Detecting elusive phase transitions with geometric entanglement

University of Waterloo
IQC

Tzu-Chieh Wei

University of British Columbia

Joint work with Román Orús

University of Queensland

arXiv:0910.2488

Workshop on Quantum Information Science and Many-Body Physics
Dec. 19, 2009

Preliminary announcement

10th Canadian Summer School on Quantum Information + Workshop

- Sat, July 17 – Fri, July 30, 2010 @ UBC, Vancouver
- Organizing committee:

  Mohammad H. S. Amin (D-Wave)

  Petr Lisonek (SFU)

  Barry Sanders (Calgary)

  Pradeep Kiran Sarvepalli (UBC)

  Robert Raussendorf (UBC)

  Tzu-Chieh Wei (UBC)

<http://www.qi10.ca> [up soon]

Outline

- ❑ Motivations
- ❑ Quantum Phase Transitions and entanglement (entropy and concurrence)
- ❑ Geometric (measure of) entanglement
- ❑ Benchmarking models: Ising spin with transverse and longitudinal fields
- ❑ XXZ and deformed AKLT
- ❑ Conclusion

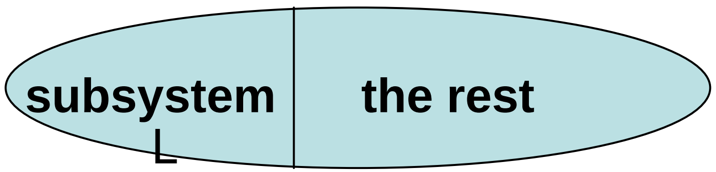
Entanglement

- Useful resource for quantum information processing (including quantum computing)
- Entanglement is common in many-body systems
- Near quantum phase transitions, behavior of entanglement is “singular”
- Can entanglement be a useful tool to distinguish quantum phase transitions?

Entanglement and criticality

- In 1D:

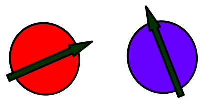
Entropy of L spins



- S ~ c/3 log L

[Vidal et al. ’03, Calabrese and Cardy ‘04]

Concurrence (nearest nb.)



- dC/dh diverges

[Osterloh, Amico, Falci & Fazio ’02, Osborne & Nielsen ‘02]

- Singularity of C is related singularity in ground-state energy→ quantum criticality

[Wu, Sarandy & Lidar ‘04]

Some remarks on concurrence and entropy

- Singularity of concurrence not necessarily gives quantum phase transition [Yang '05]

- For certain models, concurrence is maximum (but not singular) at quantum critical point, e.g. XXZ model

  [Gu, Lin & Li '03]

- Concurrence works for two qubits

- Single or few sites entropy is easy to calculate (once state is given), but not so for large block entanglement
  ➔ require knowledge of 2^L Schmidt numbers

Focus of this work: Elusive transitions

- In infinite-order transition, no divergence for correlation functions ➔ concurrence will not detect such transition
- A different type of transition: divergence in localizable entanglement length
- Will not use concurrence nor entropy
  ➔ instead use geometric measure of entanglement

Outline

- Motivations
- Quantum Phase Transitions and entanglement (entropy and concurrence)
- Geometric (measure of) entanglement
- Benchmarking models: Ising spin with transverse and longitudinal fields
- XXZ and deformed AKLT
- Conclusion

Geometric measure of entanglement

- Compare entangled state $|\Psi\rangle$

  to product states: $|\Phi\rangle \equiv \bigotimes_{i=1}^{N} |\phi^{[i]}\rangle$

- Entanglement revealed by maximum overlap

$$\Lambda_{\max}(\Psi) \equiv \max_{\Phi} |\langle\Phi|\Psi\rangle|$$

$$E_N(\Psi) \equiv -\log_2 \Lambda^2_{\max}(\Psi)$$

- As interested in thermodynamic limit
  ➔ can define entanglement density:

$$\mathcal{E} \equiv \lim_{N\to\infty} \mathcal{E}_N, \quad \mathcal{E}_N \equiv N^{-1} E_N(\Psi)$$

Geometric measure of entanglement

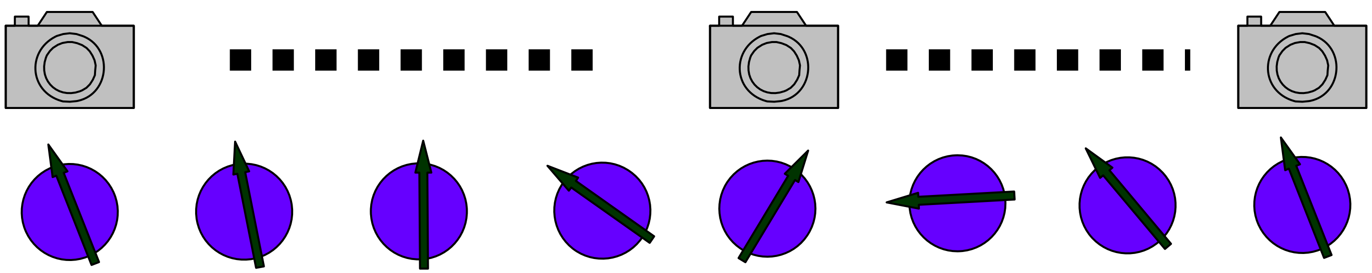
- Search for best Hartree approximation

$$|\Phi\rangle \equiv \bigotimes_{i=1}^{N} |\phi^{[i]}\rangle$$

- Maximum overlap is the probability amplitude for optimal local measurement

$$\Lambda_{\max}(\Psi) \equiv \max_{\Phi} |\langle\Phi|\Psi\rangle|, \qquad E_N(\Psi) \equiv -\log_2 \Lambda^2_{\max}(\Psi)$$

$$|\phi^{[i]}\rangle\langle\phi^{[i]}|$$



Fidelity measure

[Zanardi et al.
Prof Lin and co-workers]

$$f_N(\lambda_1, \lambda_2) \equiv \frac{1}{N} \log\langle \Psi_G(\lambda) | \Psi_G(\lambda') \rangle$$

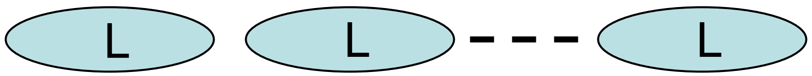
- See Prof. H-Q Lin's talk
- Overlap between different ground states
  ➔ cf. geometric measure: between ground and product states
- Not entanglement measure
  But fidelity is a very useful tool to identify critical points
- Also requires knowledge of ground states

Geometric entanglement vs. concurrence and entropy (near criticality)

Entropy

$S \sim c/3 \log L$

Blocks of L (large »ξ) spins



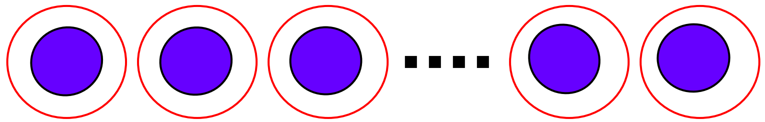
$\varepsilon^{(L)} = c/12 \log \xi$

[Orus '08]

Concurrence

dC/dh diverges

Blocks of (L=1,2) spins



$d\varepsilon^{(1,2)}/dh$ diverges

[Wei et al. '05]

Approaches to calculate geometric measure in spin models

I. Analytic? Only a few examples so far

II. Exact diagonalization to find ground states $|\Psi\rangle$

- Parametrize product states $|\Phi\rangle \equiv \bigotimes_{i=1}^{N} |\phi^{[i]}\rangle$

$$|\phi^{[j]}\rangle = \cos\theta^{[j]}|0\rangle + e^{i\phi^{[j]}}\sin\theta^{[j]}|1\rangle$$

