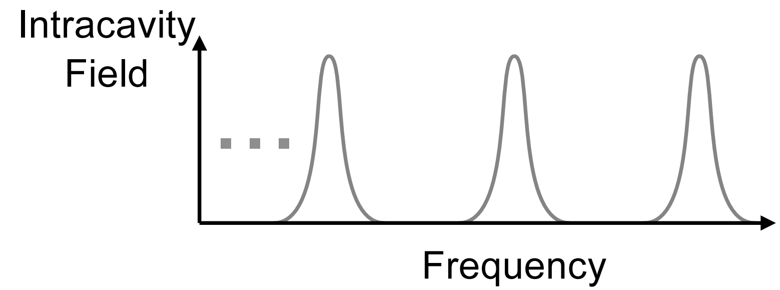


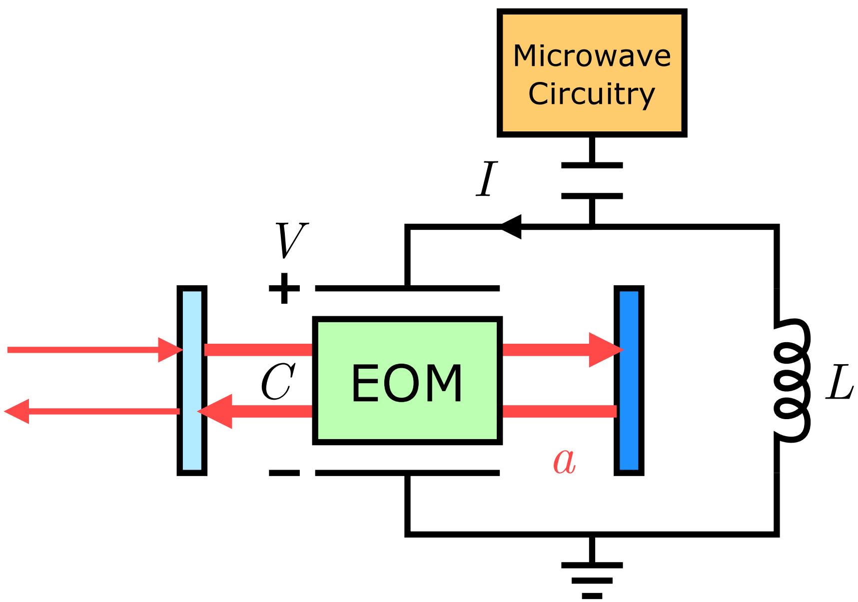
- Spatial mode matching

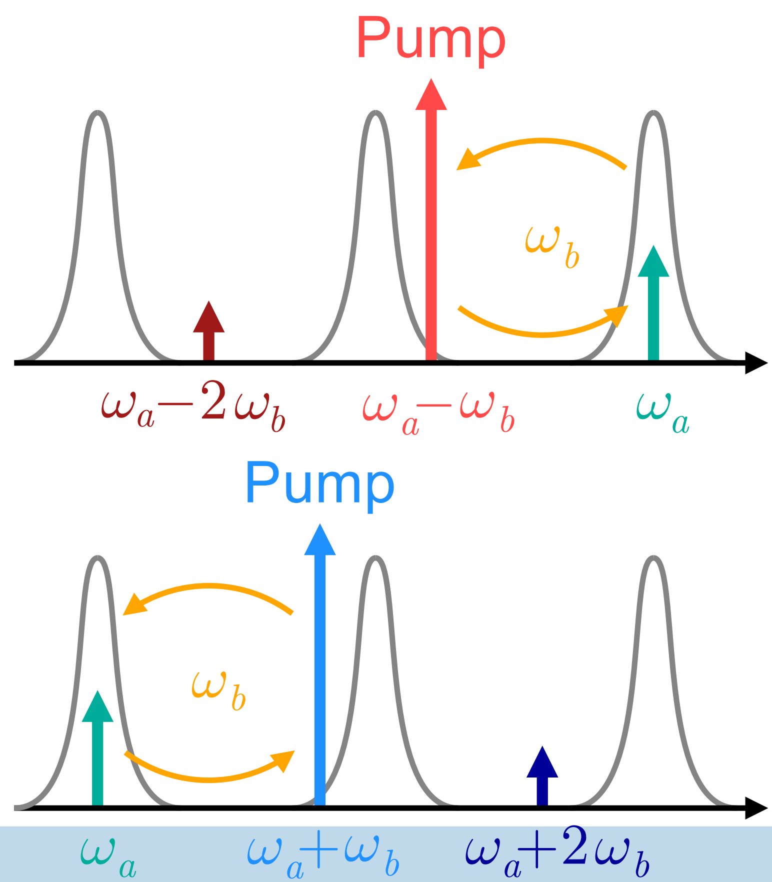
Resonant Enhancement

- How to enhance desired processes and suppress parasitic ones?



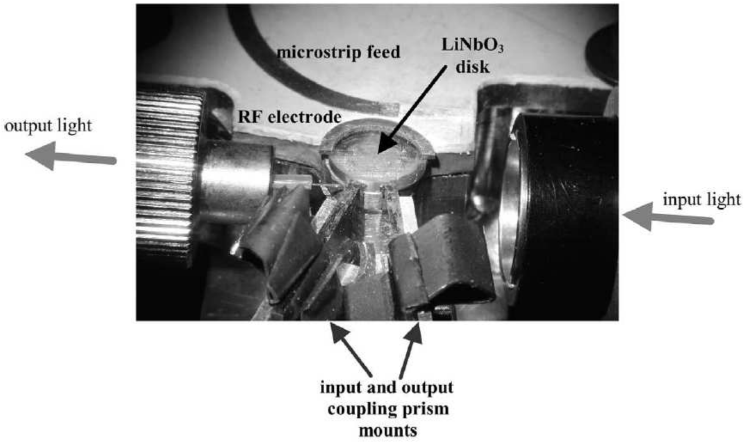






- Similar with microwave resonator

Device Geometry



Cohen *et al.* (USC), "High-Q microphotonic electro-optic modulator," Solid-State Electronics **45**, 1557 (2001)

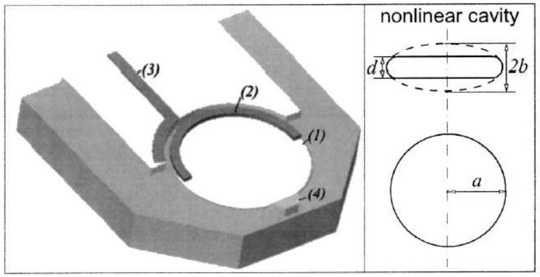
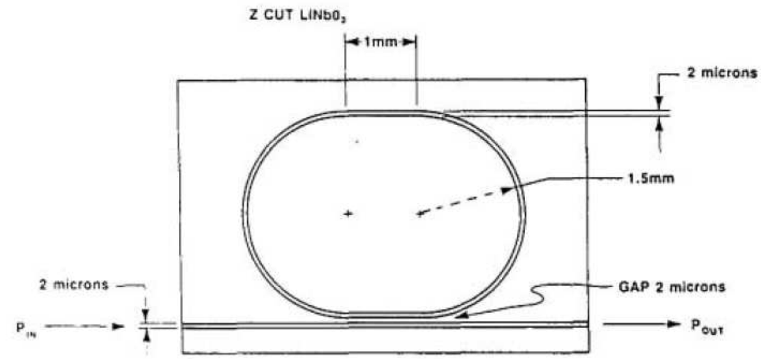
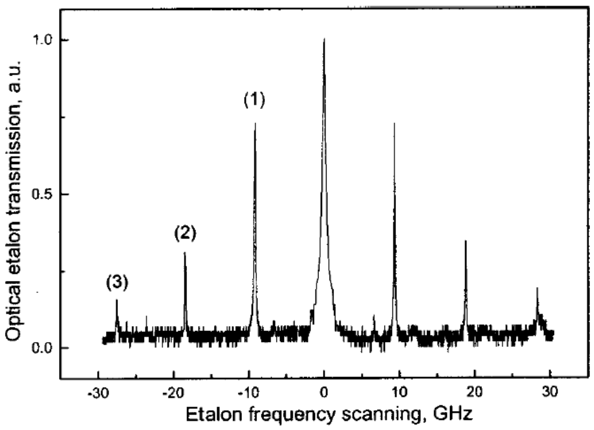


Fig. 1. Experimental setup: (1) $LiNbO_3$ optical cavity, (2) microwave resonator, (3) microwave feeding strip line, and (4) diamond coupling prism. Inset: geometric characteristics of the nonlinear optical cavity.

