Fidelity Susceptibility and Quantum Phase Transition

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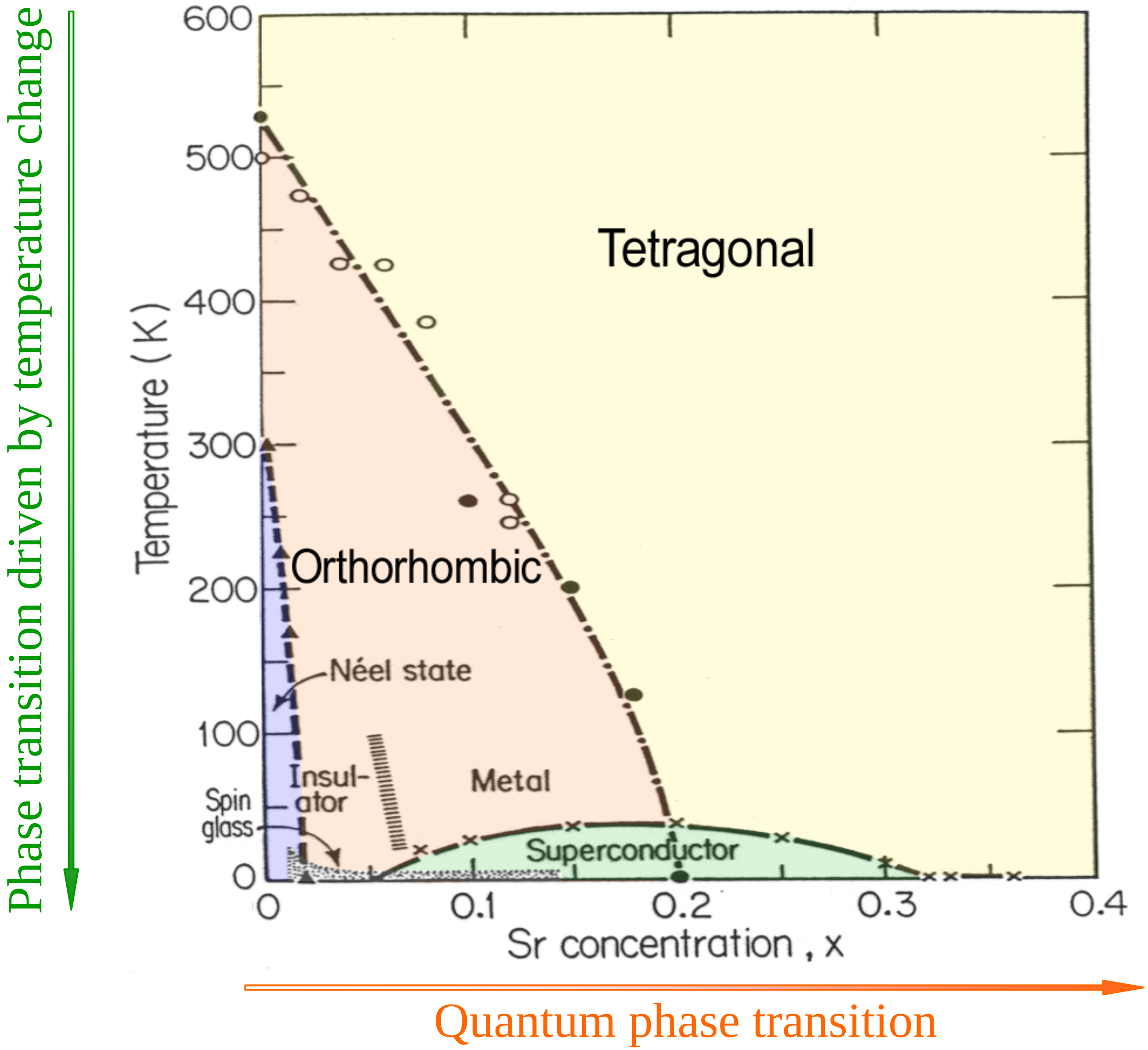
Outline

- Introduction: quantum phase transition (QPT), fidelity and fidelity susceptibility (FS).
- FS & QPT in the one-dimensional asymmetric Hubbard model (AHM)
- FS & QPT in the Kitaev honeycomb model
- FS & QPT in the **L**ipkin-**M**eshkov-**G**lick model
- Summary and discussions

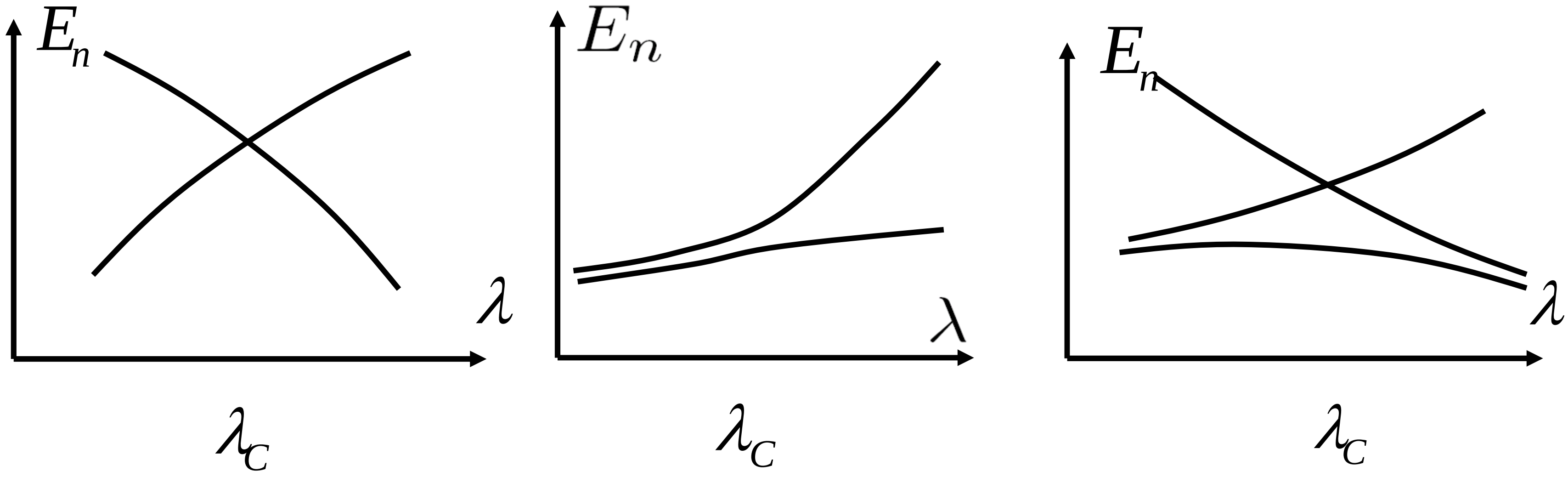
$La_{2-x}Sr_xCuO_4$ Phase Diagram



Occurrence of the quantum phase transitions

$$H(\lambda) = H_0 + \lambda H_I,$$

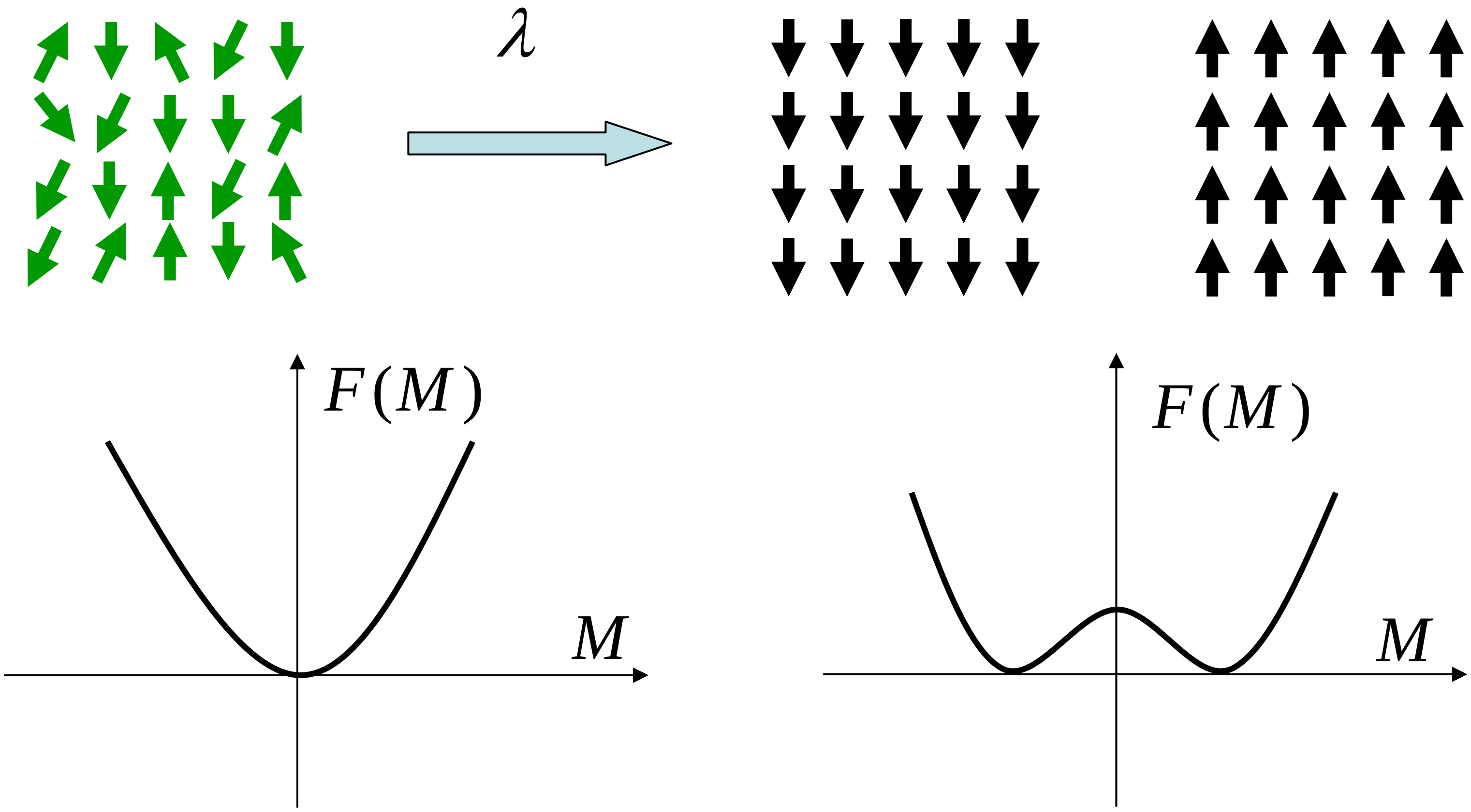
$$H(\lambda)|\Psi_n(\lambda)\rangle \;=\; E_n|\Psi_n(\lambda)\rangle$$



J_1-J_2 **Transverse Field Ising** **XXZ**

QPT: traditional view

Landau's symmetry-breaking theory



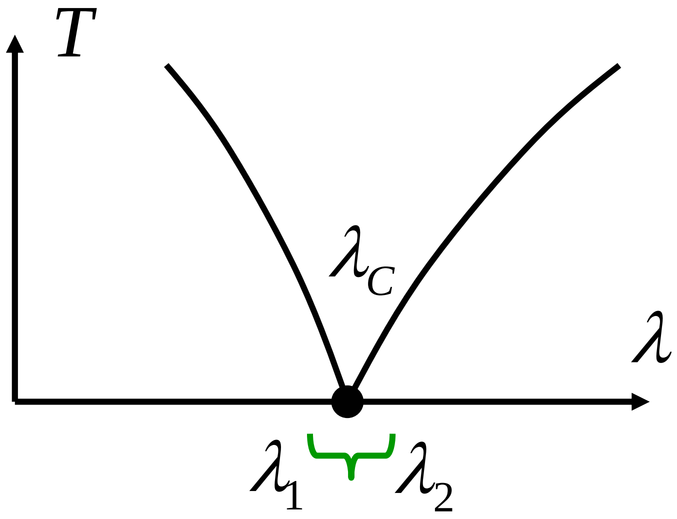
Requires some knowledge of order parameter

QPT: quantum information view?

