Efficient description of many-body systems with prjected entangled-pair states

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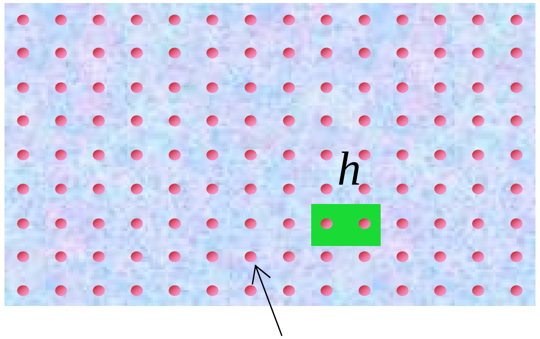
QUANTUM MANY-BODY SYSTEMS
EFFICIENT DESCRIPTIONS: LATTICES



Spins, Fermions

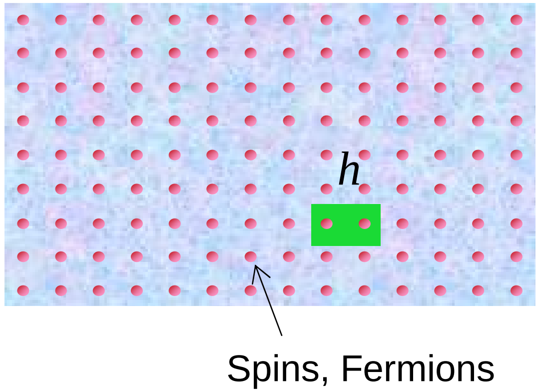
- **Hamiltonian:**

$$H = \sum_{<n,m>} h_{n,m}$$

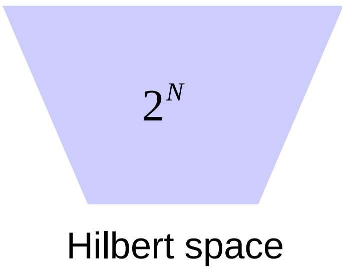
- **Questions:**
  - Ground state: $H|\Psi_0\rangle = E_0|\Psi_0\rangle$
  - Thermal state: $\rho \propto e^{-H/T}$
  - Dynamics: $|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$

Physical properties:

$\langle\sigma_n\rangle, \langle\sigma_n\sigma_m\rangle, \ldots$



- Problem:



$$c_1 |00..0\rangle + c_2 |00..1\rangle + .. + c_{2^N} |11..1\rangle$$

- Exp. number of parameters.
- Exp. number of operations.

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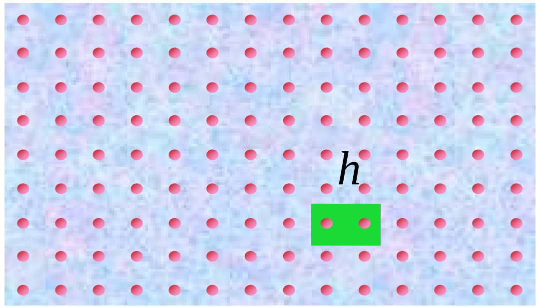

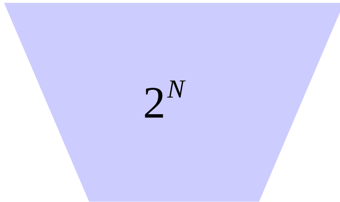

… but

$$H = H(p_1, \ldots, p_N)$$

$$\Downarrow$$

$$\Psi_0(H) = \Psi_0(p_1, \ldots, p_N)$$

depends on few parameters.

We are only interested
in some special states

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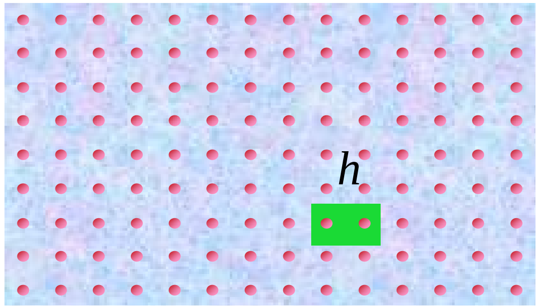
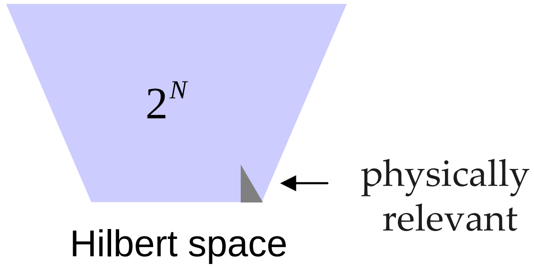
EFFICIENT DESCRIPTIONS: LATTICES





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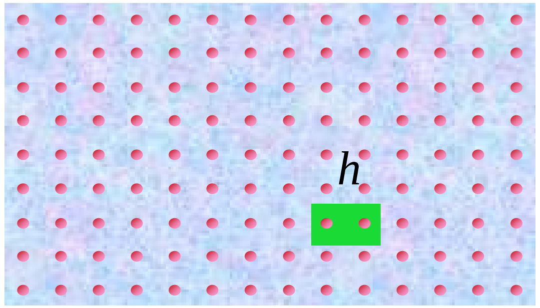
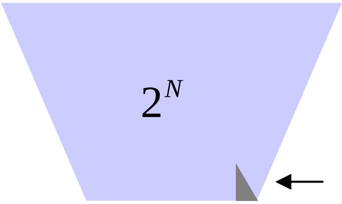
EFFICIENT DESCRIPTIONS:





- Example: spin 1/2

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

- General 2-body interactions:

$$H = \sum_{n,m=1}^{N} \sum_{\alpha,\beta=0}^{3} \lambda_{\alpha,\beta}^{n,m} \sigma_\alpha^n \otimes \sigma_\beta^m$$

there are $16N^2$ coefficients

$$\Psi_0(\lambda_{n,m}^{\alpha,\beta})$$

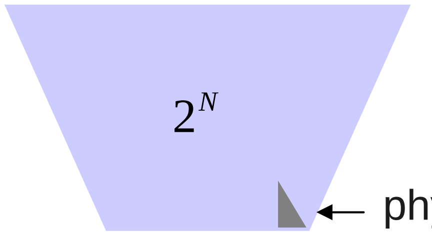
- Short-range, homogeneous

$$H = \sum_{\|n-m\|<1} \sum_{\alpha,\beta=0}^{3} \lambda_{\alpha,\beta}^{n-m} \sigma_\alpha^n \otimes \sigma_\beta^m$$

there are $16R^d$ coefficients

QUANTUM MANY-BODY SYSTEMS
EFFICIENT DESCRIPTIONS: LATTICES

- Methods:



