The Unruh effect - An intimate dialogue between QFT and Quantum Shannon Theory

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Outline

- Part I – Quantum information theory (QIT) reminder
- Part II – Classical/Quantum Shannon Theory
- Part III – The Unruh channel
- Part IV – QFT and Quantum Shannon Theory

What is the relevance of cloning channels to the Unruh effect?

Quantum information

- QIT is nothing else than QM
- QM seen from the quantum Shannon theory perspective reveals its hidden treasures

- It addresses some fundamental physical questions

Quantum information

- QM teaches that quantum dynamics is unitary
- Certainly true but we always do not have control over the whole system
- On the side of quantum states this leads to the notion of mixed states
- On the side of quantum transformations this leads to the notion of an open quantum system evolution

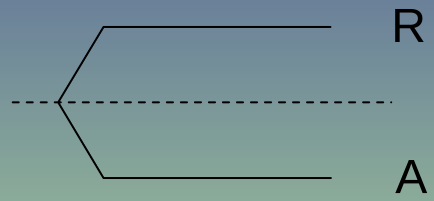
Quantum information

- QIT uses some funny names in this context and I will use them too
- PURIFICATION – A reverse operation wrt tracing over a quantum subsystem
- COMPLETELY POSITIVE MAPS (alias CP MAPS or QUANTUM CHANNELS) – Transformation mapping density matrices to density matrices

Quantum information

The reference system (another Hilbert space)

$$\varrho_A \xrightarrow{\text{purification}}$$



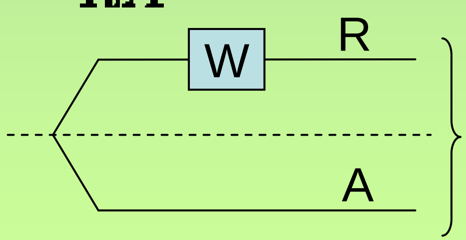
$$\Phi_{RA}$$

ϱ_A is purified by Φ_{RA} iff $\mathrm{Tr}_{\mathrm{R}}\left[\Phi_{\mathrm{RA}}\right] = \varrho_{\mathrm{A}}$

- How to?

1. Find the eigenvalues of $\varrho_A \to \sum_k \lambda_k |k\rangle\langle k|$
2. Create a pure state (Schmidt form) $|\Phi\rangle_{AR} = \sum_k \sqrt{\lambda_k}\, |k\rangle_A |\sigma_k\rangle_R$

- For any unitary W $\tilde{\Phi}_{RA}$ is also a purification of ϱ_A



$$\tilde{\Phi}_{RA}$$

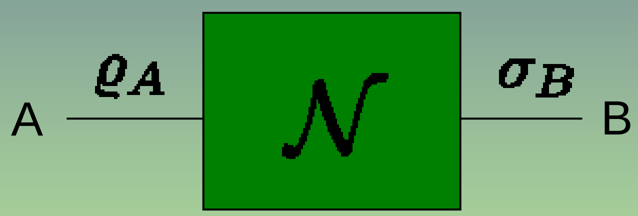
Quantum information

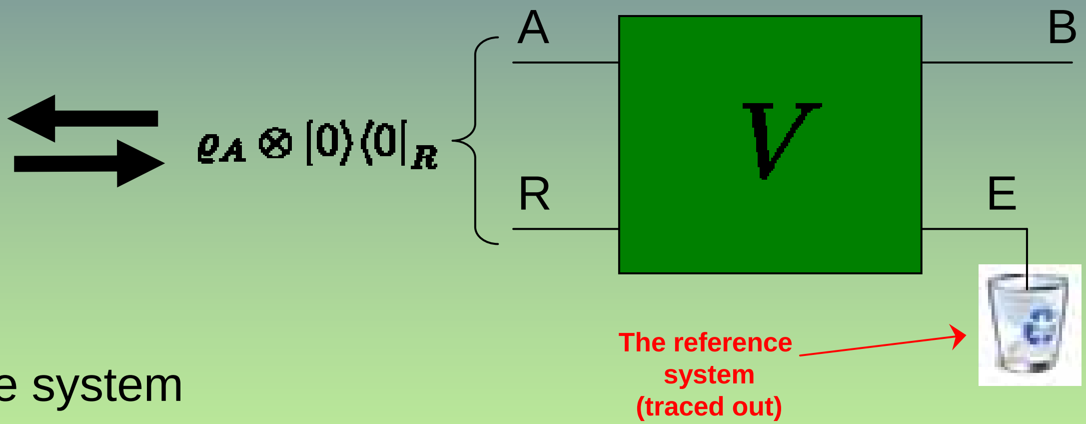
If $\mathcal{N}$ is a Trace-Preserving Completely Positive Map (a quantum channel)

$$\mathcal{N} : \varrho_A \mapsto \sigma_B$$

There exists a unitary operator (or an isometry)

$$V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$$



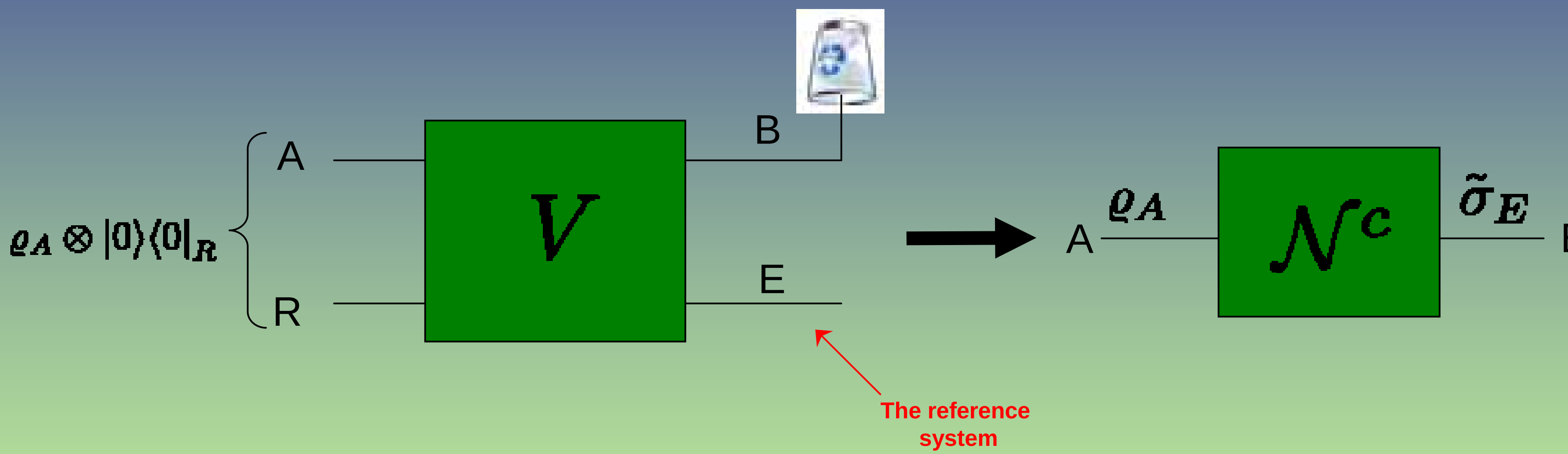


Let's trace over the reference system

$$\mathcal{N}(\varrho_A) = \mathrm{Tr}_E[V(\varrho_A \otimes |0\rangle\langle 0|_R)V^\dagger] = \sigma_B$$