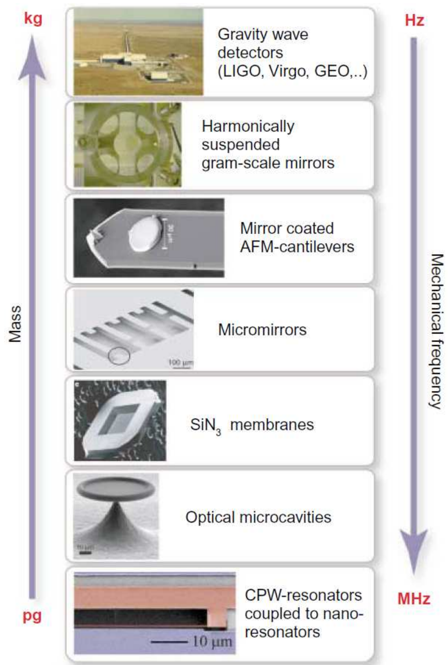
Ilchenko *et al.* (JPL), "Whispering-gallery-mode electro-optic modulator and photonic microwave receiver," J. Opt. Soc. Am. B **20**, 333 (2003), $r = 2.4$ mm, $d = 150$ μm, half-wave 9 GHz resonator



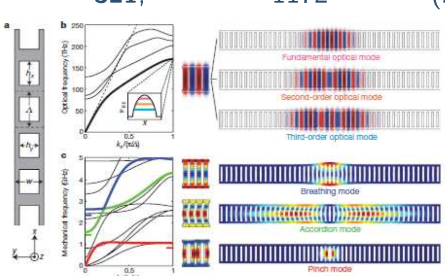
Mahapatra and Robinson, "Integrated-optic ring resonators made by proton exchange in lithium niobate," Appl. Opt. **24**, 2285 (1985).



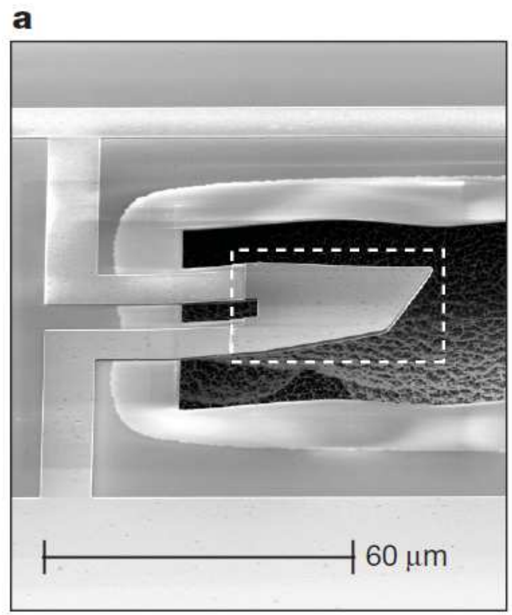
Analogy with Cavity Optomechanics



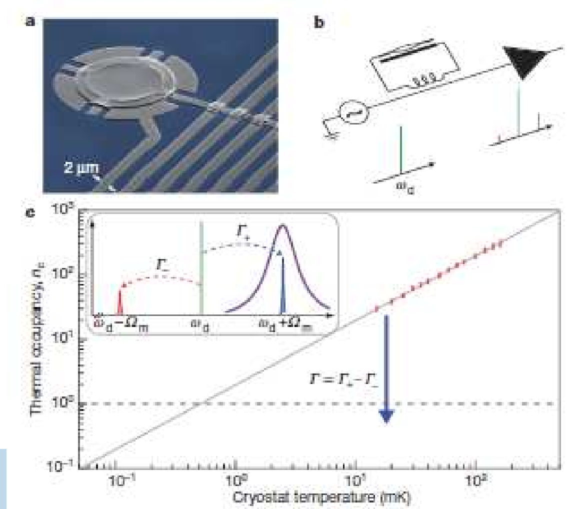
Kippenberg and Vahala, Science **321**, 1172 (2008)



Eichenfield *et al.*, Nature **462**, 78 (2009)

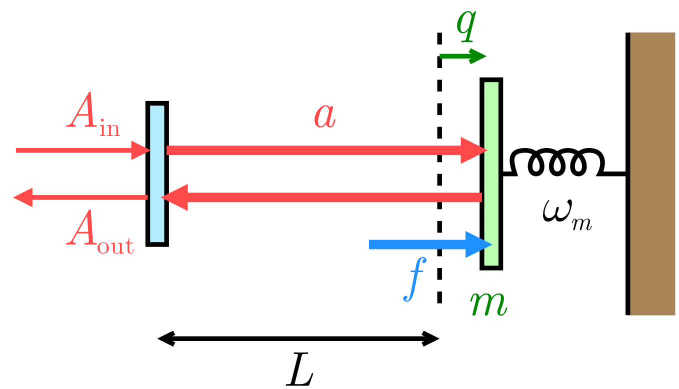


O'Connell *et al.*, Nature **464**, 697 (2010)

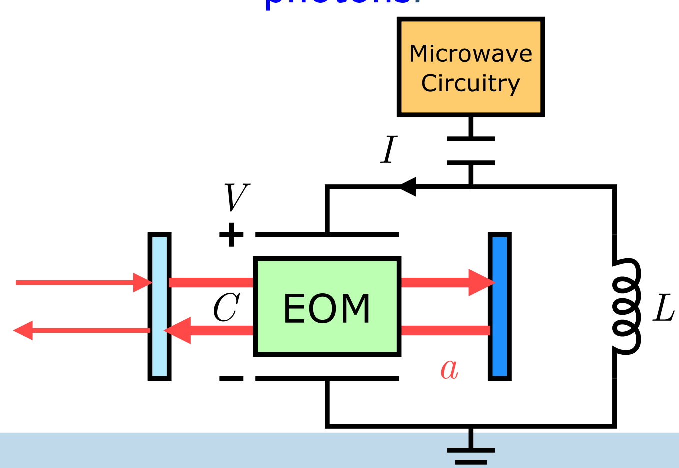


Teufel *et al.*, Nature **464**, 697 (2010)

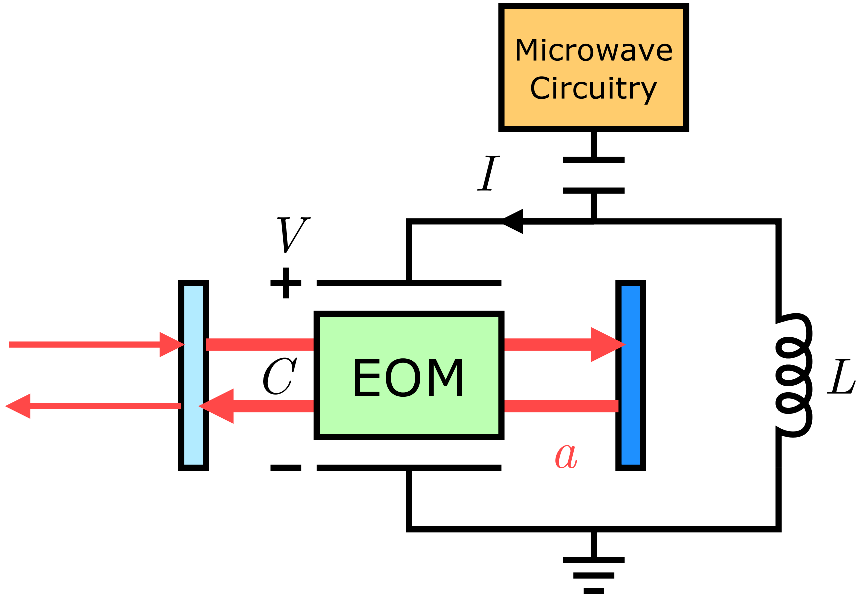
Optical/microwave photons $\leftrightarrow$ microwave/RF phonons:



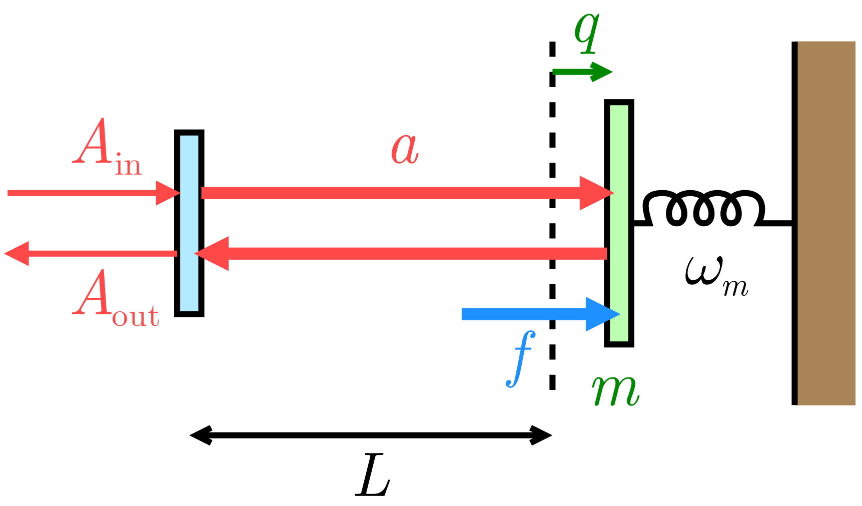
optical photons $\leftrightarrow$ microwave/RF photons:



Optical Photons ↔ Microwave/RF Photons:

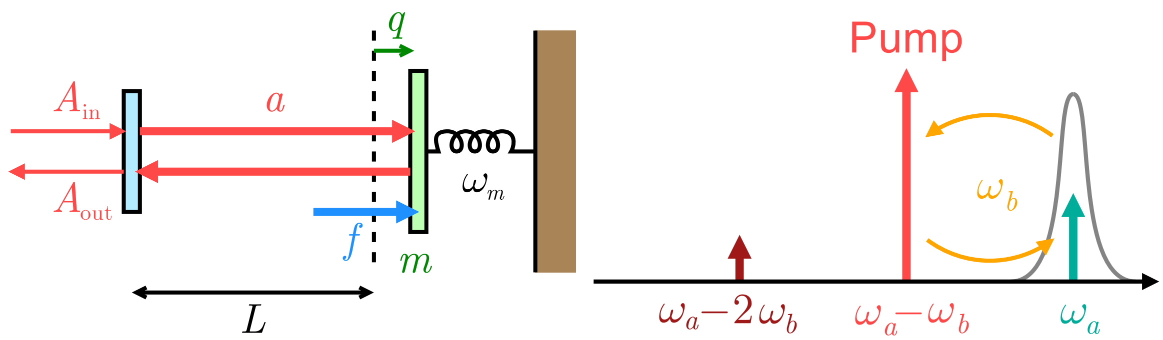


Optical Photons ↔ Microwave/RF Phonons:



$$\hat{H}_I \propto \phi(\hat{V})\hat{a}^\dagger\hat{a} \qquad \hat{H}_I \propto \phi(\hat{q})\hat{a}^\dagger\hat{a} \tag{79}$$

$$\hat{V} \propto \hat{b} + \hat{b}^\dagger \qquad \hat{q} \propto \hat{b} + \hat{b}^\dagger \tag{80}$$



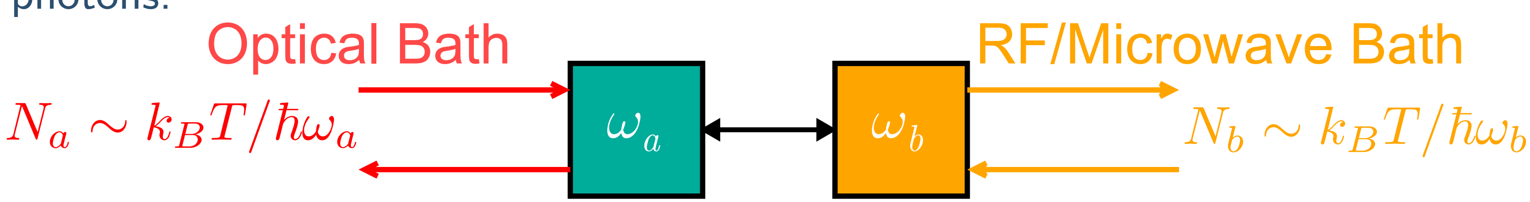
- Interaction picture:

$$H_I = \hbar g \left(b e^{-i\omega_b t} + b^\dagger e^{i\omega_b t} \right) \left[\alpha e^{i(\omega_a - \omega_b)t} + a^\dagger e^{i\omega_a t} \right] \left[\alpha e^{-i(\omega_a - \omega_b)t} + a e^{-i\omega_a t} \right] \qquad (81)$$

- Rotating-wave approximation:

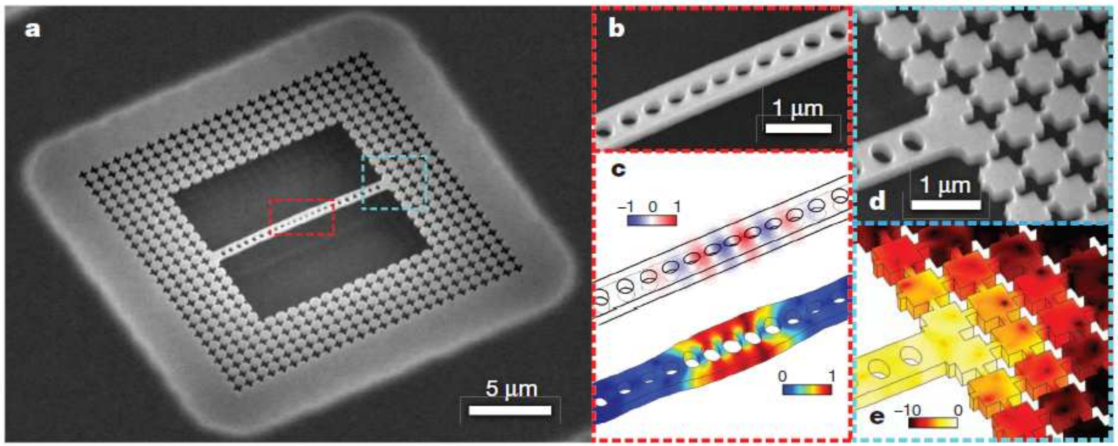
$$H_I \approx \hbar g \alpha \left(b a^\dagger + b^\dagger a \right) \qquad (82)$$

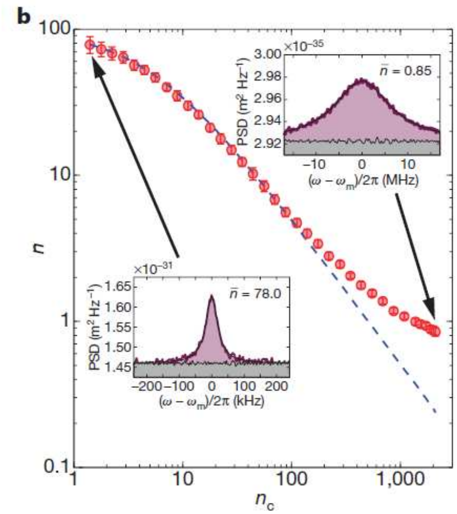
- Exchange photons:



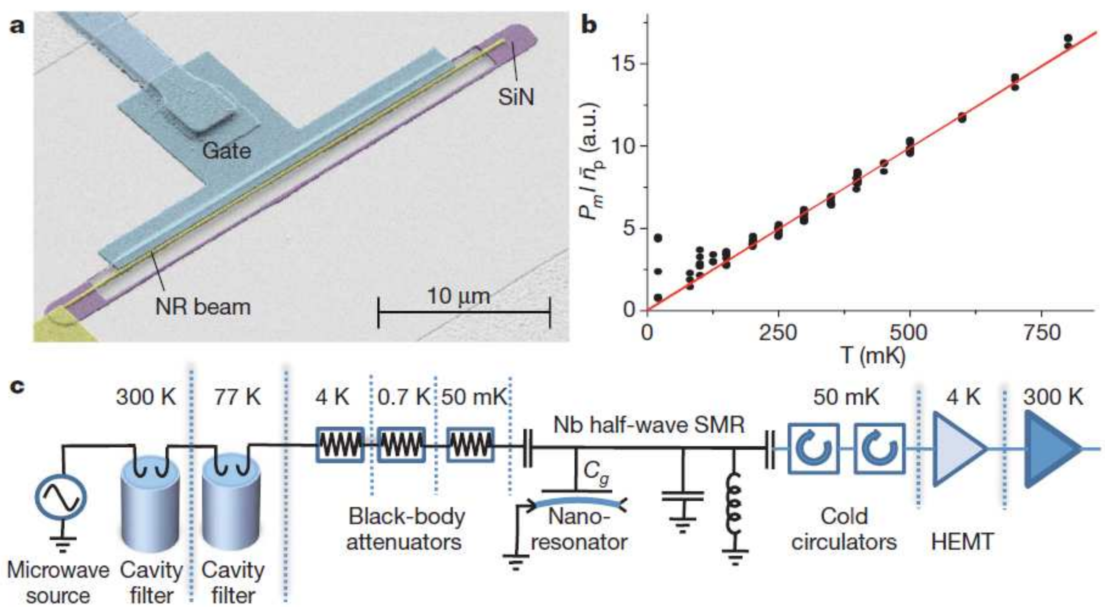
- Wilson-Rae *et al.*, PRL **99**, 093901 (2007); Marquardt *et al.*, 093902 (2007).

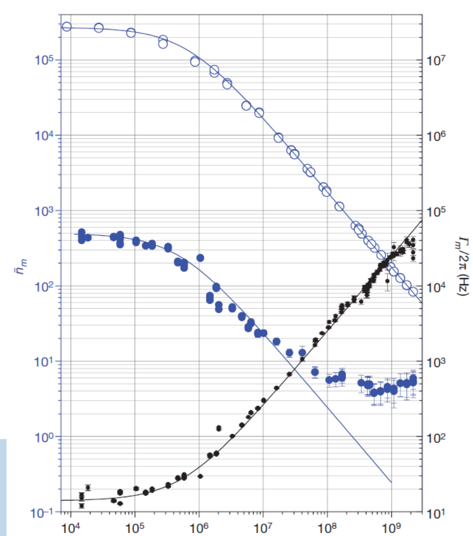
- Chan *et al.*, Nature **478**, 89 (2011)

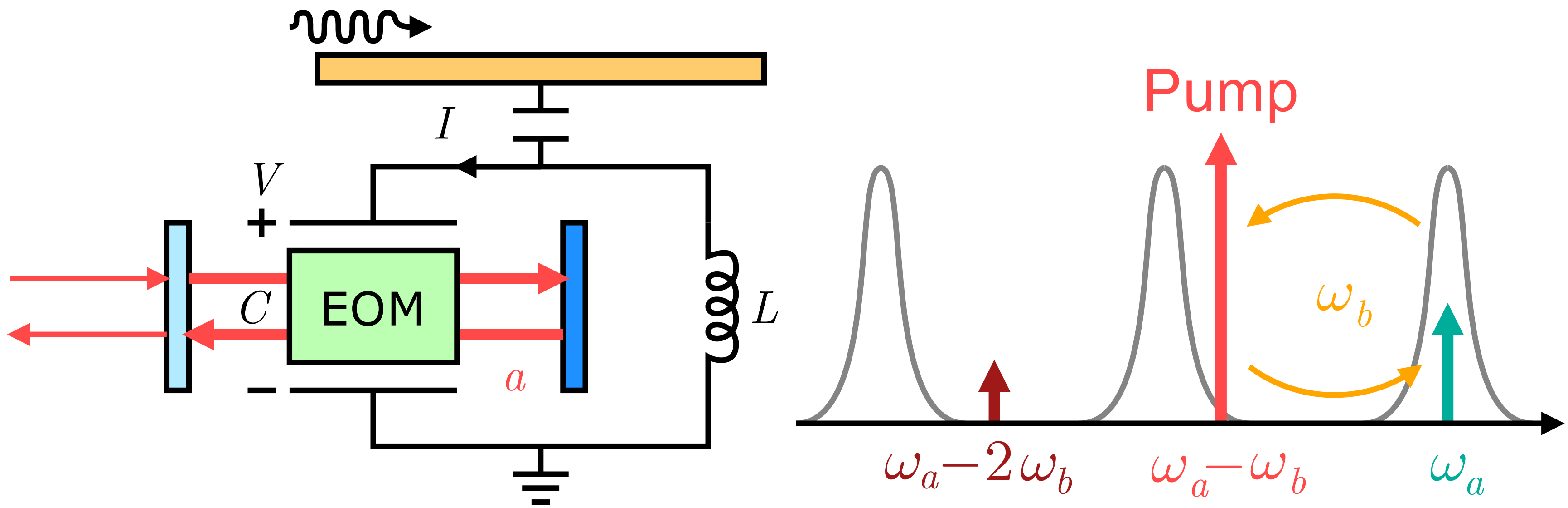




- Rocheleau *et al.*, Nature **463**, 72 (2010)







$$H_I \approx g\sqrt{N_{\text{pump}}}\left(a^\dagger b + a b^\dagger\right) \tag{83}$$

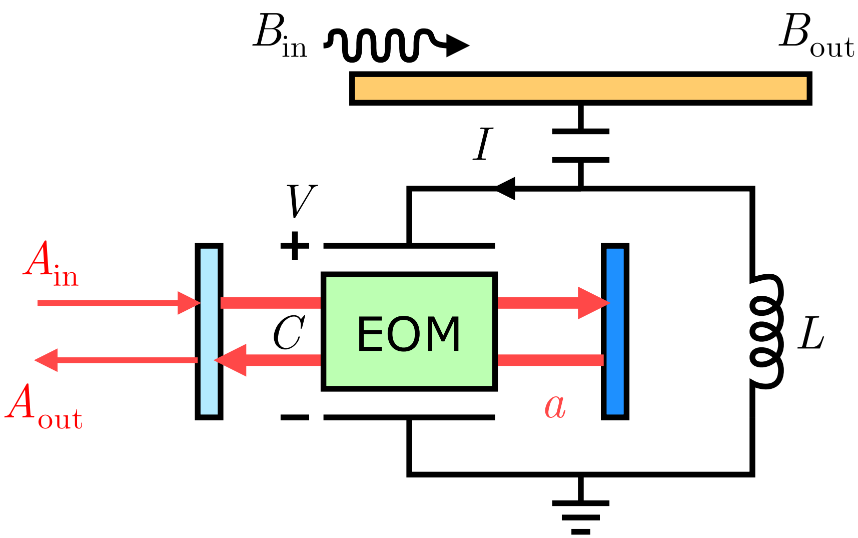
$$g = \eta \frac{\omega_a n^3 r}{2d}\sqrt{\frac{\hbar\omega_b}{2C}}, \tag{84}$$