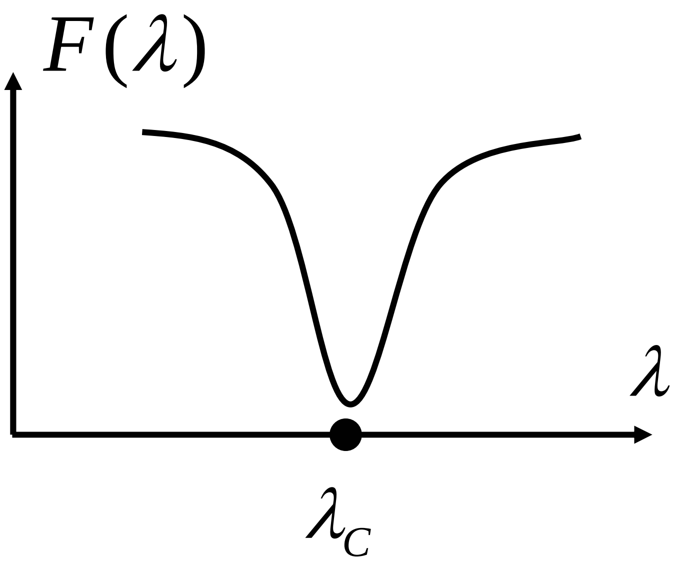
- **Could one look into QPT from quantum information point of view?**
- **Yes, there exit many works exploring relation between quantum entanglement and quantum phase transition.**
- **Most of these works used entanglement measure such as von Neumann entropy, concurrence, negativity, etc.**

QPT: the Fidelity View?

Those measures may not work for some systems. Could we look at the eigenfunction of the many-body Hamiltonian directly?



$$\psi(\lambda_1) \qquad \psi(\lambda_2)$$



$$F = |\langle \psi(\lambda_1) | \psi(\lambda_2) \rangle|$$

Questions

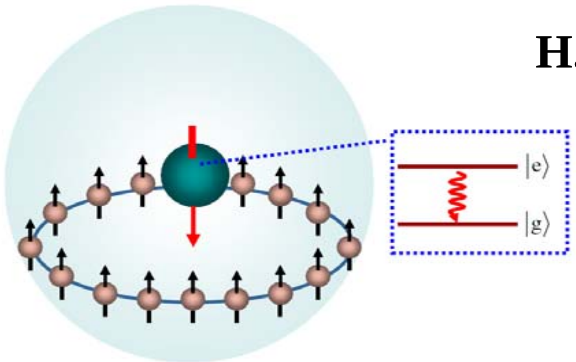
- Is there any connection between the fidelity and quantum phase transition (Landau type, KT, Topological).
- Will we also observe some kinds of universal behaviors as in the traditional approach to the critical phenomena?
- Is the fidelity a good quantity to measure for a many-body system?

Examples of QPT & Fidelity

Ising model

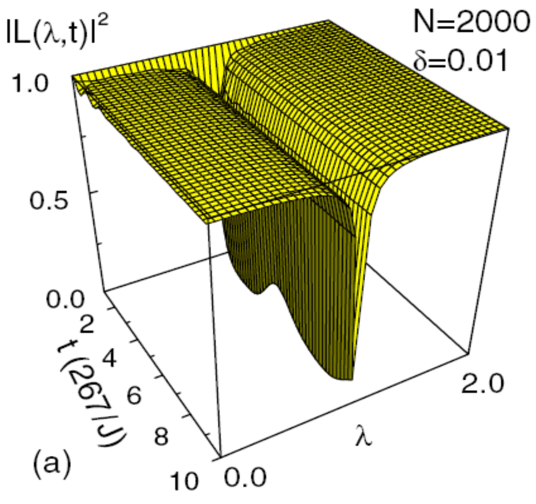
$$H(\lambda, \delta) = -J\sum_j (\sigma_j^z \sigma_{j+1}^z + \lambda\sigma_j^x + \delta|e\rangle\langle e|\sigma_j^x),$$

H. T. Quan, …, C. P. Sun, PRL 96, 140604 (2006).

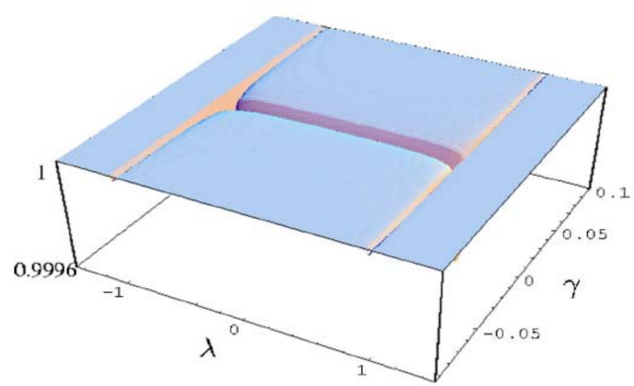


$$L(\lambda, t) = |\langle\varphi_g(t)|\varphi_e(t)\rangle|^2.$$

$$\hat{H}(\gamma,\lambda) = -\sum_{i=-M}^{M}\left(\frac{1+\gamma}{2}\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2}\hat{\sigma}_i^y\hat{\sigma}_{i+1}^y + \frac{\lambda}{2}\hat{\sigma}_i^z\right).$$



P. Zanardi and N. Paunkovijc, PRE 74, 031123 (2006).

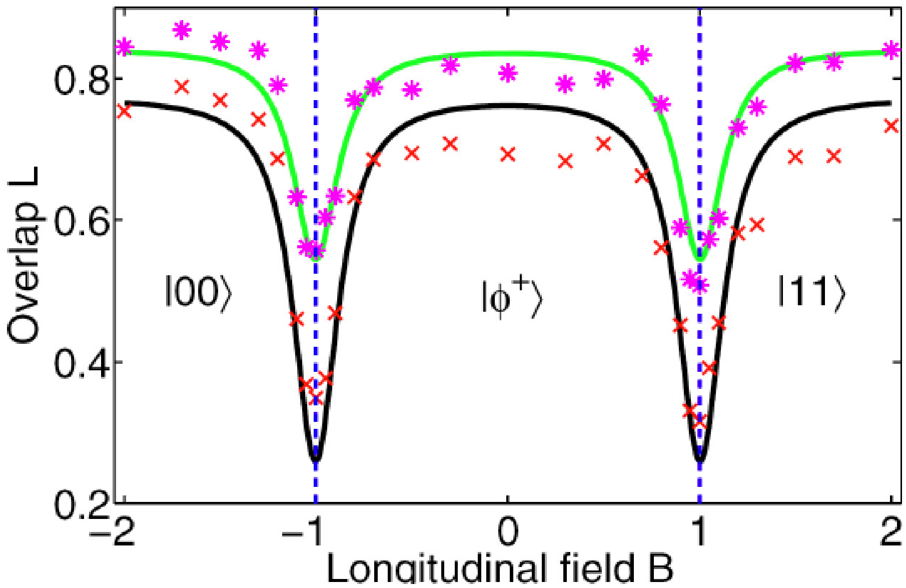


Examples of QPT & Fidelity

Nuclear-magnetic-resonance(NMR) experiments

J. Zhang, X. Peng, N. Rajendran, and D. Suter, Phys. Rev. Lett. **100**, 100501 (2008).

$$H^s = \sigma_1^z \sigma_2^z + B_x(\sigma_1^x + \sigma_2^x) + B_z(\sigma_1^z + \sigma_2^z)$$



J. Zhang, etal, Phys. Rev. A 79, 012305 (2009)

Fidelity Susceptibility

$$H(\lambda) = H_0 + \lambda H_I, \; H(\lambda)|\Psi_n(\lambda)\rangle = E_n|\Psi_n(\lambda)\rangle$$

Perturbation Theory

$$|\Psi_0(\lambda + \delta\lambda)\rangle = |\Psi_0(\lambda)\rangle + \delta\lambda \sum_{n \neq 0} \frac{H_{n0}(\lambda)|\Psi_n(\lambda)\rangle}{E_0(\lambda) - E_n(\lambda)}$$

$$H_{n0} = \langle \Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle.$$

$$F_i(\lambda, \delta) = |\langle \Psi_0(\lambda)|\Psi_0(\lambda + \delta\lambda)\rangle|$$ **Depends on both λ & $\delta\lambda$**

$$\frac{1}{F_i^2} = 1 + \delta\lambda^2 \sum_{n \neq 0} \frac{|\langle \Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle|^2}{[E_n(\lambda) - E_0(\lambda)]^2}$$

$$\chi_F \equiv \lim_{\delta\lambda \to 0} \frac{-2 \ln F_i}{\delta\lambda^2}$$

$$\chi_F(\lambda) = \sum_{n \neq 0} \frac{|\langle \Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle|^2}{[E_n(\lambda) - E_0(\lambda)]^2}$$

Fidelity Susceptibility

Physical interpretation

$$\chi_F(\omega) = \sum_{n \neq 0} \frac{|\langle \Psi_n | H_I | \Psi_0 \rangle|^2}{[E_n - E_0]^2 + \omega^2} \qquad H(\lambda) = H_0 + \lambda H_I,$$

$$\frac{\partial \chi_F(\tau)}{\partial \tau} = -\pi \left[\langle \Psi_0 | H_I(\tau) H_I(0) | \Psi_0 \rangle - \langle \Psi_0 | H_I | \Psi_0 \rangle^2 \right] \theta(\tau)$$

$$+ \pi \left[\langle \Psi_0 | H_I(0) H_I(\tau) | \Psi_0 \rangle - \langle \Psi_0 | H_I | \Psi_0 \rangle^2 \right] \theta(-\tau)$$