- CP maps are not in reality the most general transformations but let's ignore this fact for now

Quantum information



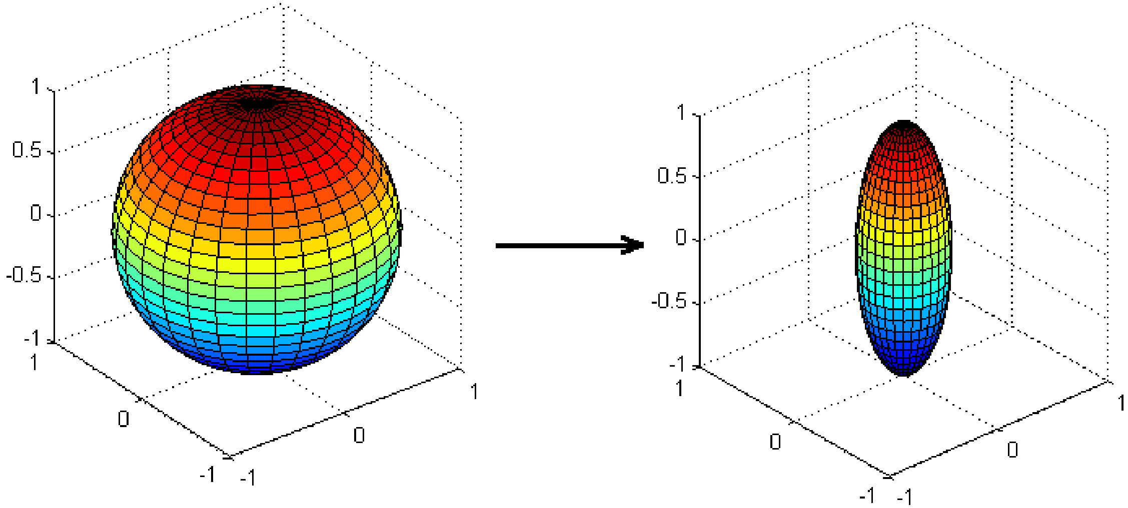
Let's not trace over the reference system but instead over the B system

$$\mathcal{N}^c(\varrho_A) = \mathrm{Tr_B}[\mathrm{V}(\varrho_\mathrm{A} \otimes |0\rangle\langle 0|_\mathrm{R})\mathrm{V}^\dagger] = \tilde{\sigma}_\mathrm{E}$$

$\mathcal{N}^c$ is called the complementary channel to $\mathcal{N}$

Quantum information

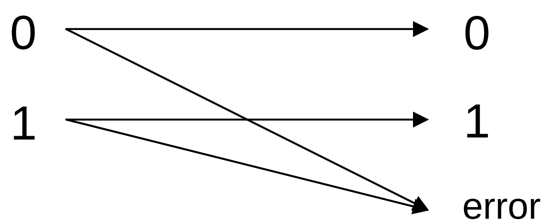
- Dephasing channel $\mathcal{D}(\varrho) \propto p\varrho + (1-p)\sigma_Z \varrho \sigma_Z$



- Erasure channel

$\mathcal{E}$)&* > q& ,)2 − q*|g⟩⟨g|

- Classically:



Classical/Quantum Communication

Questions so far?

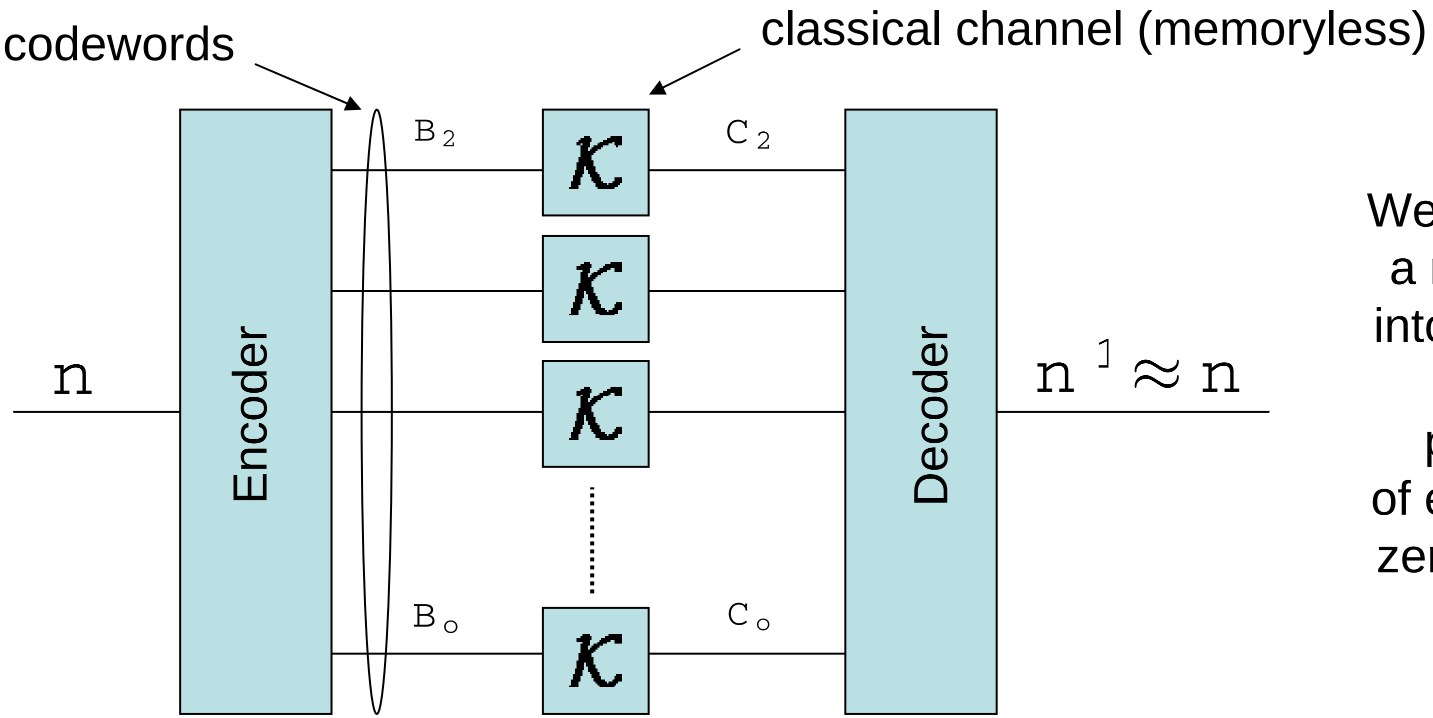
Quantum communication

- Motivated by classical communication theory (Shannon, 1948)

The important concept is the (Noisy) Channel Capacity

- How many bits per channel use can be transmitted?
- At the same time, can we do it such that the probability of error at the output can be arbitrarily small?