- Numerical optimization of $\Lambda_{\max}(\Psi) \equiv \max_{\{\theta,\phi\}} |\langle\Phi|\Psi\rangle|$

$$E_N(\Psi) \equiv -\log_2 \Lambda^2_{\max}(\Psi)$$

- Translation invariance can reduce number of parameters:

FM-like: $\theta^{[j]} = \theta, \phi^{[j]} = \phi$

AFM-like: $\theta^{[2j-1]} = \theta_1, \phi^{[2j-1]} = \phi_1$

$\theta^{[2j]} = \theta_2, \phi^{[2j]} = \phi_2$

Geometric measure for MPS

III. When ground states are approximated by MPS $|\Psi_{\text{MPS}}\rangle$

- Use results of Orús '08 (translation invariance)

$$E^{(L)}(\psi) = -\lim_{n\to\infty}\frac{1}{n}\log\frac{(d_L)^n}{\mathrm{Tr}\left(\widehat{E}^{nL}\right)}$$

$$d_L = \left|\max_{\vec{r}}\left(\vec{r}^{\,\dagger}\otimes\vec{r^*}^{\,\dagger}\,\widehat{E}^L\,\vec{r}\otimes\vec{r}^{\,*}\right)\right|$$

- In general, no translation invariance
  ➔ can use variational MPS approach to find lowest MPS with bond dimension =1 for the Hamiltonian:

$$\mathcal{H} = -|\Psi_{\text{MPS}}\rangle\langle\Psi_{\text{MPS}}|$$

➔ Efficient, easier than finding ground states

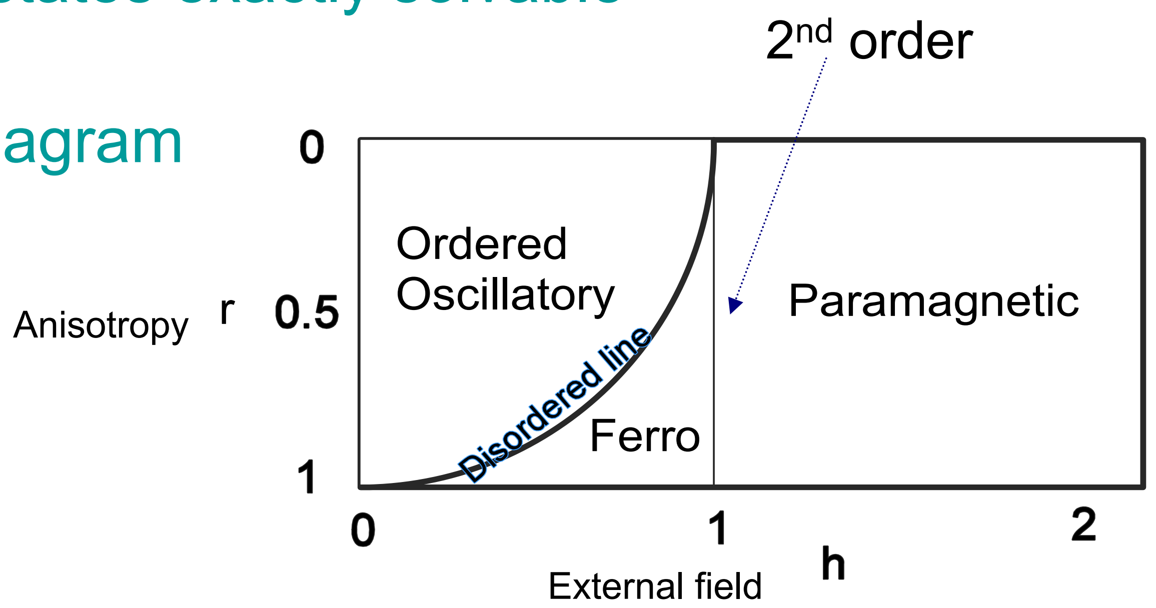
Outline

- Motivations
- Quantum Phase Transitions and entanglement (entropy and concurrence)
- Geometric (measure of) entanglement
- Benchmarking models: Ising spin with transverse and longitudinal fields
- XXZ and deformed AKLT
- Conclusion

Benchmark I:
XY models in transverse fields

$$H = -\sum_i \left(\frac{1+r}{2} \sigma_x^{[i]} \sigma_x^{[i+1]} + \frac{1-r}{2} \sigma_y^{[i]} \sigma_y^{[i+1]} + h \sigma_z^{[i]} \right)$$

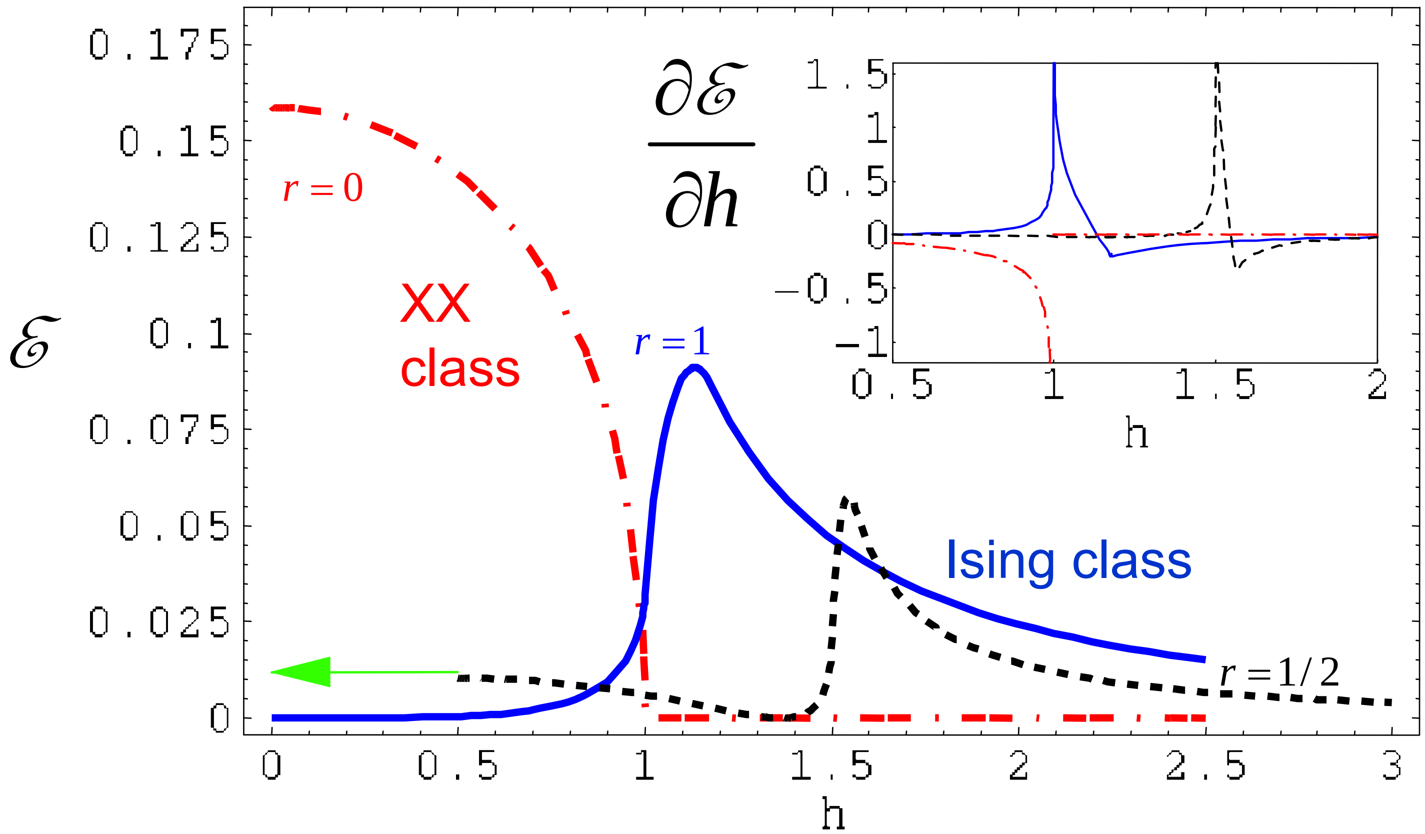
- Ground states exactly solvable
- Phase diagram



