NCKU 20th Dec 2009

Workshop on
Quantum Information Science
and Many-Body Physics

Entanglement Renormalization, critical phenomena and CFT

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Outline

MERA = **M**ulti-scale **E**ntanglement **R**enormalization **A**nsatz

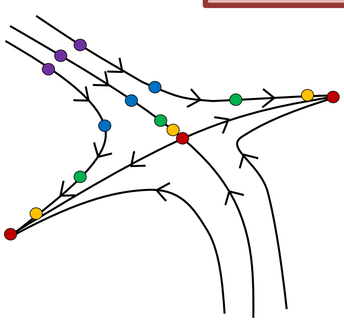
Entanglement Renormalization/MERA

- Renormalization Group (RG) transformation
- Quantum Computation

Scale invariant MERA ⇔ RG fixed point

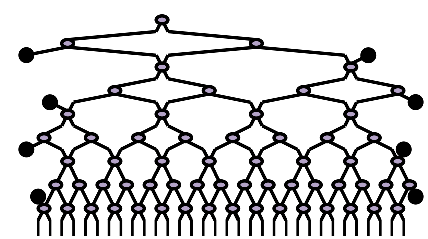
(• non-critical fixed point ⇔ topological order ⇔ TQFT)

- critical fixed point ⇔ continuous quantum phase transition ⇔ CFT
  - bulk (local/non-local)
  - boundary
  - defect



collaboration with
Glen Evenbly,
R. Pfeifer, P. Corboz,
L. Tagliacozzo, I.P. McCulloch

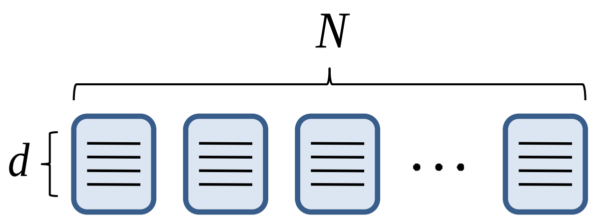
V. Pico (U. Barcelona)
S. Iblisdir (U. Barcelona)



MERA (multi-scale entanglement renormalization ansatz)

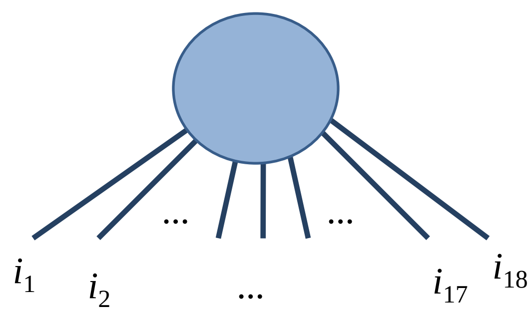
Vidal, Phys. Rev. Lett. 99, 220405 (2007); ibid 101, 110501 (2008)

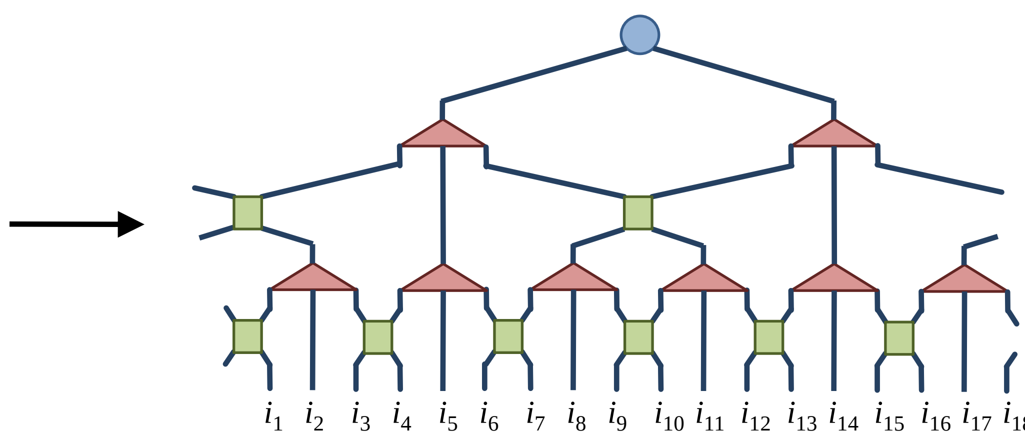
- Lattice with N sites



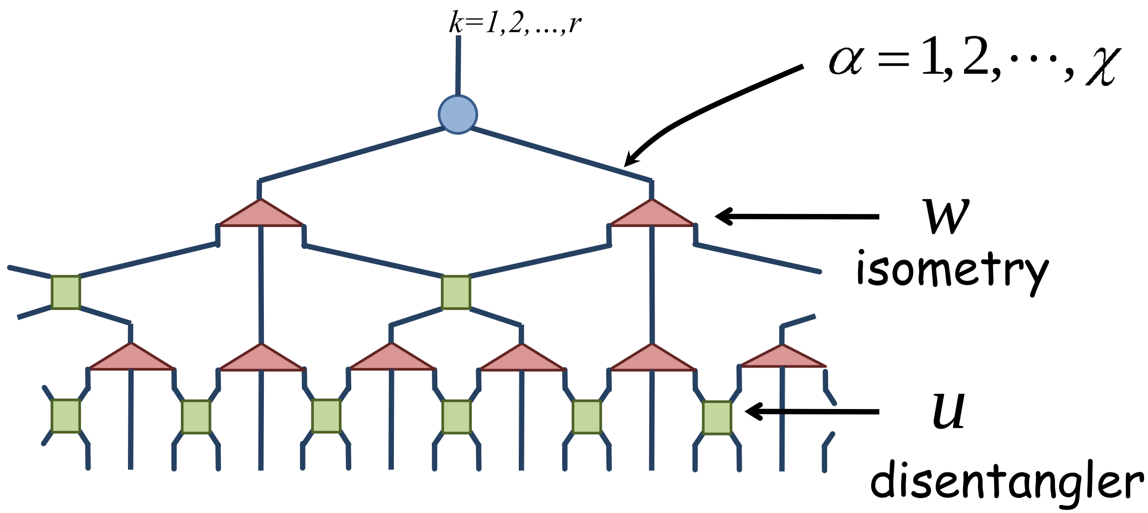
$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

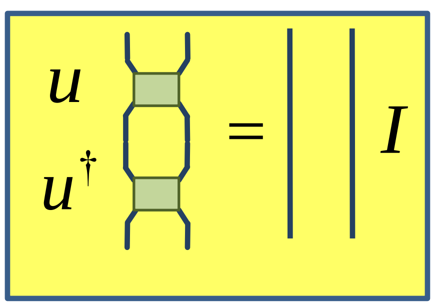
$$|\Psi_{GS}\rangle = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_N=1}^{d} c_{i_1 i_2 \ldots i_N} |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$



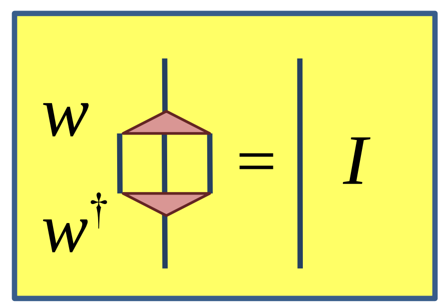


MERA (multi-scale entanglement renormalization ansatz)



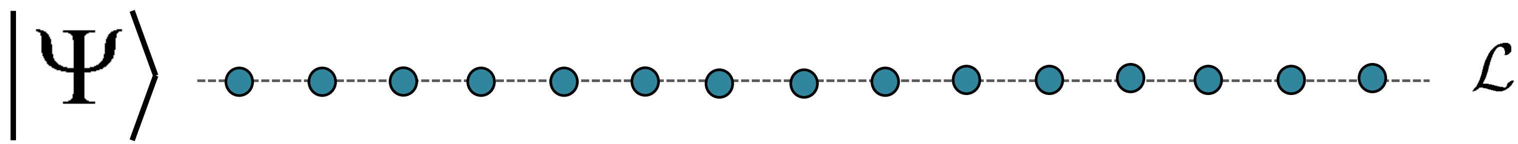


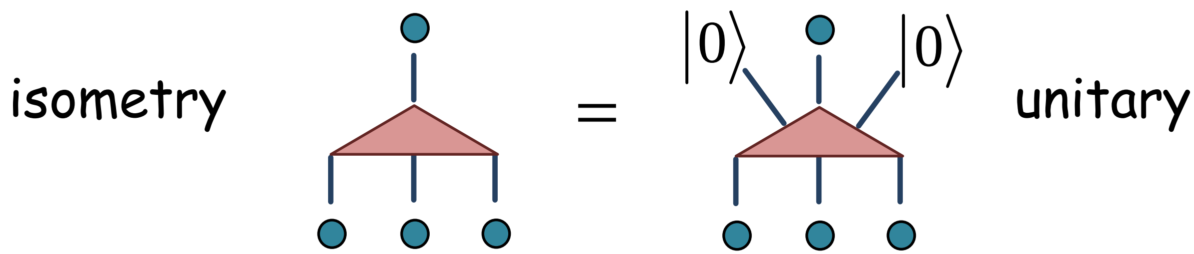
disentangler

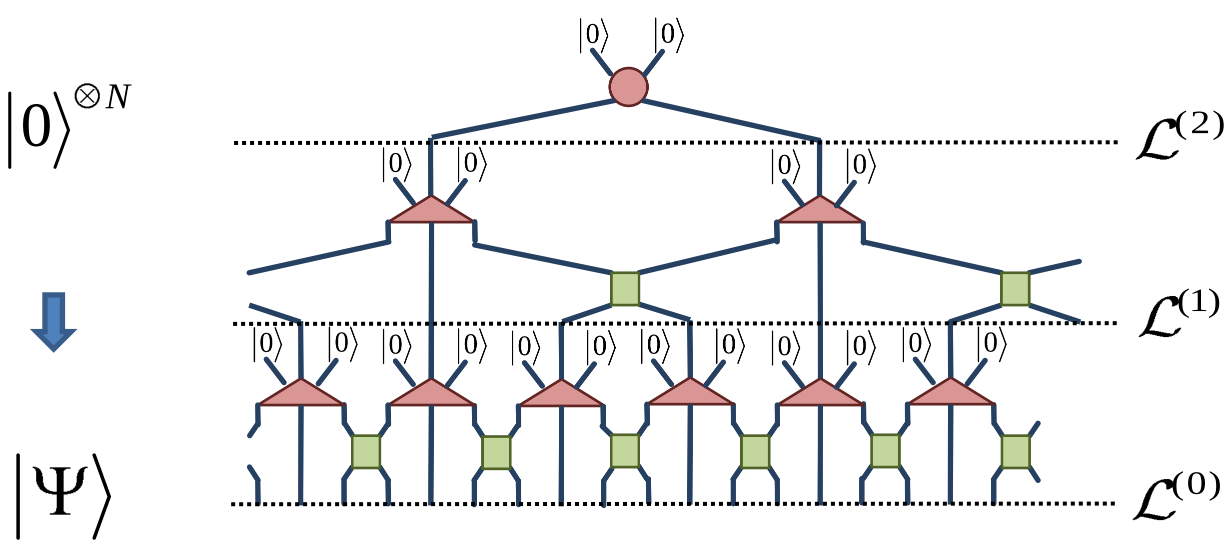


isometry

MERA as a quantum circuit

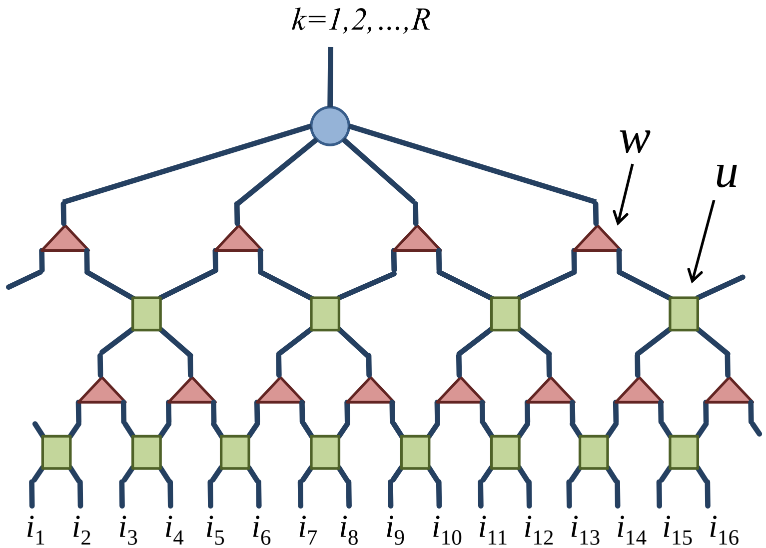




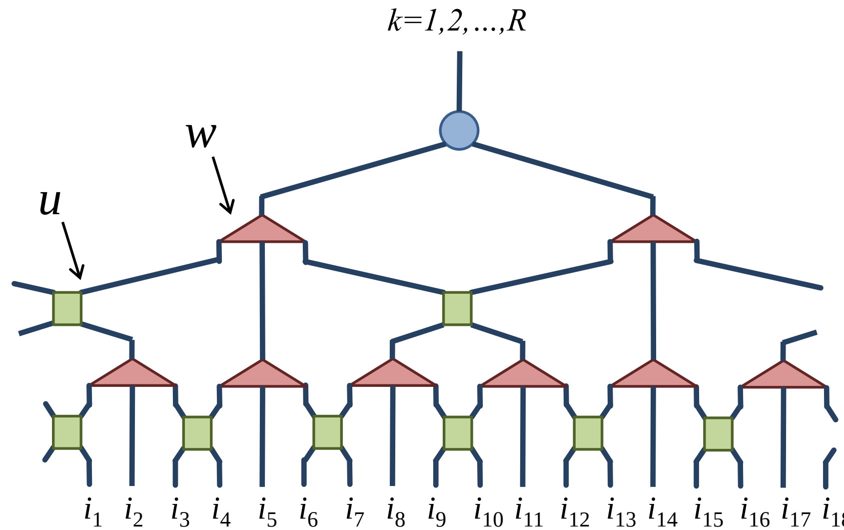


- Many possibilities

examples: binary 1D MERA

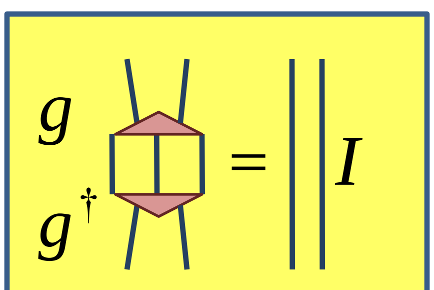


ternary 1D MERA

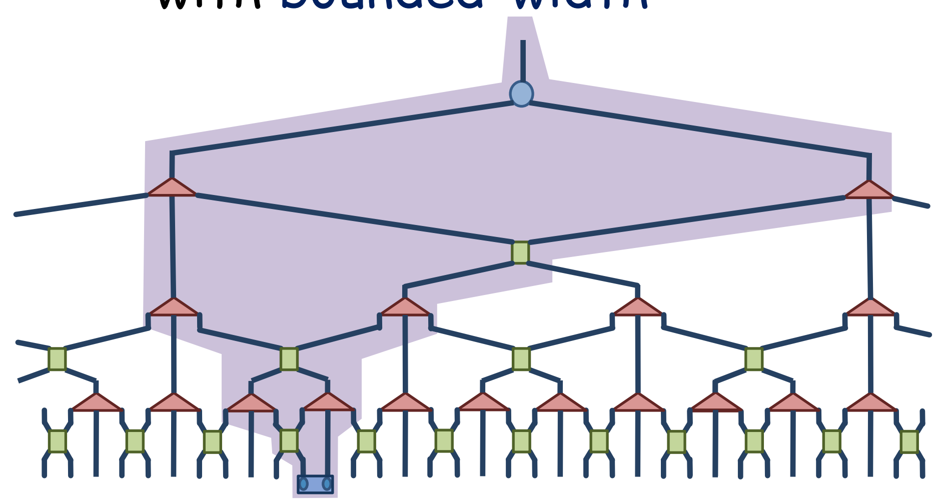


- Defining properties:

    1) isometric tensors

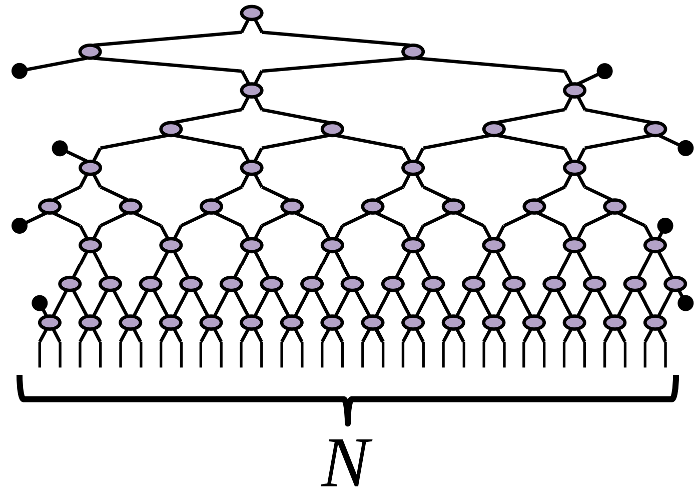


2) past causal cone with bounded 'width'



What is the MERA useful for?