Fidelity susceptibility =

dynamic structure factor of the driving term

Fidelity Susceptibility at Finite Temperatures

Extension to thermal state

$$F_i(\beta,\delta) = \frac{Z(\beta)}{\sqrt{Z(\beta-\delta\beta/2)Z(\beta+\delta\beta/2)}},$$

$$Z(\beta) = \sum_n e^{-\beta E_n} = \sum_E g(E) e^{-\beta E}.$$

Zanardi, Quan, Wang, and Sun, PRA 75, 032109 (2007)

You, Li, and Gu, PRE 76, 022101 (2007)

$$\chi_F = \left.\frac{-2\ln F_i}{\delta\beta^2}\right|_{\delta\beta\to 0} = \frac{C_v}{4\beta^2} \qquad C_v = \beta^2(\langle E^2\rangle - \langle E\rangle^2)$$

$$\chi_F = \left.\frac{-2\ln F_i}{\delta h^2}\right|_{\delta h\to 0} = \frac{\beta\chi}{4} \qquad \chi = \beta(\langle \dot{M}^2\rangle - \langle \dot{M}\rangle^2)$$

Critical Exponent & Universality Class

L. C. Venuti and P. Zanardi, Phys. Rev. Lett. **99** , 095701 (2007).

S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B **77**, 245109 (2008). arXiv:0706.2495

Similar to traditional critical phenomena

$$\frac{\chi_{F(\lambda)}(\lambda)}{N} \propto \frac{1}{|\lambda_c - \lambda|^{\alpha}}$$

Examples follow

Fidelity Susceptibility in the 1D Asymmetric Hubbard Model

- The asymmetric Hubbard model and the phase diagram
- The scaling behaviors of the fidelity susceptibility

Asymmetric Hubbard Model

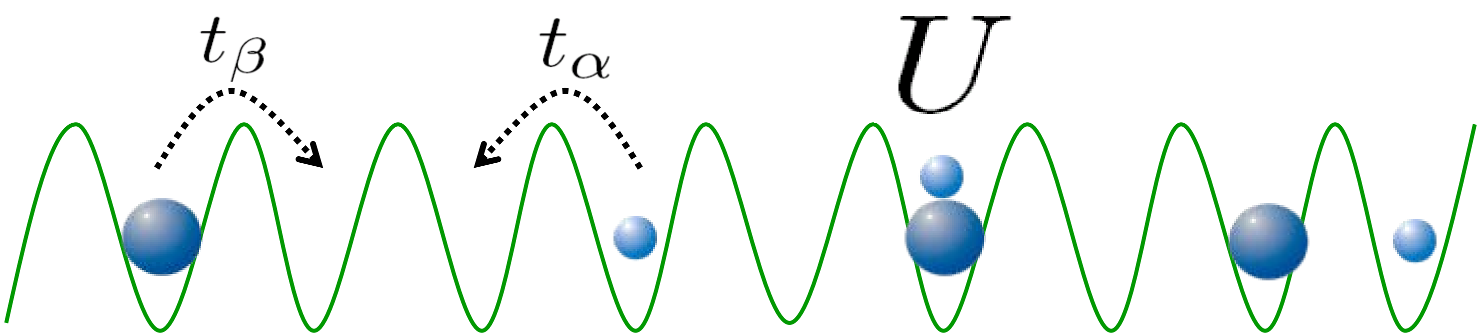
$$H = -\sum_{j=1}^{L} \sum_{\delta=\pm 1} \sum_{\sigma} t_\sigma c^\dagger_{j,\sigma} c_{j+\delta,\sigma} + U \sum_{j=1}^{L} n_{j,\alpha} n_{j,\beta}.$$

$\sigma = \alpha, \beta \quad$ (e.g., ^{6}Li and ^{40}K) $\qquad V_{\text{lat}}(x) = V_0 \sin^2(kx)$

$$\frac{t_\beta}{t_\alpha} = \frac{m_\alpha}{m_\beta}, \quad t_\alpha = 1, \; t_\beta \in [0,1]$$

$)w_\sigma(x - x_{j+1})$

$$\frac{U}{t_\alpha} = \frac{16a\sqrt{\pi m_\alpha / m_\beta}}{\lambda} \frac{v^{1/4} v_\perp^{1/2}}{(\sqrt{v} + 2\sqrt{v_\perp})} e^{\pi^2 \sqrt{v}}$$



Issues

- For two types of atoms (fermions), will they mix uniformly (**Density Wave**), or phase separated into two regions that one is dominated by heavy atoms while the other by light atoms (**Phase Separation**)?
- Transition from DW to PS?
- Relation to quantum information?

An argument for one dimensional case

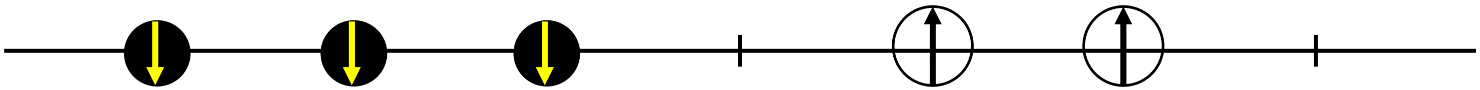
- When t_β/t_α = **1,** the **Hubbard** model, **No PS**

$$H = -\sum_{\sigma,j,\delta} t\, c^{+}_{j,\sigma} c_{j+\delta,\sigma} + U \sum_{j} n_{j,\alpha} n_{j,\beta}$$

- When t_β/t_α = **0,** the **Falicov-Kimball** model,

$$H = \sum_{<ij>} t\left(c_i^{+} c_j + c_j^{+} c_i\right) + U \sum_i n_i \cdot w_i$$

one type of fermion motionless, $U \gg t$, **Yes PS**

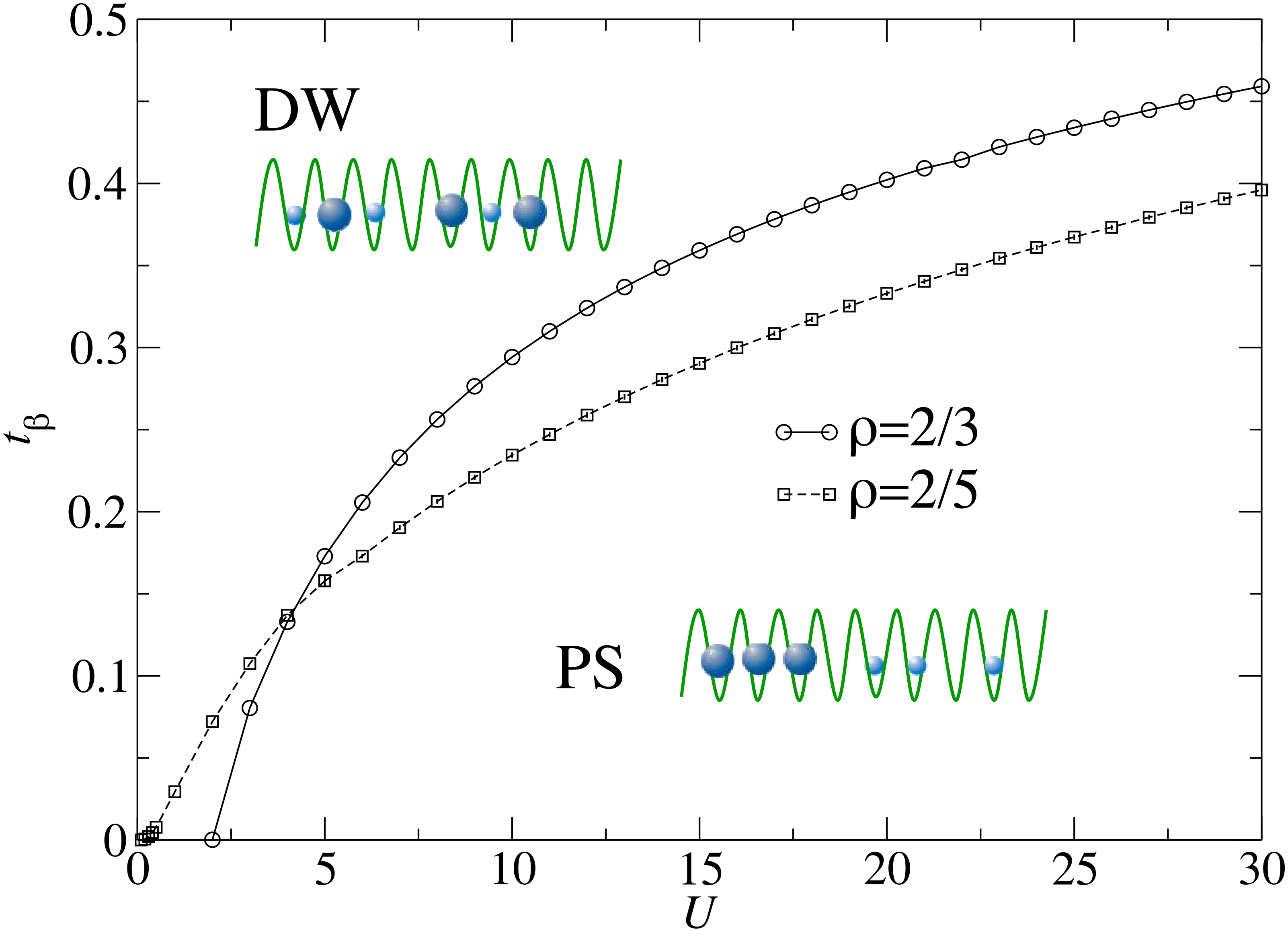


Freericks, Lieb, and Ueltsci, Phys. Rev. Lett. 88, 106401 (2002).