Geometric entanglement: analytic results

[Wei et al. '05]

$$H = -\sum_i \left(\frac{1+r}{2}\sigma_x^{[i]}\sigma_x^{[i+1]} + \frac{1-r}{2}\sigma_y^{[i]}\sigma_y^{[i+1]} + h\sigma_z^{[i]} \right)$$

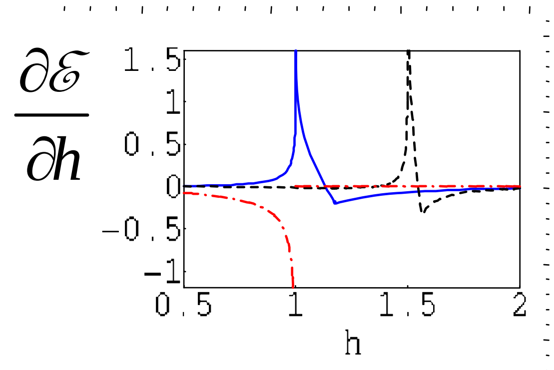


➔ Entanglement is singular across transitions

Singular behavior near critical points

- Across critical line *h*=1 the field-derivative of entanglement diverges

[Wei et al. '05]



I. Ising universality class r≠0
  ➔ ν =1

$$\frac{\partial \mathcal{E}(r,h)}{\partial h} \approx -\frac{1}{2\pi r} \log_2 |h-1| , \qquad \text{for } h \to 1$$

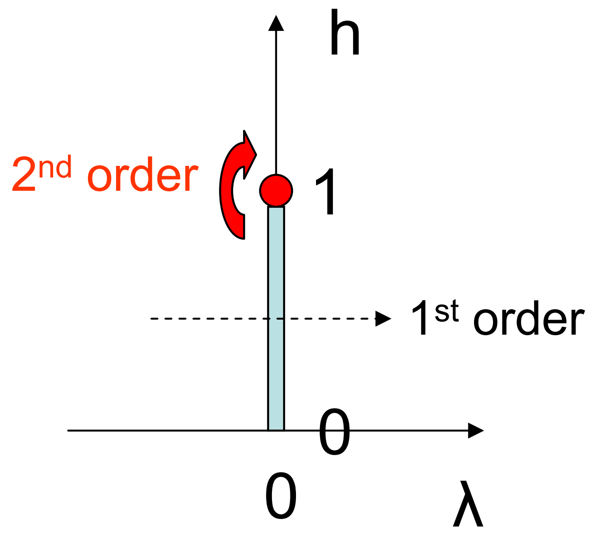
II. XX (isotropic) universality class r=0
  ➔ ν =1/2

$$\frac{\partial \mathcal{E}(0,h)}{\partial h} \approx -\frac{\log_2(\pi/2)}{\sqrt{2}\pi} \frac{1}{\sqrt{1-h}}, \quad \text{for } h \to 1^-$$

Benchmark II:Ising model in transverse and longitudinal fields

$$H = -\sum_i \left(\sigma_x^{[i]} \sigma_x^{[i+1]} + h\sigma_z^{[i]} + \lambda \sigma_x^{[i]} \right)$$

- ❖Ground states **not** exactly solvable
- ❖Phase diagram

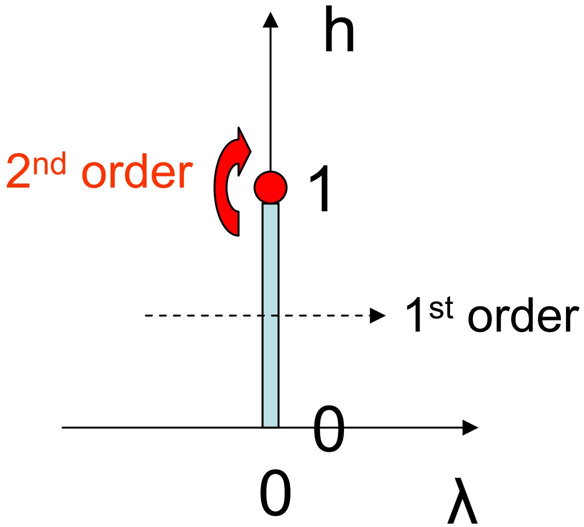


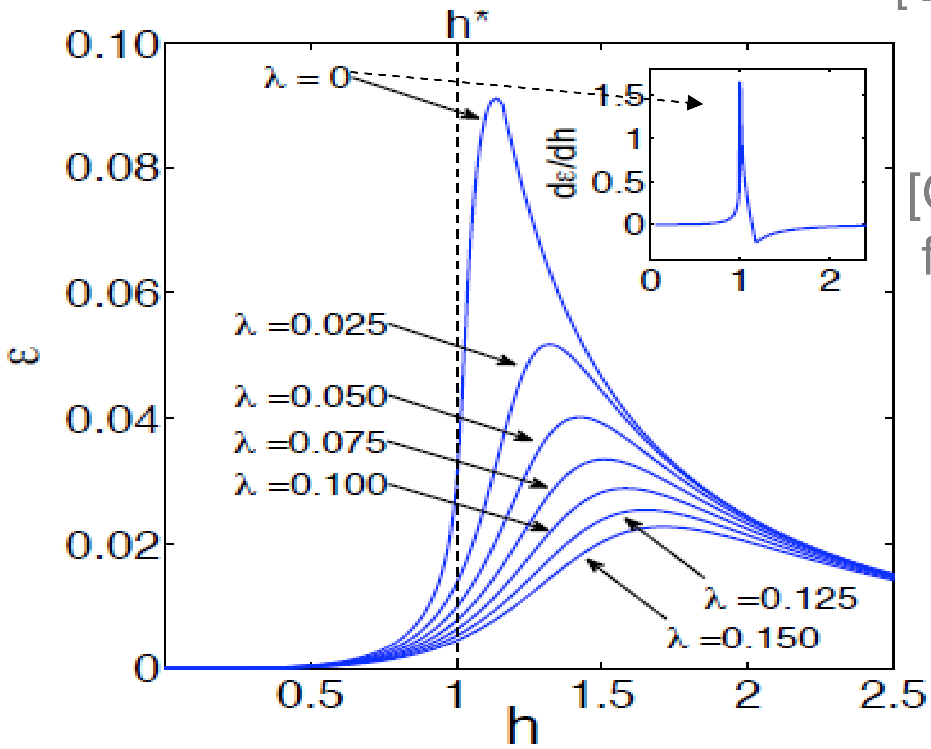
Geometric entanglement: from iTEBD

- At fixed λ,vary h

$$H = -\sum_i \left(\sigma_x^{[i]} \sigma_x^{[i+1]} + h\sigma_z^{[i]} + \lambda \sigma_x^{[i]} \right)$$

[Orús & Wei '09]





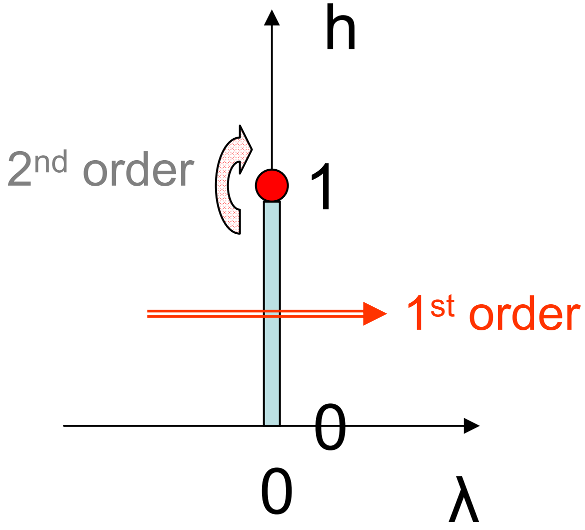
[Ground states from iTEBD]