- ground states/low energy subspaces in 1D, 2D lattices
- at criticality: critical exponents/CFT



Simulation costs:

$O(N)$ → $O(\log(N))$ → $O(1)$

$O(N)$	$O(\log(N))$	$O(1)$
$\chi^p N$ inhomogeneous	$\chi^p \log N$ translation invariance	χ^p translation & scale invariance

χ^p

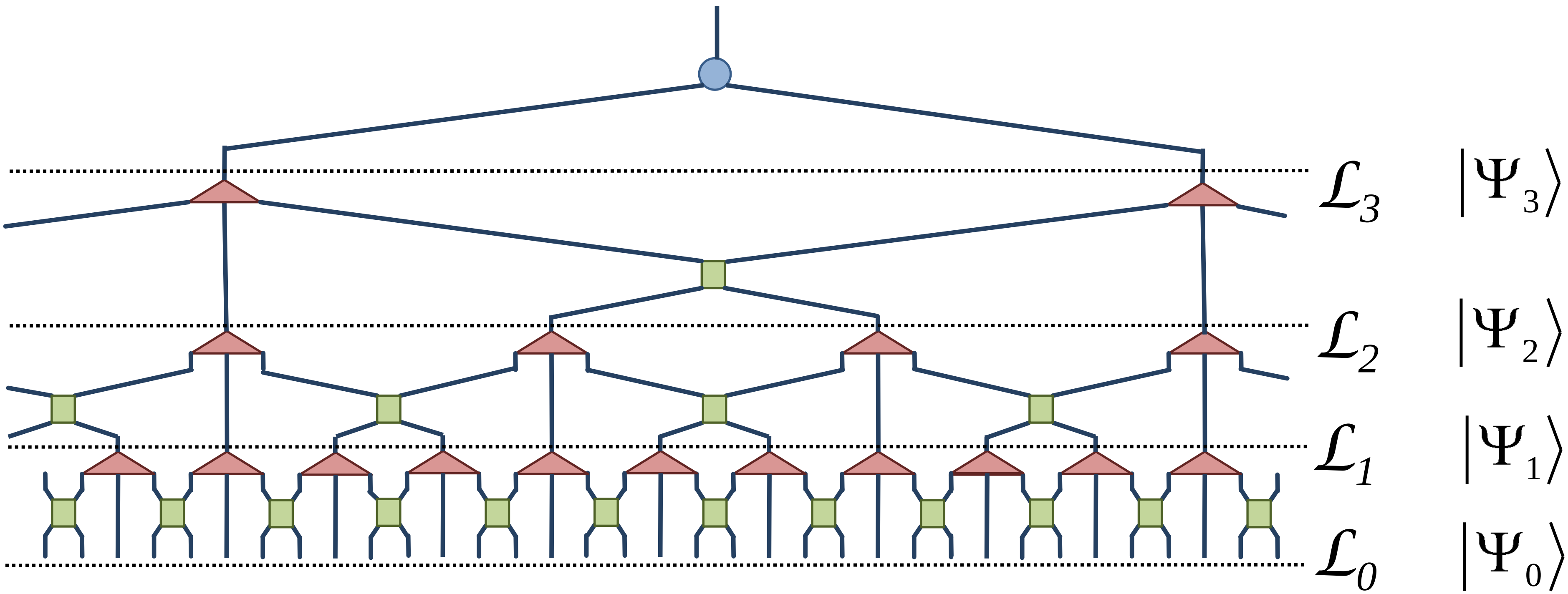
+ (scale invariant) **boundary/defect/ interface**

(no translation invariance !!!)

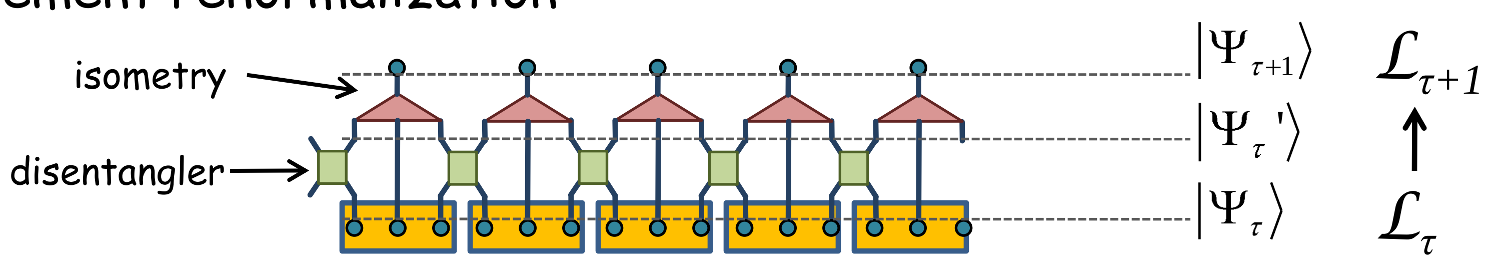
RG transformation

Vidal, Phys. Rev. Lett. 99, 220405 (2007); ibid 101, 110501 (2008)

The MERA defines a coarse-graining transformation

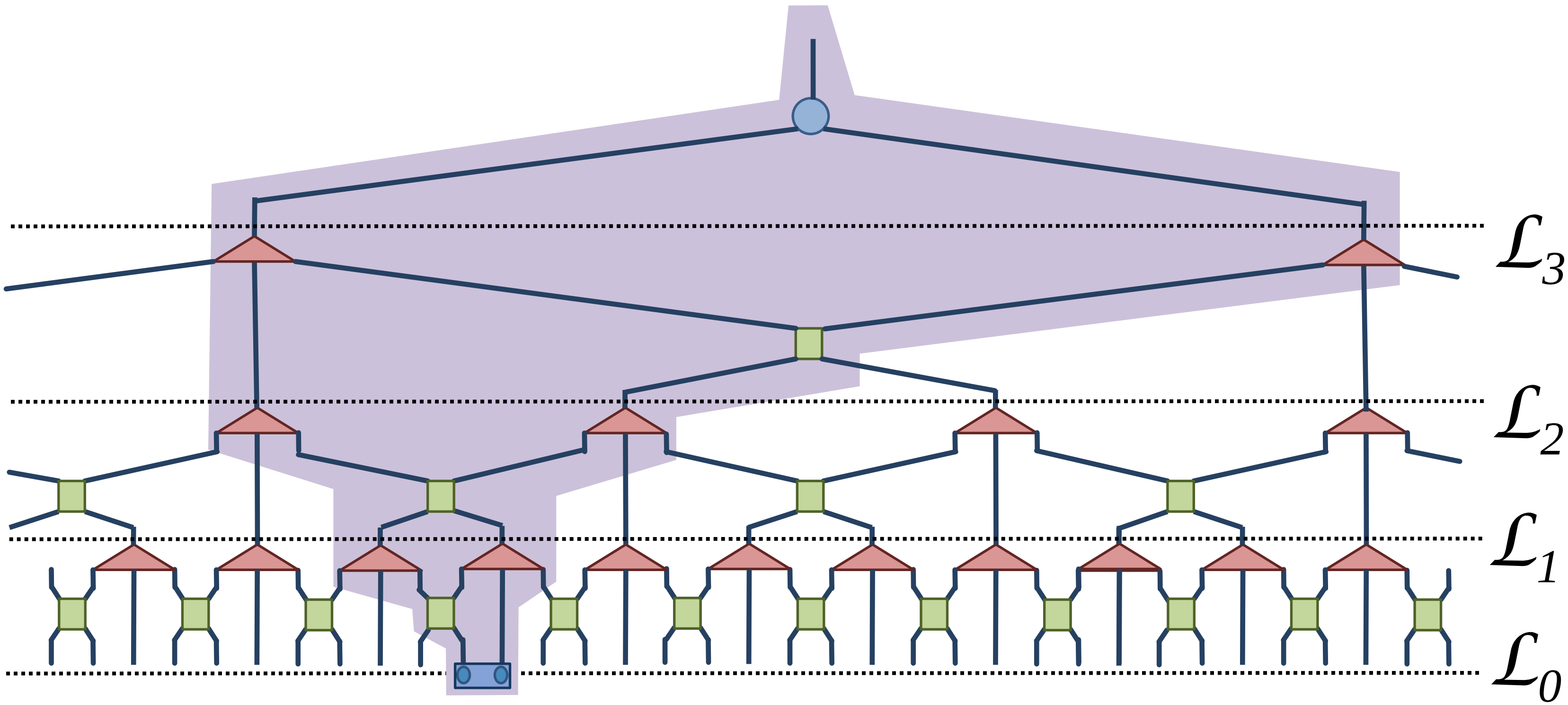


Entanglement renormalization

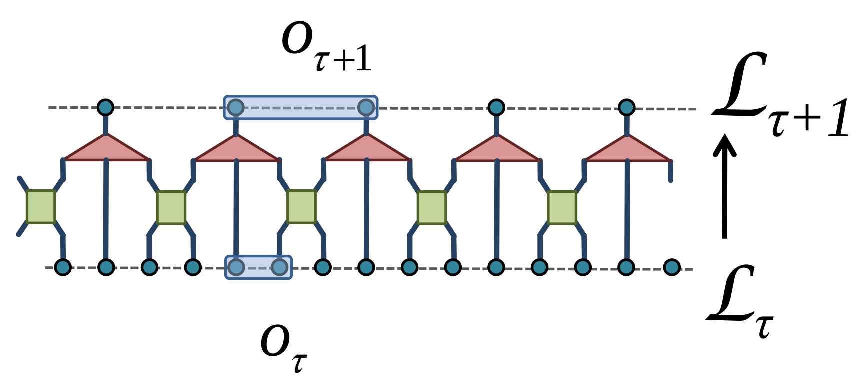


RG transformation

The MERA defines a coarse-graining transformation



- transformation of local operators

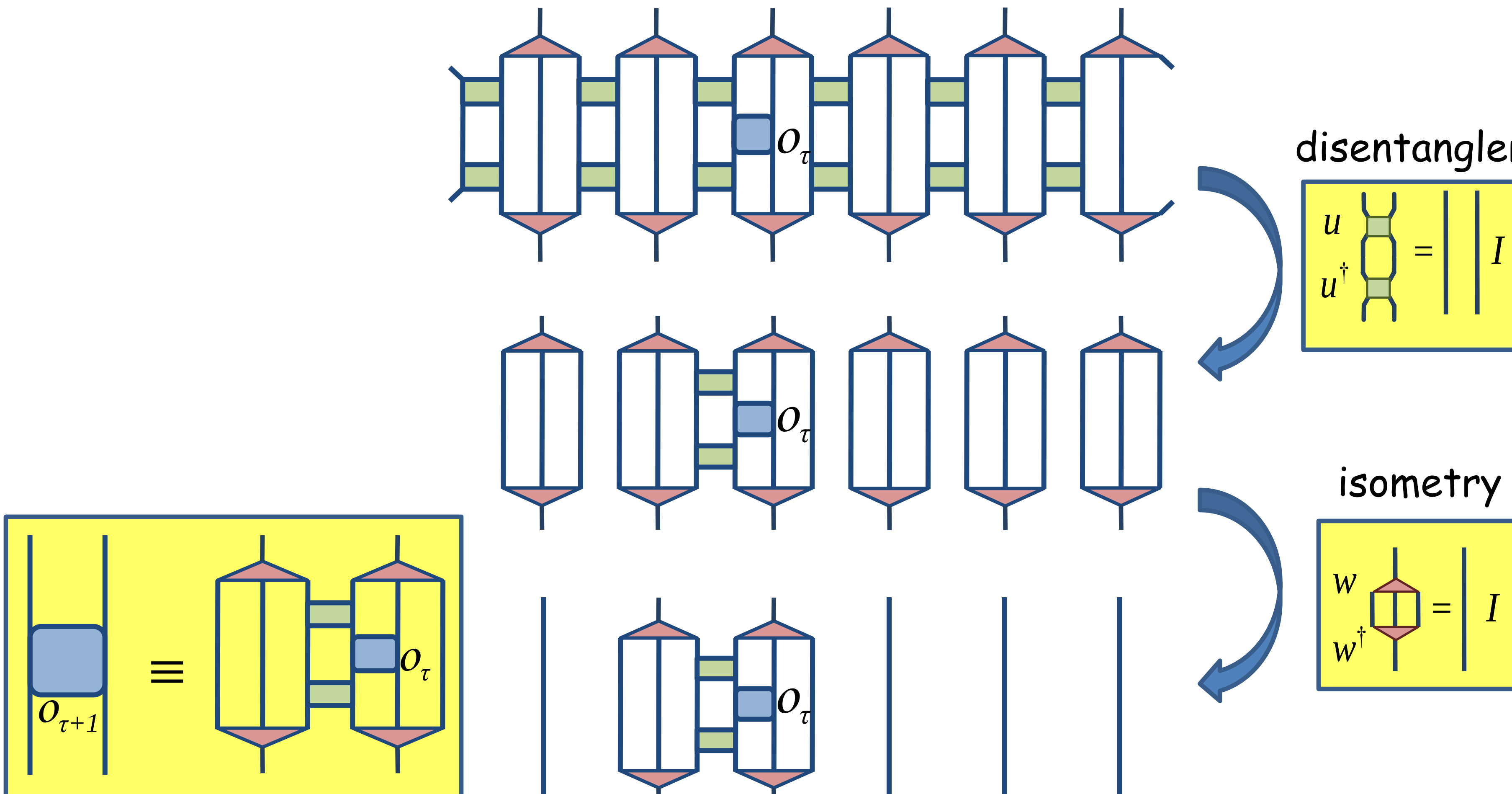


RG transformation

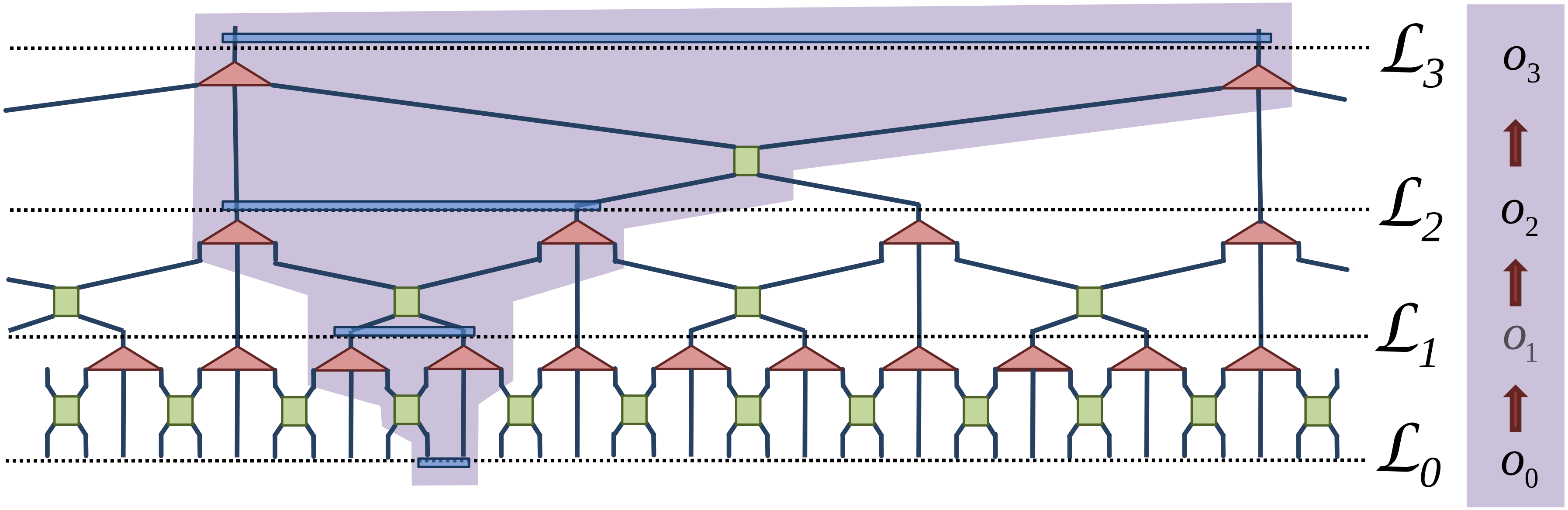
The MERA defines a coarse-graining transformation