Classical communication

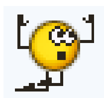


We can encode a message *m* into *n* bits such that the probability of error goes to zero as n goes to infinity

The rate is nonzero $0 \leq R \leq I(A:B) = H(B) - H(B|A)$

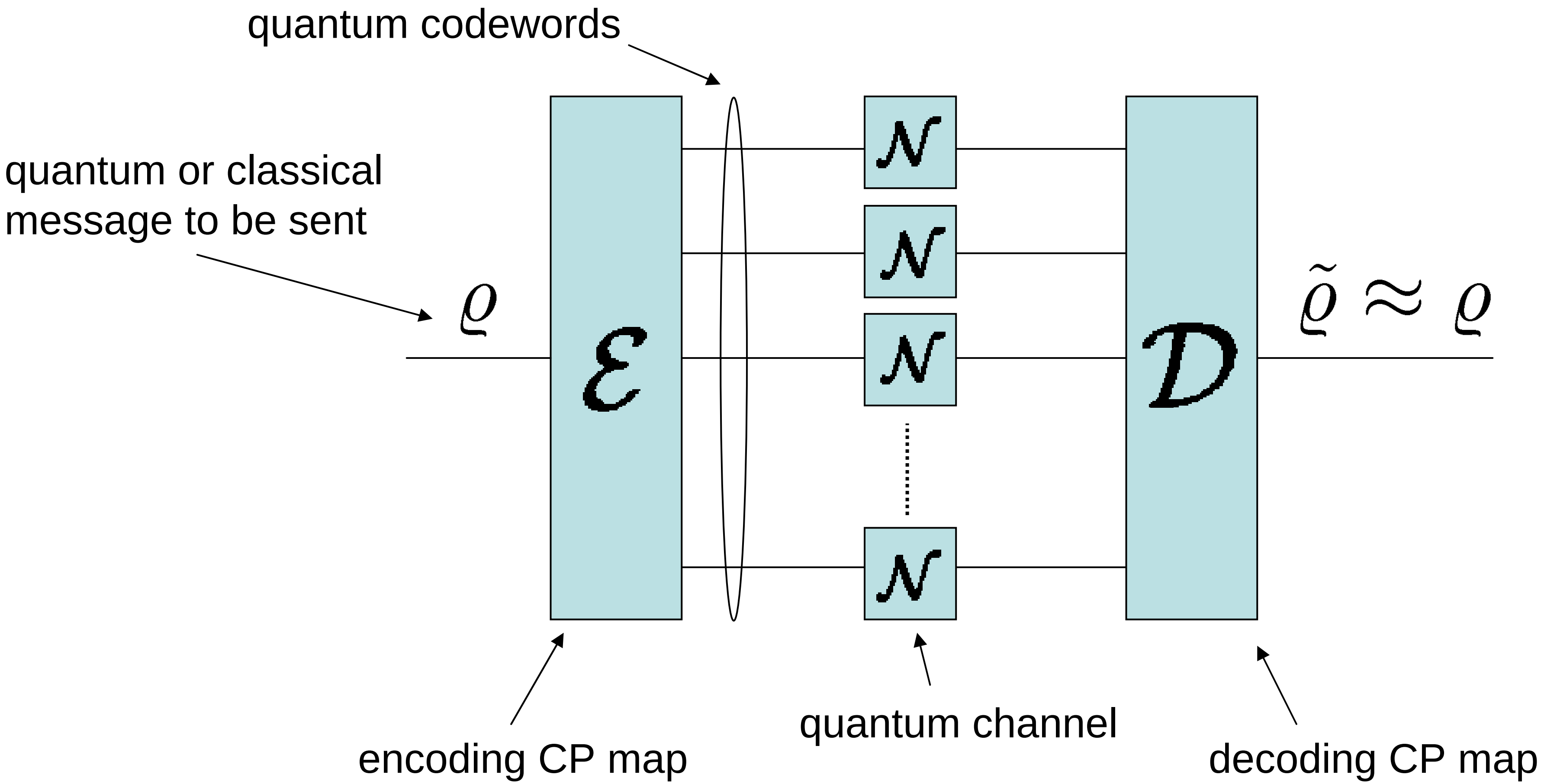
mutual information

The capacity is a supremum over all achievable rates



D > tvq J)B<C *
q)B *

Quantum communication



Classical capacity of a quantum channel

- Informally, the classical capacity of a quantum channel is the maximum number of bits per channel one can reliably transmit in an asymptotic regime

Holevo quantity for $\mathcal{N} : A \mapsto B$

$$\boxed{C_{\mathrm{Hol}}(\mathcal{N}) = \sup_{\{p_x, \sigma_x^A\}} I(X:B)_\varrho}$$

$$\sigma^{XA} = \sum_x p_x |x\rangle\langle x|^X \otimes \sigma_x^A \xrightarrow{\mathcal{N}_{A\to B}} \varrho^{XB} = \sum_x p_x |x\rangle\langle x|^X \otimes \varrho_x^B$$

- Recently there was a hope that the Holevo capacity might be additive but this has been disproven

PROBLEM $\quad$ D I pm)$\mathcal{N} \otimes \mathcal{N}$ * $\geq$ D I pm)$\mathcal{N}$ * $\otimes$ D I pm)$\mathcal{N}$ *