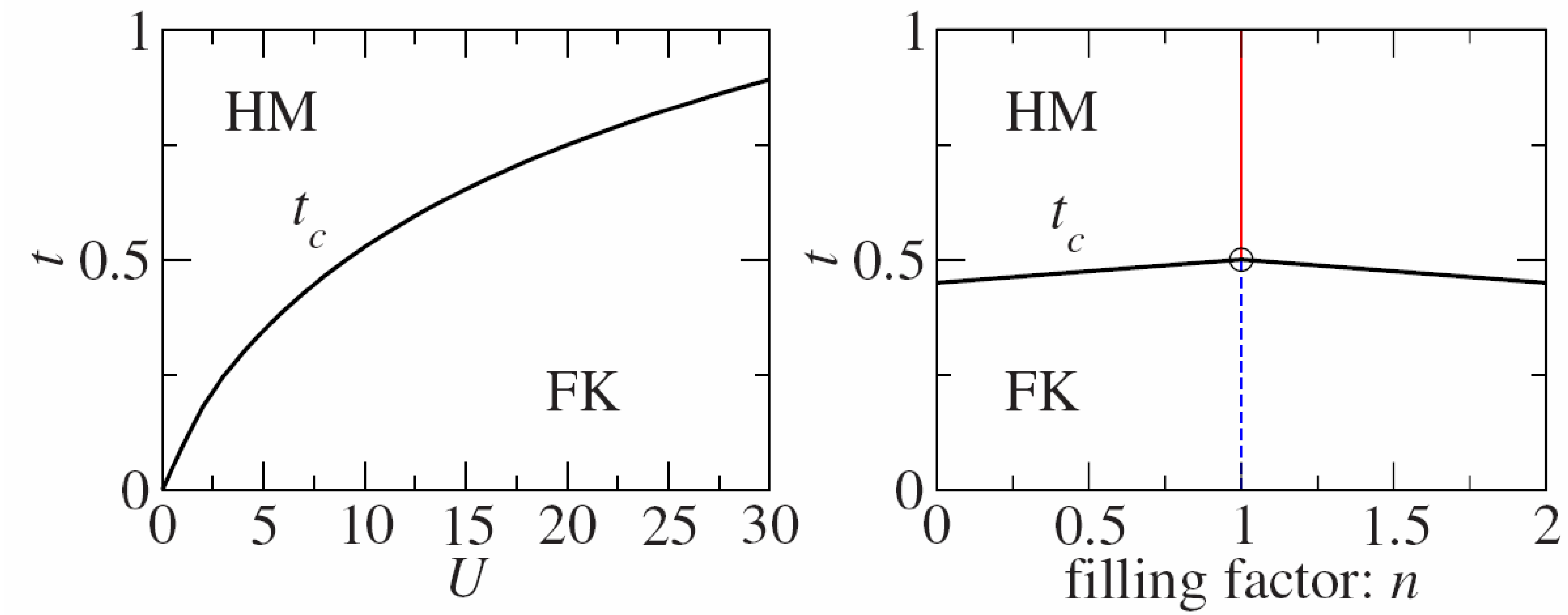
An argument for one dimensional case

$$H = -\sum_{\sigma,j,\delta} t_\sigma c^+_{j,\sigma} c_{j+\delta,\sigma} + U\sum_j n_{j,\alpha} n_{j,\beta}$$

The 1D Asymmetric Hubbard Model in its two limits: the Hubbard model ($t_\beta = t_\alpha$, **SU(2)**) and the Falicov-Kimball model ($t_\beta = 0$, $\mathbf{Z_2 \times U(1)}$) , belong to **two different universality classes**, a quantum phase transition (**QPT**) from the charge density (**DW**) state to the phase separation (**PS**) state is expected to appear on the U- t_β plane.


DW
PS
ρ=2/3
ρ=2/5
t_β
U
0
0.1
0.2
0.3
0.4
0.5
0
5
10
15
20
25
30

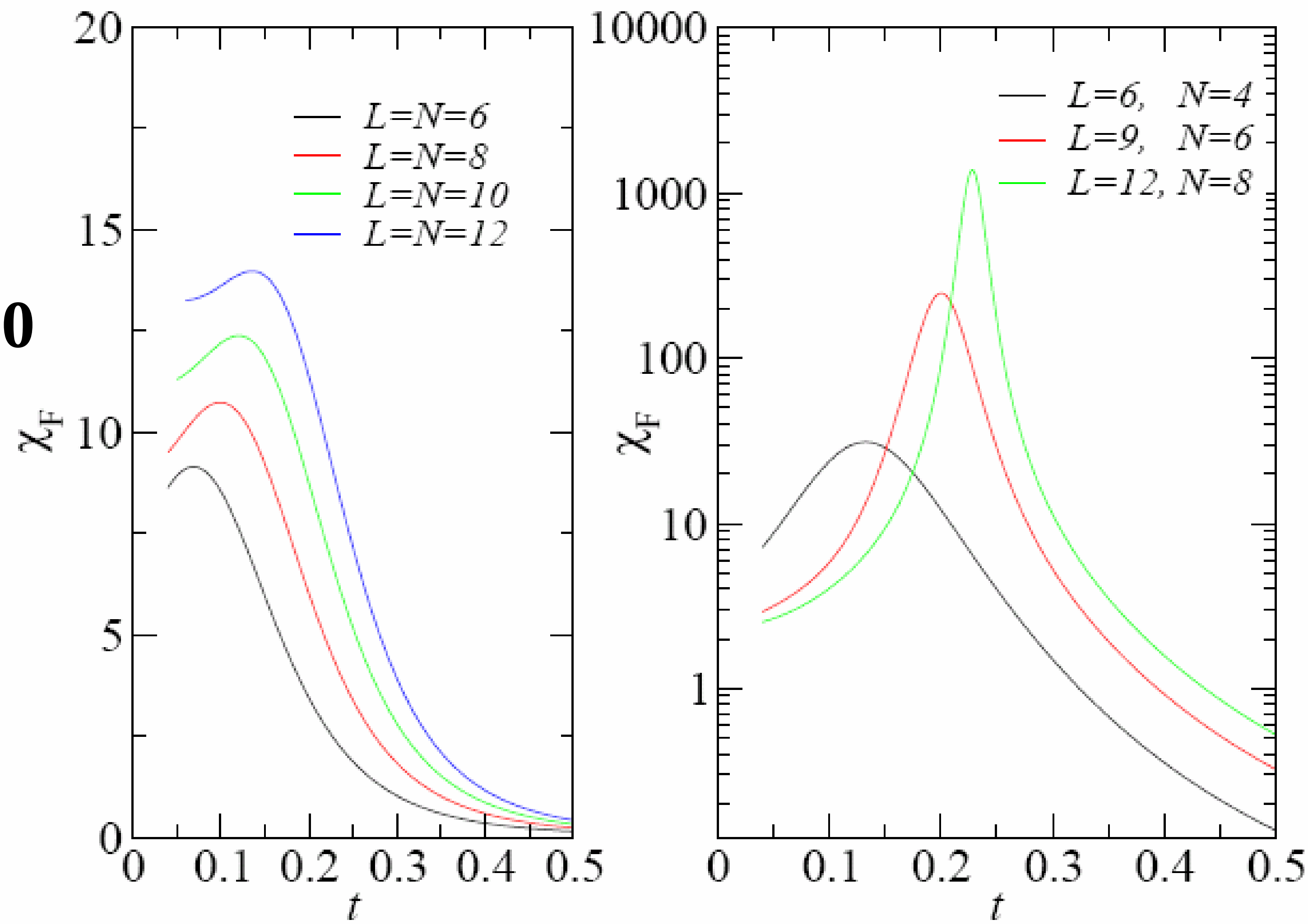
Phase Diagram



The schematic phase diagram of the 1D asymmetric Hubbard model

FS in the 1D asymmetric Hubbard model

***U*=30**



***n*=1** ***n*=2/3**

Scaling Behavior

Scaling ansatz: d (effective) dimension

$$\frac{\chi_{F(\lambda)}(\lambda)}{L} \propto \frac{1}{|\lambda_c - \lambda|^{\alpha}} \qquad \chi_{F(\lambda)}(\lambda = \lambda_{\max}) \propto L^{\mu}$$

$$\frac{\chi(\lambda, L)}{L^{d^{\pm}}} = \frac{A}{L^{-\mu+d^{\pm}} + B(\lambda - \lambda_{\max})^{\alpha^{\pm}}}$$

Universal function

$$\frac{\chi_{F(\lambda)}(\lambda = \lambda_{\max}, L) - \chi_{F(\lambda)}(\lambda, L)}{\chi_{F(\lambda)}(\lambda, L)} = f[L^{\nu}(\lambda - \lambda_{\max})]$$

Critical exponent

$$\alpha^{\pm} = \frac{\mu - d^{\pm}}{\nu}$$

Scaling Behavior

$$\frac{\chi_{F(t)}(t=t_{max}) - \chi_{F(t)}(t)}{\chi_{F(t)}(t)} = g[L^{\nu}(t-t_{max})]$$

$U = 30, \quad n = 2/3$

$$\frac{\chi(\lambda, L)}{L^{d^{\pm}}} = \frac{A}{L^{-\mu+d^{\pm}} + B(\lambda - \lambda_{\max})^{\alpha^{\pm}}}$$