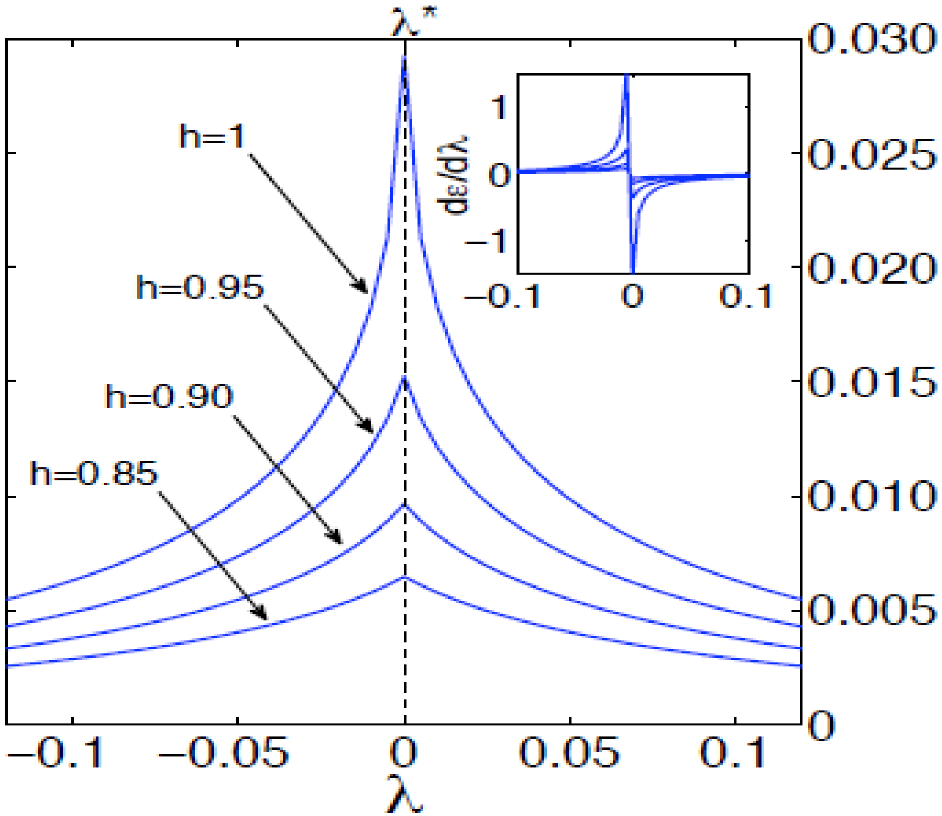
➔ Entanglement is singular across the transition (only one point)

At fixed h, vary longitudinal field λ

$$H = -\sum_i \left(\sigma_x^{[i]} \sigma_x^{[i+1]} + h\sigma_z^{[i]} + \lambda \sigma_x^{[i]} \right)$$

[Orús & Wei '09]





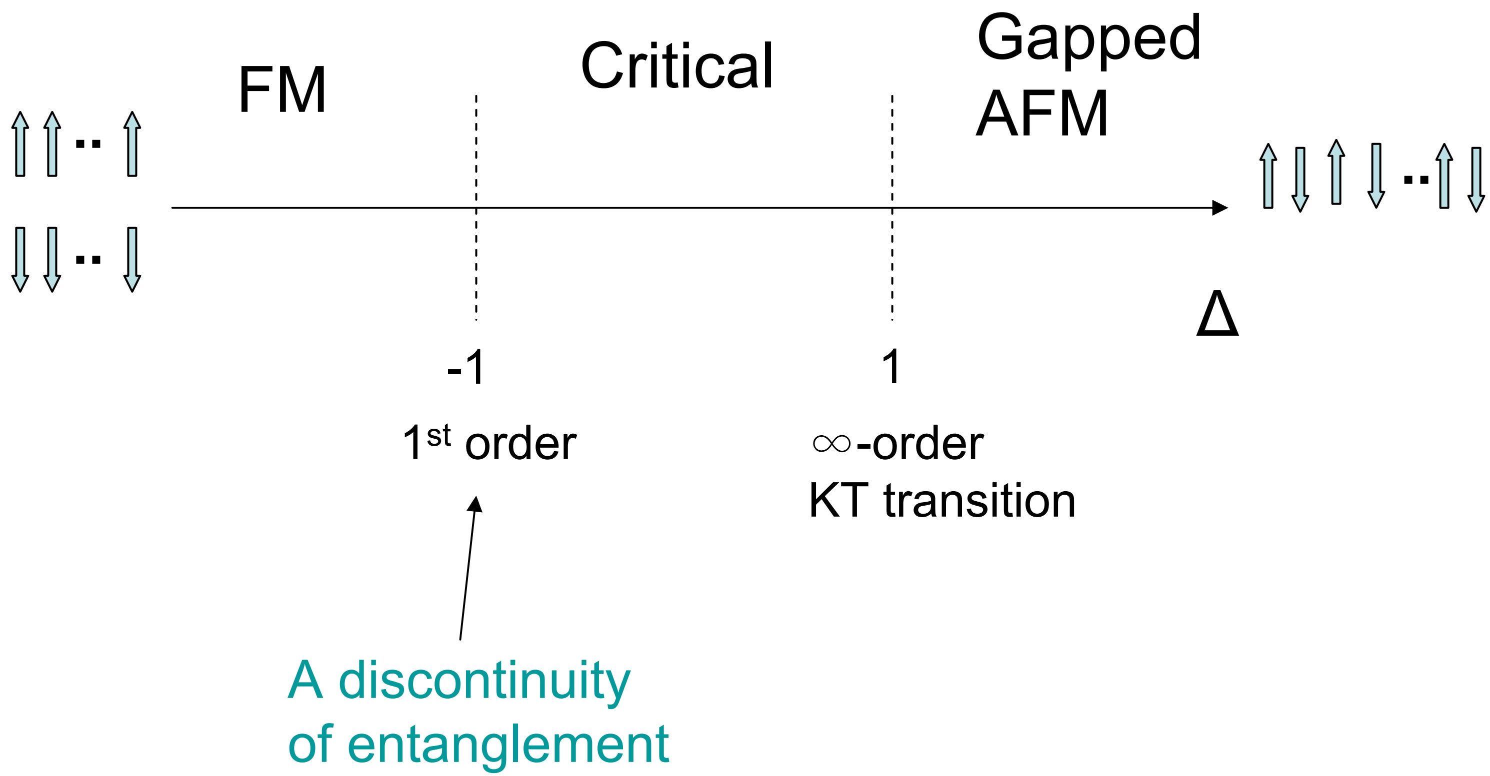
➔ Singular behavior across 1st order transition is different

Outline

- Motivations
- Quantum Phase Transitions and entanglement (entropy and concurrence)
- Geometric (measure of) entanglement
- Benchmarking models: Ising spin with transverse and longitudinal fields
- XXZ and deformed AKLT
- Conclusion

XXZ model

$$H = \sum_i \left(\sigma_x^{[i]} \sigma_x^{[i+1]} + \sigma_y^{[i]} \sigma_y^{[i+1]} + \Delta \sigma_z^{[i]} \sigma_z^{[i+1]} \right)$$