MEANING:
entangled codes help here

$$\boxed{\text{D} > \text{min}_{\text{o " 2}} \frac{2}{\text{o}} \text{D I pm} \; ¢ \, \mathcal{N}^{\text{a}} \, \text{o} \; £}$$

hard, hard, hard…

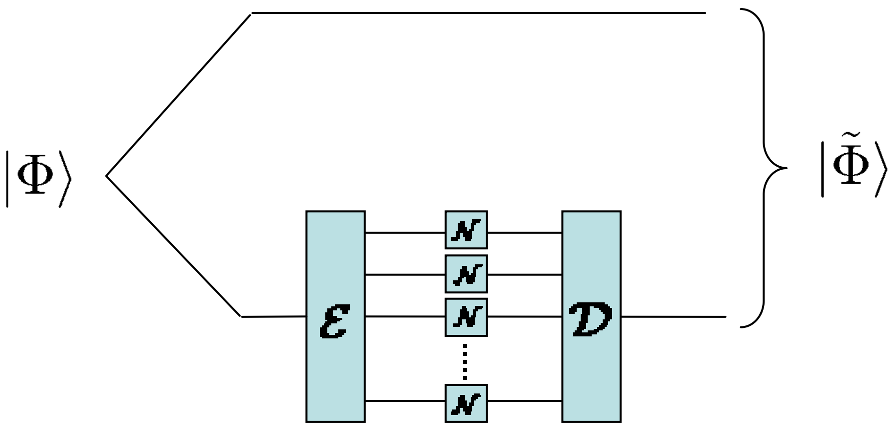
Hastings, Nature 2009

Quantum capacity

- Informally, the quantum capacity measures the ability of a channel to transmit a maximally entangled state

That is, how many qubits we are able to transmit (per channel use)

$$\left\| \left(\mathbb{1} \otimes \mathcal{D} \circ \mathcal{N}^{\otimes n} \circ \mathcal{E} \right) |\Phi\rangle - |\Phi\rangle \right\| \leq \varepsilon$$



- The achievable rate is $1 \leq S \leq J_d)\mathcal{N})^1$ **

where $$I_c(\mathcal{N}(\psi)) = H(\mathcal{N}(\psi)) - H(\mathcal{N}^c(\psi))$$

is the coherent information

$H(\varrho) = -\mathrm{Tr}\,\varrho \log \varrho$

Von Neumann entropy

Quantum capacity

- So far the same…

$$Q^{(1)}(\mathcal{N}) = \max_{\psi} \{H(\mathcal{N}(\psi)) - H(\mathcal{N}^c(\psi))\}$$

$R^{)2*})\mathcal{N}*$ is called the optimized coherent information

PROBLEM $\quad R^{)2*})\mathcal{N} \otimes \mathcal{N}* \geq R^{)2*})\mathcal{N}* \otimes R^{)2*})\mathcal{N}*$

We have to regularize

$$\boxed{R\ )\mathcal{N}* > \underset{o\ "\ 2}{\mathrm{mjn}}\ \frac{2}{o} R^{)2*})\mathcal{N}^{a\ o}*}$$

Two things to remember about the classical/quantum channel capacity

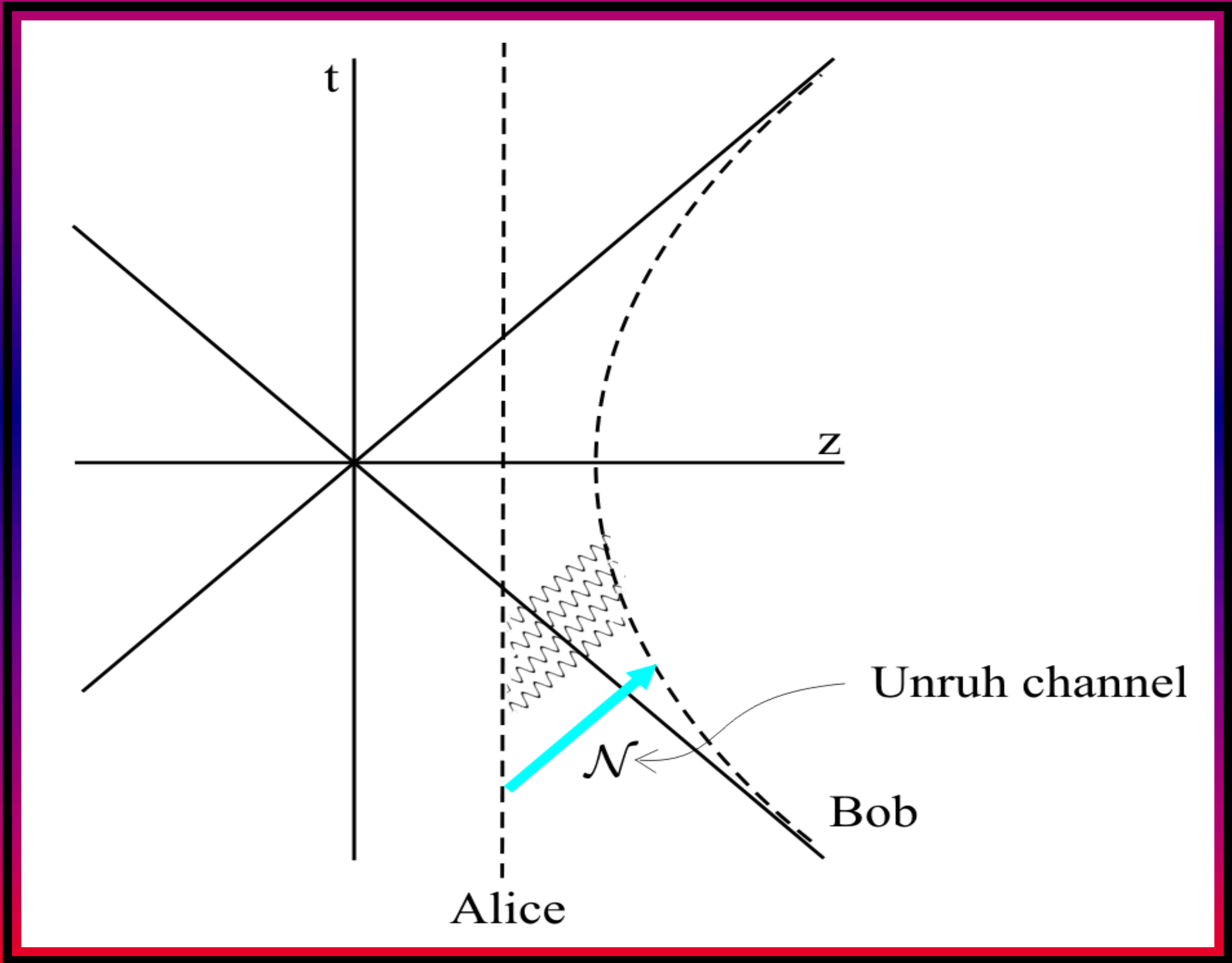
1. The meaning of the capacity number is the optimal rate, i.e. how fast (in principle) we can send bits/qubits through the channel (per channel use) such that we can recover them almost perfectly

2. In general, it is extremely hard to calculate capacities for quantum channels

The Unruh channel

Any questions?