← physically relevant

$|\Psi(p_1,\ldots,p_D)\rangle$

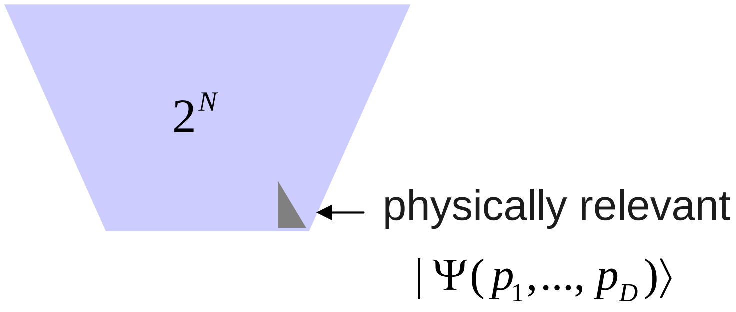
- Exact
- Monte Carlo.
- Perturbation expansions
- Density Functional Theory
- Dynamical Mean Field Theory
- …
- Variational

  - Mean Field, Hartree-Fock, BCS, Laughlin,

  - Tensor network states

QUANTUM MANY-BODY SYSTEMS
EFFICIENT DESCRIPTIONS: LATTICES

- We are interested in a „small corner“ of Hilbert space



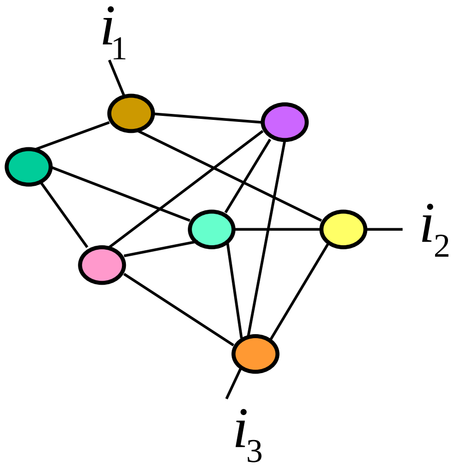
- May be one can find an efficient description of those states:
  - Few parameters (eg, scaling polynomically with N)
  - Expectation values can be efficiently calculated.

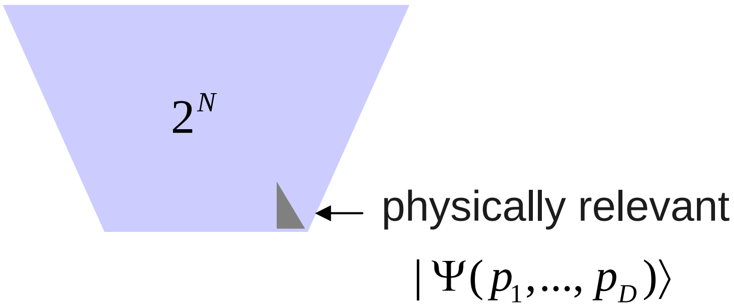
- Applications:
  - Variational family: Numerical algorithms.
  - Describe general properties.

But … what is such a description?

EFFICIENT DESCRIPTIONS: LATTICES

TENSOR NETWORKS

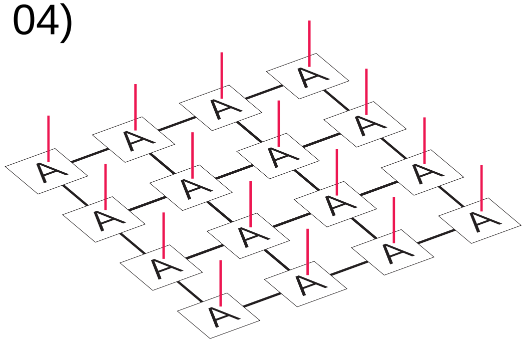




- Novel descriptions:
  - Efficient
  - Intuition: entanglement
  - No sign problem: frustrated spin and fermionic systems. Complementary to other methods (cf, Monte Carlo)
  - Extension of DMRG, NRG and other renormalization methods

PROJECTED ENTANGLED-PAIR STATES

(Verstraete, IC 04)



- Properties:
  - Build to fulfill the area law
  - Naturally arise in problems with short-range interactions.

    They contain the relevant states for short-range interactions
  - Infinite homogeneous systems:
    - Example: AKLT (in any dimension)
    - In a 2D square lattice: Vertex-Type Matrix Product Ansatz

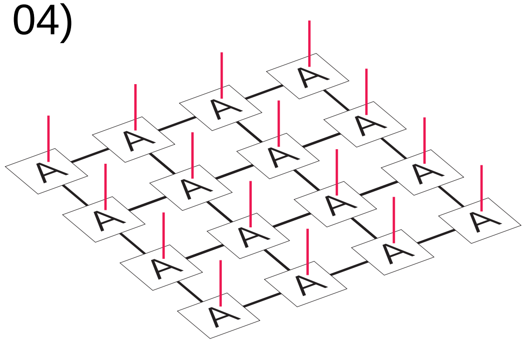
      (Sierra and Martin-Delgado, 1998)

      (different than Interaction-Round the Face used by Nishino and col)

      (TNS = TPS)
  - For 1D systems, they coincide with MPS (Zittart et al)
    - Infinite homogeneous systems FCS (Fannes, Nachtergaele and Werner 91).

PROJECTED ENTANGLED-PAIR STATES

(Verstraete, IC 04)



See: Cirac and Verstraete (J. Phys. A, 2009)
Verstraete, Murg and Cirac (Adv. Phys, 2008)

OUTLINE

- EFFICIENT DESCRIPTIONS:
  PROJECTED ENTANGLED-PAIR STATES.
- AREA LAWS
- THERMAL EQUILIBRIUM:
- SYMMETRIES
- MPS and CONFORMAL FIELD THEORY MODELS
- OTHER METHODS:
  STRING-BOND AND PLAQUTTE STATES

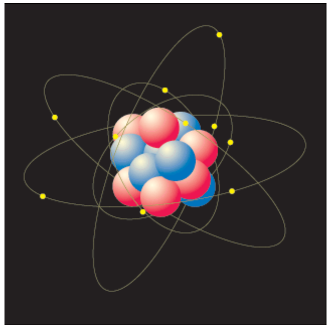
1. EFFICIENT DESCRIPTIONS:

PROJECTED ENTANGLED-PAIR STATES

Verstraete and IC 04

Let us consider our particles as composite objects

Atom



^{87}Rb

Spin 1 particle

Low energy

The three levels are entangled states of many degrees of freedom

We obtain them by projecting out the state:

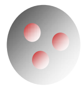
$$|s\rangle = P|\Psi\rangle_{el-pos} \otimes |\Psi\rangle_{el-spin} \otimes |\Psi\rangle_{nuc-pos} \otimes |\Psi\rangle_{nuc-spin}$$

$$P_n : H_{el-pos} \otimes H_{el-spin} \otimes H_{nuc-pos} \otimes H_{nuc-spin} \rightarrow C^3$$

PEPS
DEFINITION

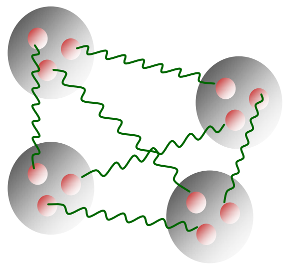
Verstraete and IC 04

Let us consider our particles as composite objects



Spin ½ atoms is composed of electrons, protons, etc

- Auxiliary particles have different dimensions: D
- We only „see“ the macroscopic objects:
  We project onto a 2-dimensional subspace.
- The auxiliary particles are in a very simple state: maximally entangled.