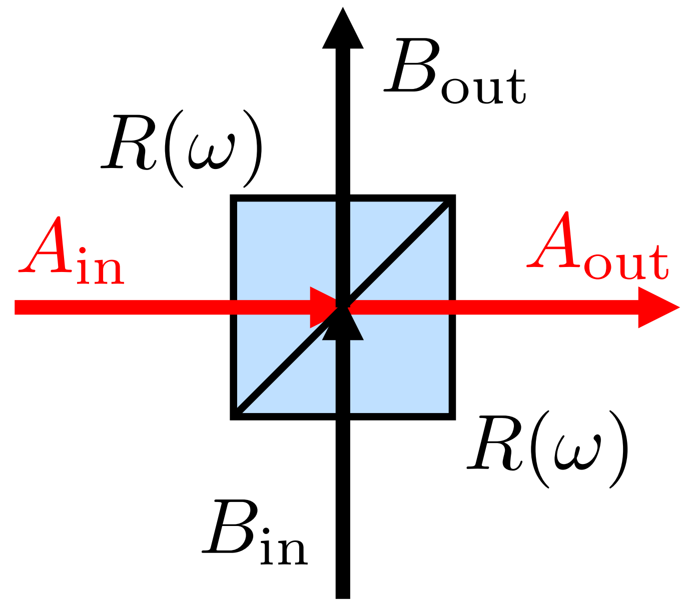
$$\boxed{G \equiv \frac{g^2 N_{\text{pump}}}{\gamma_a \gamma_b}} \tag{85}$$

$$\text{Cooling} : G \gg 1 \tag{86}$$

$$\text{Conversion} : G = 1 \tag{87}$$

Noiseless Frequency Conversion





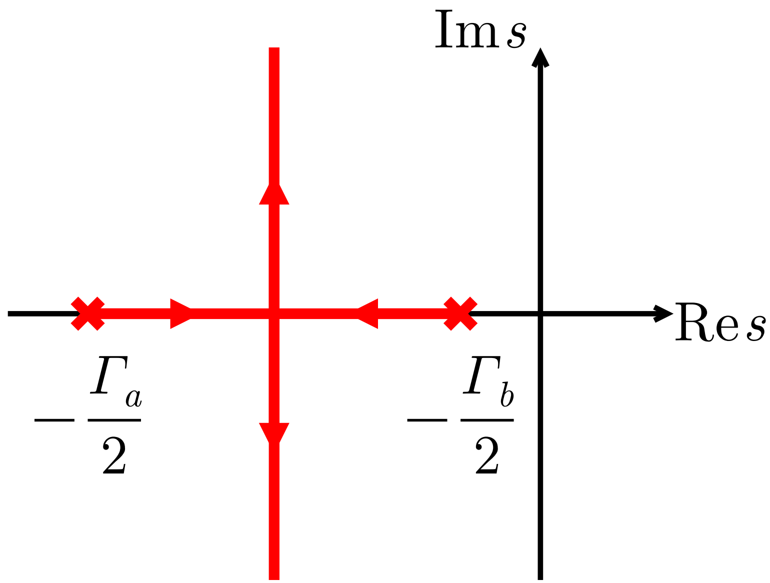
$$\frac{da}{dt} = ig\alpha b - \frac{\Gamma_a}{2} a + \sqrt{\gamma_a} A_{\text{in}} + \sqrt{\gamma_a'} A', \tag{88}$$

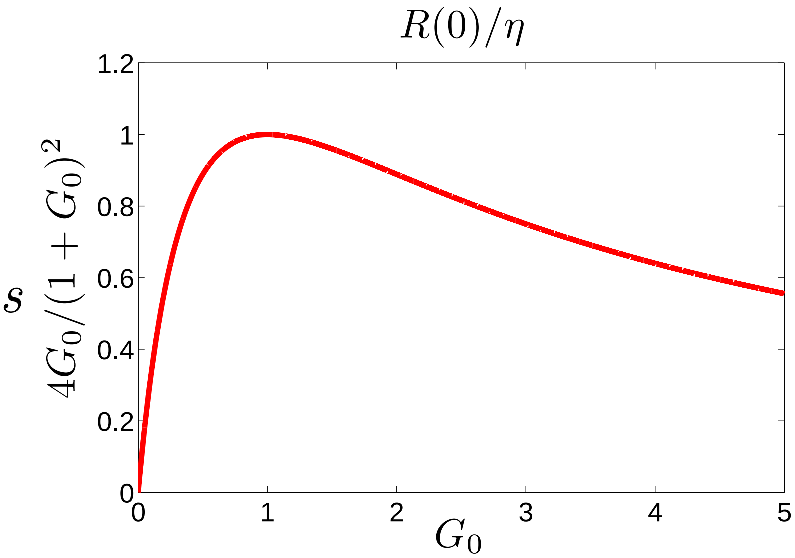
$$\frac{db}{dt} = ig\alpha^* a - \frac{\Gamma_b}{2} b + \sqrt{\gamma_b} B_{\text{in}} + \sqrt{\gamma_b'} B', \tag{89}$$

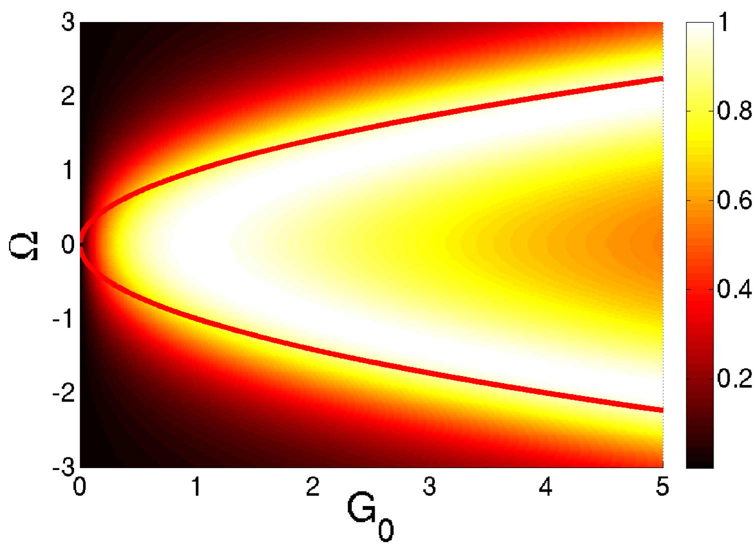
$$A_{\text{out}} = \sqrt{\gamma_a} a - A_{\text{in}}, \quad B_{\text{out}} = \sqrt{\gamma_b} b - B_{\text{in}}, \tag{90}$$

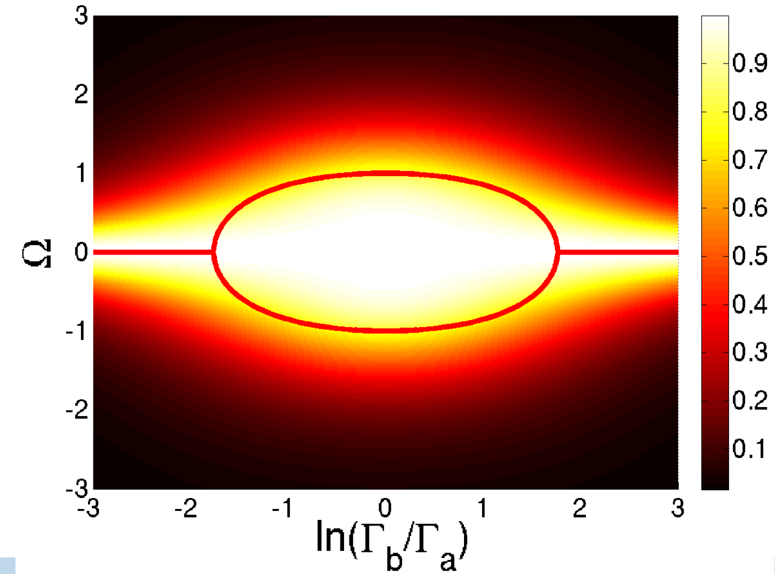
$$\Gamma_{a,b} = \gamma_{a,b} + \gamma_{a,b}', \quad G_0 \equiv \frac{g^2 N_{\text{pump}}}{\Gamma_a \Gamma_b}, \quad \eta \equiv \frac{\gamma_a \gamma_b}{\Gamma_a \Gamma_b}. \tag{91}$$