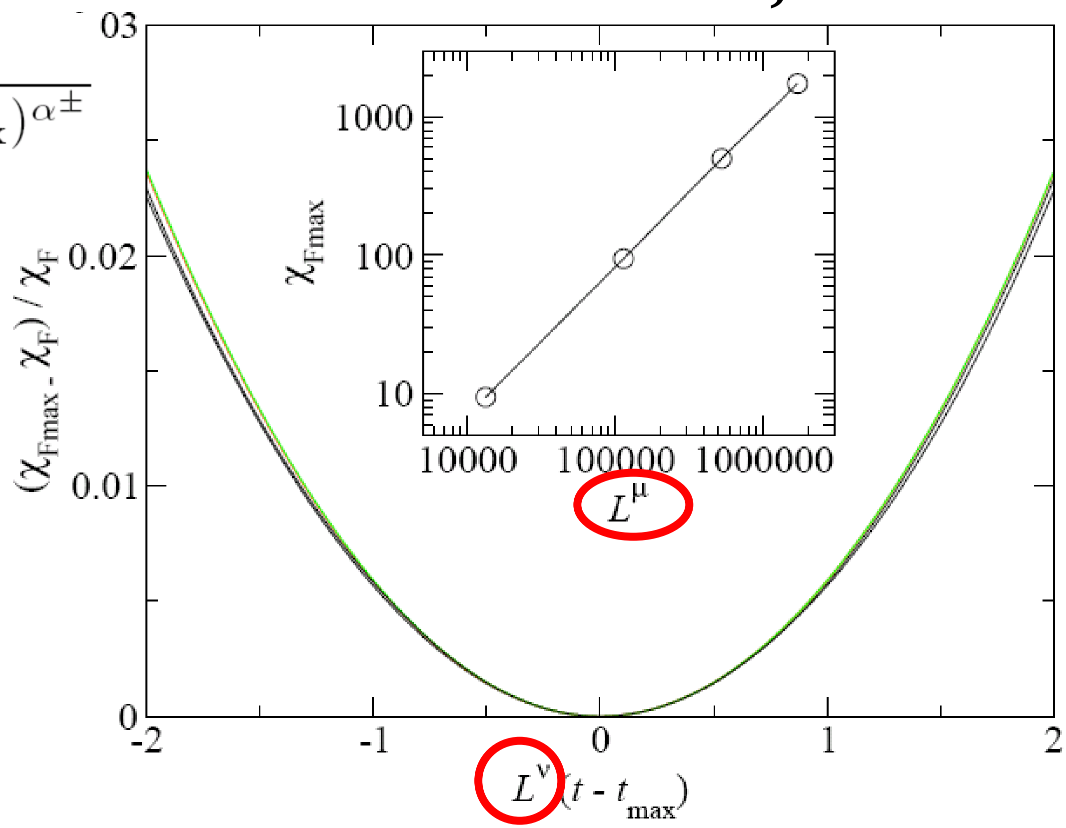
$\mu = 5.30,$

$\nu = 2.65,$

$\alpha = (\mu - 1)/\nu$

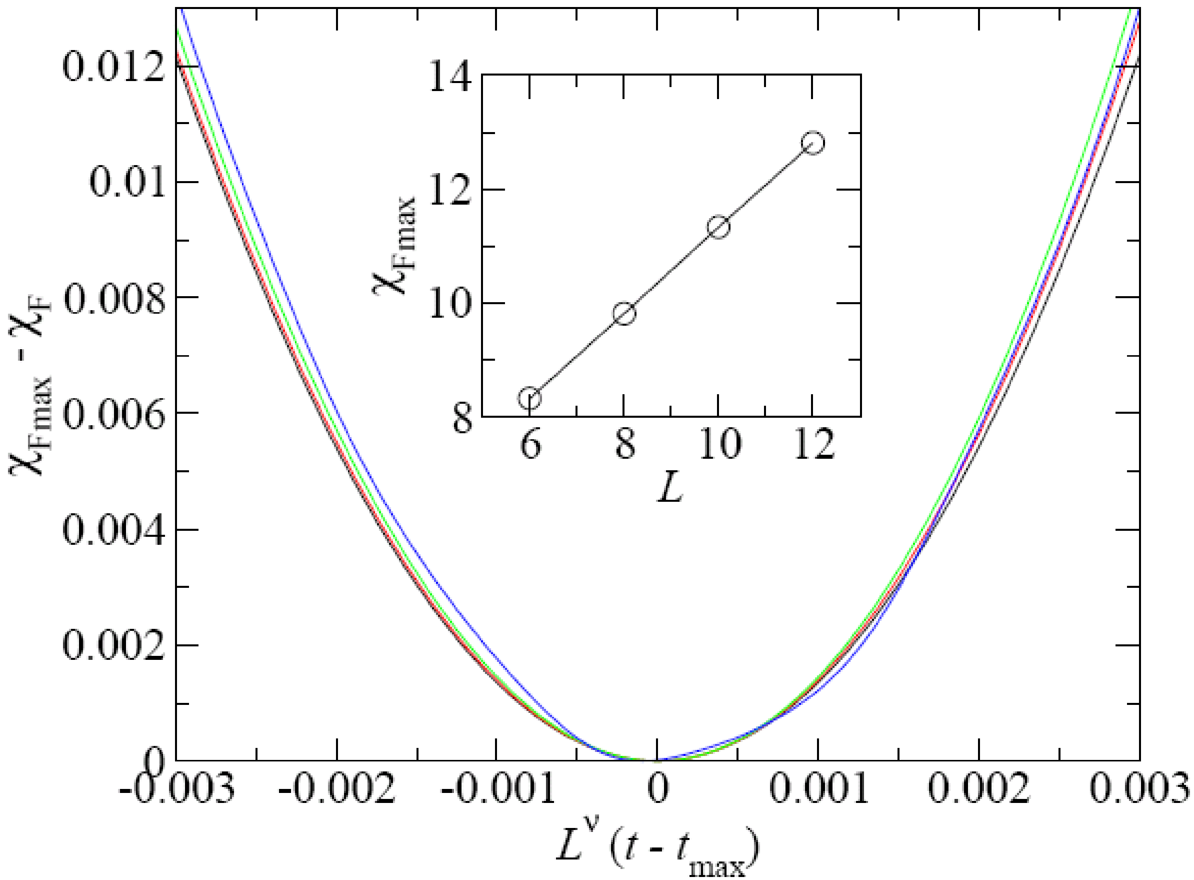
$= 0.60$

$$\chi_{F(\lambda)}(\lambda = \lambda_{\max}) \propto L^{\mu}$$



Increases with *L* exponentially (Landau type)

Scaling Behavior



***U* = 30, *n* = 1**

$$\chi_{F(t)}(t) \simeq 3.86 + 0.75L + 1350L^{-1/2}(t - t_{\max})^2$$

Increases with *L* linearly (KT type)

Brief Summary

- In connection to cold atom systems, we studied the fidelity susceptibility (FS) and its scaling behavior in the 1D AHM.
- We showed that FS can help us to identify Landau symmetry breaking and Kosterlize-Thouless phase transitions.
- We also demonstrated that the FS can be used to characterize the universality class in quantum critical phenomena. Exponents were obtained without *a priori* knowledge of order parameter.

Fidelity Susceptibility in the Kitaev Honeycomb Model

- The model and motivation
- The Fidelity susceptibility and its scaling behavior
- The bond-bond correlation

Introduction: QPT & quantum information theory

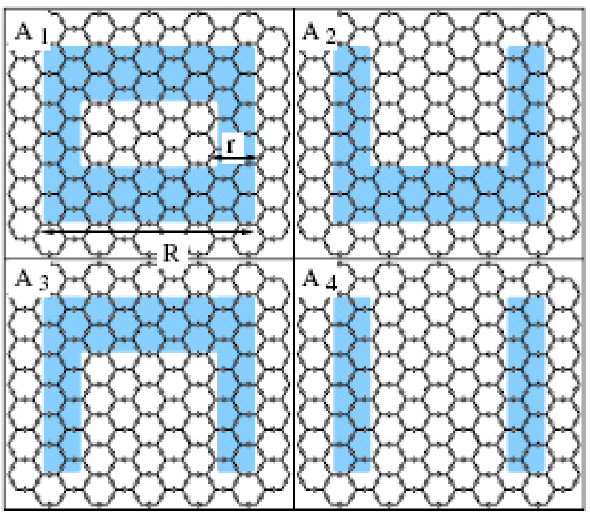
Detecting Topological Order in a Ground State Wave Function

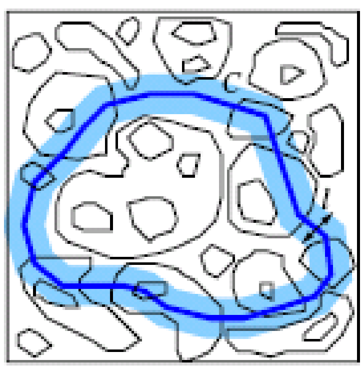
Michael Levin and Xiao-Gang Wen

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 25 October 2005; published 24 March 2006)

A large class of topological orders can be understood and classified using the string-net condensation picture. These topological orders can be characterized by a set of data $(N, d_i, F^{ijk}_{lmn}, \delta_{ijk})$. We describe a way to detect this kind of topological order using only the ground state wave function. The method involves computing a quantity called the "topological entropy" which directly measures the total quantum dimension $D = \sum_i d_i^2$.





$$(S_1 - S_2) - (S_3 - S_4)$$

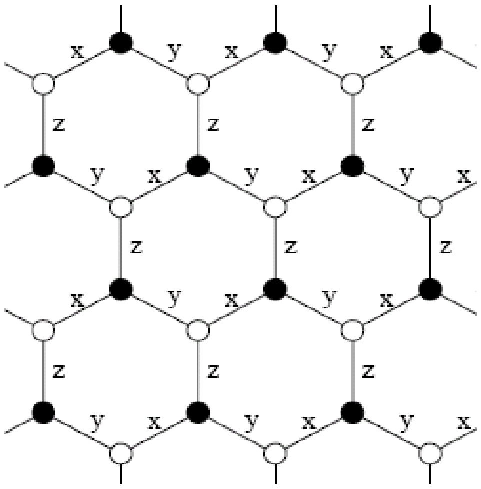
Motivation

- **Fidelity susceptibility** **witness ?**
  - Related to the structure factor (fluctuation) of the driving term in the Hamiltonian
  - Related to correlation functions and local order parameter
- **Topological phase transition**
  - Beyond Landau symmetry-breaking theory
  - No broken symmetry
  - No local order parameters

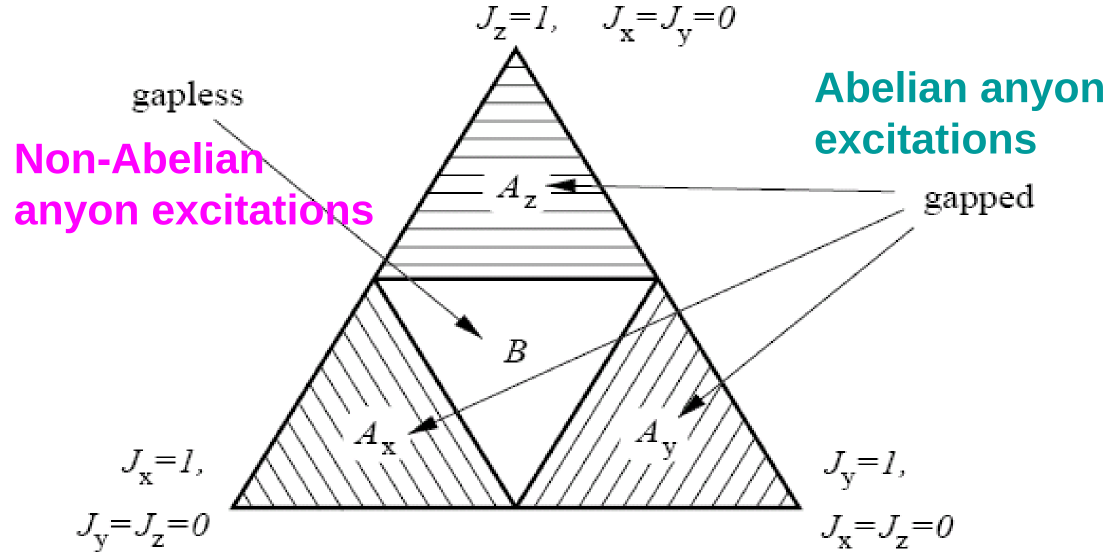
The Kitaev Honeycomb Model

A. Kitaev, Ann. Phys. 303, 2(2003); 321, 2(2006).

$$H = -J_x \sum_{x\text{-bonds}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-bonds}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-bonds}} \sigma_j^z \sigma_k^z,$$



$$J_x + J_y + J_z = 1$$



Topological order & quantum computing

X. G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University, New York, 2004).

Diagonalizing the Kitaev Model

$$\sigma^x = \mathrm{i}b^x c,\ \sigma^y = \mathrm{i}b^y c,\ \sigma^z = \mathrm{i}b^z c.$$