The overall picture



The Unruh effect

- Let's express Minkowski vacuum in terms of Rindler states

$$\begin{pmatrix} b^{R}_{\Omega,\mathbf{k}} \\ b^{L\dagger}_{\Omega,-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \begin{pmatrix} d_{-\Omega,\mathbf{k}} \\ d^{\dagger}{}_{\Omega,-\mathbf{k}} \end{pmatrix}$$

Bogoliubov transformation

$$\sinh r = \sqrt{e^{2\pi\Omega}/(e^{2\pi\Omega}-1)}$$
$$\cosh r = \sqrt{1/(e^{2\pi\Omega}-1)}$$

- Responsible for the following transformation on states

$$\mathcal{O} : |n\rangle_{Mink} \mapsto \frac{1}{\cosh^{1+n} r} \sum_{m=0}^{\infty} \binom{n+m}{n}^{1/2} \tanh^{n} r \, |(n+m)^{L}_{\Omega,-\mathbf{k}}\rangle_{Rin} \, |m^{R}_{\Omega,\mathbf{k}}\rangle_{Rin}$$

- Is there a user-friendly way to work with $\mathcal{O}$?

The Unruh effect – general view

Yes, we can simulate the action of $\mathcal{O}$
with the help of

$$U_{ac}(r) = \exp\left[r(a^\dagger c^\dagger + ac)\right]$$

- The unitary operator captures the complete evolution of an input state
- The reference system lives beyond the horizon
- In the language of QIT the Unruh channel is a trace-preserving completely positive map realizing the Unruh effect

The Unruh effect – disclaimer

- To avoid working with unnormalized (unphysical) states, one can use the wave packet approach – yields the same qualitative answer
- Everything indicates that we can keep calculating with Fock states but then transfer the results to the wave packet scenario
- Operational definition much more realistic – finite time acceleration, detector models, switching functions, etc..

Audretsch/Muller PRD 1994, Takagi PTP Suppl. 1986 Lin/Hu, PRD 2007 KB, PRA 2007

The Unruh channel – construction

- Let's introduce the **dual-rail encoding**
- as two spatially/momentum distinguished modes

$$|\varphi\rangle_{12} = (\beta a_1^\dagger + \alpha a_2^\dagger)\,|0\rangle$$

- or using polarizations

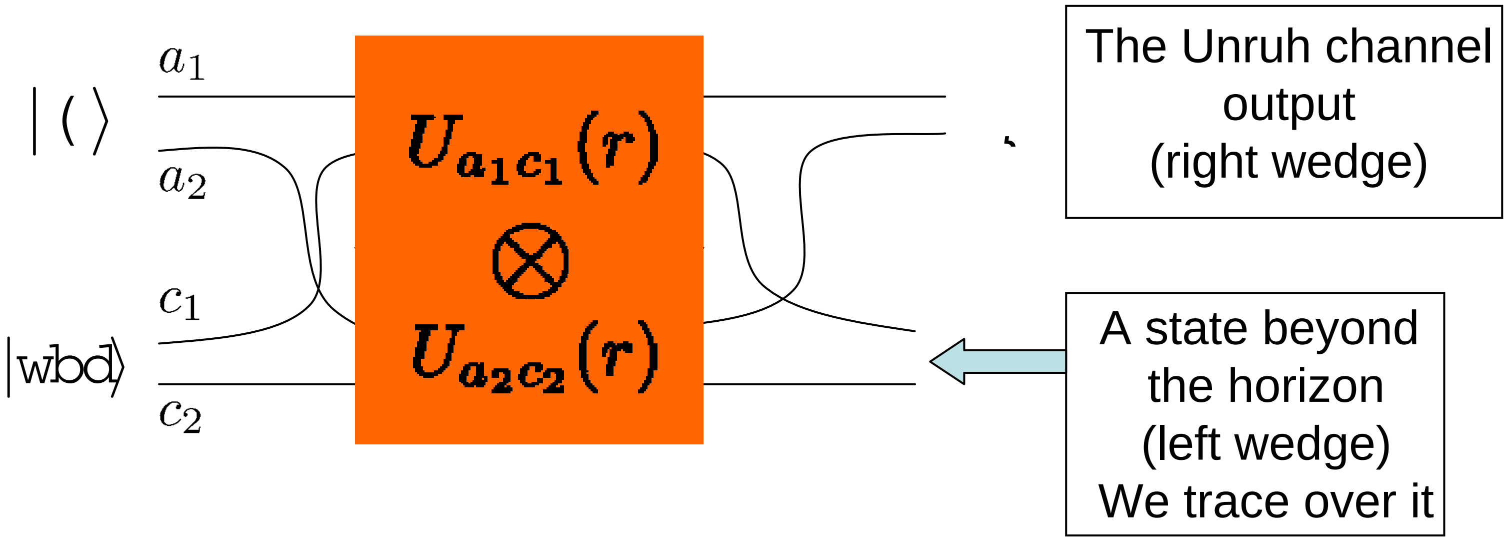
$$|\varphi\rangle = (\beta a_V^\dagger + \alpha a_H^\dagger)\,|0\rangle$$

- Note that we use a sector of the input Fock space spanned by $\{|01\rangle_{12}, |10\rangle_{12}\}$
- Then, the unitary is

$$U_{a_1 c_1}(r) \otimes U_{a_2 c_2}(r) =$$

$$\frac{1}{\cosh^2 r} e^{\tanh r (a_1^\dagger c_1^\dagger + a_2^\dagger c_2^\dagger)} e^{-\ln\cosh r (a_1^\dagger a_1 + a_2^\dagger a_2 + c_1^\dagger c_1 + c_2^\dagger c_2)} e^{-\tanh r (a_1 c_1 + a_2 c_2)}$$

The Unruh channel – construction



The Unruh channel $\mathcal{N}$ produces

$$\mathcal{N} : |\varphi\rangle_{a_1 a_2} \mapsto \sigma = 1/2(1-z^3)\bigoplus_{\ell=2}^{\infty} \ell(\ell-1) z^{\ell-2} \varepsilon_\ell \qquad \{ > \text{uboi}^3\ \text{s}$$

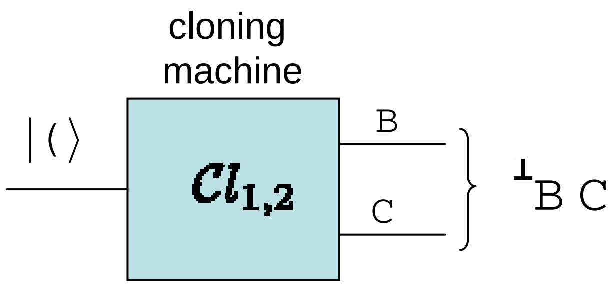
to be identified later

Originally introduced to study whether two inertial observers can privately communicate in the presence of a non-inertial eavesdropper

KB/Hayden/Panangaden, JHEP 2009

Detour - Cloning machines

- Cloning of an unknown pure qubit is impossible $|\varphi\rangle \not\to |\varphi\rangle_A |\varphi\rangle_B$
- The no-go theorem is a hallmark of quantum mechanics
- What is the best we can do?
- What 'the best' means?