Plots



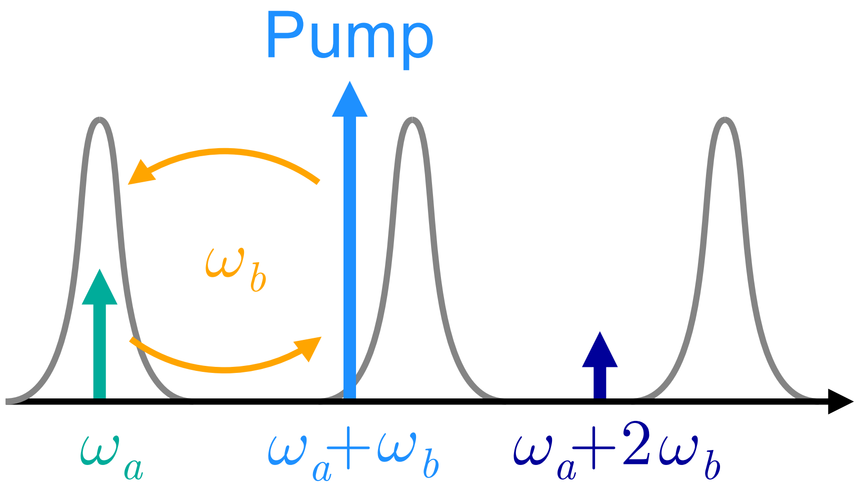


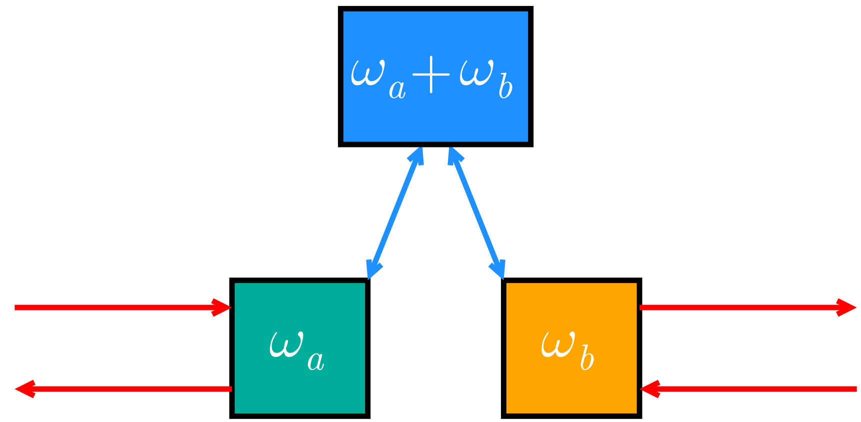




M. Tsang, PRA **84**, 043845 (2011)

Parametric Amplification/Oscillation





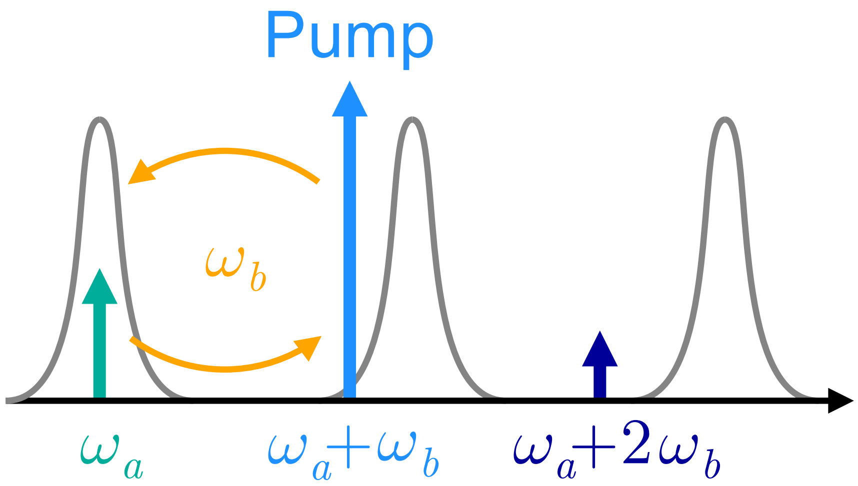
- Photon-pair creation/annihilation:

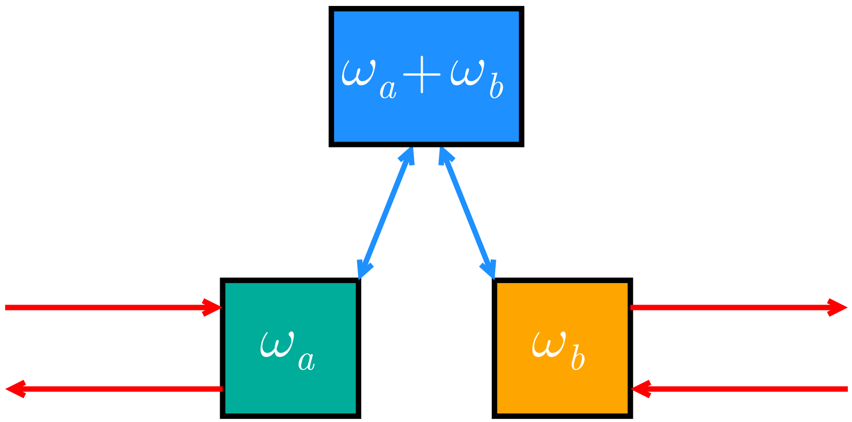
$$H_I \approx g\sqrt{N_{\text{pump}}}\left(a^\dagger b^\dagger + ab\right), \quad \boxed{G \equiv \frac{g^2 N_{\text{pump}}}{\gamma_a \gamma_b}} \tag{92}$$

$$\text{Oscillation} : G \geq 1, \tag{93}$$

$$\text{Entangled Photon Pairs} : G \ll 1 \tag{94}$$

Parametric Amplification/Oscillation





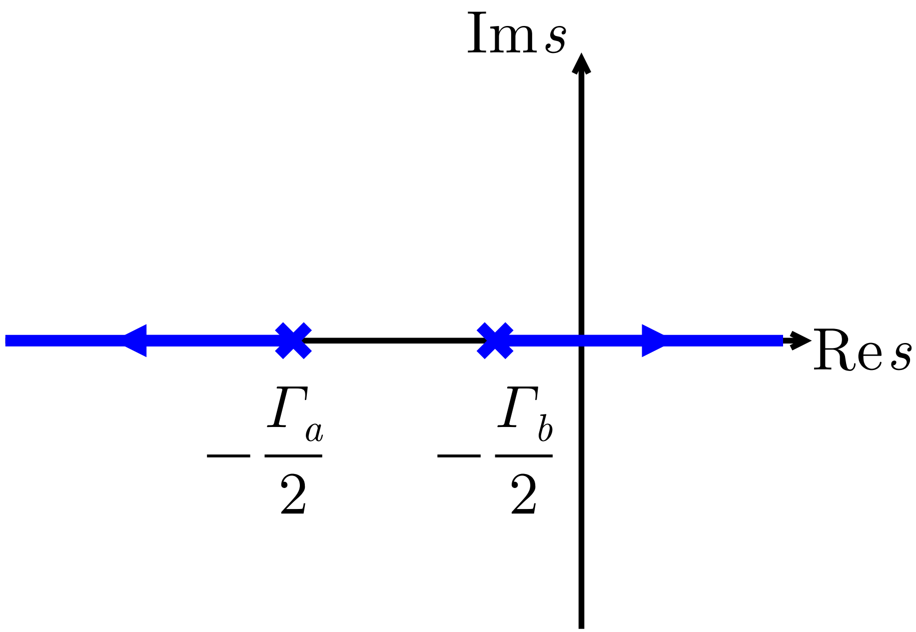
$$\frac{da}{dt} = ig\alpha b^\dagger - \frac{\Gamma_a}{2}a + \sqrt{\gamma_a}A_{\text{in}} + \sqrt{\gamma_a'}A', \tag{95}$$

$$\frac{db}{dt} = ig\alpha a^\dagger - \frac{\Gamma_b}{2}b + \sqrt{\gamma_b}B_{\text{in}} + \sqrt{\gamma_b'}B', \tag{96}$$