Majorana fermion *c*

$$H = \frac{\mathrm{i}}{2}\sum_{j,k}\widehat{u}_{jk}J_{a_{jk}}c_j c_k \qquad \hat{u}_{jk} = ib_j^{a_{jk}}b_k^{a_{jk}} \qquad \hat{u}_{jk}^2 = 1$$

$$H = \sum_{\mathbf{q}}\begin{pmatrix} a_{-\mathbf{q},1} \\ a_{-\mathbf{q},2}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} 0 & \mathrm{i}f(\mathbf{q}) \\ -\mathrm{i}f(\mathbf{q})^* & 0\end{pmatrix}\begin{pmatrix} a_{\mathbf{q},1} \\ a_{\mathbf{q},2}\end{pmatrix}$$

$$F^2 = \prod_{\mathbf{q}}\frac{1}{2}\left(1 + \frac{\Delta_{\mathbf{q}}\Delta'_{\mathbf{q}} + \epsilon_{\mathbf{q}}\epsilon'_{\mathbf{q}}}{E_{\mathbf{q}}E'_{\mathbf{q}}}\right)$$

$$\begin{aligned} f(\mathbf{q}) &= \epsilon_{\mathbf{q}} + \mathrm{i}\Delta_{\mathbf{q}}, \\ \epsilon_{\mathbf{q}} &= J_x\cos q_x + J_y\cos q_y + J_z \\ \Delta_{\mathbf{q}} &= J_x\sin q_x + J_y\sin q_y. \end{aligned}$$

The Fidelity Susceptibility

$$|\Psi_0\rangle = \prod_{\mathbf{q}} C^\dagger_{\mathbf{q},2}|0\rangle \qquad E_0 = -\sum_{\mathbf{q}} \sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}.$$

$$= \prod_{\mathbf{q}} \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}}{\Delta_{\mathbf{q}} + i\epsilon_{\mathbf{q}}} a_{-\mathbf{q},1} + a_{-\mathbf{q},2} \right) |0\rangle$$

The fidelity of the two ground states at λ and λ'

$$F^2 = \prod_{\mathbf{q}} \frac{1}{2} \left(1 + \frac{\Delta_{\mathbf{q}}\Delta'_{\mathbf{q}} + \epsilon_{\mathbf{q}}\epsilon'_{\mathbf{q}}}{E_{\mathbf{q}}E'_{\mathbf{q}}} \right) = \prod_{\mathbf{q}} \cos^2\left(\theta_{\mathbf{q}} - \theta'_{\mathbf{q}}\right).$$

with
$$\cos(2\theta_{\mathbf{q}}) = \frac{\epsilon_{\mathbf{q}}}{E_{\mathbf{q}}}, \sin(2\theta_{\mathbf{q}}) = \frac{\Delta_{\mathbf{q}}}{E_{\mathbf{q}}},$$
$$\cos(2\theta'_{\mathbf{q}}) = \frac{\epsilon'_{\mathbf{q}}}{E'_{\mathbf{q}}}, \sin(2\theta'_{\mathbf{q}}) = \frac{\Delta'_{\mathbf{q}}}{E'_{\mathbf{q}}}.$$

The Fidelity Susceptibility

$$\chi_F = \lim_{\delta\lambda\to 0} \frac{-2\ln F_i}{\delta\lambda^2} = \sum_{ab} g_{ab} n^a n^b,$$

$$g^{ab} = \sum_{\mathbf{q}} \left(\frac{\partial\theta_{\mathbf{q}}}{\partial J_a}\right)\left(\frac{\partial\theta_{\mathbf{q}}}{\partial J_b}\right) \qquad n^a = \partial J_a / \partial\lambda$$

$$\frac{\partial\left(2\theta_{\mathbf{q}}\right)}{\partial J_x} = \frac{J_z \sin q_x + J_y \sin\left(q_x - q_y\right)}{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2} \cdot \frac{\Delta_{\mathbf{q}}}{|\Delta_{\mathbf{q}}|},$$

$$\frac{\partial\left(2\theta_{\mathbf{q}}\right)}{\partial J_y} = -\frac{J_x \sin\left(q_x - q_y\right) - J_z \sin q_y}{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2} \cdot \frac{\Delta_{\mathbf{q}}}{|\Delta_{\mathbf{q}}|},$$

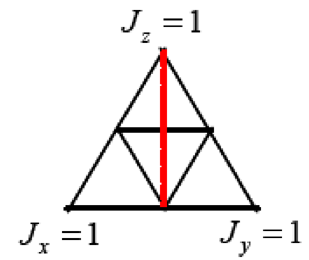
$$\frac{\partial\left(2\theta_{\mathbf{q}}\right)}{\partial J_z} = -\frac{J_x \sin q_x + J_y \sin q_y}{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2} \cdot \frac{\Delta_{\mathbf{q}}}{|\Delta_{\mathbf{q}}|}.$$

FS depends on the evolution path

along $J_x = J_y = (1 - J_z)/2$ line

$$\chi_F = \frac{1}{4} g^{xx} + \frac{1}{4} g^{yy} + g^{zz} + \frac{1}{4}(g^{xy} + g^{yx}) - \frac{1}{2}(g^{xz} + g^{zx}) - \frac{1}{2}(g^{yz} + g^{zy})$$

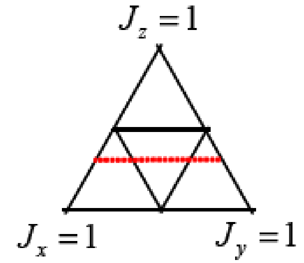
$$= \frac{1}{16} \sum_q \left[\frac{\sin q_x + \sin q_y}{\epsilon_q^2 + \Delta_q^2} \right]^2$$

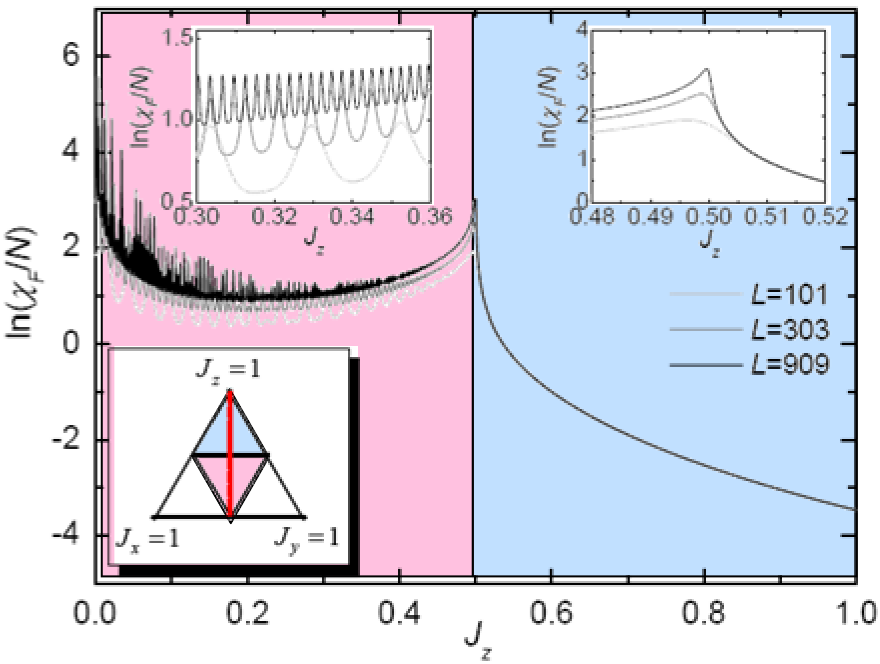


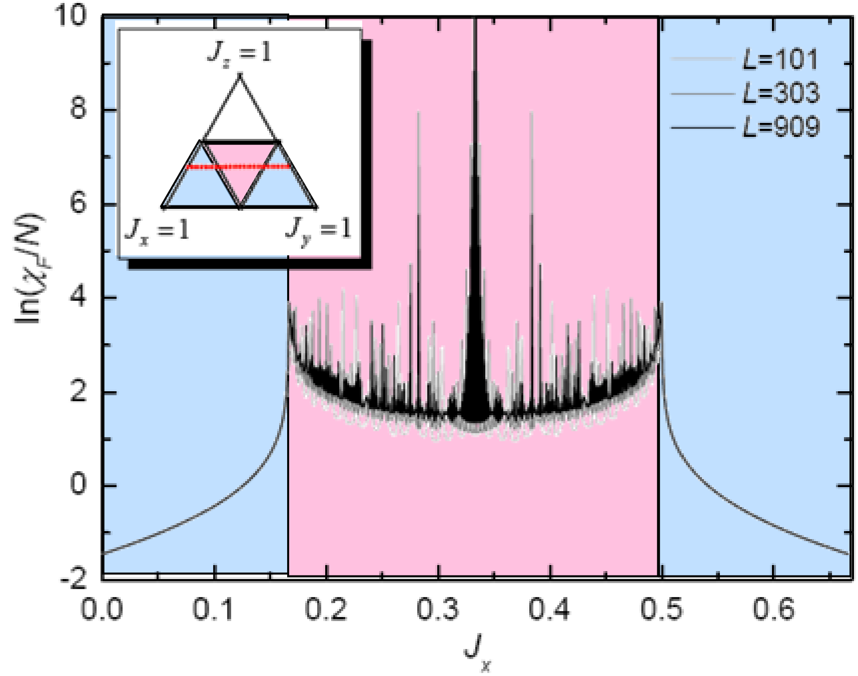
along $J_x + J_y = 1 - J_z = const$ line $\quad J_z = 1/3$,

$$\chi_F = g^{xx} + g^{yy} - (g^{xy} + g^{yx})$$

$$= \frac{1}{36} \sum_q \left[\frac{(\sin q_x - \sin q_y) + 2\sin(q_x - q_y)}{\epsilon_q^2 + \Delta_q^2} \right]^2$$







A phase χ_F ***/N*** is an intensive quantity (***N*=*L* × *L***)