$$\mathcal{L}_\tau \longrightarrow \mathcal{L}_{\tau+1}$$

$$o_\tau \rightarrow o_{\tau+1}$$

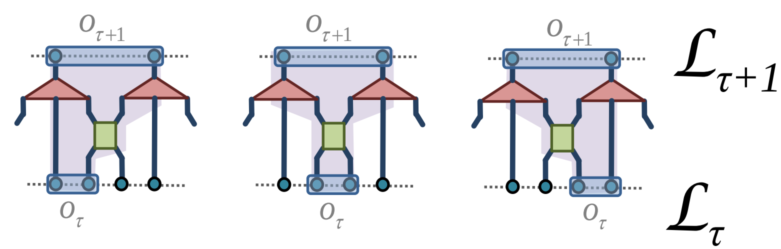


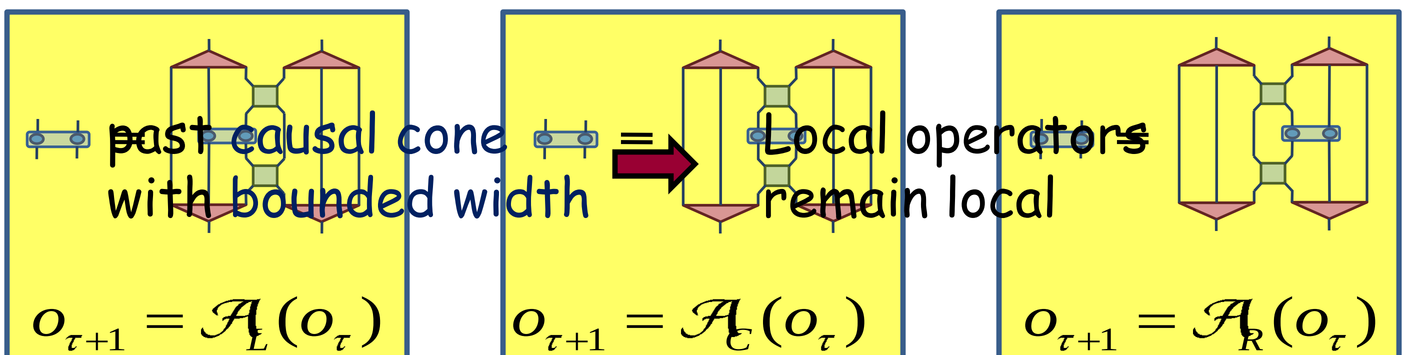
Ascending superoperator

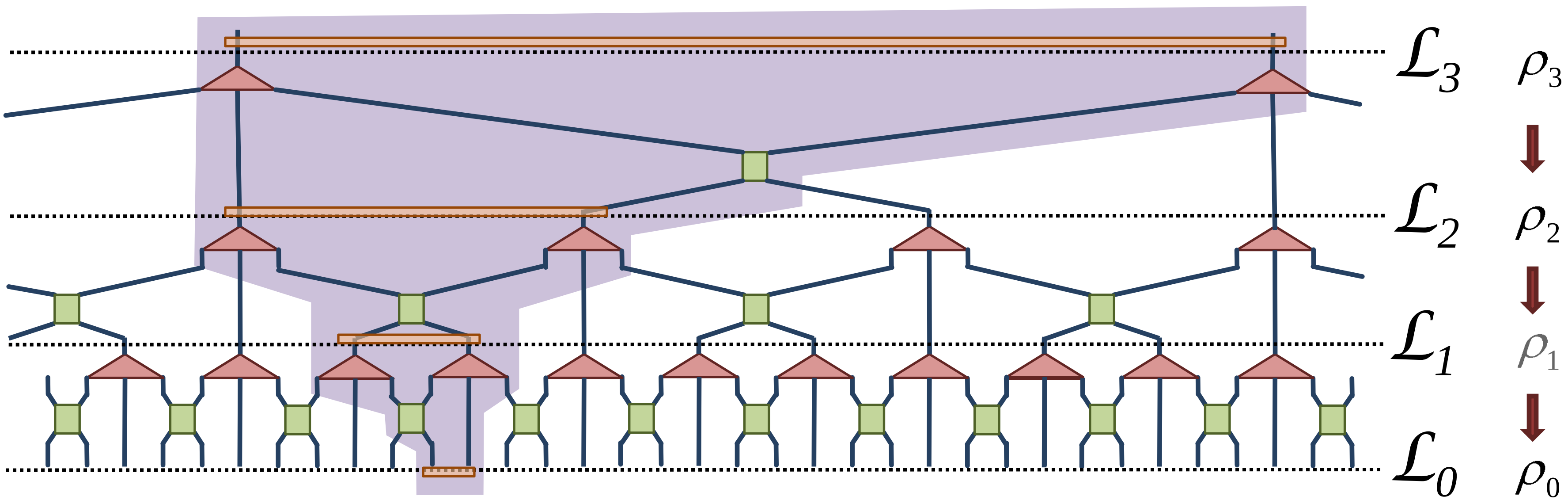


- ascending superoperator $o_{\tau+1} = \mathcal{A}(o_\tau)$

$o_0 \rightarrow o_1 \rightarrow o_2 \rightarrow \dots$

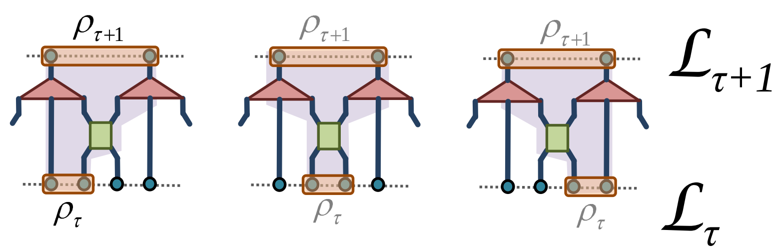


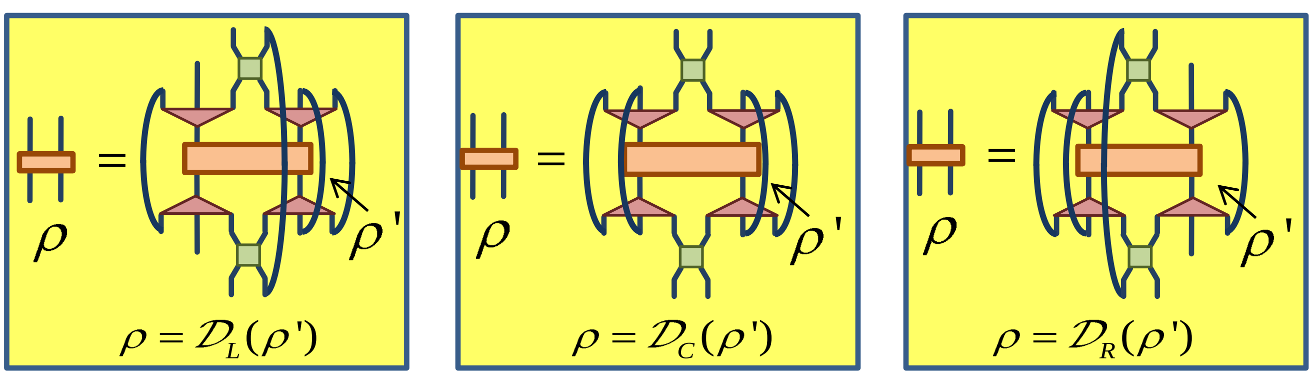




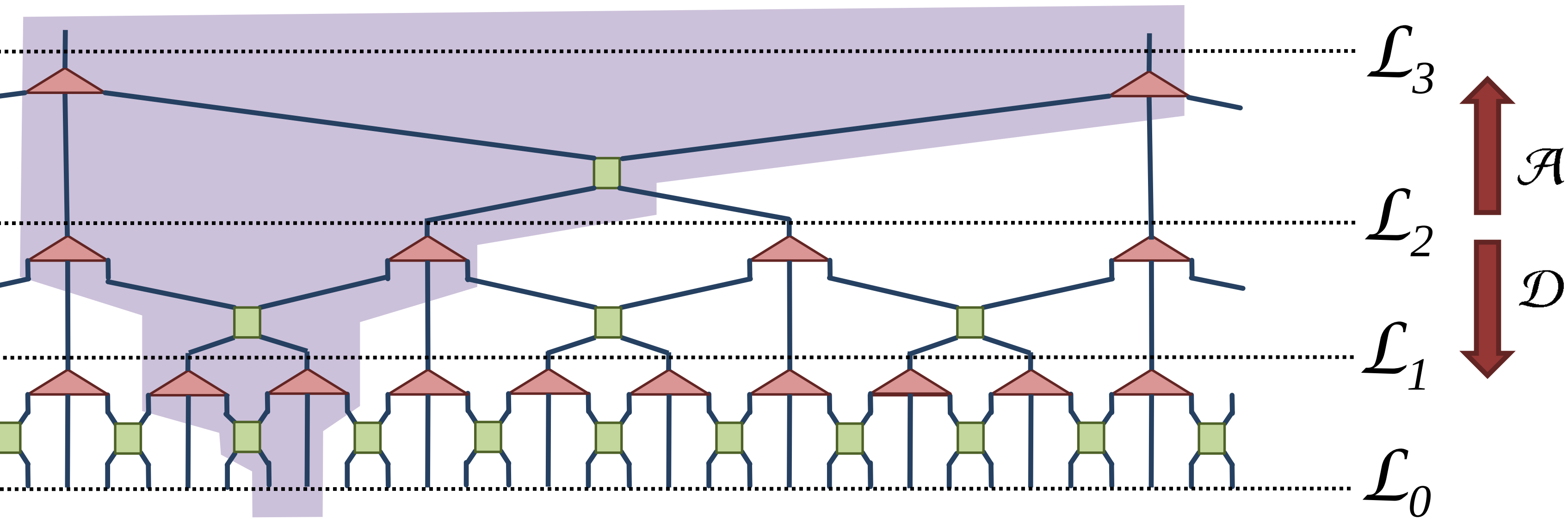
• descending superoperator $\rho_\tau = \mathcal{D}(\rho_{\tau+1})$

$\rho_0 \Rightarrow \rho_1 \Rightarrow \rho_2 \Rightarrow \ldots$

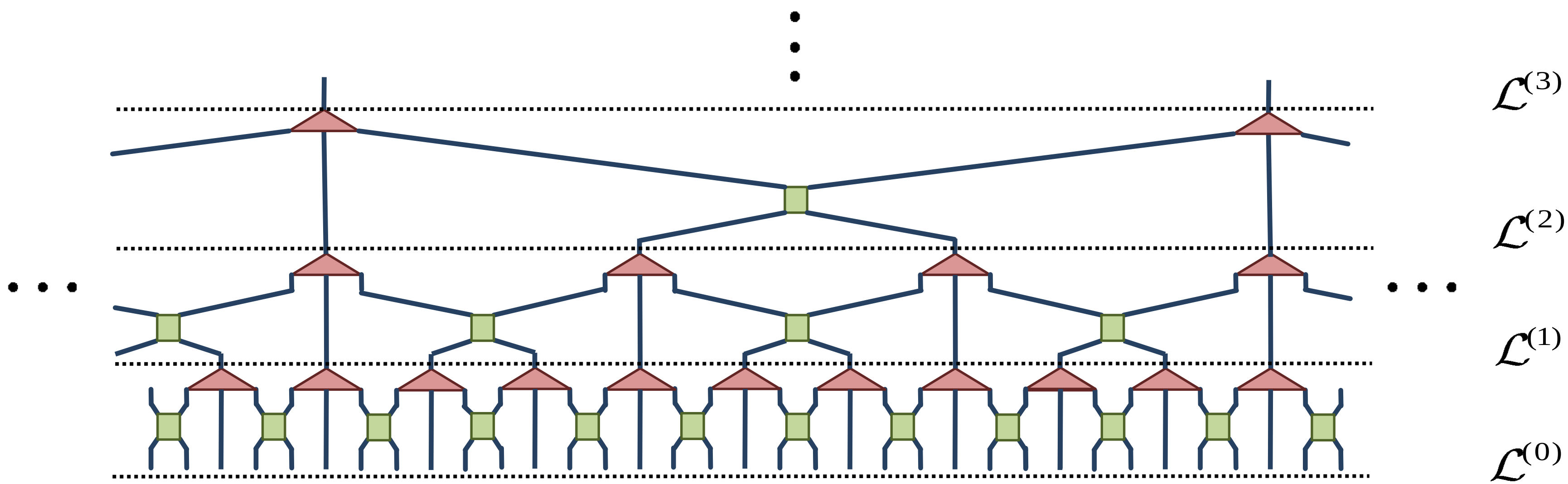




- Description of the system at different length scales
- Ascending and descending super-operators: change of length scale (or time in a quantum computation)



Scale invariant MERA



critical systems (1D)

Vidal, Phys. Rev. Lett. 99, 220405 (2007)
Vidal, Phys. Rev. Lett. 101, 110501 (2008)
Evenbly, Vidal, arXiv:0710.0692
Evenbly, Vidal, arXiv:0801.2449

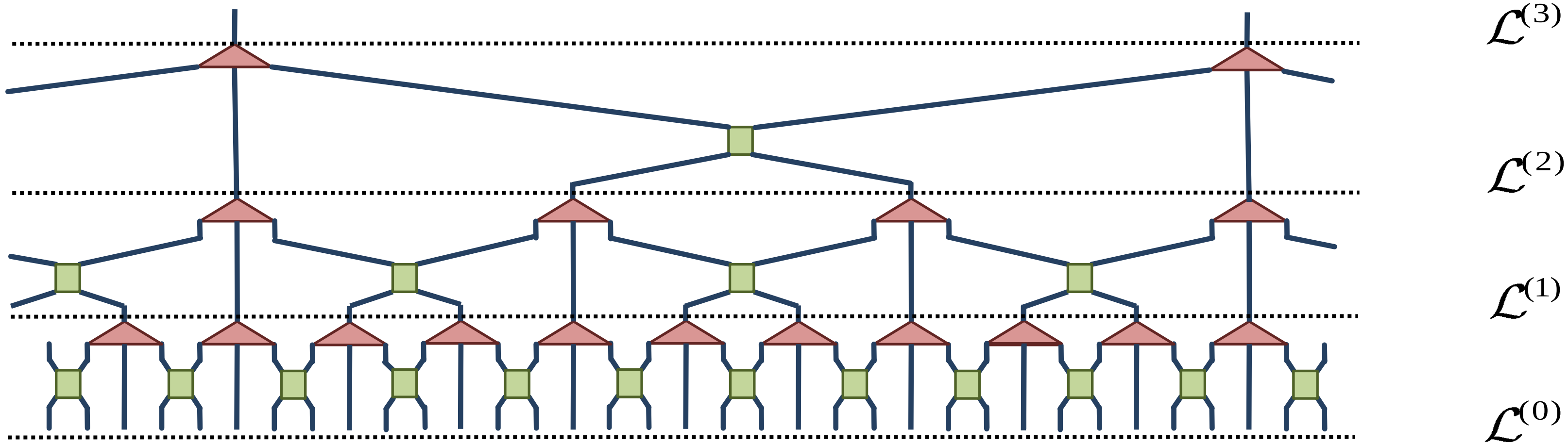
critical exponents → **Giovannetti, Montangero, Fazio**, Phys. Rev. Lett. 101, 180503 (2008)
OPE, CFT → **Pfeifer**, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)
Montangero, **Rizzi**, Giovannetti, Fazio, Phys. Rev. B 80, 113103 (2009)
Giovannetti, Montangero, Rizzi, Fazio, Phys. Rev. A 79, 052314(2009)

boundary & defects → Evenbly, Pfeifer, **Pico**, **Iblisdir**, **Tagliacozzo**, **McCulloch**, Vidal, arXiv:0912.1642
non-local operators → Evenbly, **Corboz**, Vidal, arXiv: 0912.2166
Silvi, Giovannetti, **Calabrese**, **Santoro**, Fazio, arXiv: 0912.2893

topologically ordered systems (2D)