The goal is to prepare a machine (cloning machine) producing approximate copies with the best possible quality measured by a suitable **figure of merit**

The cloning machine is supposed to be:

- **Universal** i.e. *SU(2)* covariant and therefore independent on φ
- **Symmetric** $\varrho_A = \mathrm{Tr}_{\mathrm{B}} \tau_{\mathrm{AB}} = \mathrm{Tr}_{\mathrm{A}} \tau_{\mathrm{AB}} = \varrho_{\mathrm{B}}$
- **Optimal** $F(\varphi, \varrho_A) = \max\limits_{\mathrm{CP\ maps}} \langle\varphi| \varrho_A |\varphi\rangle$

The figure of merit to maximize is the fidelity (QM inner product)

Buzek/Werner/Hillery/many others 1996-present

Cloning machines/channels

The 1->2 cloning machine (alias 1->2 cloning channel) explicitly reads

$$E_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

$$\&_{\mathrm{B}} \equiv \&_{\mathrm{C}} > \frac{6}{7} | (\rangle \langle (| \;,\; \frac{2}{7} \text{-}(^{@} \;\text{``}\; (^{@} \text{-}$$

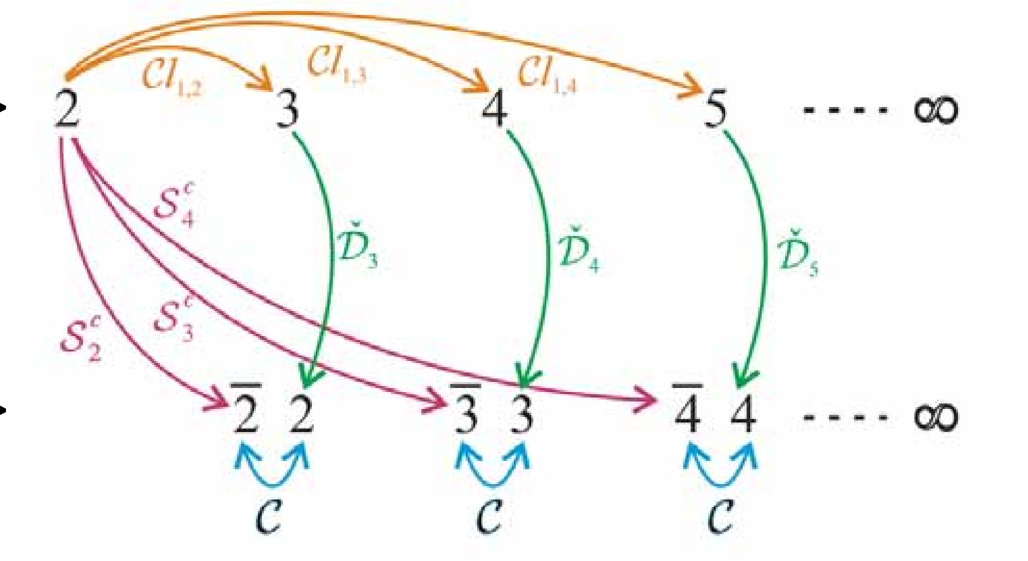
with fidelity $F(\varphi, \varrho_A) = \frac{5}{6}$

The Unruh channel – structure

Recall the Unruh channel $\mathcal{N}:|\varphi\rangle \mapsto \sigma = 1/2(1-z^3)\bigoplus_{\ell=2}^{\infty}\ell(\ell-1)z^{\ell-2}\varepsilon_\ell$

channel output dimension (lives here)

complementary channel output dimension



Cloning channels

Their complementary channels

Conjugate degrading maps

Complex conjugation

The Unruh channel – properties

Pauli matrices

Let $(> 2>3\,,\ o \cdot K^{)3*}$ be an input qubit

then the $2 \to)a - 2*$ cloning channel output can be written

$$\mathcal{C}m_{2<a\text{¢}\,2})(*> \frac{3}{a)a-2*}^{\circledR}\, 2^{)a*})a-2*>3\,,\ \underset{j>\,y<z<\{}{Y}\ o_j K_j^{)a*} \equiv \#_a$$

ℓ-dimensional representation of the generators of the $tv)3*$ algebra

$|o| \leq 2$

The Unruh channel can be written as

$$\boxed{\mathcal{N})(*> \underset{a>\,3}{\overset{N^2}{}}\ q_a \mathcal{C}m_{2<a\text{¢}\,2})(*}$$

The corresponding complementary channel $\mathcal{S}^{d}_{a\text{¢}\,2}$

yields

$$\mathcal{S}^{d}_{a\text{¢}\,2})(*> \frac{3}{a)a-2*}^{\circledR}\, 2^{)a\text{¢}\,2*}a>3\,,\ \underset{j>\,y<z<\{}{Y}\ o_j K_j^{)a\text{¢}\,2*}$$

$o_y > o_y$

$o_z > -o_z$

$o_\{ > o_\{$

This channel is entanglement-breaking!

Quantum channel capacities

- Some channels have the Holevo capacity additive

$$\mathrm{D}\ )\mathcal{N}* > \ \mathrm{D}_{\mathrm{I\ pm}})\mathcal{N}* \quad !$$

- For us, the important class of channels satisfying the additivity condition are entanglement-breaking channels

$$(\mathbb{1}_A \otimes \mathcal{N}_B)(\Phi_{AB}) = \sigma_{AB}$$

separable for all c_{BC}

- Why did we talk about conjugate degradable channels? Because for them

$$\mathrm{R}\ )\mathcal{N}* > \ \mathrm{R}^{)2*})\mathcal{N}* \quad !$$

- Quantum capacity of conjugate degradable channels can be calculated (we say that it single-letterizes the capacity formula)