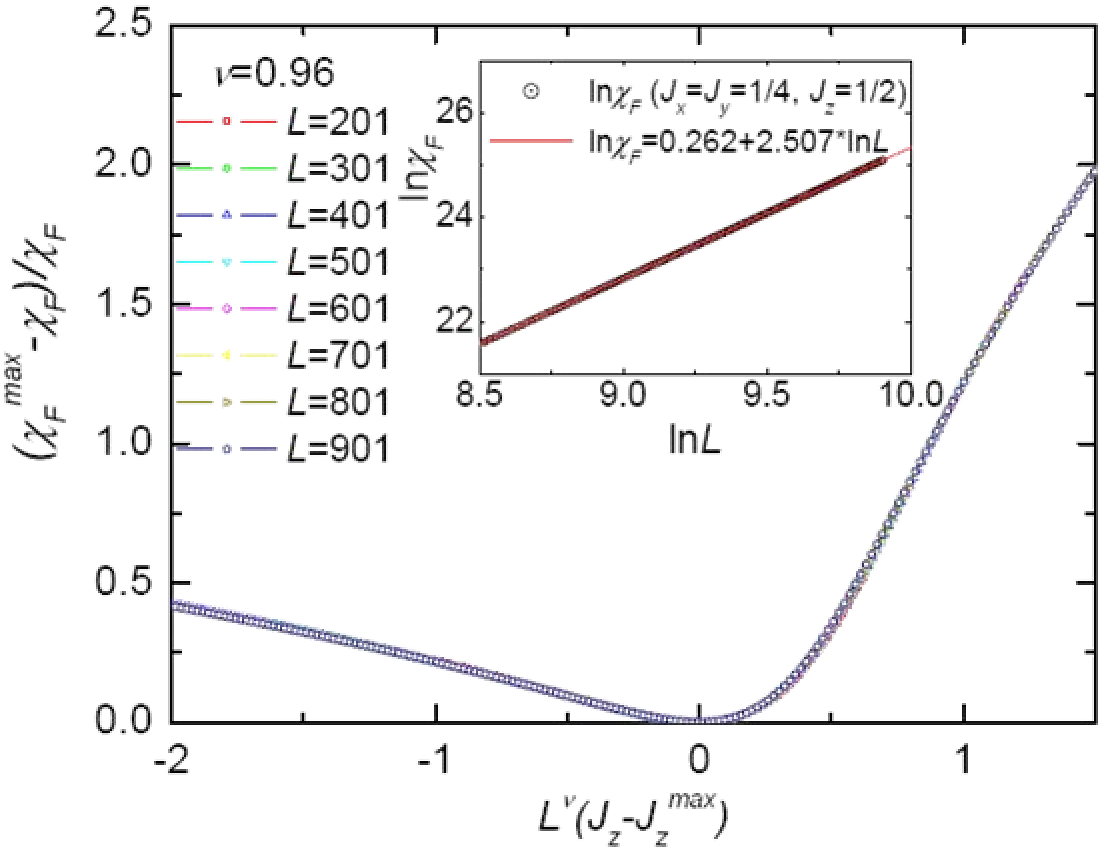
B phase many peaks

the number of peaks linearly increases with system size L

χ_F ***/N*** diverges with increasing system size.

Scaling Behavior

$$\frac{\chi_F^{\max} - \chi_F}{\chi_F} = f[L^\nu (J_z - J_z^{\max})]$$



$$\mu = 0.507 \pm 0.0001$$

$$\nu = 0.96 \pm 0.005.$$

$$\alpha = \frac{\mu}{\nu} = 0.528 \pm 0.001.$$

Hidden Correlation?

- **The fidelity susceptibility can be expressed as**

$$\chi_F = \int \tau \left[\langle \Psi_0 | H_I(\tau) H_I(0) | \Psi_0 \rangle - \langle \Psi_0 | H_I | \Psi_0 \rangle^2 \right] d\tau$$

You, Li, and Gu, PRE 76, 022101 (2007).

- **Therefore, the divergence of the FS at the critical point implies the existence of some kind of long-range correlation.**

The Bond-Bond Correlation

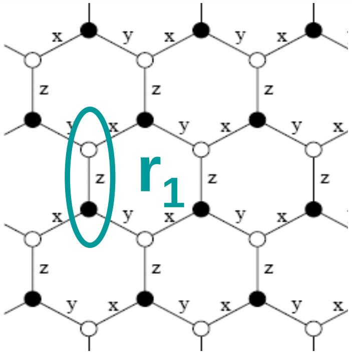
Bond-bond correlation function

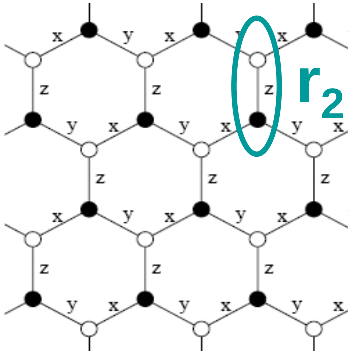
$$C(\mathbf{r}_1,\mathbf{r}_2) = \langle \sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}\rangle - \langle \sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\rangle\langle \sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}\rangle$$

Unlike usual point-point correlation (two operators)

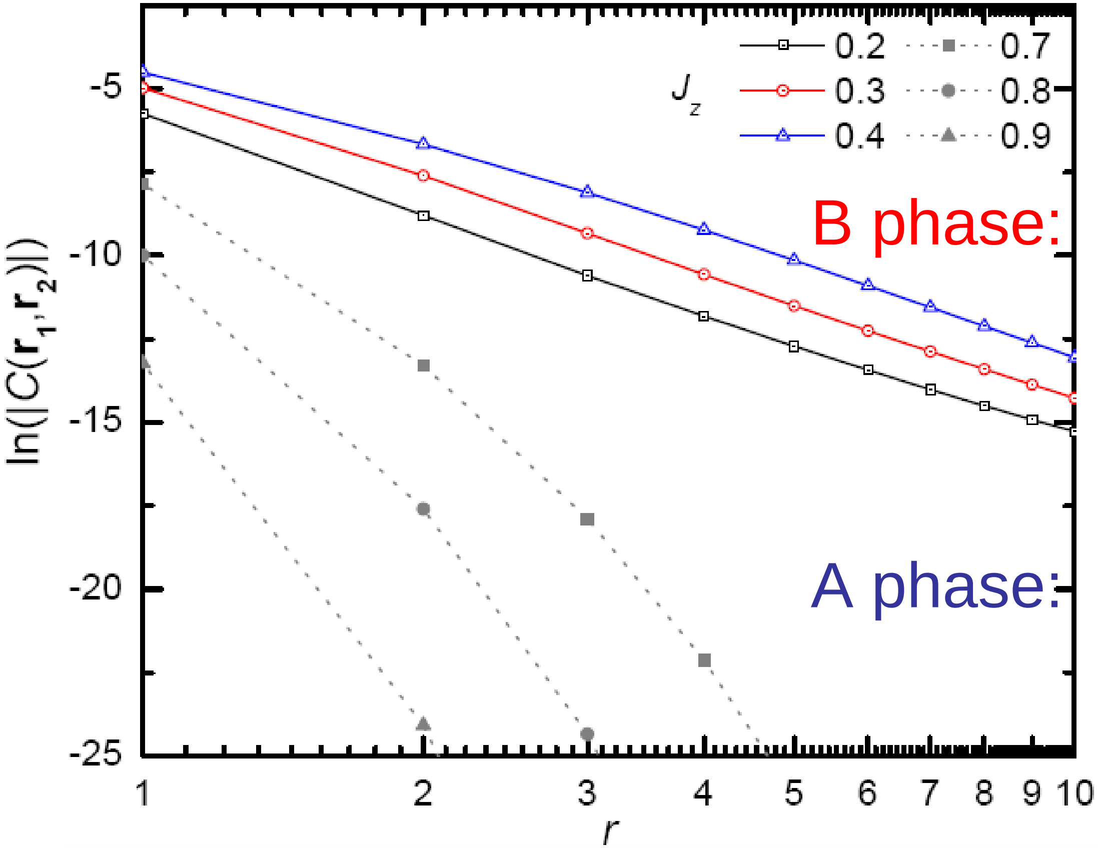
$$\langle \sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\rangle = \langle \sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}\rangle = \frac{1}{N}\sum_{\mathbf{q}}\frac{\epsilon_{\mathbf{q}}}{E_{\mathbf{q}}}$$

$$\langle\Psi_0|\sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}|\Psi_0\rangle = \frac{1}{N^2}\sum_{\mathbf{q},\mathbf{q}'}\left\{\cos\left[(\mathbf{q}-\mathbf{q}')(\mathbf{r}_1-\mathbf{r}_2)\right]-1\right\}\frac{(\Delta_{\mathbf{q}}\Delta_{\mathbf{q}'}-\epsilon_{\mathbf{q}}\epsilon_{\mathbf{q}'})}{E_{\mathbf{q}}E_{\mathbf{q}'}}$$





The Bond-Bond Correlation



B phase: decays algebraically

$$C(\mathbf{r}_1, \mathbf{r}_2) \propto \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|^4}.$$

A phase: decays exponentially

$$\frac{1}{\xi} = 2\sinh^{-1}\frac{\sqrt{2J_z - 1}}{1 - J_z}.$$

the correlation length becomes divergent as

$$J_z \rightarrow 0.5^+$$

Signal quantum phase transition

Brief Summary

- In connection to the topological order, we studied the fidelity susceptibility and its scaling behavior in the Kitaev model.
- We showed that the fidelity susceptibility can be used to witness the topological quantum phase transition in the Kitaev model.
- Suggested by the divergence observed in the fidelity susceptibility, we found a bond-bond correlation which is associated with the topological quantum phase transition in the Kitaev model.

Fidelity Susceptibility in the Lipkin-Meshkov-Glick Model

- The LMG model and its eigenstates
- The FS and its scaling behaviors

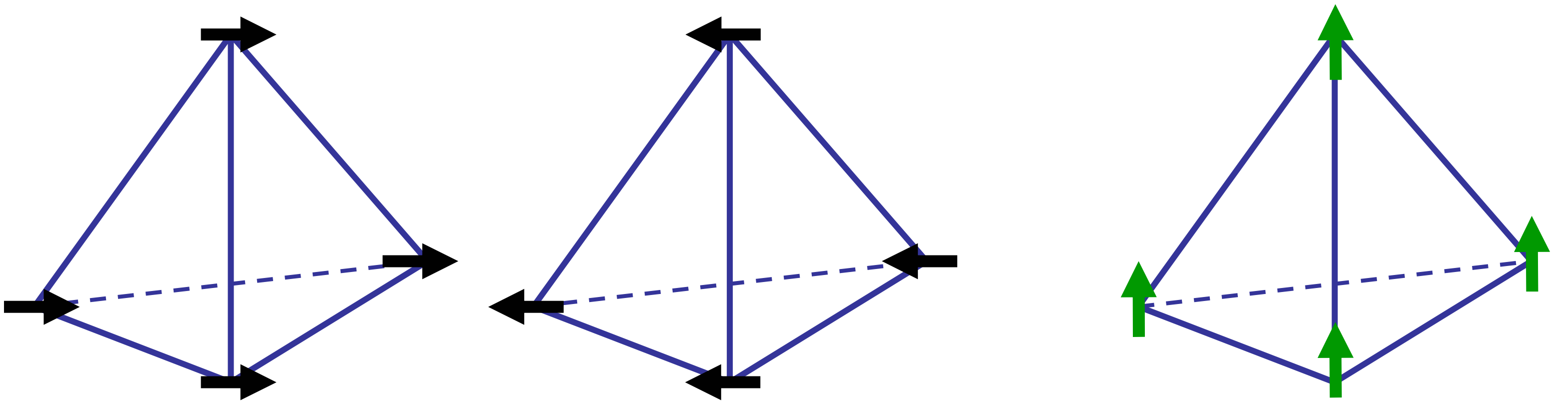
The Lipkin-Meshkov-Glick Model

Model Hamiltonian

$$H = -\frac{\lambda}{N}\sum_{i<j}\left(\sigma_x^i\sigma_x^j + \gamma\sigma_y^i\sigma_y^j\right) - h\sum_i \sigma_z^i,$$

Ground state phases (ferromagnetic)

$h<1$ $\qquad\qquad$ $h>1$



H.J. Lipkin, N. Meshkov, and A.J. Glick, Nucl. Phys. 62, 188 (1965).