Aguado, Vidal, Phys. Rev. Lett. 100, 070404 (2008)
Koenig, Reichardt, Vidal, Phys. Rev. B 79, 195123 (2009)

Scale invariant MERA



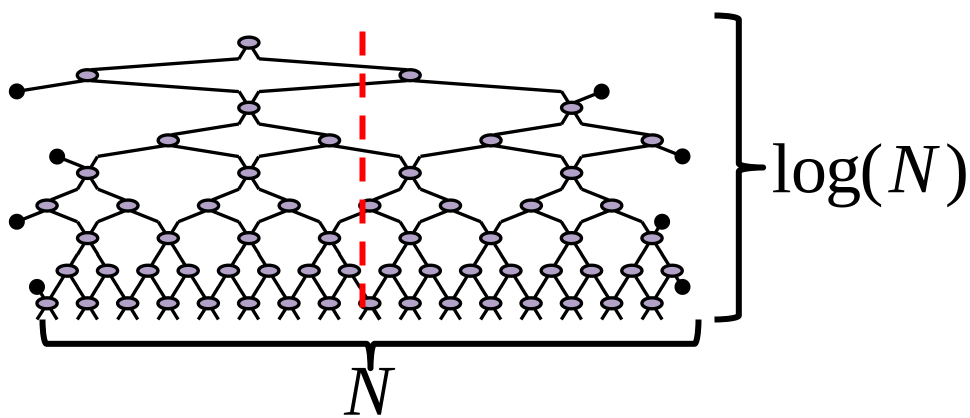
Vidal, Phys. Rev. Lett. 99, 220405 (2007)
Vidal, Phys. Rev. Lett. 101, 110501 (2008)

Evenbly, Vidal, arXiv:0710.0692
Evenbly, Vidal, arXiv:0801.2449

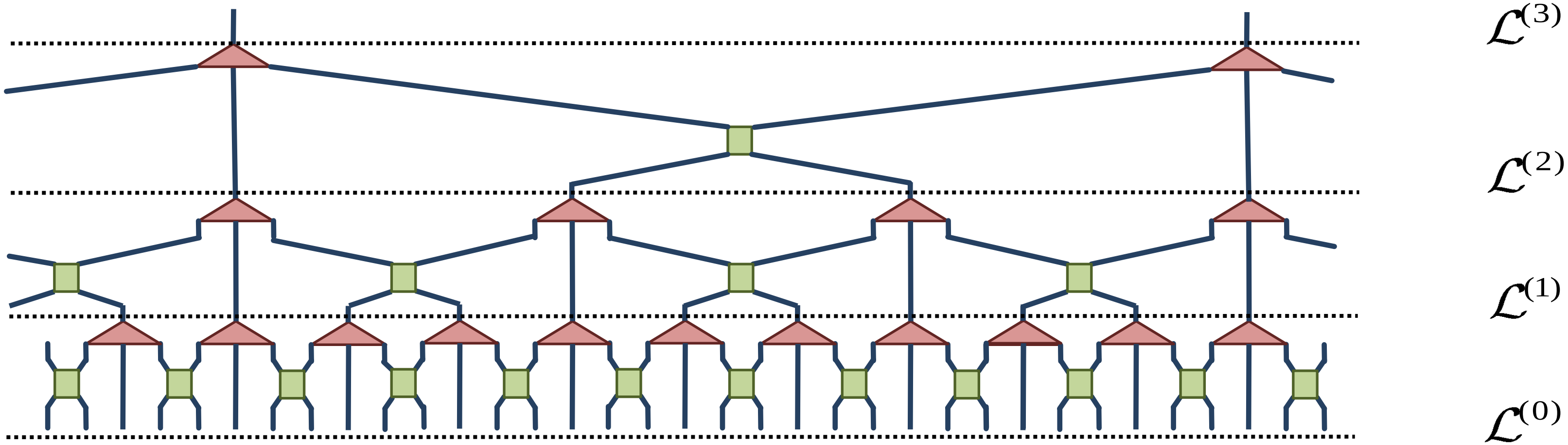
MERA is a good ansatz for ground state at quantum critical point

- it recovers scale invariance! same state and Hamiltonian at each level
  [shown for Ising model, free fermions, free bosons]
- proper scaling of entanglement entropy

$$S_{N/2} \sim \log(N)$$



Scale invariant MERA



Vidal, Phys. Rev. Lett. 99, 220405 (2007)
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Evenbly, Vidal, arXiv:0710.0692
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MERA is a good ansatz for ground state at quantum critical point

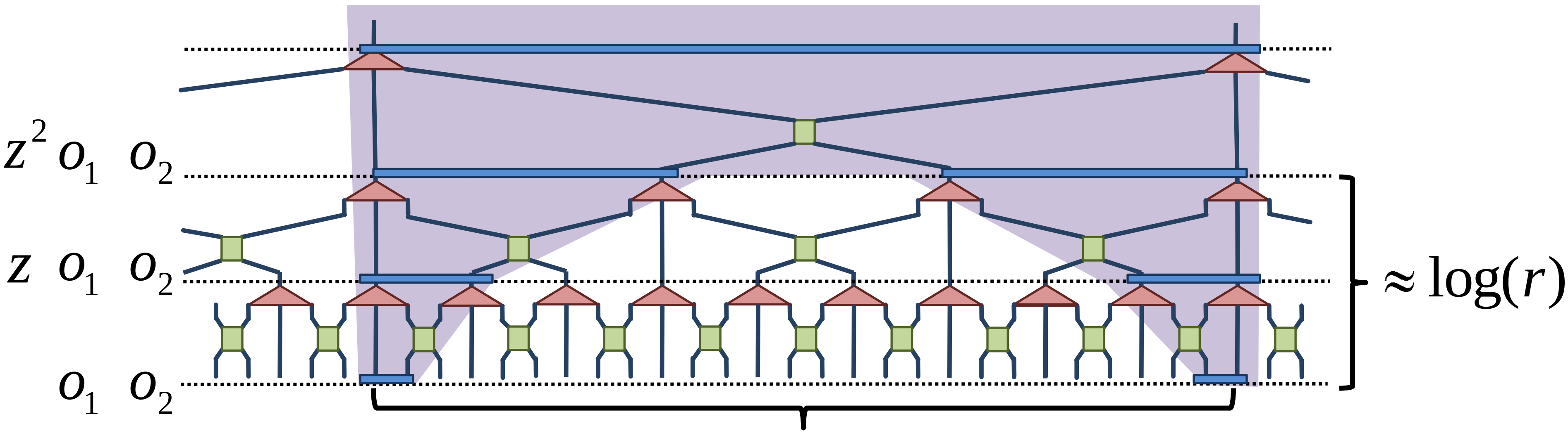
- it recovers scale invariance! same state and Hamiltonian at each level [shown for Ising model, free fermions, free bosons]
- proper scaling of entanglement entropy
- polynomial decay of correlations

$$o' = \mathcal{A}(o) \equiv \mathcal{S}(o)$$

$$o \rightarrow o' \rightarrow o'' \rightarrow \ldots$$

Scale invariant MERA

- polynomial decay of correlations Vidal, Phys. Rev. Lett. 101, 110501 (2008)



$$C_2(s_1, s_2) \approx z^{\log(r)} = r^{\log(z)} = r^{-q}, \qquad q \equiv -\log(z)$$

$$\mathcal{S}(o_1) = \sqrt{z} o_1 \qquad \mathcal{S}(o_2) = \sqrt{z} o_2$$

- Eigenvalue decomposition Giovannetti, Montangero, Fazio, Phys. Rev. Lett. 101, 180503 (2008)

$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

ϕ_α scaling operator $\qquad \phi_\alpha \to 3^{-\Delta_\alpha} \phi_\alpha$

Δ_α scaling dimension $\qquad \Delta_\alpha \equiv -\log_3 \lambda_\alpha$

- Eigenvalue decomposition Giovannetti, Montangero, Fazio, Phys. Rev. Lett. 101, 180503 (2008)

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ϕ_α scaling operator $\phi_\alpha \to 3^{-\Delta_\alpha} \phi_\alpha$

Δ_α scaling dimension $\Delta_\alpha \equiv -\log_3 \lambda_\alpha$

Critical exponents can be extracted from the scaling superoperator
(= quMERA channel, MERA transfer map)

- Connection to Conformal Field Theory Pfeifer, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)

spin lattice at quantum critical point → CFT

- central charge c
- primary fields ϕ_α^p

  conformal dimensions $(h_\alpha^p, \bar{h}_\alpha^p)$

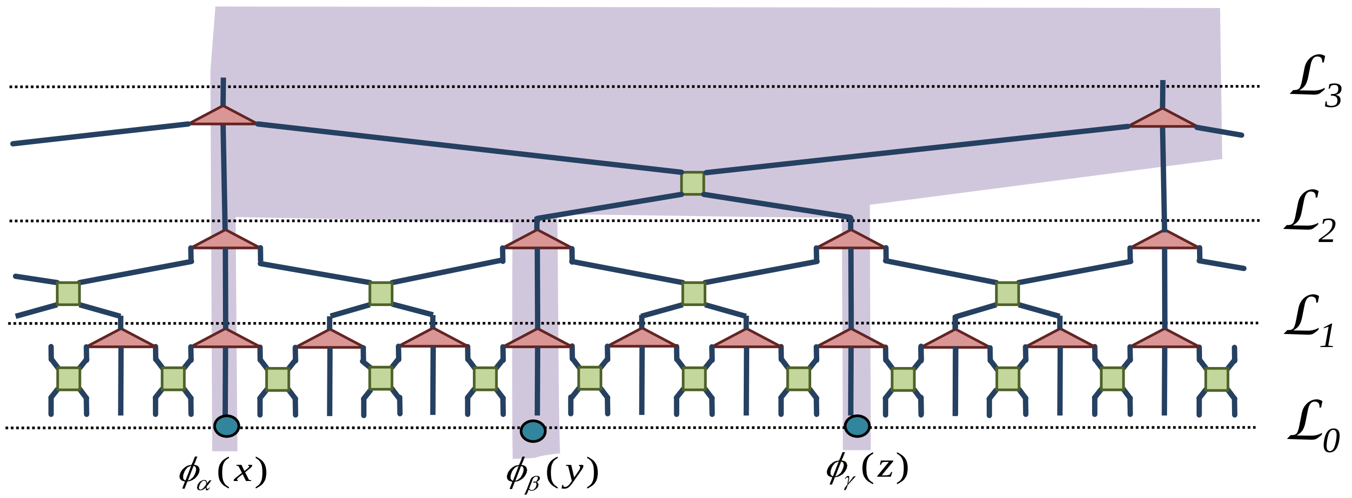
  $\Delta_\alpha^p = h_\alpha^p + \bar{h}_\alpha^p$
- operator product expansion OPE

$$\phi_\alpha^p \times \phi_\beta^p \approx C_{\alpha\beta\gamma} \phi_\gamma^p$$

Pfeifer, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)

- operator product expansion OPE from three point correlators

$$\phi_\alpha^p \times \phi_\beta^p \approx C_{\alpha\beta\gamma}\phi_\gamma^p$$



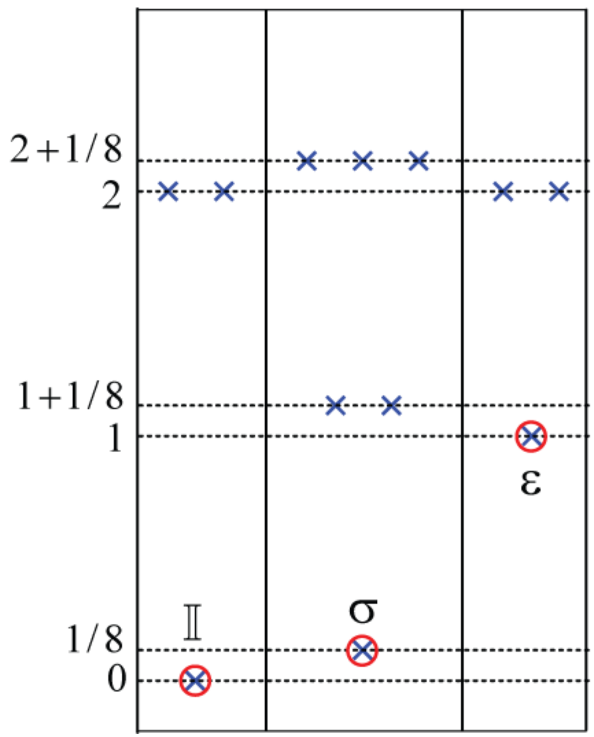
$$\left\langle \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z)\right\rangle = \frac{C_{\alpha\beta\gamma}}{|x-y|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}|y-z|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}|z-x|^{\Delta_\gamma+\Delta_\alpha-\Delta_\beta}}$$

Scale invariant MERA (bulk)

- Example: Ising model   Pfeifer, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)

χ=36 $\tilde{\chi}$=20

scaling operators/dimensions:



	scaling dimension (exact)	scaling dimension (MERA)	error
identity I	0	0	----
spin σ	0.125	0.124997	0.003%
energy ε	1	0.99993	0.007%
	1.125	1.12495	0.005%
	1.125	1.12499	0.001%
	2	1.99956	0.022%
	2	1.99985	0.007%
	2	1.99994	0.003%
	2	2.00057	0.03%

- Operator product expansion (OPE):

$$C_{\alpha\beta\mathbb{I}} = \delta_{\alpha\beta} \qquad C_{\sigma\sigma\varepsilon} = 1/2$$

$$C_{\sigma\varepsilon\varepsilon} = C_{\sigma\sigma\sigma} = C_{\varepsilon\varepsilon\varepsilon} = 0 \qquad (\pm 5\times 10^{-4})$$

$\Longrightarrow$

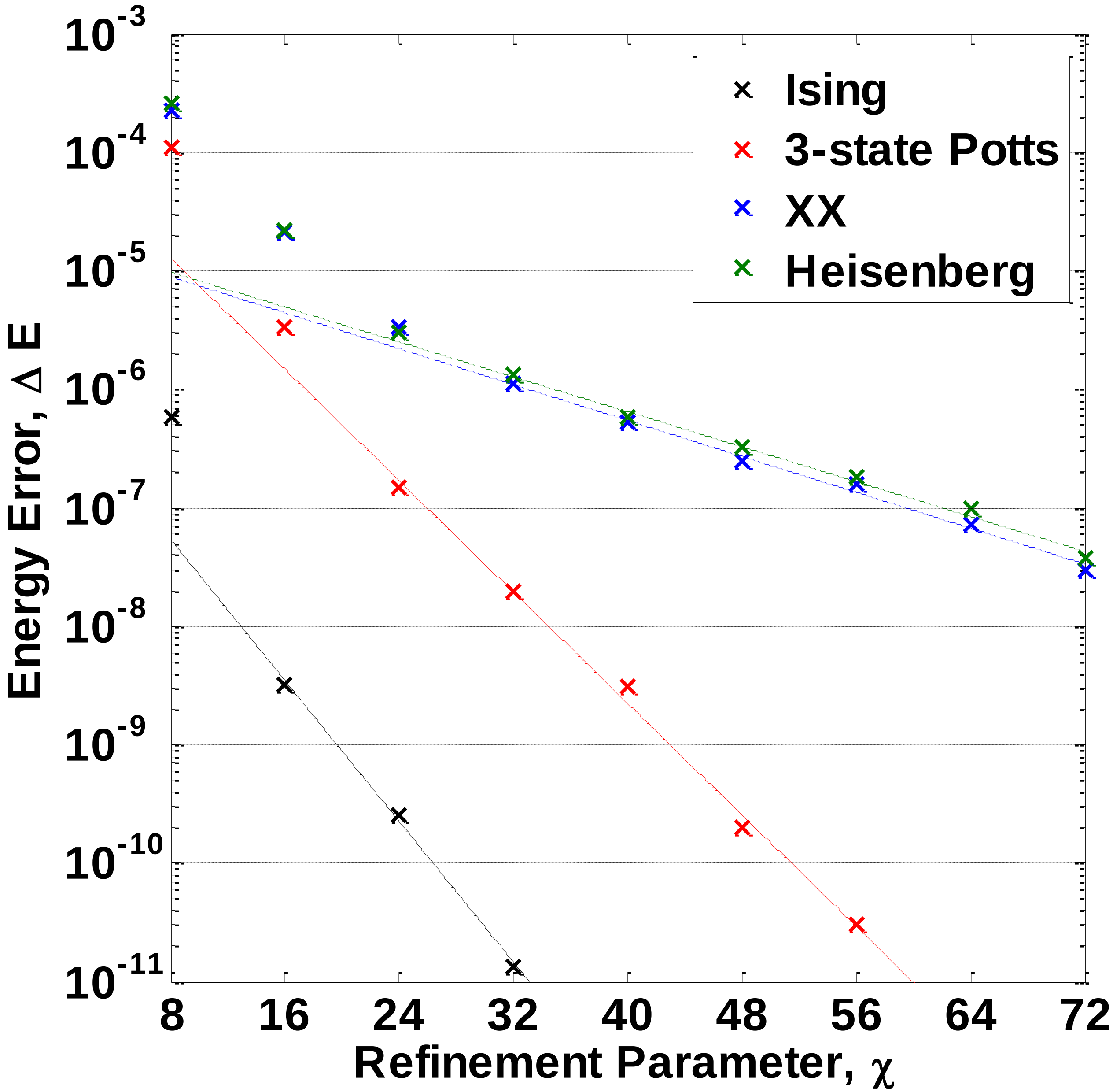
fusion rules

$$\varepsilon \times \varepsilon = \mathrm{I}$$

$$\sigma \times \sigma = \mathrm{I} + \varepsilon$$

$$\sigma \times \varepsilon = \sigma$$

• Example: other models



Recent developments:

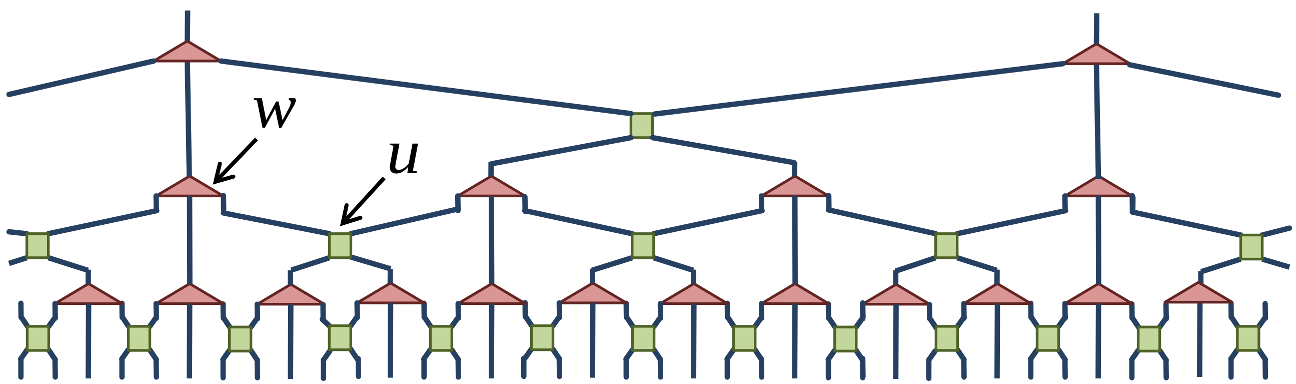
- non-local scaling operators (bulk)
- boundary critical phenomena
- defects (in bulk)

Non-local scaling operators

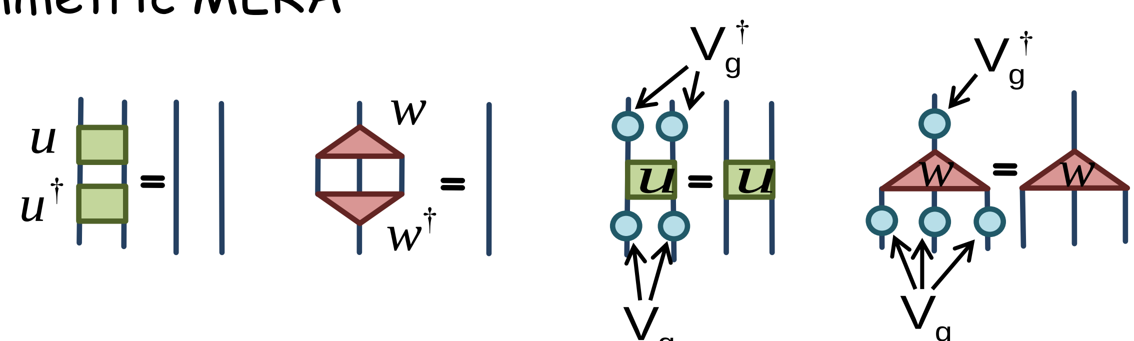
Evenbly, Corboz, Vidal, arXiv: 0912.2166

- global symmetry G $\quad V_g^{\otimes N} \, H \, V_g^{\dagger \, \otimes N} = H \qquad g \in G$

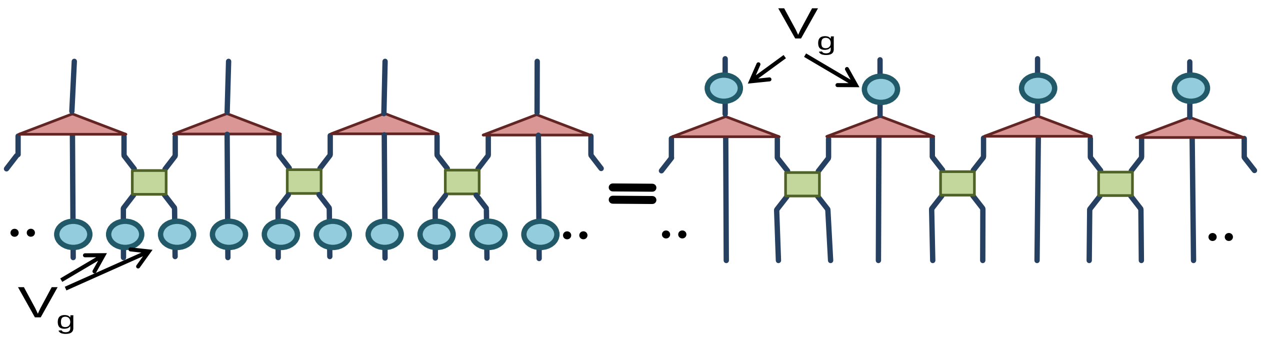
[example $\quad H_{\text{Ising}} \equiv -\sum X_i X_{i+1} - \sum Z_i \qquad Z^{\otimes N} \, H_{\text{Ising}} \, Z^{\otimes N} = H_{\text{Ising}}$]



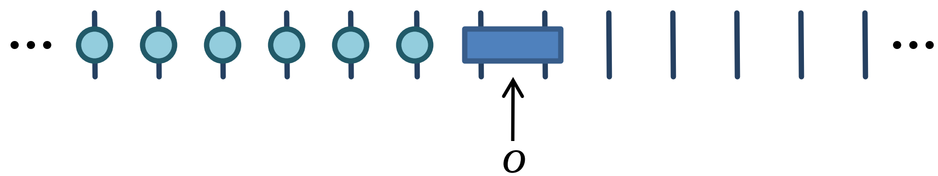
- G-symmetric MERA



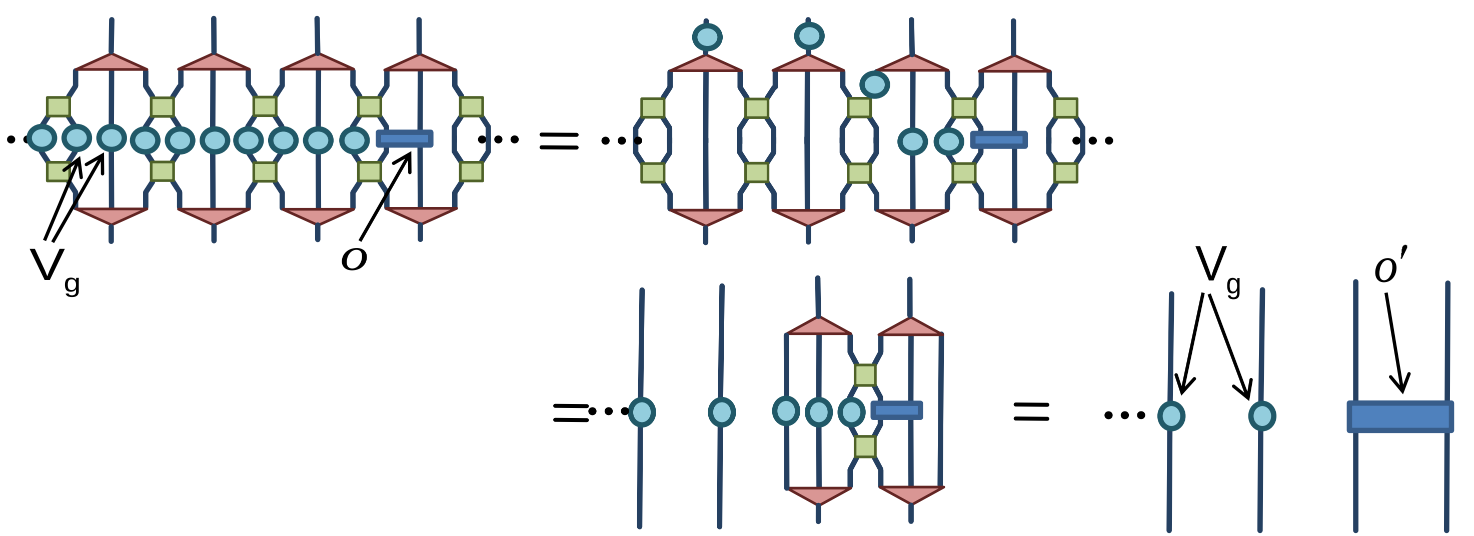
- “the symmetry commutes with the coarse-graining”



- non-local operators of the form



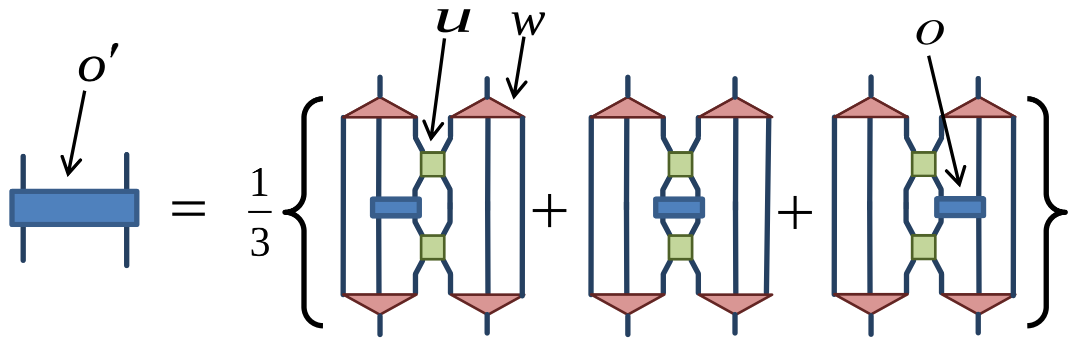
can be “locally” coarse-grained



- local scaling operators

$$o' = \mathcal{S}(o)$$

scaling superoperator



$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

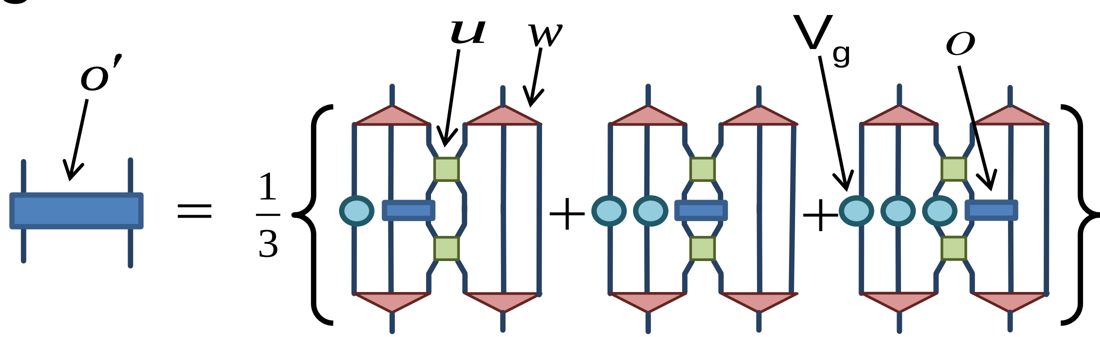
ϕ_α scaling operator $\qquad \phi_\alpha \rightarrow 3^{-\Delta_\alpha} \phi_\alpha$

Δ_α scaling dimension $\qquad \Delta_\alpha \equiv -\log_3 \lambda_\alpha$

- non-local scaling operators

$$o' = \mathcal{S}_g(o)$$

modified scaling superoperator



$$\mathcal{S}_g(\phi_{g,\alpha}) = \lambda_{g,\alpha} \phi_{g,\alpha}$$

$\phi_{g,\alpha}$ non-local scaling operator $\qquad \phi_{g,\alpha} \rightarrow 3^{-\Delta_{g,\alpha}} \phi_{g,\alpha}$

$\Delta_{g,\alpha}$ scaling dimension $\qquad \Delta_{g,\alpha} \equiv -\log_3 \lambda_{g,\alpha}$

Non-local scaling operators
Scale invariant MERA (bulk)

- Example: Ising model

$\mathbb{I}$ identity

σ spin

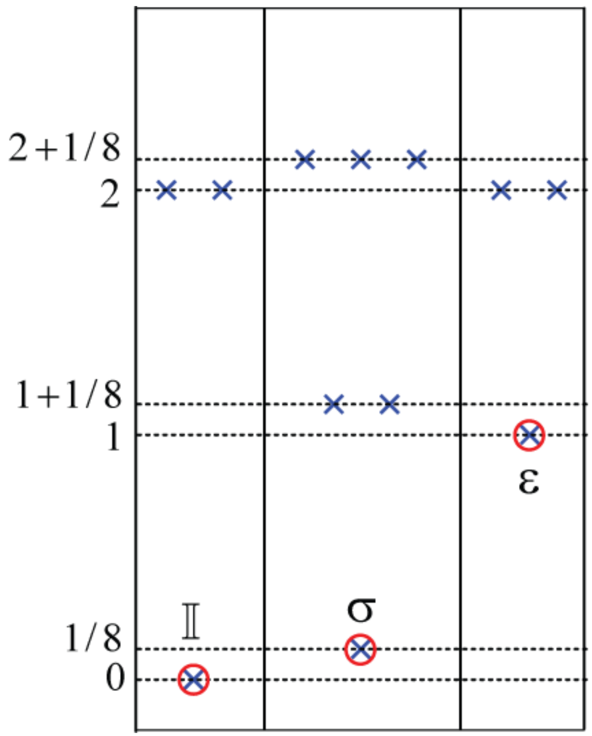
ε energy

μ disorder

ψ fermions

$\bar{\psi}$

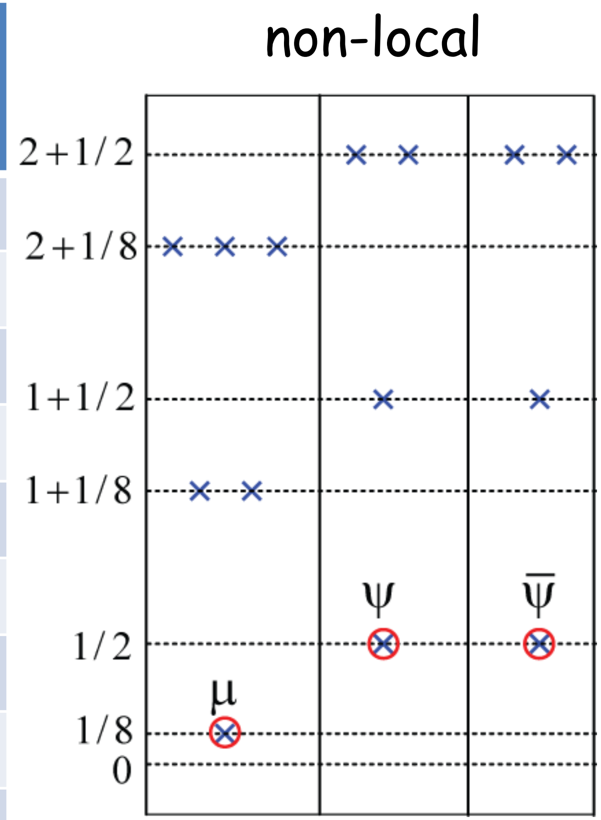
local



Non-local scaling operators
Scale invariant MERA (bulk)