KB/Dutil/Hayden/Muhammad, arXiv:0909.3297, accepted in JMP

KB, arXiv:0903.1638

Classical capacity of cloning channels

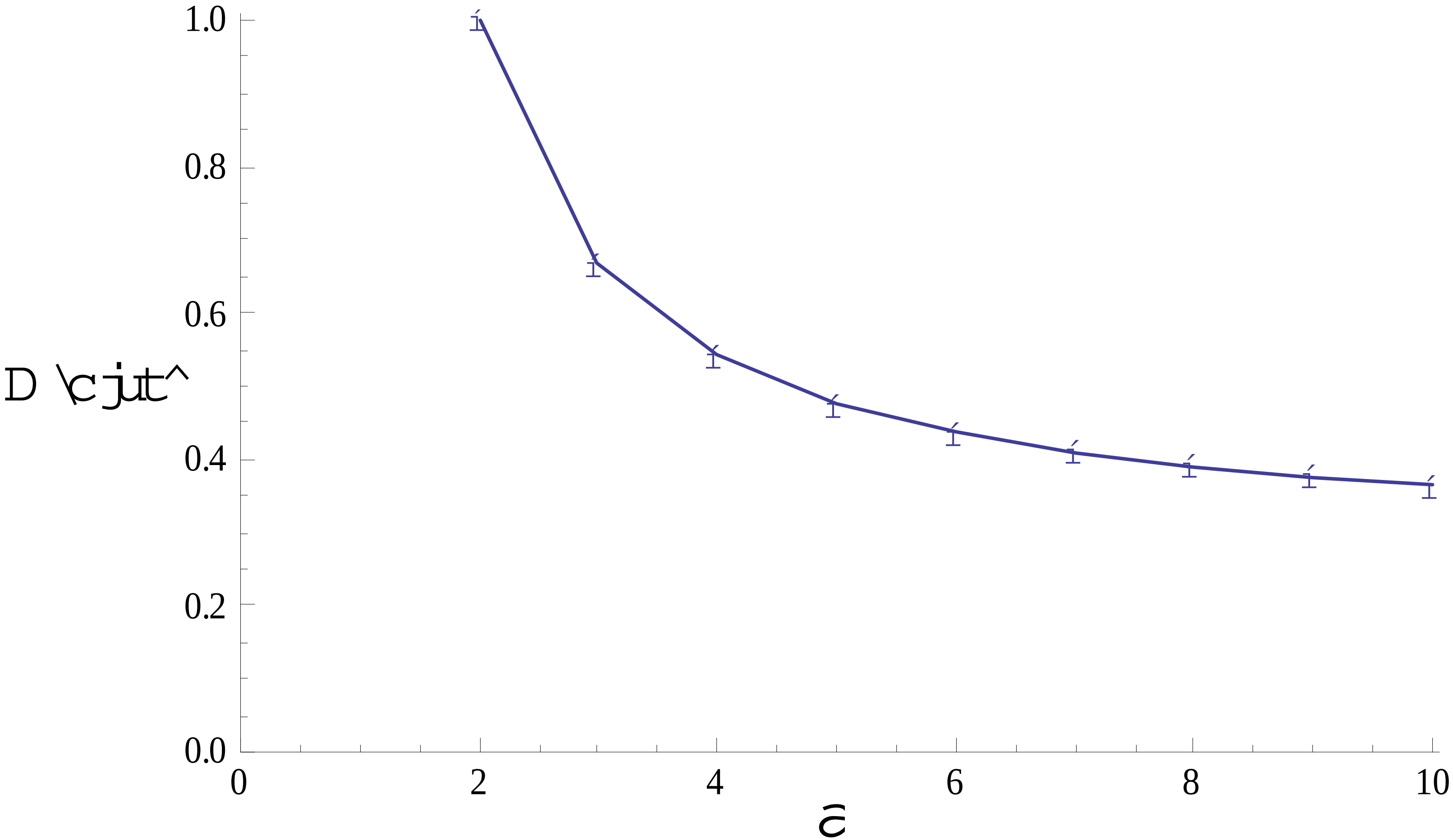
- All $1 \to N+1$ cloning channels have the Holevo capacity additive since their complementary channel is entanglement-breaking
- The classical capacity reads

$$C(\mathcal{E}_{1,N+1}) = 1 - \log(N+1) + \frac{1}{M}\sum_{k=0}^{N+1} k \log k \qquad M = N(N+1)/2$$

- The Unruh channel has the Holevo capacity additive too

Classical capacity of cloning channels

2 → ɘ − 2 Cloning channels 𝒞m2<a¢ 2



Quantum capacity of cloning channels

- Cloning channels are conjugate degradable
- Consequently, we only maximize the coherent information and the resulting quantum capacity is

R)𝒞m 2<a¢ 2 *> mph a − mph)a − 2*

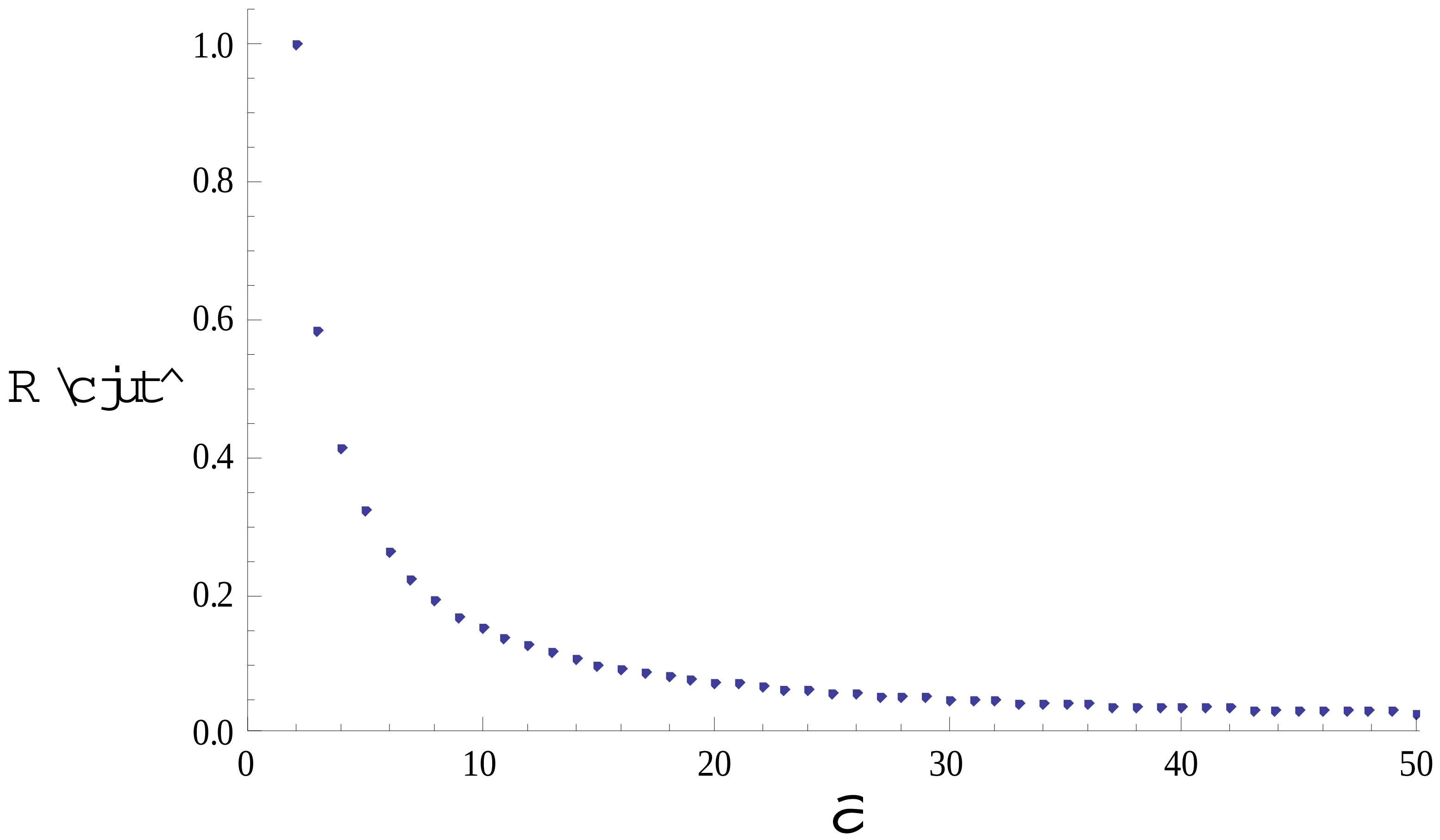
- The quantum capacity for the Unruh channel