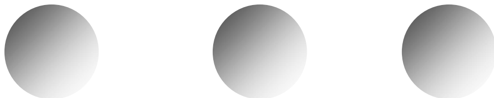


- The state is determined by the way we project onto the spin 1/2 space.
  The state is completely characterized by N projectors.

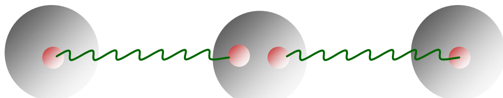
PEPS
EXAMPLE

- Example: GHZ state

$$|\Psi\rangle = |1,1,1\rangle + |2,2,2\rangle$$



Auxiliary construction:



$$|1,1\rangle + |2,2\rangle \quad |1,1\rangle + |2,2\rangle$$

$$P = |1\rangle\langle 1,1| + |2\rangle\langle 2,2|$$

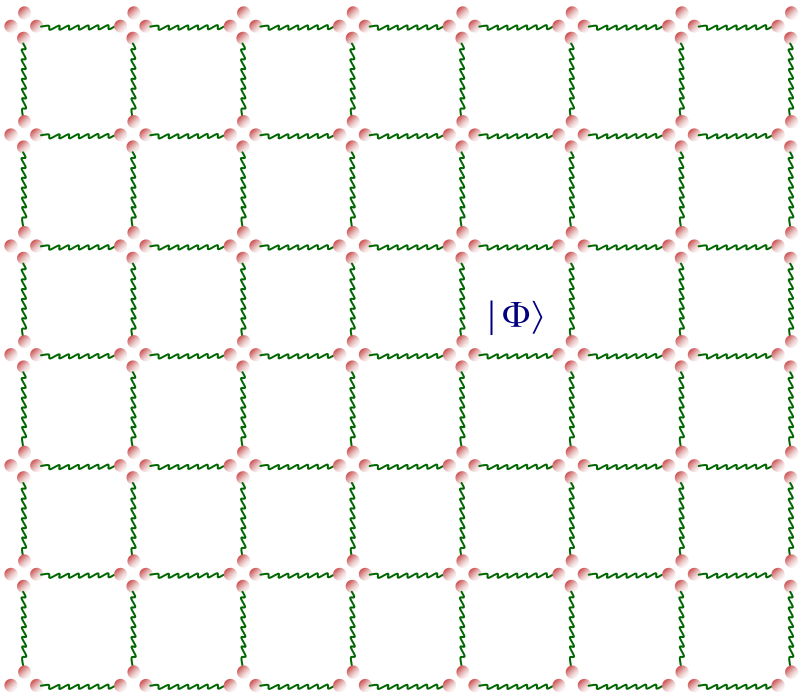
$$\underbrace{|1,1\rangle|1,1\rangle}_{|1,1,1\rangle} + |1,1\rangle|2,2\rangle + |2,2\rangle|1,1\rangle + \underbrace{|2,2\rangle|2,2\rangle}_{|2,2,2\rangle}$$

We can specify the state by just giving $P = |1\rangle\langle 1,1| + |2\rangle\langle 2,2|$