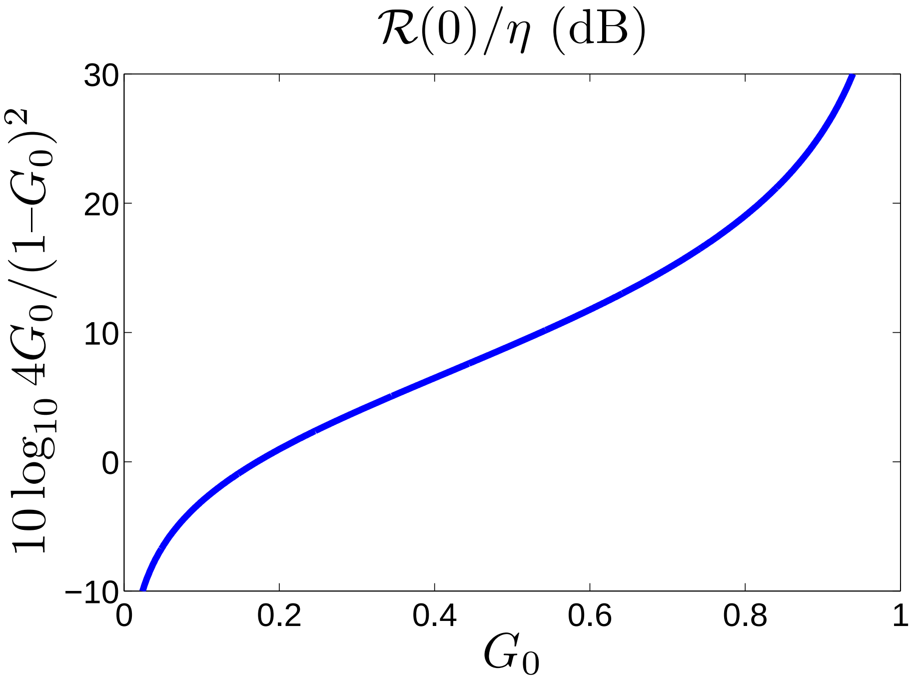
$$A_{\text{out}} = \sqrt{\gamma_a}a - A_{\text{in}}, \tag{97}$$

$$B_{\text{out}} = \sqrt{\gamma_b}b - B_{\text{in}}. \tag{98}$$

Plots

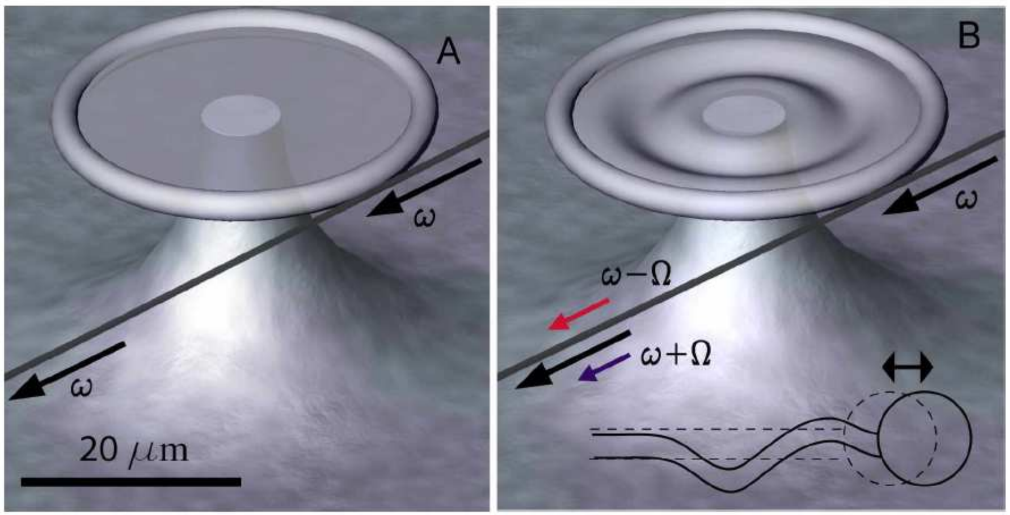


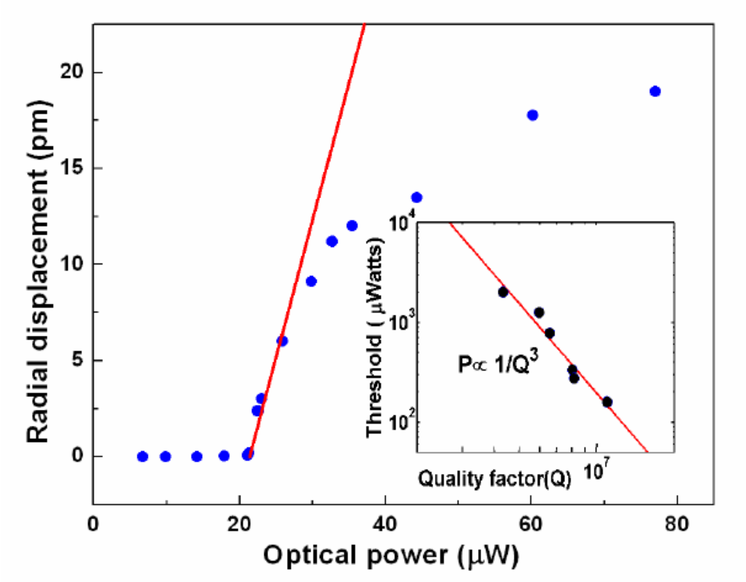
$\mathrm{Im}\, s$
$\mathrm{Re}\, s$
$-\frac{\Gamma_a}{2}$
$-\frac{\Gamma_b}{2}$



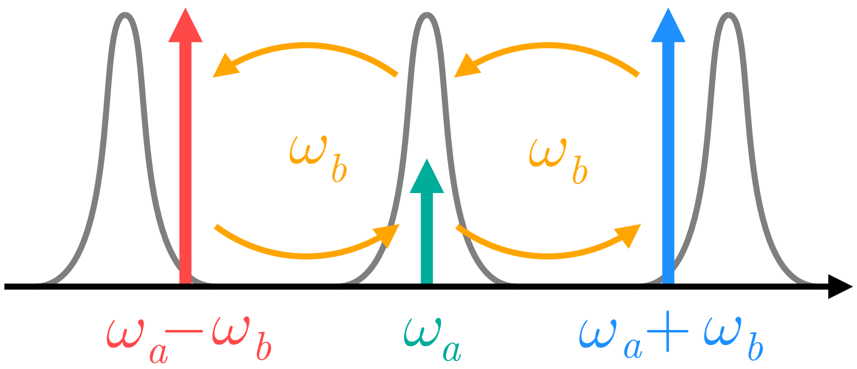
$\mathcal{R}(0)/\eta$ (dB)
$10 \log_{10} 4G_0/(1-G_0)^2$
30
20
10
0
−10
0
0.2
0.4
0.6
0.8
1
G_0

- Rokhsari *et al.*, Opt. Express **13**, 5293 (2005):





Double-Sideband Pumping



- Double-sideband pumping: backaction-evading microwave quadrature measurement [Thorne *et al.*, PRL **40**, 667 (1978); electromechanics experiment: Hertzberg *et al.*, Nature Phys. **6**, 213 (2010)].

$$H_I \approx g\sqrt{N_{\rm pump}}\left(e^{-i\bar{\theta}}a + e^{i\bar{\theta}}a^\dagger\right)\left(e^{-i\delta}b + e^{i\delta}b^\dagger\right), \tag{99}$$

$$\bar{\theta} \equiv \frac{\theta_+ + \theta_-}{2}, \quad \delta \equiv \frac{\theta_+ - \theta_-}{2}. \tag{100}$$

- $\theta_\pm$ are phases of the pump beams and control which quadratures are coupled.
- $\chi^{(3)}$ (Kerr): $\phi(V) \propto \chi^{(3)}V^2$, backaction-evading microwave energy measurement:

$$H_I \propto \chi^{(3)}V^2 a^\dagger a. \tag{101}$$

Coupling Strength

$$G = \frac{g^2 N_{\text{pump}}}{\gamma_a \gamma_b}, \qquad g = \eta \frac{\omega_a n^3 r}{2d} \sqrt{\frac{\hbar \omega_b}{2C}}. \tag{102}$$

- Ilchenko *et al.*, JOSAB 20, 333 (2003) ($\gamma_a \approx 2\pi \times 90$ MHz, $\gamma_b \approx 2\pi \times 50$ MHz, $d \approx 150\mu$m):