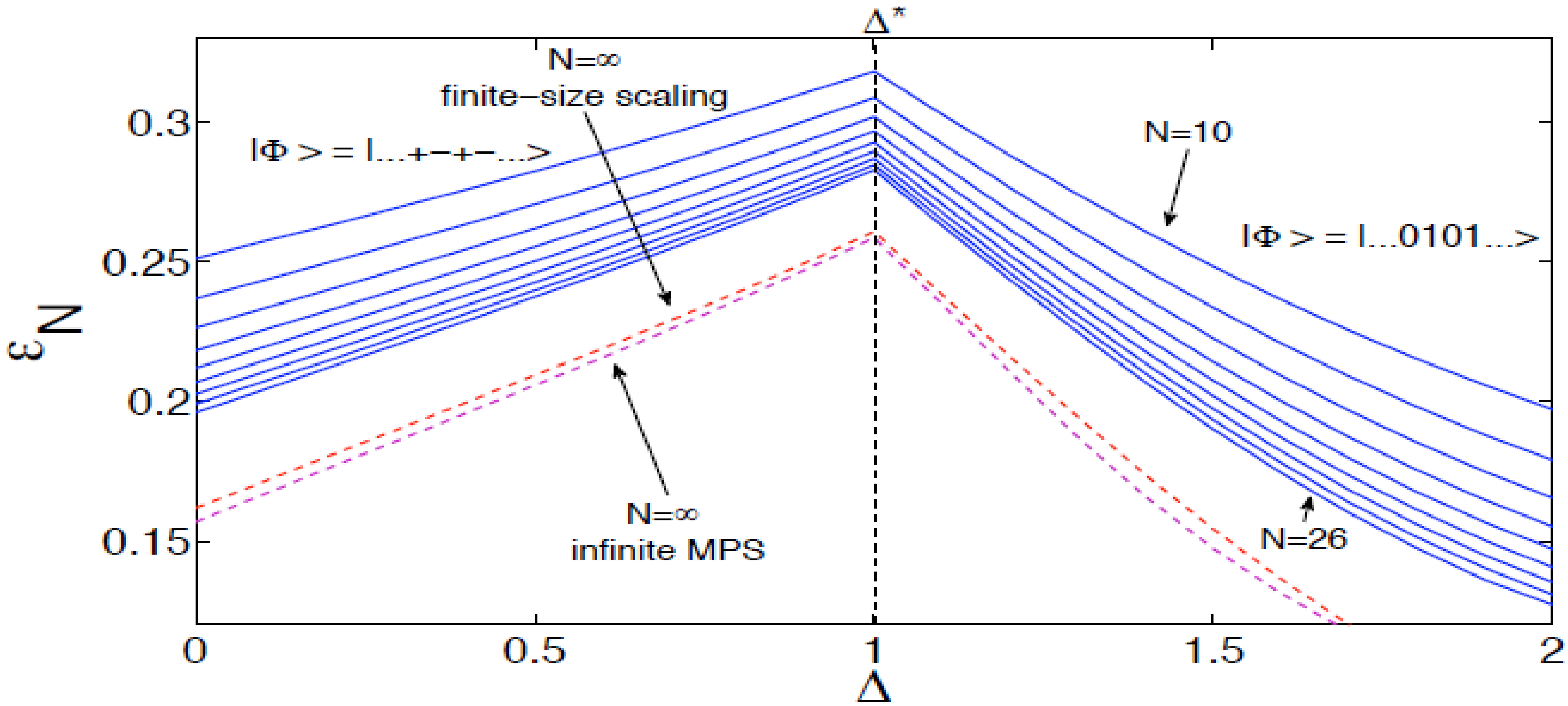


XXZ: Near KT transition

$$H = \sum_i \left(\sigma_x^{[i]} \sigma_x^{[i+1]} + \sigma_y^{[i]} \sigma_y^{[i+1]} + \Delta \sigma_z^{[i]} \sigma_z^{[i+1]} \right)$$

- Geometric entanglement

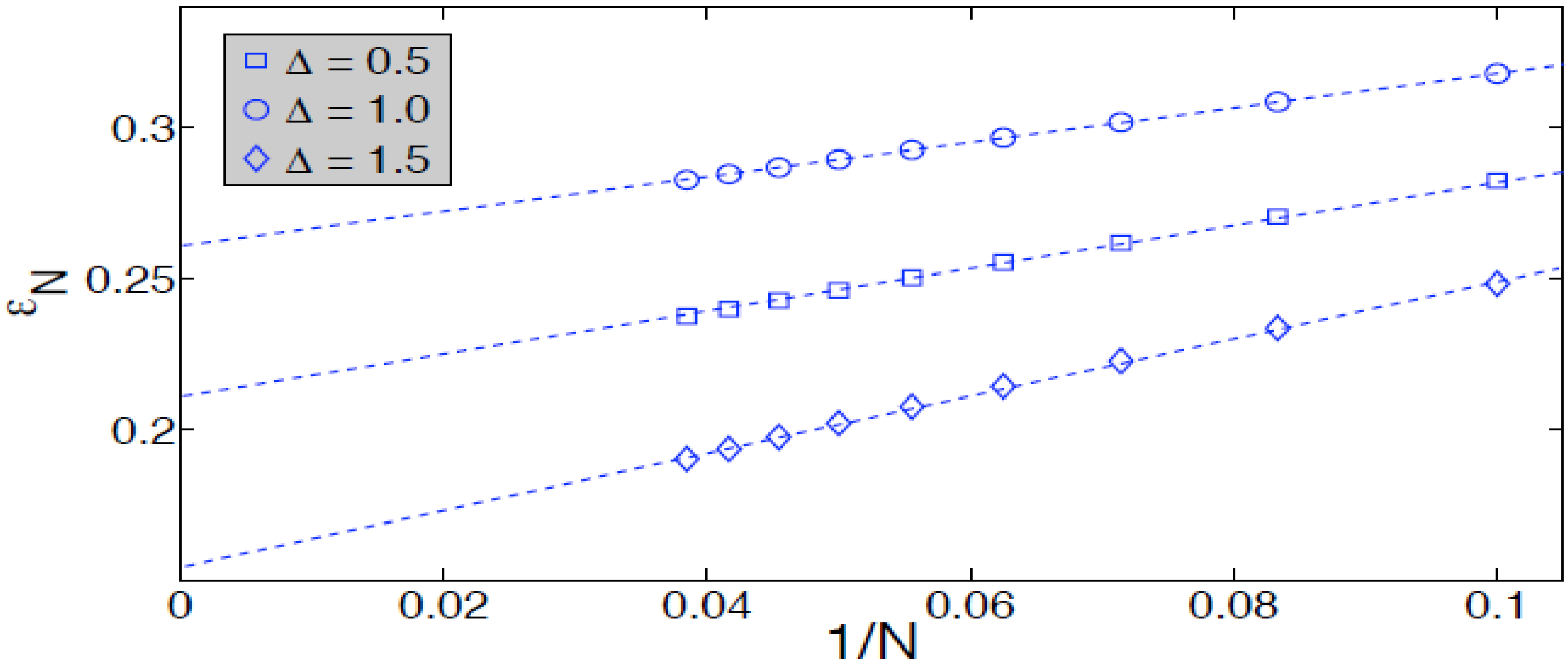
[Orús & Wei '09]



XXZ: Near KT transition

$$H = \sum_i \left(\sigma_x^{[i]} \sigma_x^{[i+1]} + \sigma_y^{[i]} \sigma_y^{[i+1]} + \Delta \sigma_z^{[i]} \sigma_z^{[i+1]} \right)$$

- Finite-size scaling $\quad \mathcal{E}_N(\Delta) \sim \mathcal{E}(\Delta) + \frac{b(\Delta)}{N}$ [Orús & Wei ’09]