PEPS
EXAMPLE

Verstraete and IC 04

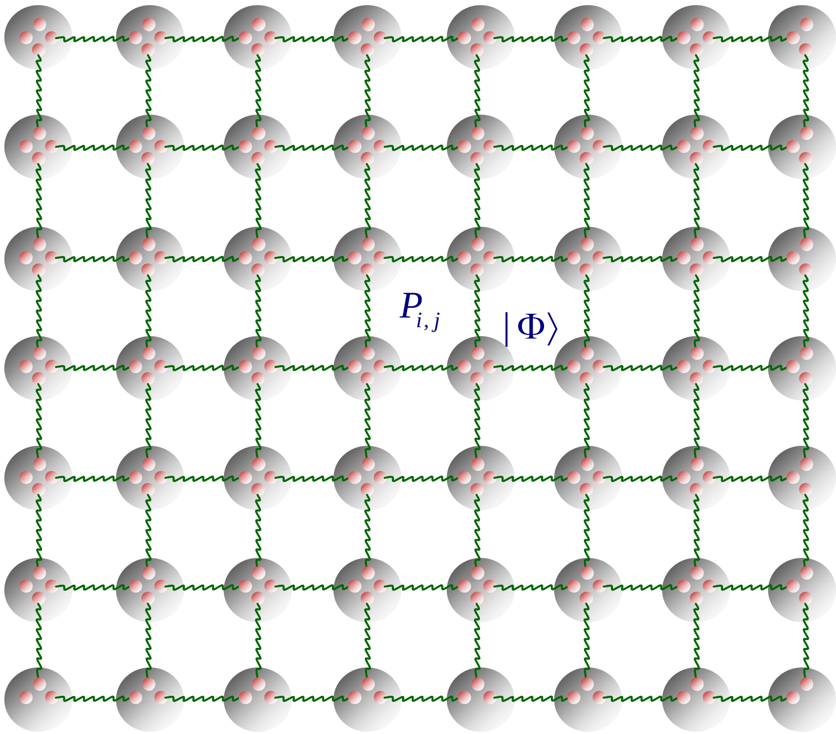
- Example: 2D lattice



PEPS
EXAMPLE

Verstraete and IC 04

Example: 2D lattice

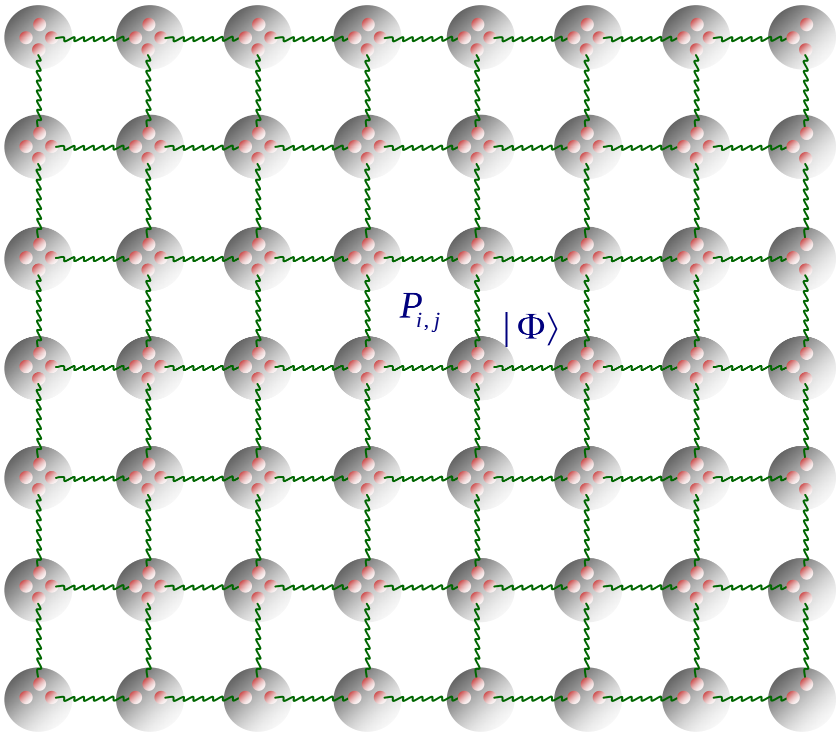


$$P_n : C^D \otimes C^D \otimes C^D \otimes C^D \to C^2$$

PEPS
EXAMPLE

Verstraete and IC 04

Example: 2D lattice



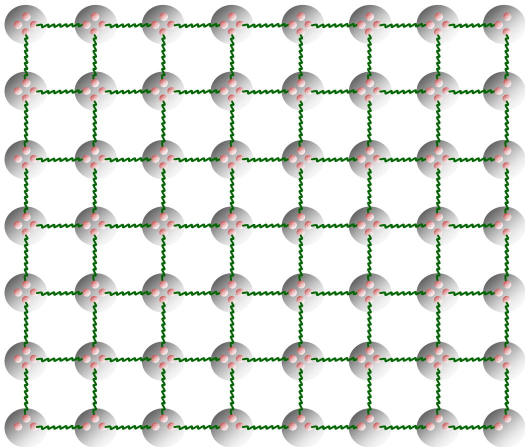
$$P_n : C^D \otimes C^D \otimes C^D \otimes C^D \to C^2$$

- All the information is contained in the P's
- There are $2D^4N$ parameters

PEPS
EXAMPLE

Verstraete and IC 04

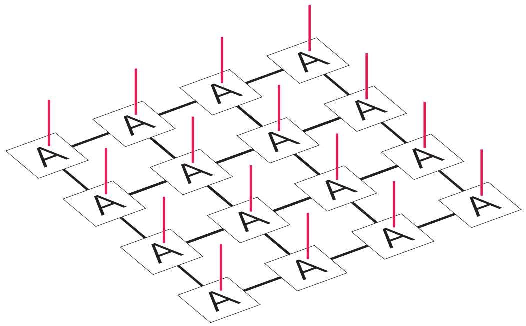
Example: 2D lattice



$$P = \sum A^{i}_{\alpha\beta\gamma\delta} \,|\, i \rangle\langle \alpha\beta\gamma\delta \,|$$

written in a basis:

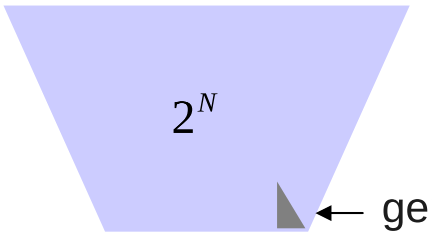
$$|\,\Psi\rangle = \sum_{i} c^{i_1,\ldots,i_N} \,|\, i_1,\ldots,i_N \rangle$$



- One can define them in any lattice in any dimension
- One can define them for Fermions too (Kraus, Schuch, Verstraete, IC, 09)

2. AREA LAWS:
PROJECTED ENTANGLED-PAIR STATES

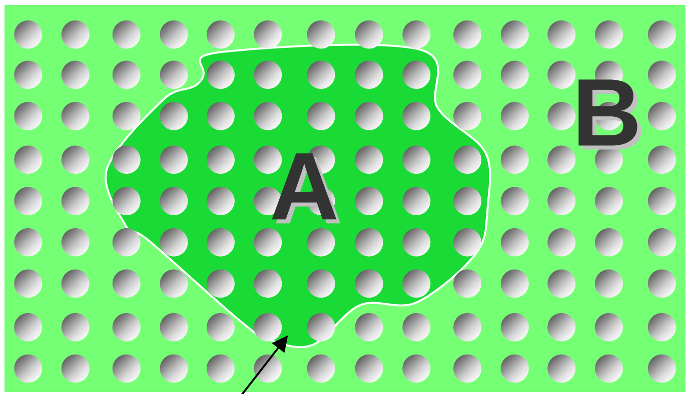
AREA LAWS
EFFICIENT DESCRIPTIONS



← general properties:

- There is a spatial structure.
- Hamiltonian has short-range interactions, etc
- Low temperatures

Guiding principle: Area law (Sredniky 93)



$$\rho_A = tr_B |\Psi\rangle\langle\Psi|$$

- Area law:

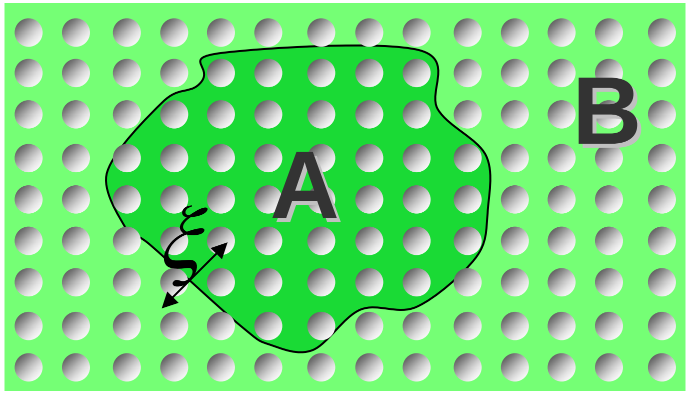
$$S(\rho_A) < O(L^{d-1})$$

- Conjecture:

  All physically relevant states at T=0 fulfill this law (up to log corrections).

AREA LAWS
CORRELATIONS

- Idea: at long distances, particles are uncorrelated.