- Example: Ising model

	scaling dimension (exact)	scaling dimension (MERA)	error
disorder μ	1/8	0.1250002	0.0002%
fermions ψ	1/2	0.5	<10−8%
$\bar{\psi}$	1/2	0.5	<10−8%
	1+1/8	1.124937	0.006 %
	1+1/2	1.49999	< 10−5%
	1+1/2	1.49999	< 10−5%
	2+1/8	2.123237	0.083 %
	2+1/8	2.124866	0.006 %
	2+1/8	2.125487	0.023 %



Non-local scaling operators
Scale invariant MERA (bulk)

- Example: Ising model

OPE for local & non-local primary fields

$$C_{\varepsilon\sigma\sigma} = 1/2 \qquad C_{\varepsilon\psi\bar{\psi}} = i$$

$$C_{\varepsilon\mu\mu} = -1/2 \qquad C_{\varepsilon\bar{\psi}\psi} = -i$$

$$C_{\psi\mu\sigma} = e^{-i\pi/4}/\sqrt{2}$$

$$(\pm 6\times 10^{-4})$$

$$C_{\bar{\psi}\mu\sigma} = e^{i\pi/4}/\sqrt{2}$$

$$\Longrightarrow$$

fusion rules

$$\varepsilon\times\varepsilon = \mathrm{I}$$
$$\sigma\times\sigma = \mathrm{I}+\varepsilon$$
$$\sigma\times\varepsilon = \sigma$$
$$\mu\times\mu = \mathrm{I}+\varepsilon$$
$$\mu\times\varepsilon = \mu$$
$$\psi\times\psi = \mathrm{I}$$
$$\bar{\psi}\times\bar{\psi} = \mathrm{I}$$
$$\psi\times\bar{\psi} = \varepsilon$$
$$\psi\times\varepsilon = \bar{\psi}$$
$$\bar{\psi}\times\varepsilon = \psi$$
$$\cdots$$

$$\{\mathrm{I},\varepsilon,\sigma,\mu,\psi,\bar{\psi}\}$$

local and semi-local subalgebras

$$\{\mathrm{I},\varepsilon\} \qquad \{\mathrm{I},\varepsilon,\sigma\}$$

$$\{\mathrm{I},\varepsilon,\mu\} \qquad \{\mathrm{I},\varepsilon,\psi,\bar{\psi}\}$$

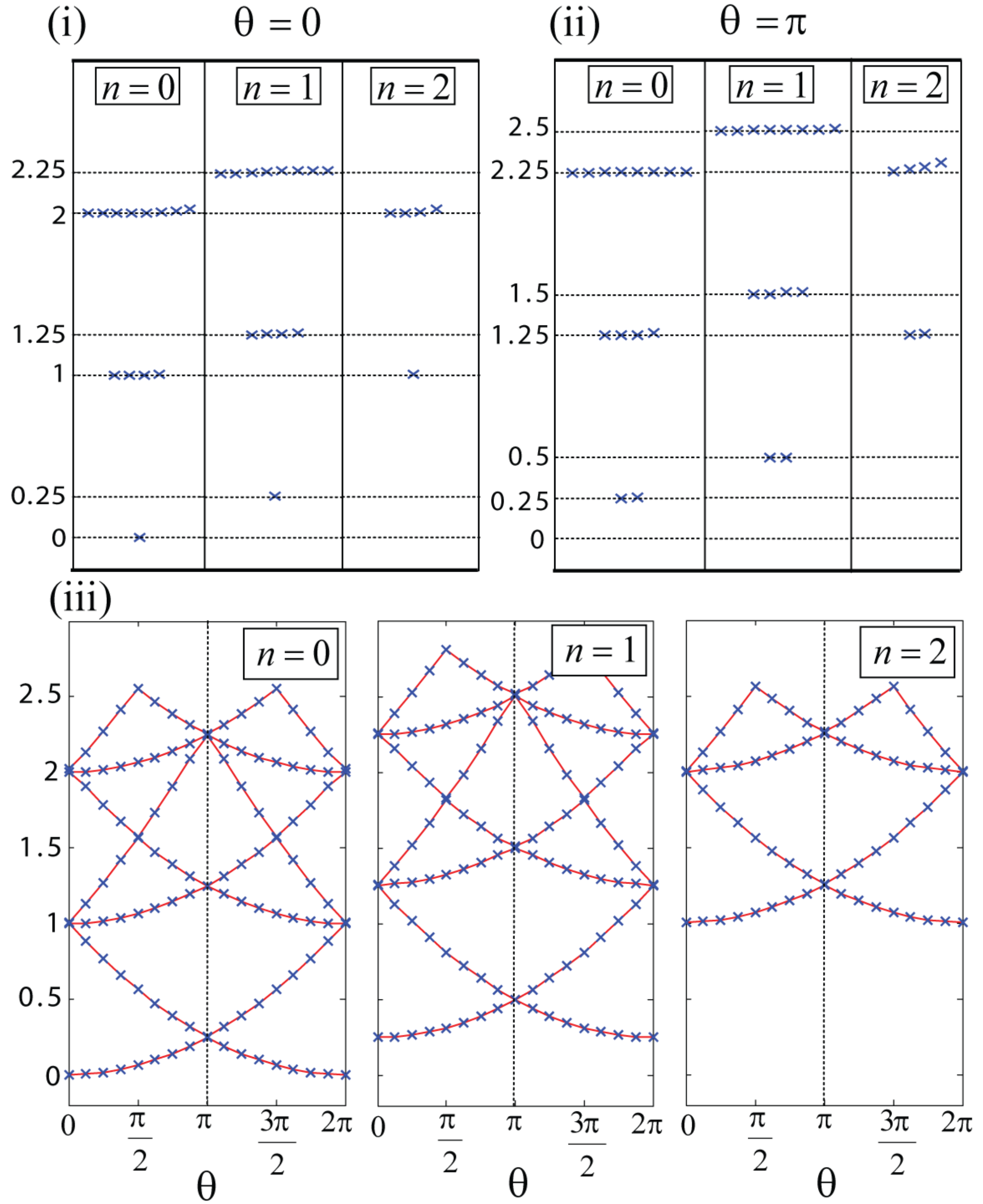
- Example:
  quantum XX model

$$H_{XX} \equiv -\sum (X_i X_{i+1} + Y_i Y_{i+1})$$

G = U(1) symmetry

χ=54 $\tilde{\chi}$=32

(exploiting U(1) symmetry)

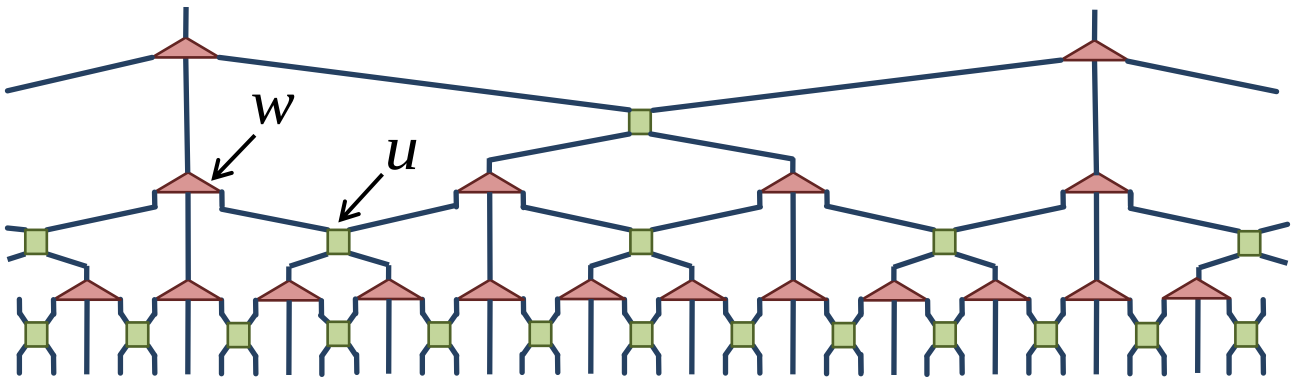


Boundary critical phenomena

Evenbly, Pfeifer, Pico, Iblisdir, Tagliacozzo, McCulloch, Vidal, arXiv:0912.1642

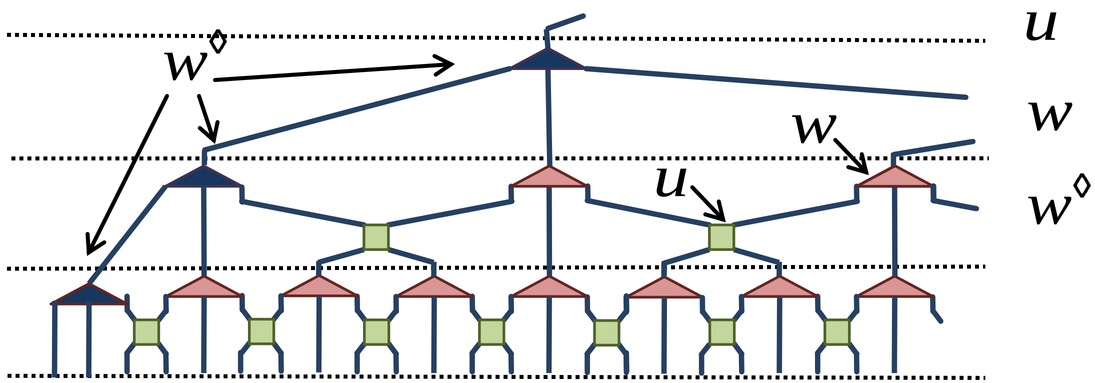
see also Silvi, Giovannetti, Calabrese, Santoro, Fazio, arXiv: 0912.2893

- infinite chain (bulk)



bulk MERA

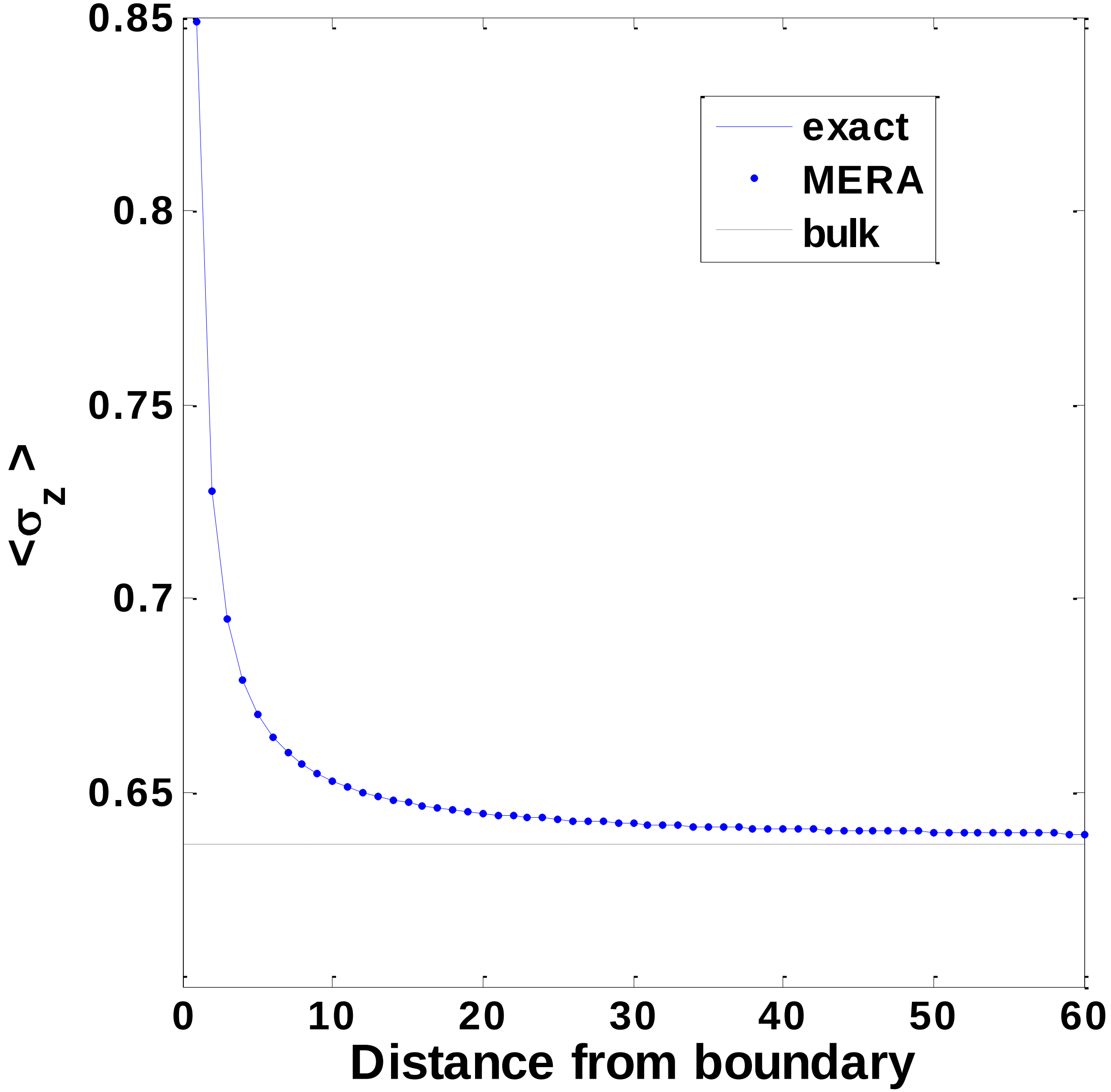
- semi-infinite chain (boundary)