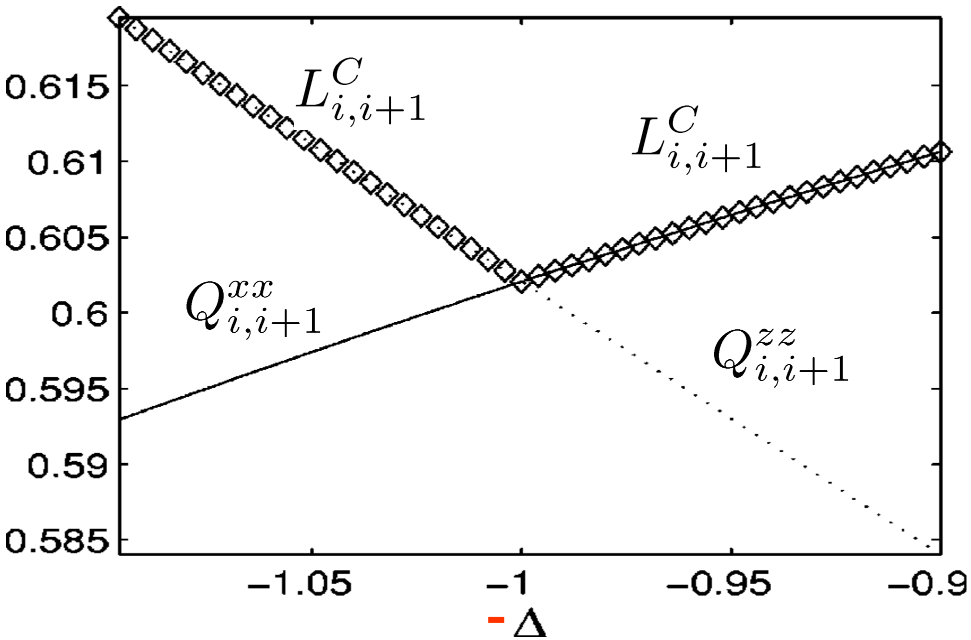


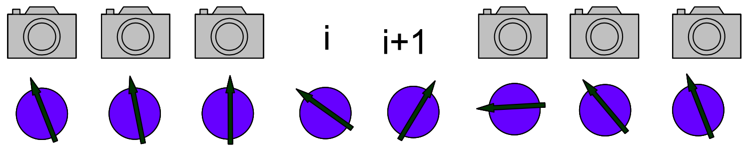
XXZ: Near KT transition

$$H = \sum_i \left(\sigma_x^{[i]} \sigma_x^{[i+1]} + \sigma_y^{[i]} \sigma_y^{[i+1]} + \Delta \sigma_z^{[i]} \sigma_z^{[i+1]} \right)$$

- Localizable entanglement [Verstraete, Martín-Delgado & Cirac ’04]

[Popp, Verstraete, Martín-Delgado & Cirac ’05]





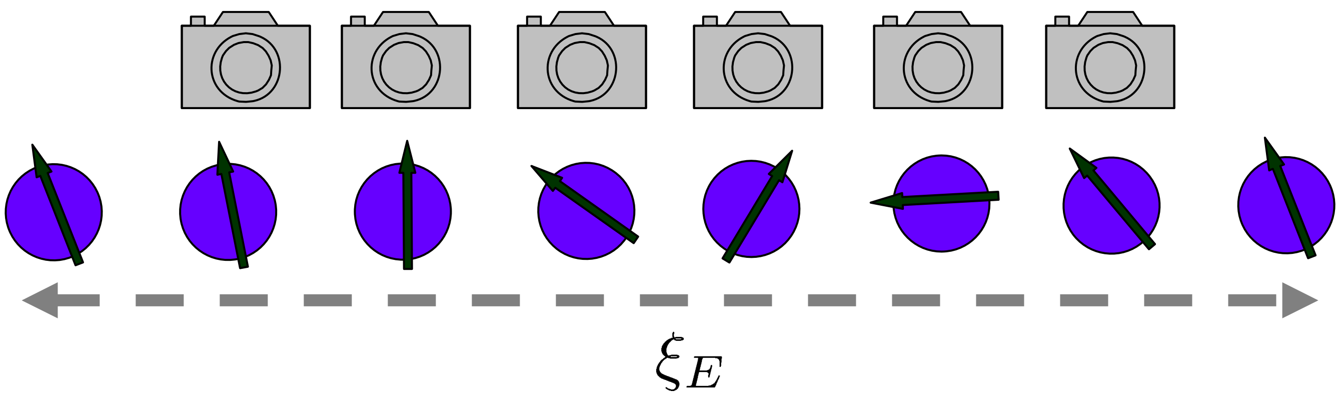
Deformed AKLT model (spin 1)

$$H(\mu) = \sum_i \left((\Sigma_\mu^{[i]})^{-1} \otimes \Sigma_\mu^{[i+1]} \right) X_{\mathrm{AKLT}}^{[i,i+1]} \left((\Sigma_\mu^{[i]})^{-1} \otimes \Sigma_\mu^{[i+1]} \right)$$

$$X_{\mathrm{AKLT}}^{[i,i+1]} = \left(\vec{S}^{[i]} \vec{S}^{[i+1]} + \frac{1}{3} (\vec{S}^{[i]} \vec{S}^{[i+1]})^2 + \frac{2}{3} \right)$$

$$\Sigma_\mu^{[i]} = I^{[i]} + \sinh(\mu) S_z^{[i]} + (\cosh(\mu) - 1)(S_z^{[i]})^2$$

- When μ=0 it's AKLT model
- Subtle transition: an transition in localizable entanglement (from finite entanglement length to infinite at transition point)

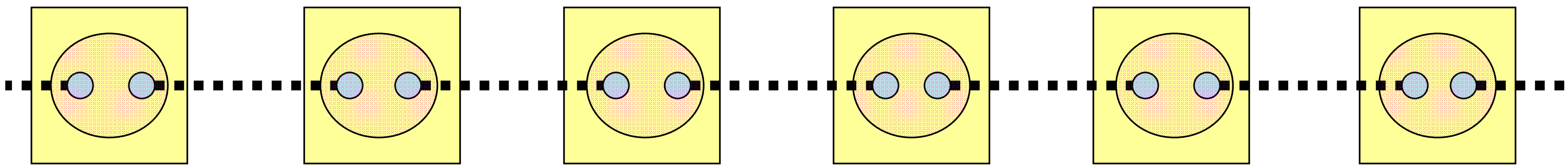


Deformed AKLT model (spin 1)

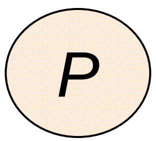
$$H(\mu) = \sum_i \left((\Sigma_\mu^{[i]})^{-1} \otimes \Sigma_\mu^{[i+1]} \right) X_{\text{AKLT}}^{[i,i+1]} \left((\Sigma_\mu^{[i]})^{-1} \otimes \Sigma_\mu^{[i+1]} \right)$$

$$\Sigma_\mu^{[i]} = I^{[i]} + \sinh(\mu) S_z^{[i]} + (\cosh(\mu) - 1)(S_z^{[i]})^2$$