AREA LAWS
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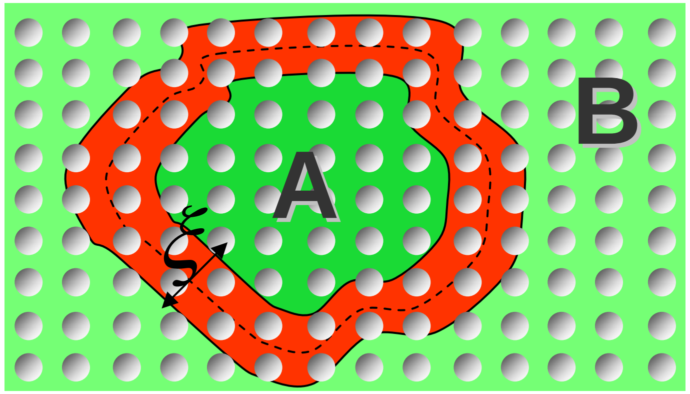


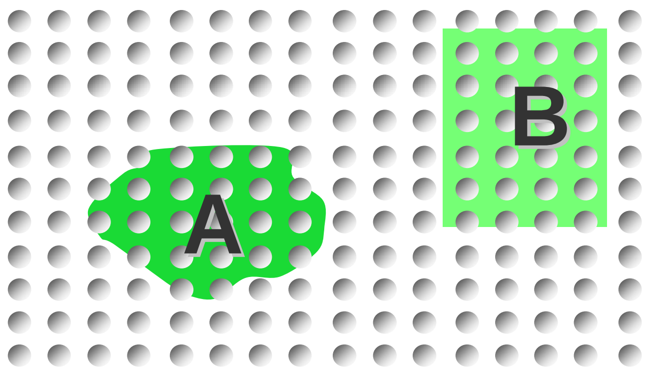
$$S_A \approx E(A,B) \leq \xi N_\partial$$

In some way, the existence of a correlation length should give rise to the area law.

AREA LAWS
MUTUAL INFORMATION

(Wolf, Hastings, Verstraete, Cirac 08)

- Finite T: quantum mutual information



$$I(A:B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

- It measures correlations, but is „stronger“:

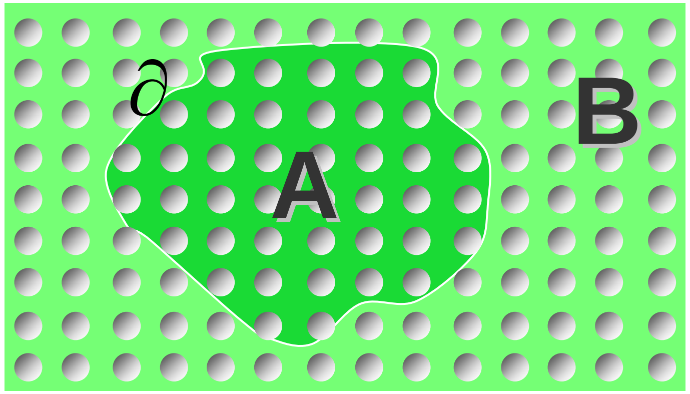
$$I(A:B) \geq \tfrac{1}{2} \| \rho_{AB} - \rho_A \otimes \rho_B \|_1^2 \Rightarrow I(A:B) \geq \frac{|\langle X_A \otimes Y_B \rangle - \langle X_A \rangle \langle Y_B \rangle|^2}{2 \| X_A \|^2 \| Y_B \|^2}$$

- The mutual information does not „overlook“ correlations (cf data-hidding states)
- it coincides with the entanglement entropy for T=0.

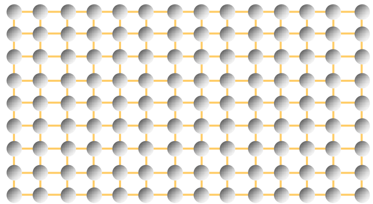
- It fulfills area laws for Gibbs states.
- Area law is a consequence of the decay of correlations.

AREA LAWS
THERMAL EQUILIBRIUM

(Wolf, Hastings, Verstraete, Cirac 08)



i) Hamiltonian with short-range interactions: $H = \sum_{\langle i,j \rangle} h_{i,j}$

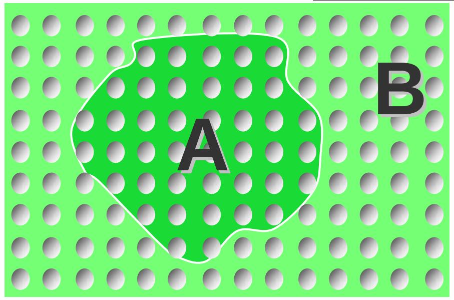


ii) Finite temperature. $\rho_T = \frac{1}{Z} e^{-H/k_B T}$

$I(A:B) \leq \frac{\| h \|_{op}}{k_B T} N_\partial$ where $N_\partial =$ # particles at the border A-B

AREA LAWS
THERMAL EQUILIBRIUM

PROOF:

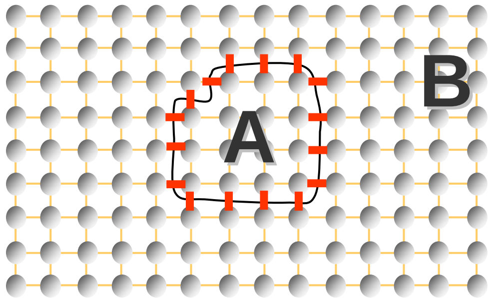


- Main tool: FREE ENERGY: $F(\rho) := \langle H \rangle_\rho - k_B T S(\rho)$
- Free energy is minimized by the Gibbs state:

$$\min F(\rho) = F(\rho_T) \quad \text{where} \quad \rho_T = \frac{1}{Z} e^{-H/k_B T}$$

$$F(\rho_{AB}) \leq F(\rho_A \otimes \rho_B)$$

$$H = H_A + H_B + H_{ab}$$



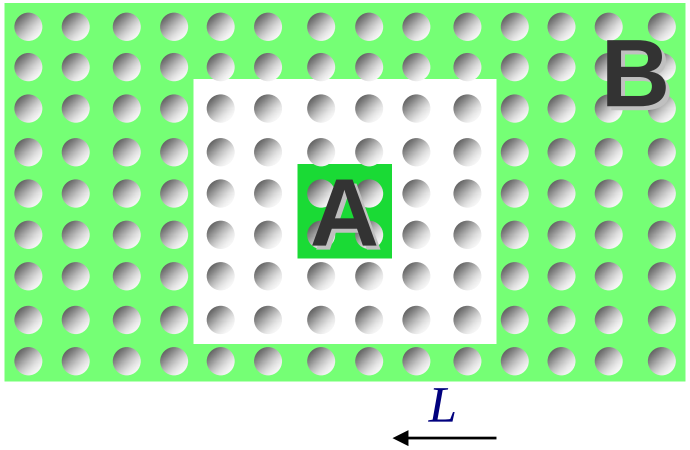
$$\underbrace{S(\rho_A \otimes \rho_B) - S(\rho_{AB})}_{I(A:B)} \leq \underbrace{\frac{\langle H \rangle_{\rho_A \otimes \rho_B} - \langle H \rangle_{\rho_{AB}}}{k_B T}}$$

$$\leq \frac{\langle H_{ab} \rangle_{\rho_A \otimes \rho_B} - \langle H_{ab} \rangle_{\rho_{AB}}}{k_B T}$$

$$\leq \frac{\| h \|_{op}}{k_B T} N_\partial$$

AREA LAWS
CORRELATIONS

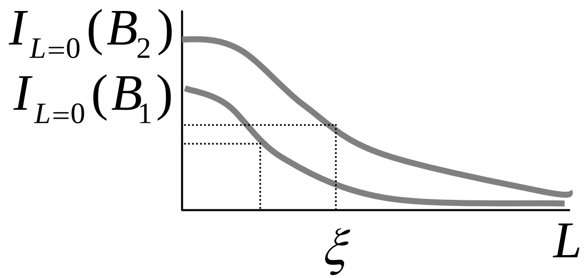
(Wolf, Hastings, Verstraete, Cirac 08)