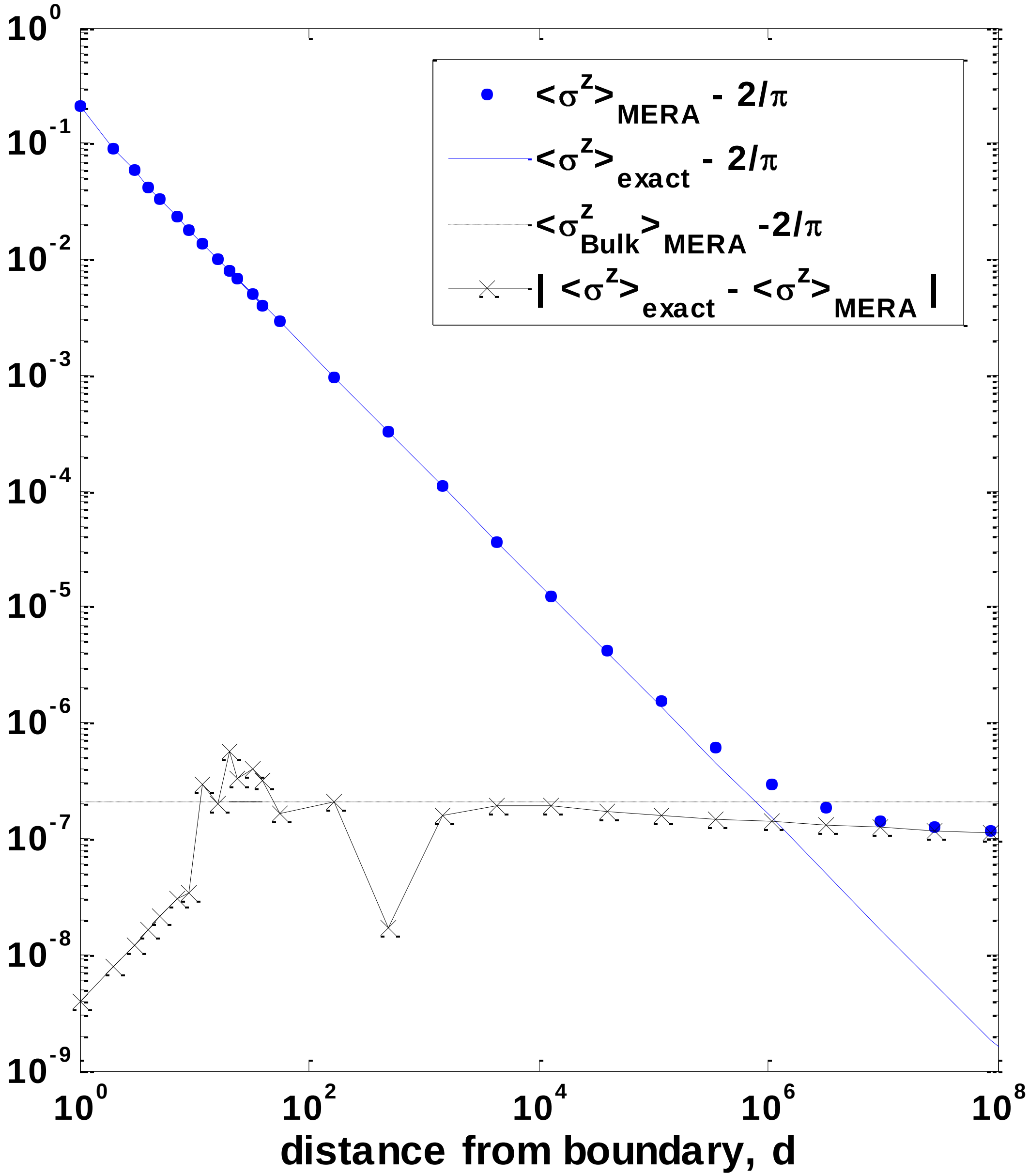


MERA with boundary

- Example: Ising model

Free boundary conditions: local magnetization





- Boundary effects still noticeable far away from the boundary

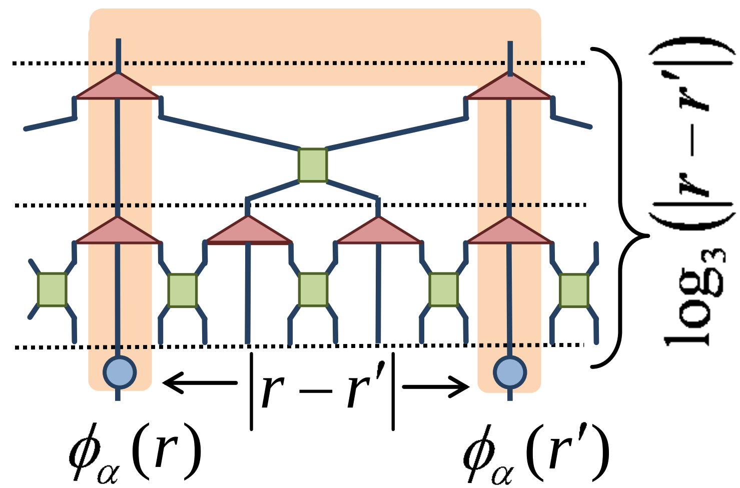
- MERA gets correct magnetization everywhere (approx. same accuracy as with bulk MERA without boundary)

Bulk expectation values in the presence of a boundary

- bulk

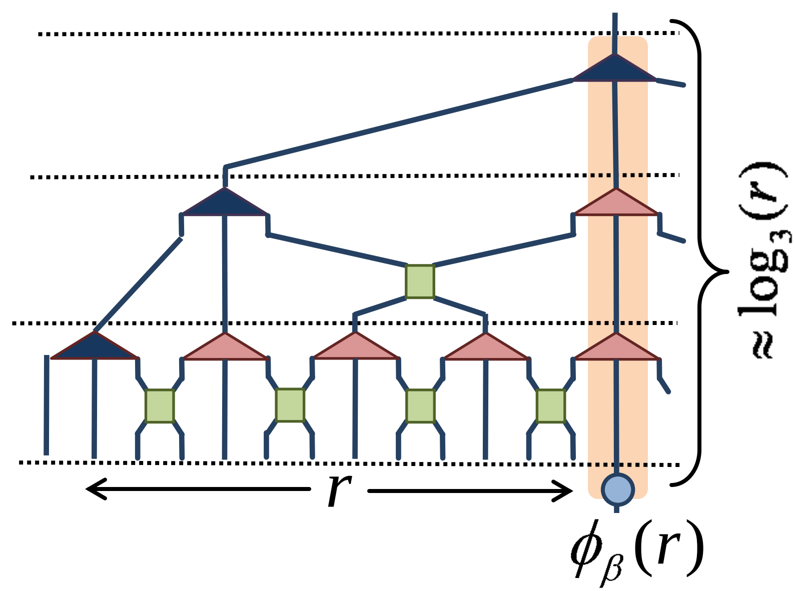
$$\langle \phi_\alpha(r) \rangle = 0$$

$$\langle \phi_\alpha(r)\phi_\alpha(r') \rangle = \frac{1}{|r-r'|^{2\Delta_\alpha}}$$

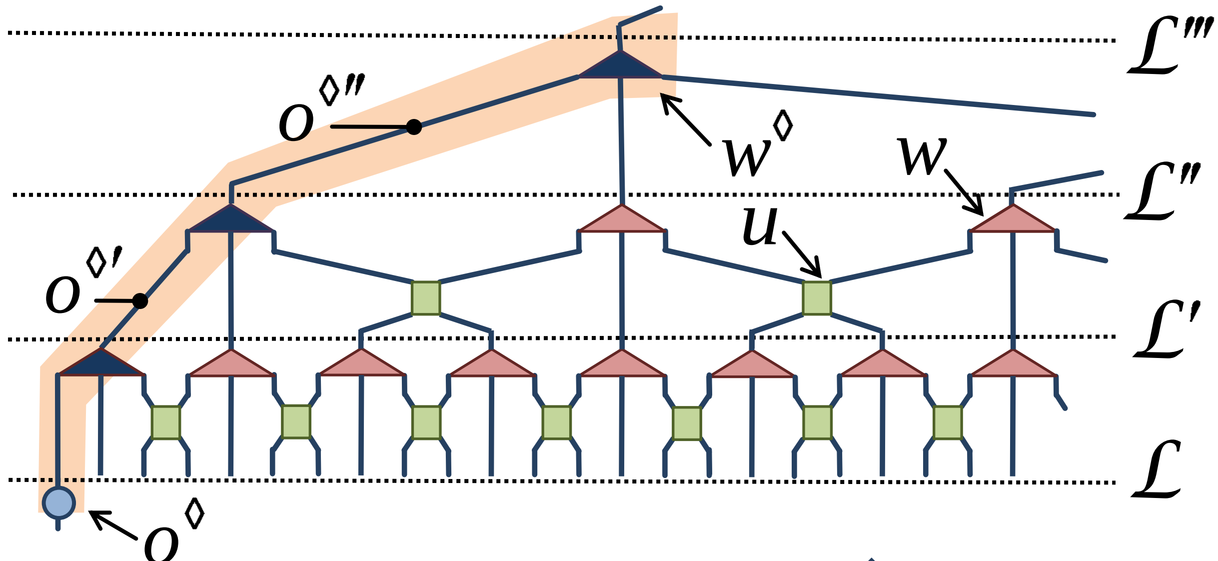


- boundary

$$\langle \phi_\alpha(r) \rangle \approx \frac{1}{|r|^{\Delta_\alpha}}$$

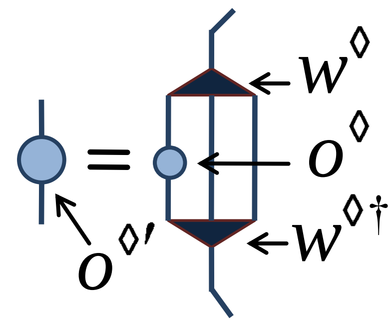


Boundary scaling operators/dimensions



boundary scaling superoperator

$$o^{\Diamond'} = \mathcal{S}^{\Diamond}(o^{\Diamond})$$



$$\mathcal{S}^{\Diamond}(\phi_{\alpha}^{\Diamond}) = \lambda_{\alpha}^{\Diamond}\phi_{\alpha}^{\Diamond}$$

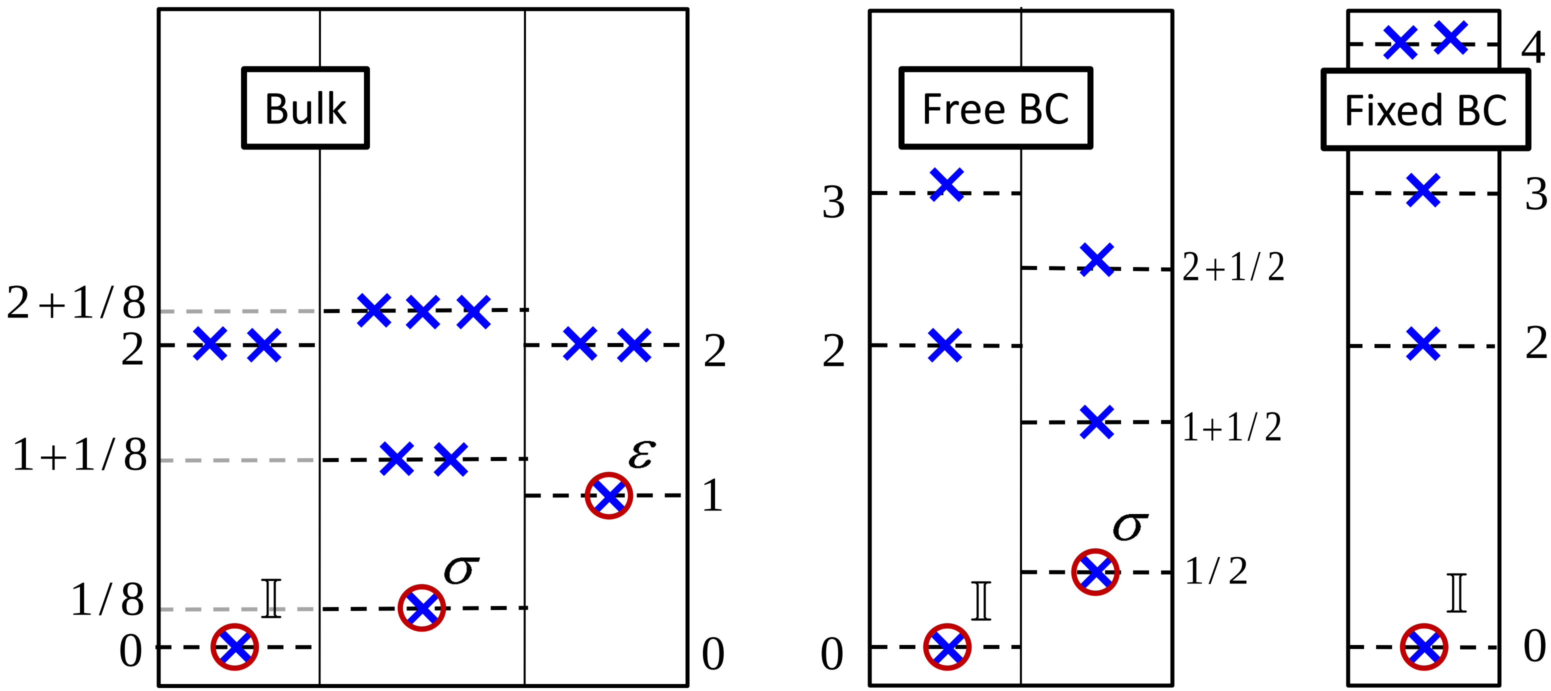
⇒

$\phi_{\alpha}^{\Diamond}$ boundary scaling operator $\qquad \phi_{\alpha}^{\Diamond} \rightarrow 3^{-\Delta_{\alpha}^{\Diamond}}\phi_{\alpha}^{\Diamond}$

$\Delta_{\alpha}^{\Diamond}$ boundary scaling dimension $\qquad \Delta_{\alpha}^{\Diamond} \equiv -\log_3 \lambda_{\alpha}^{\Diamond}$

Boundary scaling operators/dimensions

- Example:
  Ising model



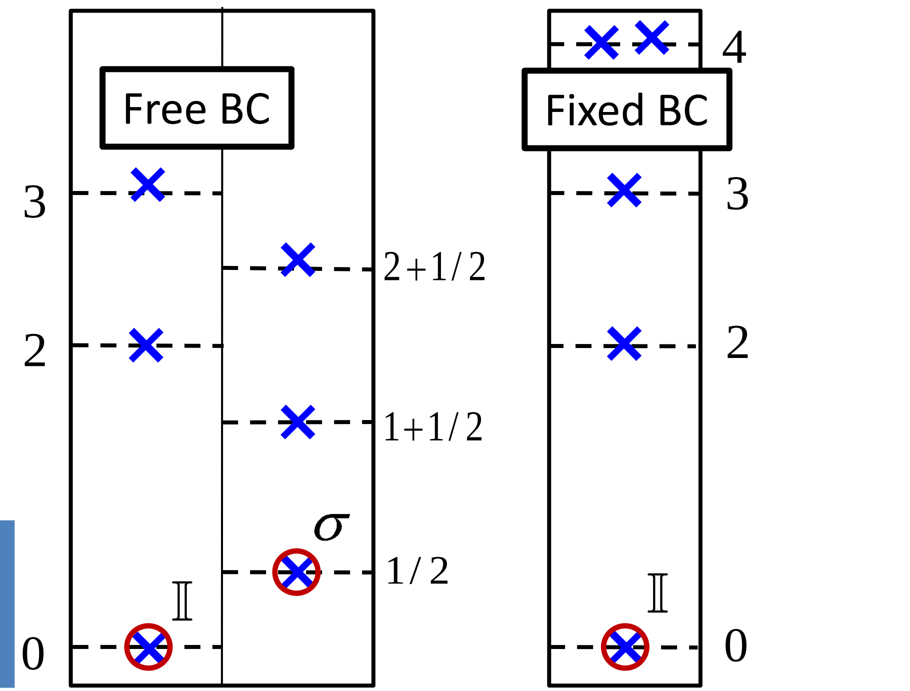
Boundary scaling operators/dimensions

Free BC

	scaling dimension (exact)	scaling dimension (MERA)	error
identity $\mathbb{I}$	0	0	---
spin σ	1/2	0.499	0.2%
	1+1/2	1.503	0.18%
	2	2.001	0.07 %
	2+1/2	2.553	2.1%

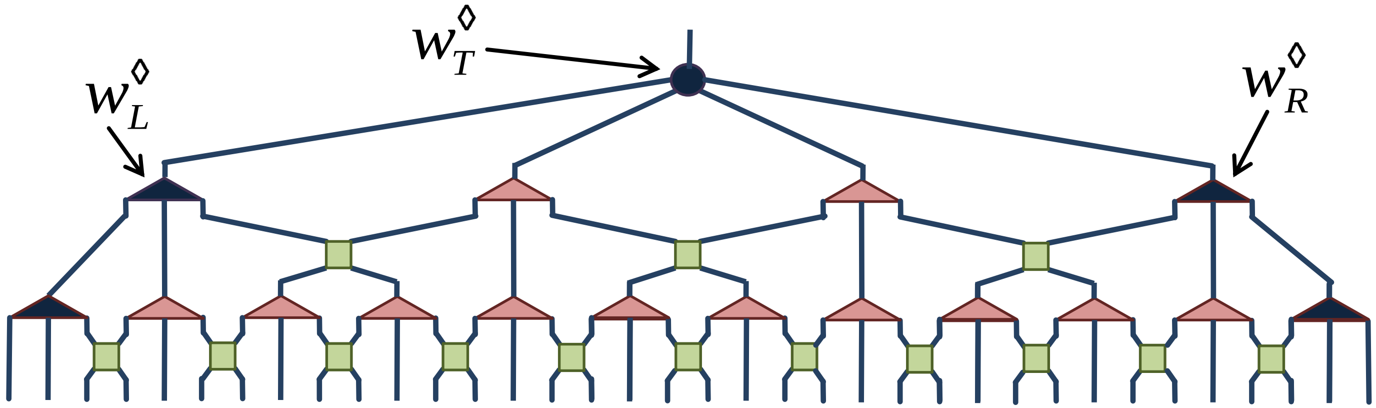
Fixed BC

	scaling dimension (exact)	scaling dimension (MERA)	error
identity $\mathbb{I}$	0	0	---
	2	1.992	0.4%
	3	2.998	0.07%
	4	4.005	0.12 %
	4	4.062	1.5%