Diagonalizing the LMG model

If *h* > 1

$$S_z = S - a^\dagger a,$$
$$S_+ = (2S - a^\dagger a)^{1/2} a$$

The Hamiltonian in terms of bosons

$$H = -hN + [2(h-1) + \eta] a^\dagger a - \frac{\eta}{2}\left(a^{\dagger 2} + a^2\right)$$

$$a^\dagger = \cosh(\Theta/2) b^\dagger + \sinh(\Theta/2) b,$$
$$a = \sinh(\Theta/2) b^\dagger + \cosh(\Theta/2) b,$$

The diagonalized form

$$H = -h(N+1) + 2\sqrt{(h-1)(h-1+\eta)}\left(b^\dagger b + \frac{1}{2}\right)$$

Fidelity Susceptibility of the LMG Model

If *h* > 1

$$-\sum_i \sigma_z^i = -2S_z$$

$$\chi_{F(h)}(\eta, h > 1) = \frac{\eta^2}{32(h-1)^2(h-1+\eta)^2}$$

If *h* < 1

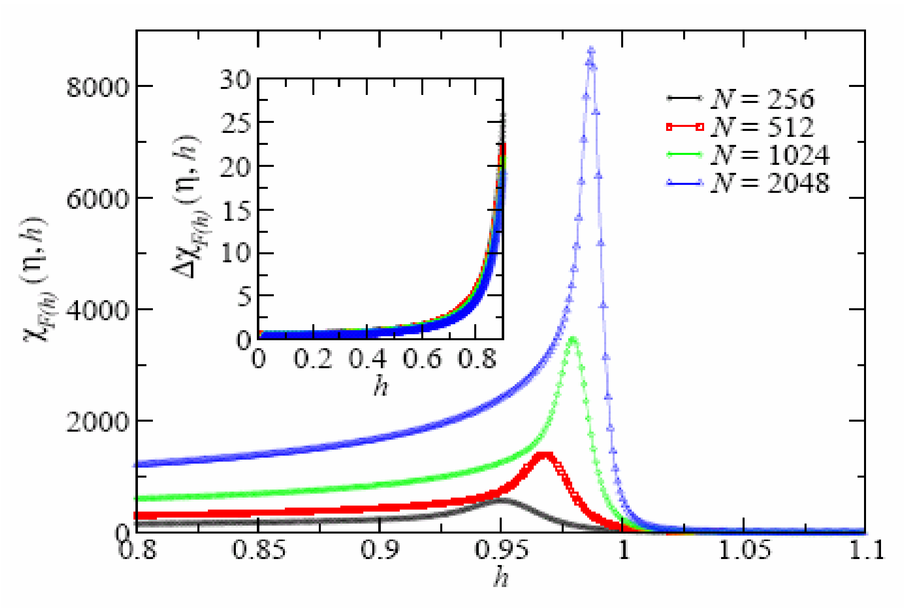
$$\frac{\chi_{F(h)}(\eta, h < 1)}{N} = \frac{1}{4\sqrt{(1-h^2)\eta}}$$

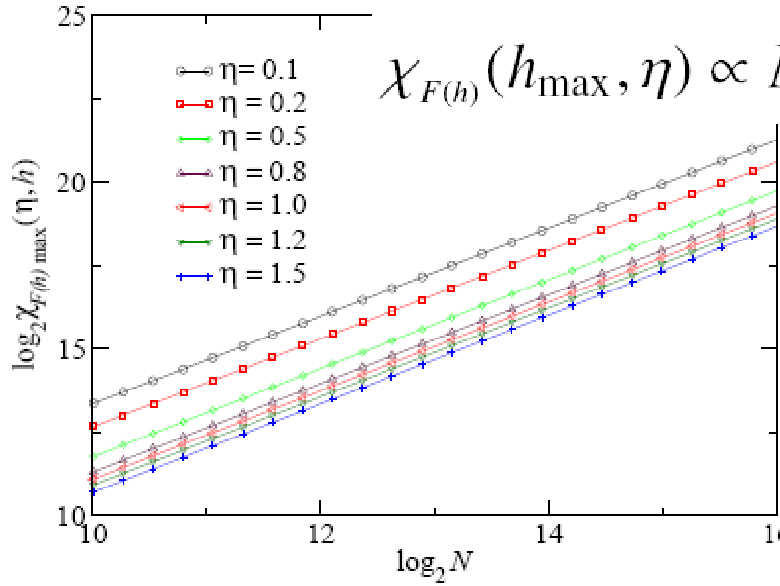
Exponents

$$\alpha = \begin{cases} 2, h > 1 \\ \frac{1}{2}, 0 \leq h < 1 \end{cases}$$

Scaling Behavior of the FS in the LMG Model

$$H = -\frac{\lambda}{N}\sum_{i<j}\left(\sigma_x^i\sigma_x^j + \gamma\sigma_y^i\sigma_y^j\right) - h\sum_i \sigma_z^i,$$





$$\chi_{F(h)}(h_{\max},\eta) \propto N^{\mu}$$

$$\frac{\chi_{F(h)}(h_{\max},\eta) - \chi_{F(h)}(h,\eta)}{\chi_{F(h)}(h,\eta)} = f[L^{\nu}(h - h_{\max})]$$

Scaling Behavior of the FS in the LMG Model

$$\frac{\chi_{F(h)}(h_{\max},\eta)-\chi_{F(h)}(h,\eta)}{\chi_{F(h)}(h,\eta)} = f[L^{\nu}(h-h_{\max})]$$

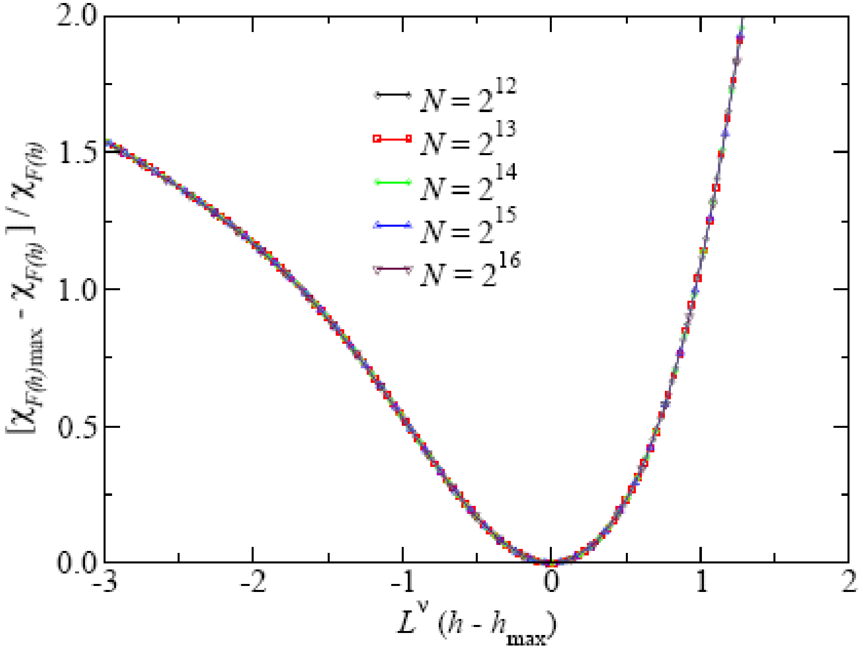
$$\chi_{F(h)}(h_{\max},\eta) \propto N^{\mu}$$

$$\frac{\chi_{F(h)}}{N} \propto N^{(\mu-1)}$$

$$\alpha = \frac{\mu}{\nu}$$

$$\alpha = \begin{cases} 2, h > 1 \\ \frac{1}{2}, 0 \le h < 1 \end{cases}$$

$$\nu \simeq 0.665. \qquad \mu = 1.33$$



Brief Summary

- The LMG model has the advantage that it can be solved exactly which makes it possible to do detailed finite size scaling.
- We analyzed scaling behavior of the fidelity susceptibility of LMG model and calculated various critical exponents.
- We demonstrated that the fidelity susceptibility exhibits universality in the LMG model.

Quantum criticality in terms of fidelity susceptibility

Model		μ	ν	d^+	α^+	d^-	α^-
1D Ising model($h_c = 1$)	**[1]**	2	1	1	1	1	1
Lipkin-Meshkov-Glick model($h_c = 1$)	**[2]**	4/3	2/3	0	2	1	1/2
Kitaev honeycomb model($J_{z,c} = 1/2$)	**[3,4]**	2.50	1	2	1	2+ln	1/2-ln
Deformed Kitaev toric model [$\lambda_c = \frac{1}{2}\ln(\sqrt{2}+1)$]	**[5]**	ln	1	1	ln	1	ln
1D AHM ($t_c = 0.456$ for $n = 2/3$)	**[6]**	5.3	2.65	-	-	1	1.6
Luttinger model($\lambda_c = 1$ of XXZ model)		-	-	-	-	-	1
Luttinger model($\lambda_c = -1$ of XXZ model)	**[7]**	-	-	-	1	-	-

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2.H. M. Kwok, W. Q. Ning, S. J. Gu, and H. Q. Lin, Phys. Rev. E 78, 032103 (2008).
3.S. Yang, S. J. Gu, C. P. Sun, and H. Q. Lin, Phys. Rev. A 78, 012304 (2008).
4.S. J. Gu and H. Q. Lin, arXiv:0807.3491.
5.D. F. Abasto, A. Hamma, and P. Zanardi, Phys. Rev. A 78, 010301(R) (2008).
6.S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B 77, 245109 (2008).
7.M. F. Yang, Phys. Rev. B 76, 180403 (R) (2007).

Summary and Discussions

- We showed that fidelity susceptibility is a better quantity to measure and we have investigated its scaling behavior in three widely studied models. Various critical exponents were obtained.
- We showed that the fidelity susceptibility can be used to identify the universality class of the quantum phase transitions without *a priori* knowledge of the order parameter.
- We also showed that the fidelity susceptibility can be used to identify the topological phase transition and found a new correlation.


Thanks
谢谢！