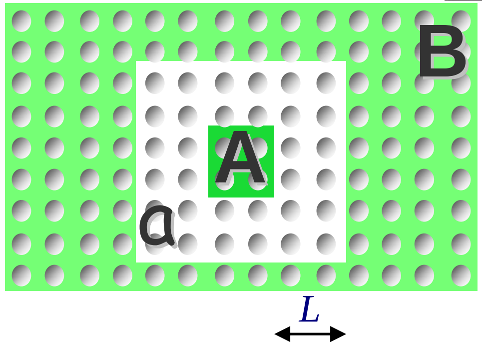
- For fixed B, $I_L(B) := I(A:B)$ decreases with L.
- We denote by ξ, the length L such that for all B (sufficiently large)

$$I_{L=\xi}(B) \leq \tfrac{1}{2} I_{L=0}(B)$$

(i.e, ξ is the correlation length)



If ξ is finite then $I_{L=0}(B) \leq 4\xi N_\partial$



PROOF:

- The mutual information decreases with L.

$$I(A:B) \le I(Aa:B)$$

- But cannot decrease too much:

$$I(Aa:B) \le I(A:B) + 2S_a$$

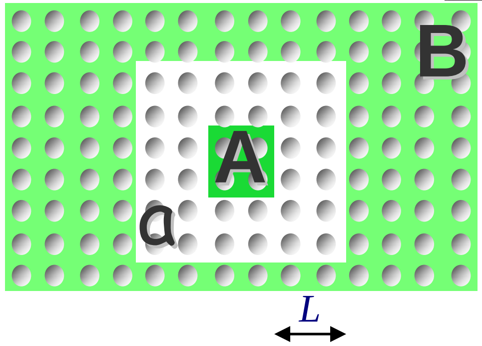
We choose $L = \xi$

$$I_{L=0}(B) \le I_{L=\xi}(B) + 2LN_{\partial} \le \tfrac{1}{2} I_{L=0}(B) + 2LN_{\partial}$$

$$I_{L=0}(B) \le 4\xi N_{\partial}$$

AREA LAWS

T=0



$$I(A:B) \leq \frac{\| h \|_{op}}{k_B T} N_{\partial}$$

- 1D systems:

$$S_\alpha(\rho_L) \equiv \frac{1}{1-\alpha} \log_2[\mathrm{tr}(\rho^\alpha)]$$ Renyi entropies:

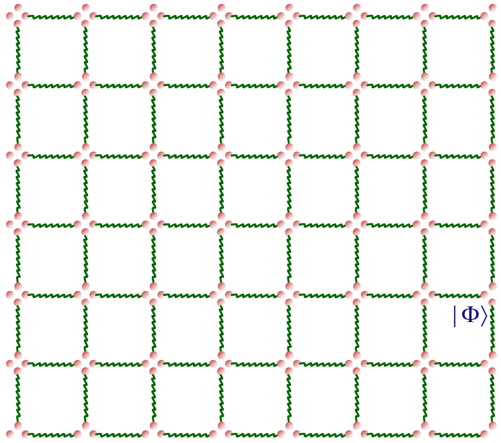
- Critical systems: CFT: $S_\alpha(\rho_L) = \frac{c+\bar{c}}{12}\left[1+\frac{1}{\alpha}\right]\log_2 L$ (Vidal, Rico, Kitaev, Latorre) (Calbrese, Cardy, Korepin,..)
- Non-critical: $S_\alpha(\rho_L) < C_\alpha$

Higher dimensions:

- Gaussian theories (Eisert et al)
- Spin/Fermionic systems (Wolf, Perez-Garcia et al, …)

AREA LAWS
PEPS

- PEPS is the natural family fulfilling the area law:



The mixed state version, also fulfills the area law.

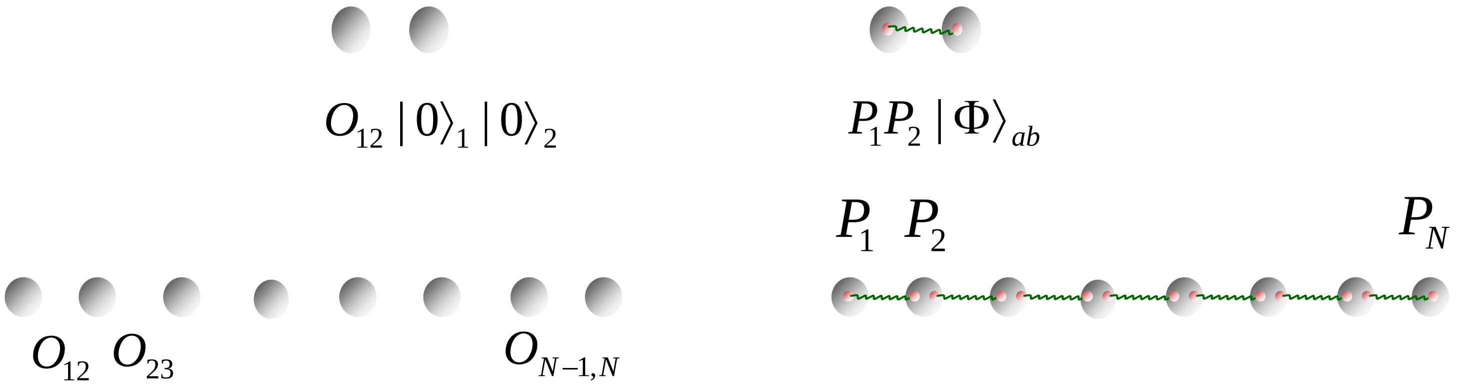
They provide the natural way of expressing the area law

3. THERMAL EQUILIBRIUM: PROJECTED ENTANGLED-PAIR STATES

THERMAL EQUILIBRIUM
PEPS

- Any action between two systems can be described in terms of ancillas:

(Kraus, Dur, Lewenstein, and IC 01, Verstraete and IC 04)