$$\mathrm{Li}(s,z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

$$Q(\mathcal{N}(z)) = \frac{(1-z)^3}{2} \frac{\partial^2}{\partial z^2} \frac{\partial}{\partial s} \left[(z-1)\mathrm{Li}(\mathrm{s},\mathrm{z})\right]_{s=0}$$

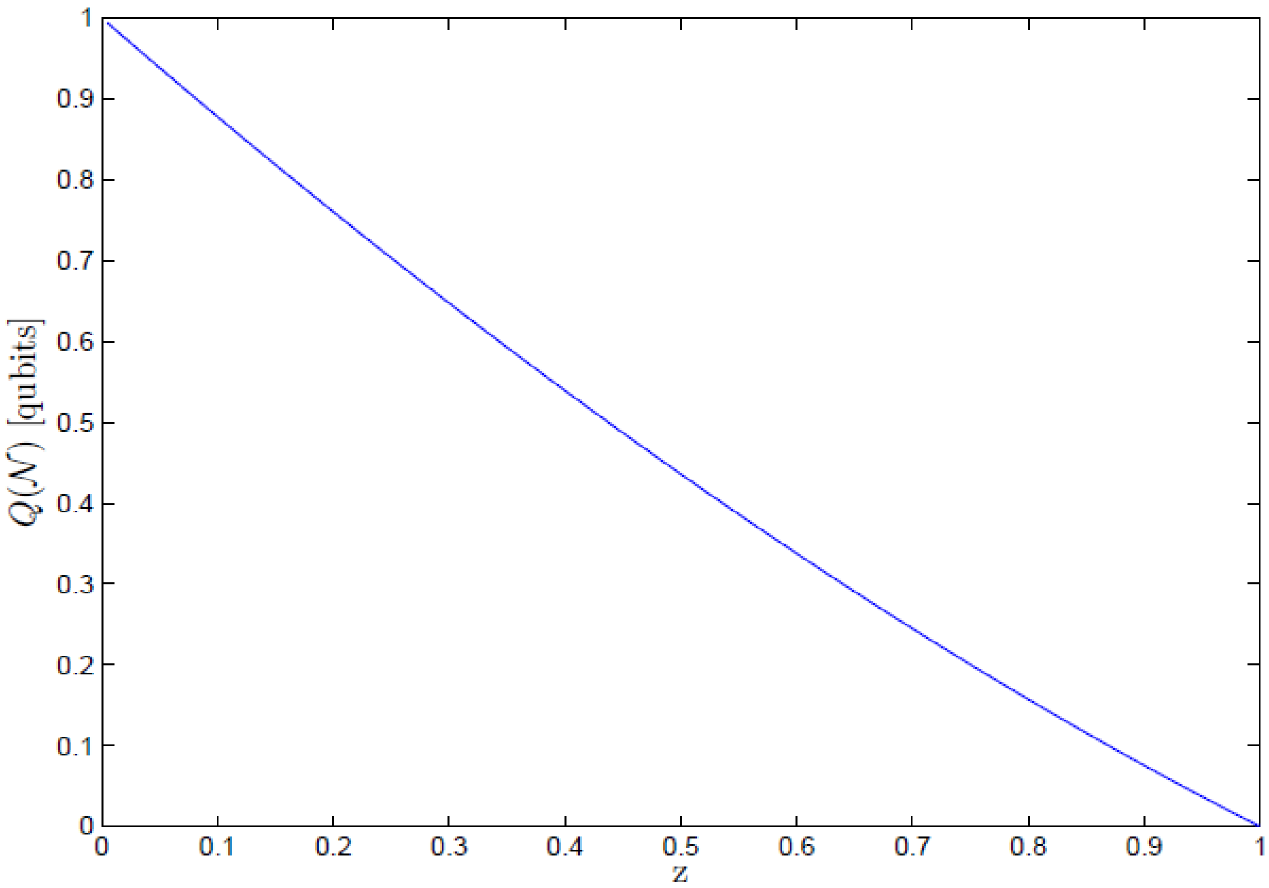
Quantum capacity of cloning channels

2 → ɘ − 2 Cloning channels $\mathcal{C}$m$_{2<a¢\ 2}$



Quantum capacity

The Unruh channel



Quantum capacity of the Unruh channel - Consequences

Relativistic communication protocols studied in the past were too pessimistic

Example: Teleportation into a non-inertial frame

Originally, concluded impossible to be done perfectly

The above quantum capacity results show that this is can be circumvented

Alsing/Milburn PRL 2003

Conclusions for QIT

- We found an infinite sequence of physically relevant channels with both capacities having single-letter formulas
- Qudit generalization on the way!
- We found a (tentatively) new class of channels with a single-letter formula for the quantum capacity – conjugate degradable channels

  However, it is not clear whether conjugate degradability is different from degradability (conjectured)
- If not, it is still a useful notion since for some channels conjugate degradability is easier to prove
- Cloning channels and the Unruh channel are members of a larger channel family – Hadamard channels

KB/Hayden/Panangaden, soon to appear

KB/Hayden/Touchette/Wilde, arXiv:1001.1732 accepted in PRA

Conclusions for QFT

- The Unruh channel is the same one as the channel describing the black hole stimulated emission process
- Since the quantum capacity is nonzero the Rindler observer can simulate (?) the Minkowski observer's physics
- The Unruh complementary channel (the channel going beyond the horizon) is entanglement-breaking and therefore has zero quantum capacity

  Does it mean something?
- Is it just a coincidence that cloning channels have appeared?
- By studying the Unruh effect for fermions we find a different channel family

KB et al, in progress

Adami/Ver Steeg, arXiv:0601065

Thank you
for your attention