- Ground state



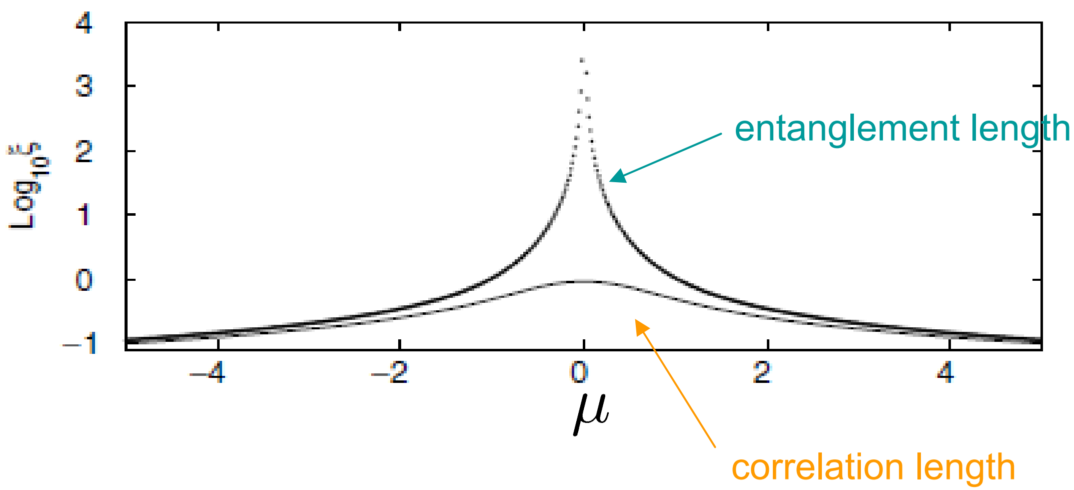
$$\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle$$



$$= e^{\mu} \left|+1\right\rangle\left\langle\uparrow\uparrow\right| + e^{-\mu} \left|-1\right\rangle\left\langle\downarrow\downarrow\right| + \frac{e^{\mu}}{\sqrt{2}} \left|0\right\rangle\left\langle\uparrow\downarrow\right| + \frac{e^{-\mu}}{\sqrt{2}} \left|0\right\rangle\left\langle\downarrow\uparrow\right|$$

Deformed AKLT model

- Subtle transition: an transition in localizable entanglement (from finite to infinite)
  - At AKLT: perfect end-end teleportation (ent. swapping)
  - Away from AKLT, teleportation by non-maximally entangled states ➔ fidelity decreases w. distance



[Verstraete, Martin-Delgado & Cirac ’04]

- Not detectable by statistical mechanical approaches

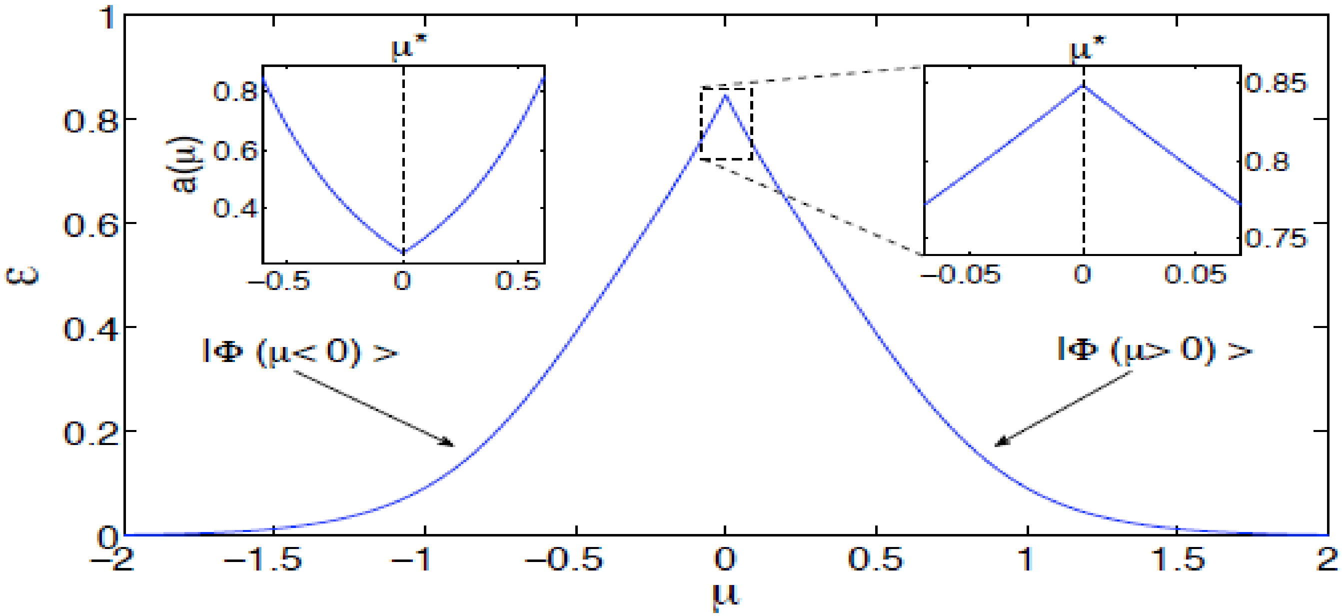
Deformed AKLT model: geometric entanglement model

- Choose product state to be product of nearest-two-site states $$|\Phi\rangle \equiv \bigotimes_{j=1}^{(N/2)} |\phi^{[2j-1,2j]}\rangle$$

[Orús & Wei '09]

➔ Closest product state

$$|\Phi^*(\mu)\rangle = \left[C(\mu)\left(a(\mu)|0,0\rangle + \frac{1}{2}|x(\mu)\rangle\right)\right]^{\otimes\infty}$$



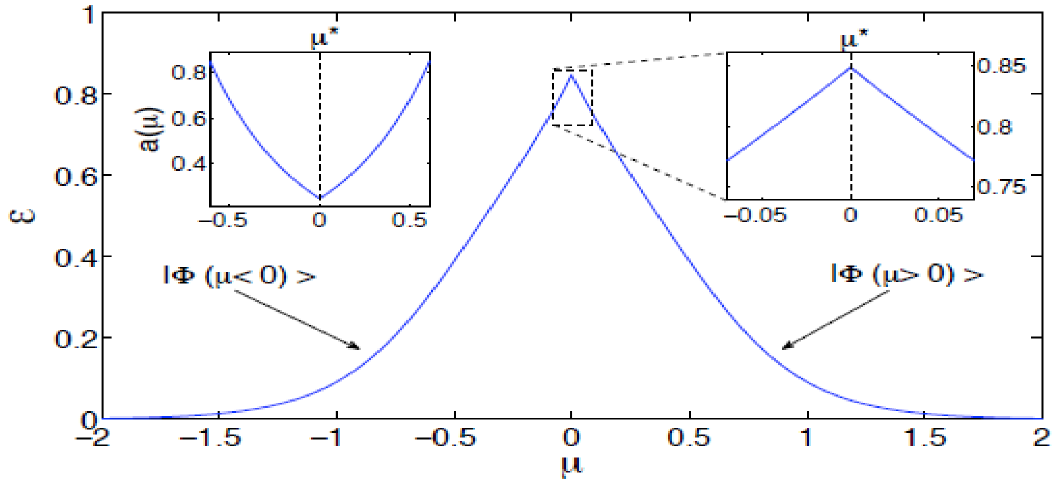
$$|x(\mu \geq 0)\rangle = |1,1\rangle$$

$$|x(\mu \leq 0)\rangle = |-1,-1\rangle$$

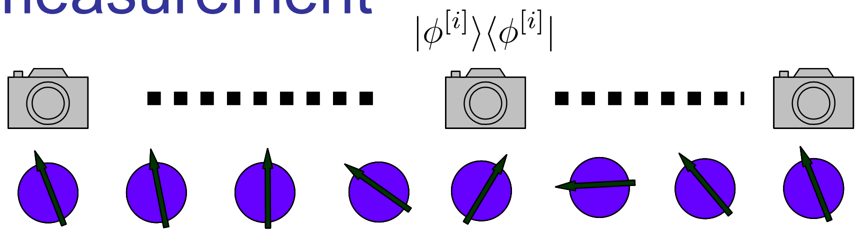
Deformed AKLT model: geometric entanglement model

- Geometric entanglement is singular across the localizable-entanglement transition

[Orús & Wei '09]