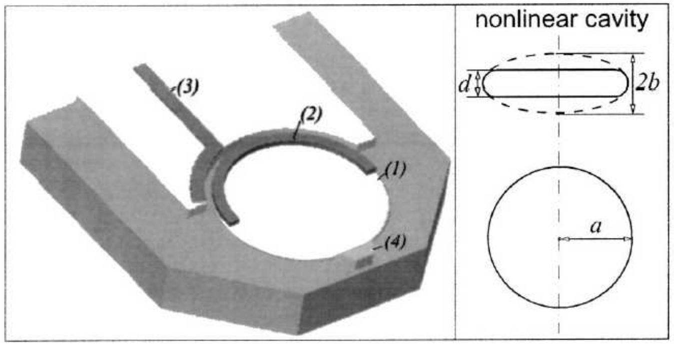
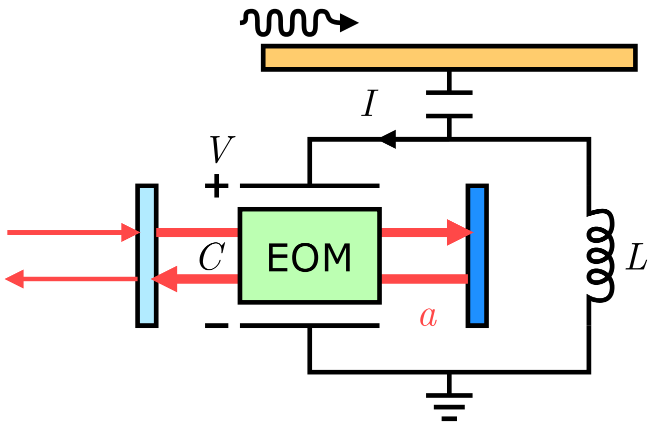


Fig. 1. Experimental setup: (1) $LiNbO_3$ optical cavity, (2) microwave resonator, (3) microwave feeding strip line, and (4) diamond coupling prism. Inset: geometric characteristics of the nonlinear optical cavity.

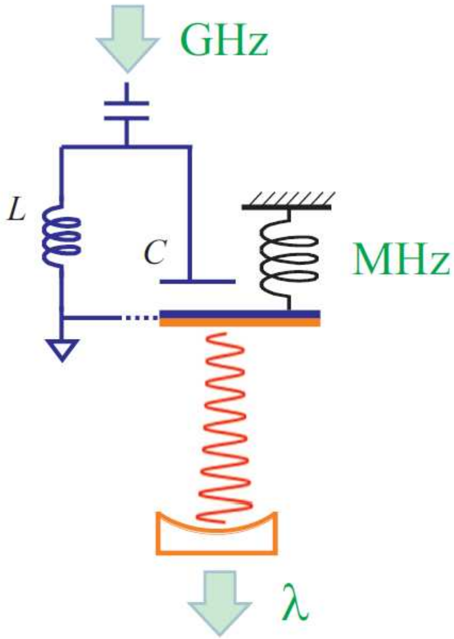
$$g \approx 20 \text{ Hz}, \qquad G \approx 2 \times 10^{-5} \text{ at 2 mW pump} \tag{103}$$

- g can be improved by $\sim 10^1 - 10^2$, γ_b reduced by $\sim 10^3$ using superconducting microwave resonator
- r in $BaTiO_3$ and KTN is higher than $LiNbO_3$ by $10^1 - 10^2$

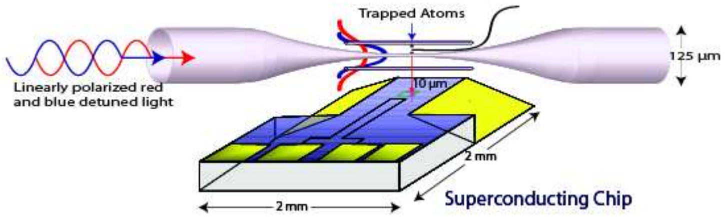
Competition: Electro-optomechanics, Atoms



M. Tsang, PRA **81**, 063837 (2010); **84**, 043845 (2011).



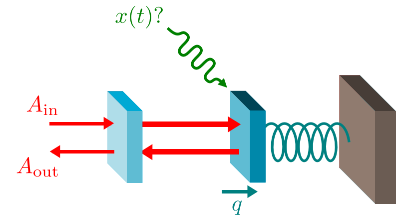
Regal and Lehnert, J. Phys.: Conf. Series **264**, 012025 (2011); Safavi-Naeini and Painter, NJP 13, 013017 (2011); Taylor *et al.*, PRL 107, 273601 (2011)

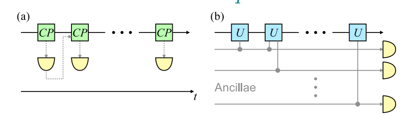


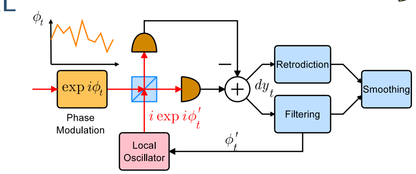
Hoffman *et al.*, arXiv:1108.4153 (2011); Hafezi *et al.*, PRA 85, 020302(R) (2012)

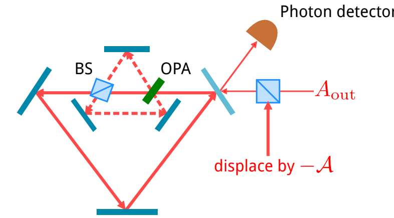
	Electro-optics	Mechanics	Atoms
Effect	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(3)}$
Pumps	optical	optical + microw.	optical + microw.
Resonators	microw. + optical	microw. + optical + mech.	microw. + optical + atoms
Experiment	$g = 20$ Hz (Ilchenko)	N/A	N/A

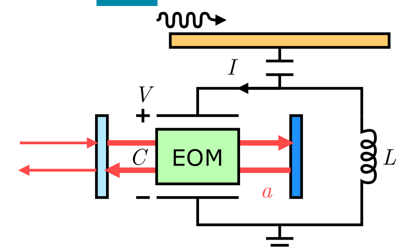
- **Waveform Detection**
  - Estimation: M. Tsang, PRL **108**, 170502 (2012).
  - Fundamental Limits/Control: M. Tsang and R. Nair, PRA **86**, 042115 (2012).
- **Waveform Estimation**
  - Fundamental Limits: M. Tsang, H. M. Wiseman, and C. M. Caves, PRL **106**, 090401 (2011).
  - Estimation: M. Tsang, J. H. Shapiro, and S. Lloyd, PRA **78**, 053820 (2008); **79**, 053843 (2009); M. Tsang, PRL **102**, 250403 (2009); PRA **80**, 033840 (2009); **81**, 013824 (2010).
  - Control: M. Tsang and C. M. Caves, PRL **105**, 123601 (2010); PRX **2**, 031016 (2012).
- **Parameter Estimation Beyond CRB**
  - Quantum Ziv-Zakai Bound: M. Tsang, PRL **108**, 230401 (2012).
  - Rate Distortion: R. Nair, arXiv:1204.3761.
- **Quantum Electro-optics**
  - M. Tsang, PRA **81**, 063837 (2010); **84**, 043845 (2011).
- **Open-System Quantum Metrology**
  - M. Tsang, e-print arXiv:1301.5733v3 (2013).
- **Imaging**
  - Superresolution: M. Tsang, PRA **75**, 043813 (2007); PRL **101**, 033602 (2008); PRL **102**, 253601 (2009).
  - Stellar Interferometry: M. Tsang, PRL **107**, 270402 (2011).
  - Computational Imaging: L. Waller, M. Tsang *et al.*, Opt. Express **19**, 2805 (2011).
  - Metamaterials: M. Tsang and D. Psaltis, Opt. Express **15**, 11959 (2007); PRB **77**, 035122 (2008).

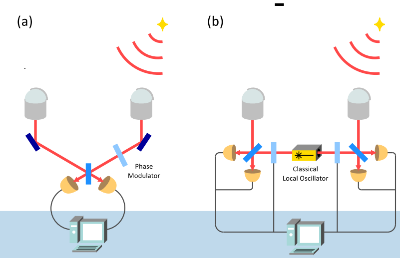












Collaborations

- **Cavity electro-optics**: with Aaron Danner at National U. Singapore
- **Parameter estimation for optomechanical force sensing**: with Warwick Bowen at U. Queensland
- **Time-varying optical phase estimation**: with Hidehiro Yonezawa at U. Tokyo, Elanor Huntington *et al.* at U. New South Wales
  - `http://mankei.tsang.googlepages.com/`
  - `eletmk@nus.edu.sg`
- Supported by Singapore National Research Foundation Fellowship (NRF-NRFF2011-07).