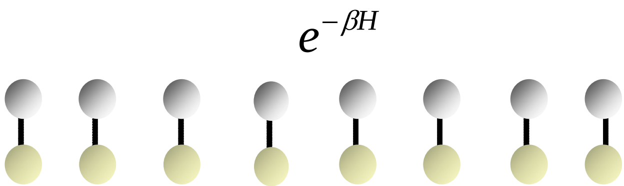


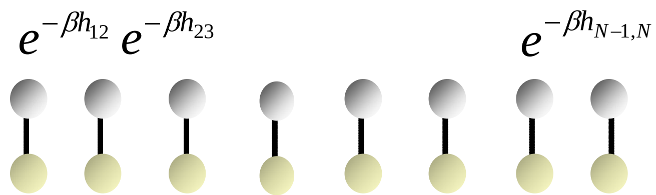
- Thermal states can be written in terms of a purification:

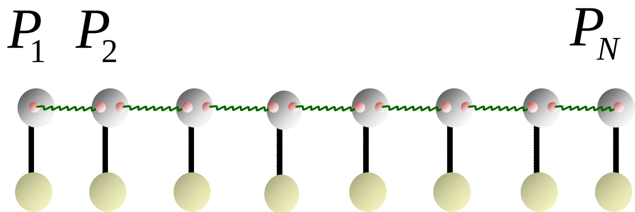
$$e^{-\beta H} = \mathrm{tr}(|\Psi\rangle\langle\Psi|) \text{ with } |\Psi\rangle = \left[e^{-\beta H/2}\otimes 1\right]|\Phi\rangle^{\otimes N}$$

THERMAL EQUILIBRIUM
PEPS

- Consider local Hamiltonians: $H = \sum_n h_{n,n+1}$
  - Let us assume that the h‘s commute: $e^{-\beta H} = e^{-\beta h_{12}} e^{-\beta h_{23}} \ldots e^{-\beta h_{N-1,N}}$







  - If the h‘s do not commute: $e^{-\beta H} = \left[e^{-\beta h_{12}/M} e^{-\beta h_{23}/M} \ldots e^{-\beta h_{N-1,N}/M} \right]^M$

    There are M bonds between neighbors, but they are weakly entangled

    It can be approximated with a single bond of dimension D.

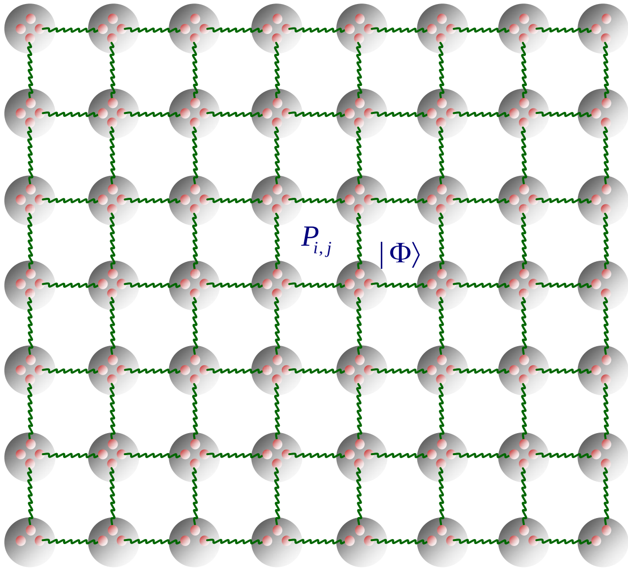
- This argument has been made rigorous by Hasting (Hastings 06).
- It also works for Fermions (Kraus, Schuch, Verstraete, and IC 09).

4. SYMMETRIES:

PROJECTED ENTANGLED-PAIR STATES

SYMMETRIES
TRANSLATIONS

- Translational: $T|\Psi\rangle = |\Psi\rangle$



- We choose all the P equal:

  $P: C^D \otimes C^D \otimes C^D \otimes C^D \to C^2$

- A single operator P describes the whole system.
- All physical properties, including symmeries, are encapsultated in P.

SYMMETRIES

LOCAL

- Local symmetries: $u_g^{\otimes N} |\Psi\rangle = e^{i\theta_g} |\Psi\rangle$ where $g \in G$

  - For the PEPS: $u_g P V_g^{\otimes 4} = e^{i\Phi_g} P$ where V_g is a D-dim representation.

    $P : C^D \otimes C^D \otimes C^D \otimes C^D \to C^2$

- Applications:

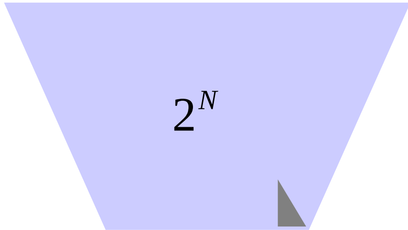
  - Lieb-Schultz-Mattis theorem: $G = su(2)$ (Sanz, Perez-Garcia, Wolf, IC 08)

    - For a 1D systems and semi-integer spin, V_g must be reducible.
    - If reducible, then for any H for which Ψ is the ground state, it must be degenerate.
    - It also applies to higher dimensions.

  - Extensions to other grups: $u(1), Z^d, etc$

  - String order, topological order, etc. (Perez-Garcia, Sanz, Wolf, Verstraete, IC 08)

SYMMETRIES
LOCAL



- Since PEPS represent the states that appear in Nature, we should develop the theory of those states.

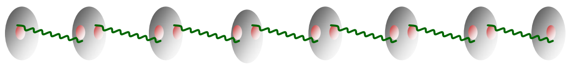
- Whatever we can show to hold for PEPS, it should be true in general (?).

5. CONFORMAL FIELD THEORIES: MATRIX PRODUCT STATES

(In collaboration with German Sierra, Madrid)

MATRIX PRODUCT STATES
1 DIMENSION

- 1D systems:



$$P_n : C^D \otimes C^D \to C^2$$

- In a basis: $P_n = \sum_{x,y=1}^{D} \sum_{s=-1}^{1} [A_n]^s_{x,y} \, | s \rangle \langle x, y |$

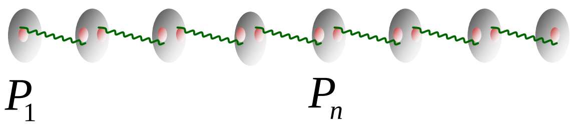
- Matrix Product States:
  (Fannes, Narchtengaele, Werner 08)

$$| \Psi \rangle = \sum_{s_1 \ldots s_N = \pm 1} \mathrm{tr}(A_1^{s_1} \ldots A_N^{s_N}) \, | s_1, \ldots, s_N \rangle$$

- For finite D, all correlation functions decay exponentially.
- In order to describe exactly a critical system, we have to take $D \to \infty$
- We have taken a QFT for the auxiliary particles.