- Geometric entanglement is related to optimal local measurement



Summary

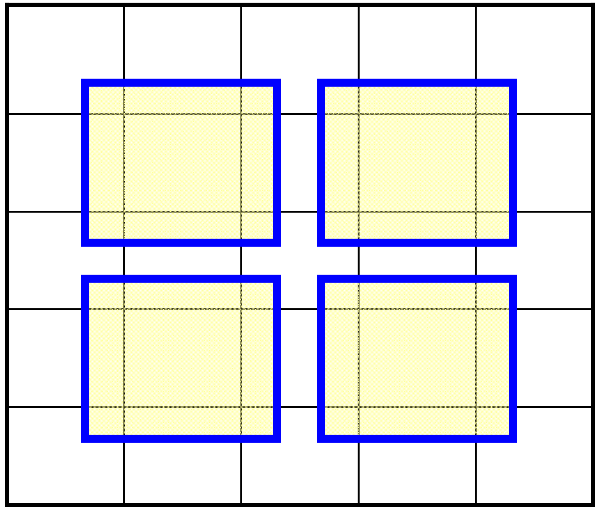
- Benchmark geometric entanglement for Ising models with transverse and longitudinal fields via analytic & iTEBD
  ➔ 2nd order and 1st quantum phase transitions have different singular behaviors

- XXZ model: Geometric entanglement shows jump at 1st order FM transition (Δ= -1) and cusp at ∞-order transition from critical phase to gapped AFM (Δ=1)
  ➔ similar to localizable entanglement

- Deformed AKLT: finite-to-infinite entanglement length can also be detected by geometric entanglement

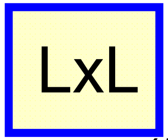
Outlook

- Two or higher dimensions; see results by Huang & Lin '09
- Area law via entanglement entropy, is there area law using geometric entanglement?



$|\Psi\rangle$: entangled state with NxN sites

$$|\Phi_{\text{product}}\rangle = \bigotimes_{i,j=1}^{N/L}$$



(i,j)-th block

$$\mathcal{E}_{L\times L} \sim L?$$

- Detecting topological phases?

Preliminary announcement

10th Canadian Summer School on Quantum Information	+ Workshop

- Sat, July 17 – Fri, July 30, 2010 @ UBC, Vancouver
- Organizing committee:

  Mohammad H. S. Amin (D-Wave)

  Petr Lisonek (SFU)

  Barry Sanders (Calgary)

  Pradeep Kiran Sarvepalli (UBC)

  Robert Raussendorf (UBC)

  Tzu-Chieh Wei (UBC)

<http://www.qi10.ca> [under construction]

Overlap and correlation functions

- A single spin state $\alpha|\uparrow\rangle+\beta|\downarrow\rangle$ can be expressed in terms of a density matrix

$$\rho=\frac{1}{2}\left(I+\vec{r}\cdot\vec{\sigma}\right),\qquad \text{with } \left|r\right|=1$$

- N-spin separable pure state:

$$\left|\phi_S\right\rangle\left\langle\phi_S\right|=\frac{1}{2}\left(I+\vec{r}_1\cdot\vec{\sigma}\right)\otimes\frac{1}{2}\left(I+\vec{r}_2\cdot\vec{\sigma}\right)\otimes\cdots\otimes\frac{1}{2}\left(I+\vec{r}_N\cdot\vec{\sigma}\right)$$

- The overlap square

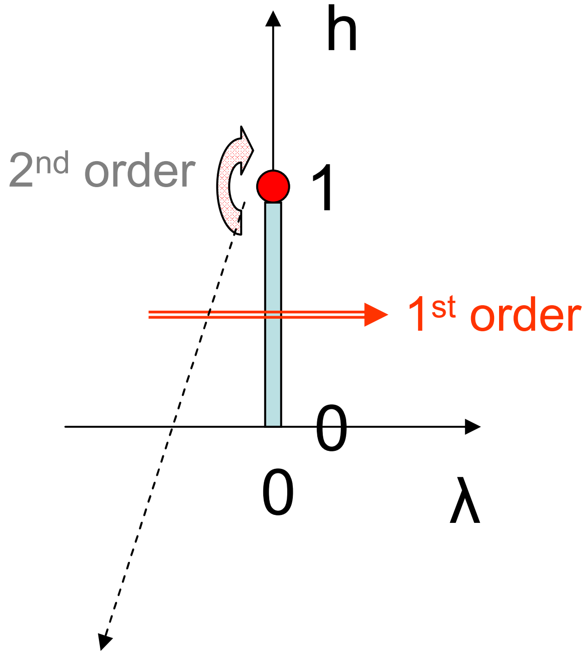
$$\Lambda^2=\left\langle\left(\left|\phi_S\right\rangle\left\langle\phi_S\right|\right)\right\rangle_\psi$$

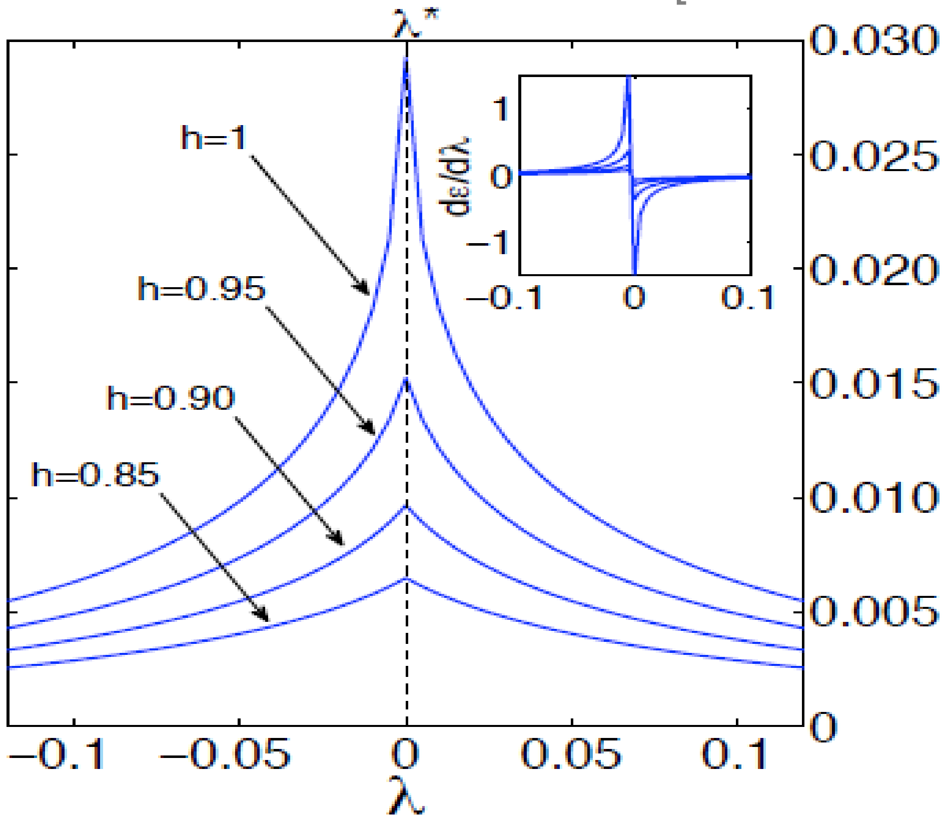
Is a linear combination of all correlation functions

At fixed h, vary longitudinal field λ

$$H = -\sum_i \left(\sigma_x^{[i]}\sigma_x^{[i+1]} + h\sigma_z^{[i]} + \lambda\sigma_x^{[i]}\right)$$

[Orús & Wei '09]





$$\partial\mathcal{E}/\partial\lambda(\lambda, h=1) \approx \mp 6.5(1)\log_2|\lambda| \text{ for } |\lambda| \ll 1$$