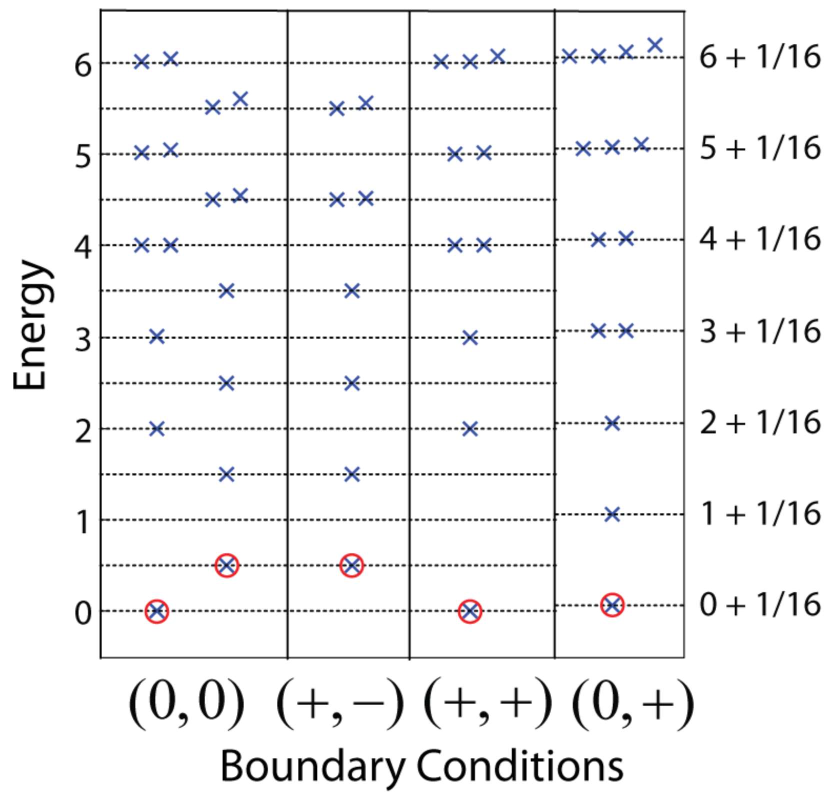


Finite system with two boundaries

- finite MERA with two boundaries

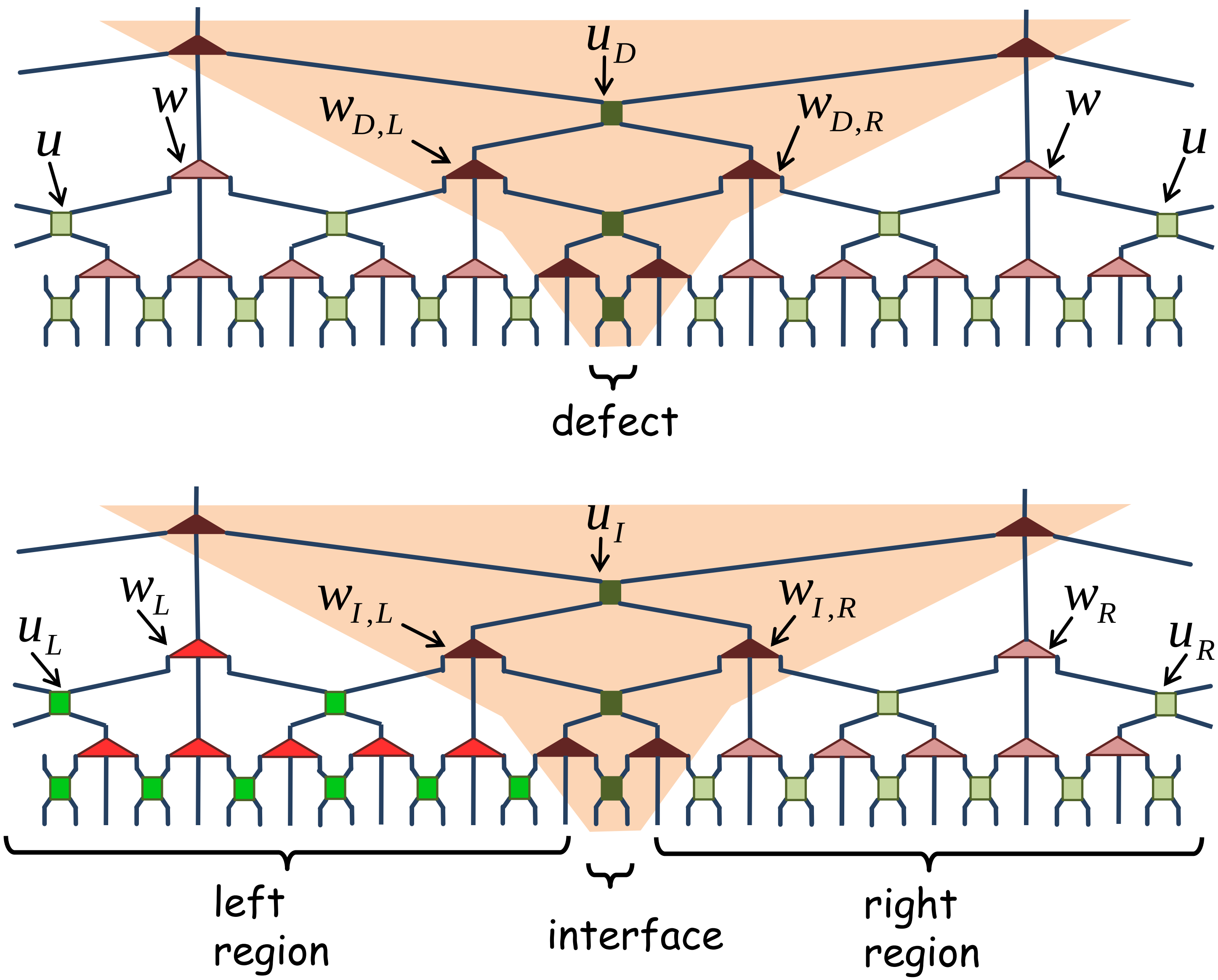


- Energy spectrum

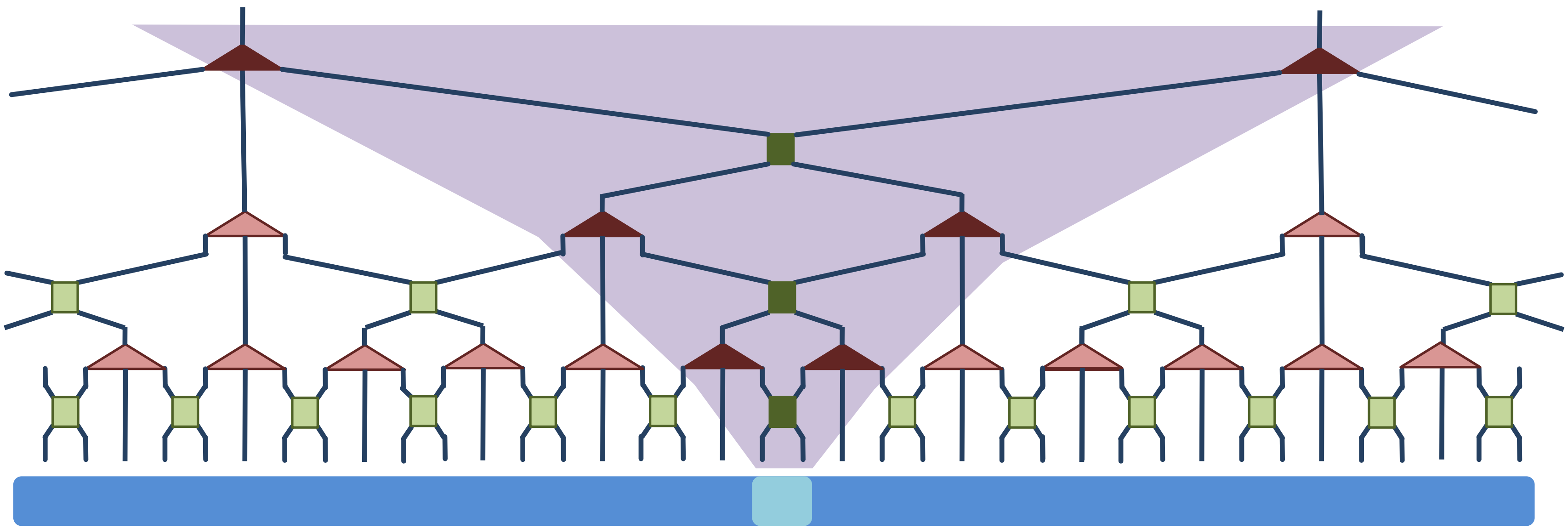


defect / interface

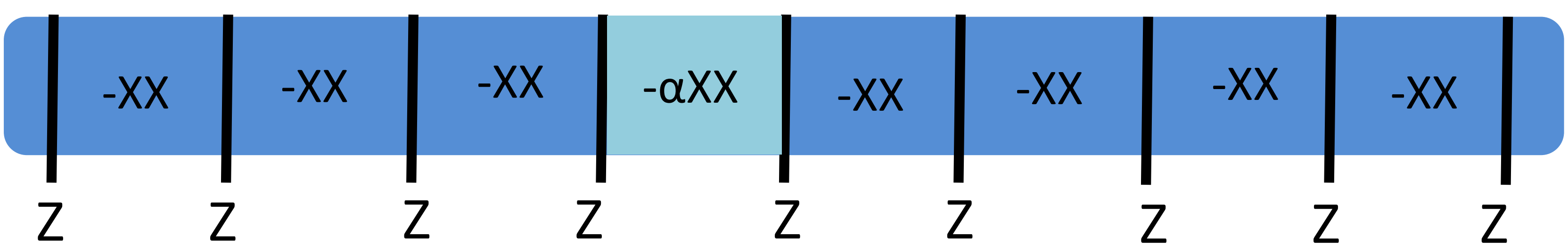
Evenbly, Pfeifer, Pico, Iblisdir, Tagliacozzo, McCulloch, Vidal, arXiv:0912.1642

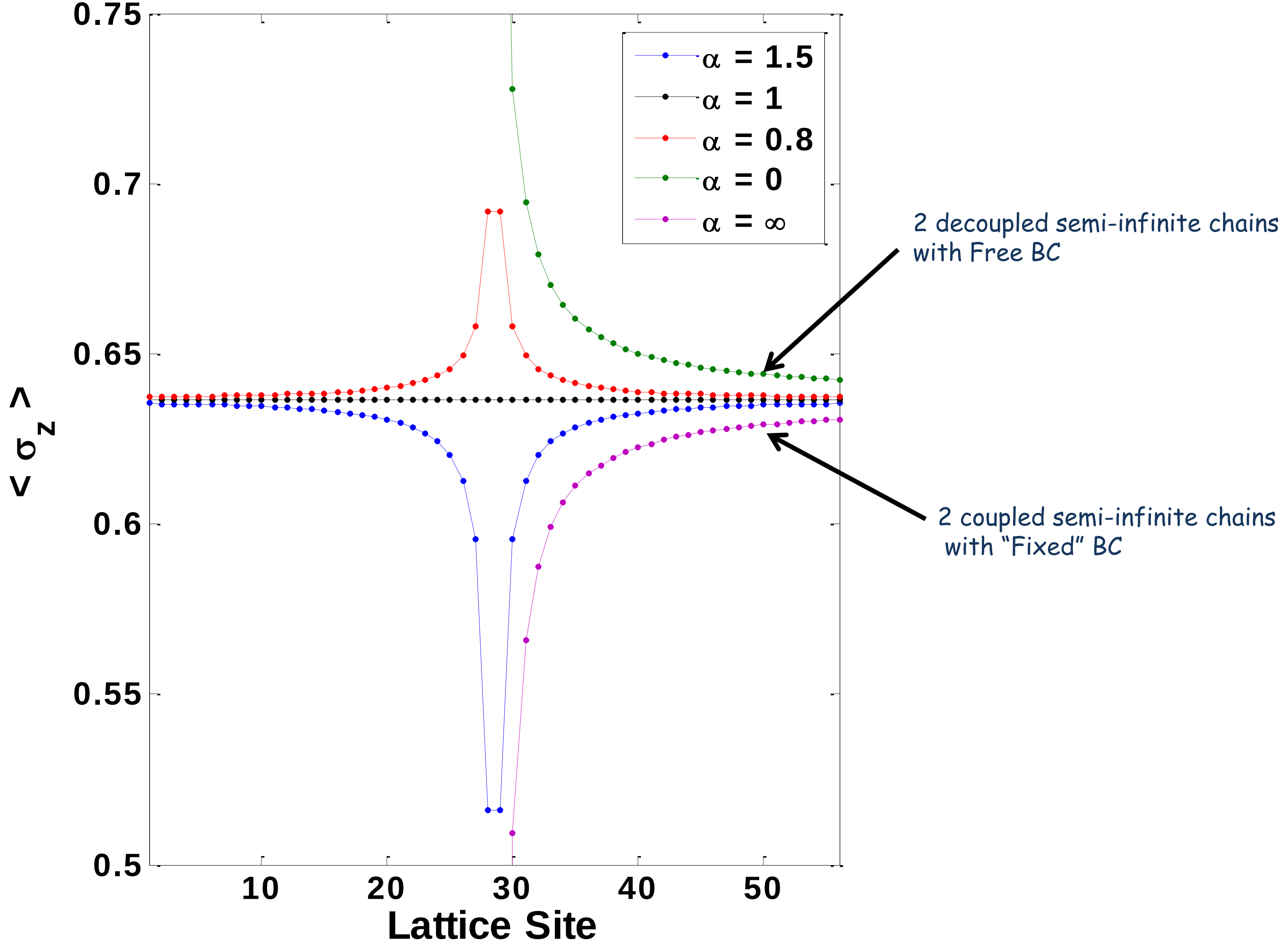


Lattice Defects:



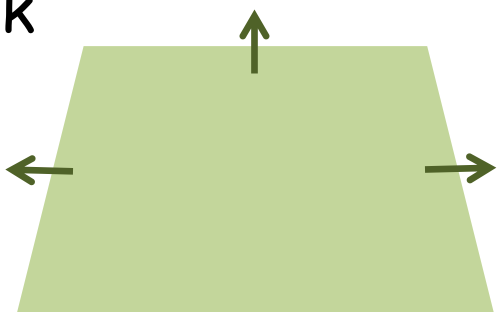
Hamiltonian:




α = 1.5
α = 1
α = 0.8
α = 0
α = ∞
2 decoupled semi-infinite chains with Free BC
2 coupled semi-infinite chains with "Fixed" BC
< σ z >
0.75
0.7
0.65
0.6
0.55
0.5
10
20
30
40
50
Lattice Site

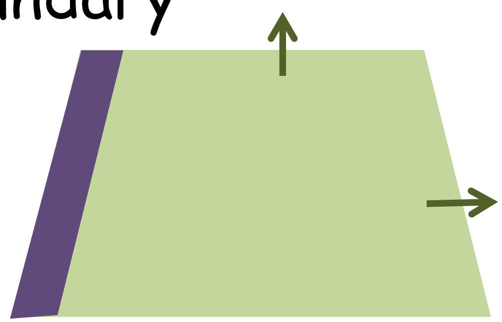
Summary: scale invariant MERA, critical phenomena, and CFT

- Bulk

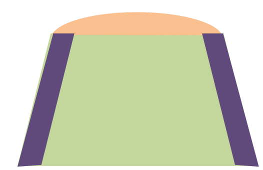


- (local & non-local) scaling operators/dimensions
- CFT: primary fields and OPE

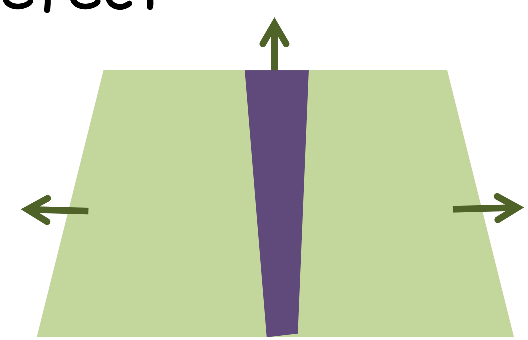
- Boundary



- boundary scaling operators
- BCFT: primary fields and OPE
- finite system with two boundaries



- Defect



- defect scaling operators/dimensions
- interface