MATRIX PRODUCT STATES
CFT

- 1D systems:



$$P_n : C^D \otimes C^D \rightarrow C^2$$

$$P_n = \sum_{x,y=1}^{D} \sum_{s=-1}^{1} [A_n]^s_{x,y} \, |s\rangle\langle x,y|$$

$$V_s(\alpha, z_n) =: e^{i\sqrt{\alpha} s \phi(z_n)} :$$

vertex operator

$\phi(z)$ chiral free boson field

z: complex number

- State: $|\Psi\rangle = \sum_{s_1 \dots s_N = \pm 1} c_{s_1 .. s_N}(\alpha, z_1, \dots, z_N) \, | s_1 \dots s_N \rangle$

Variational parameters (arrow pointing to $c_{s_1 .. s_N}(\alpha, z_1, \dots, z_N)$)

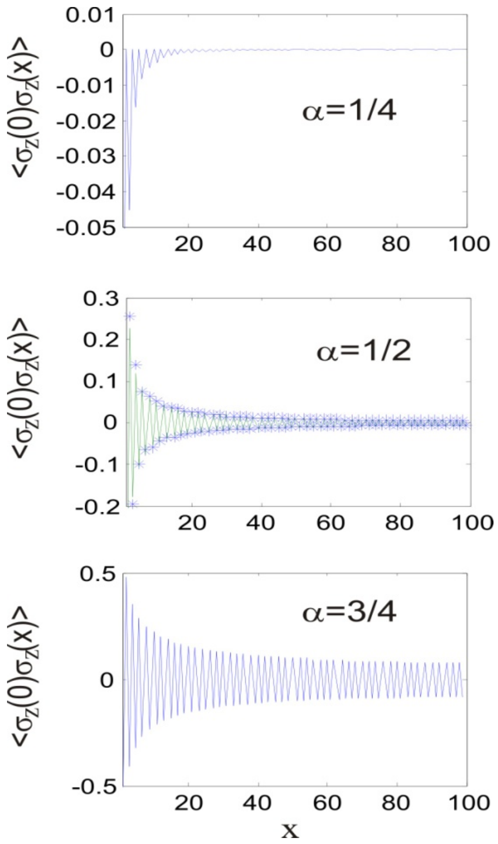
$$c_{s_1 .. s_N}(\alpha, z_1, \dots, z_N) = \langle V_{s_1}(\alpha, z_1) \dots V_{s_N}(\alpha, z_N) \rangle_{\text{vac}}$$

- The auxiliary particles correspond to a critical theory (CFT c=1).
- The spin theory, for some values of α, corresponds to a critical theory (c=1)
- We can use the „technology“ of CFT.

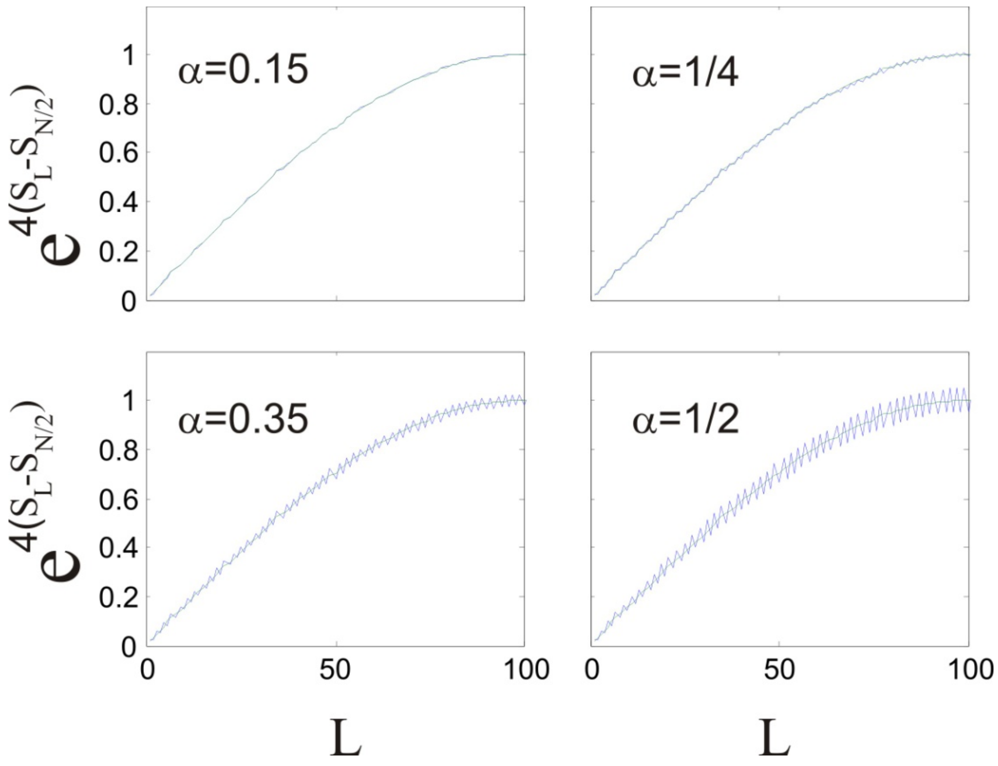
MATRIX PRODUCT STATES
CRITICALITY

- Translationally invariant: z equidistant in the unit circle.

Correlation functions



Area law (Renyi entropy)

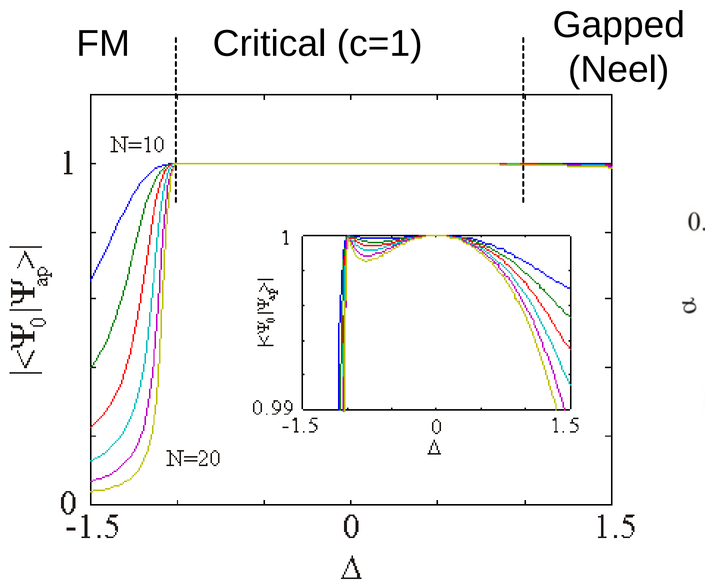


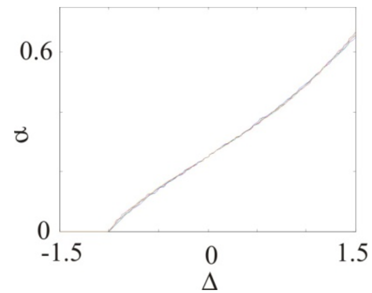
- $\alpha \in (0, \frac{1}{2}]$ critical with c=1
- Non-critical otherwise.

MATRIX PRODUCT STATES
VARIATIONAL CALCULATIONS

- Anisotropic Heisenberg Model: z equidistant in the unit circle.

$$H = \sum_{1=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

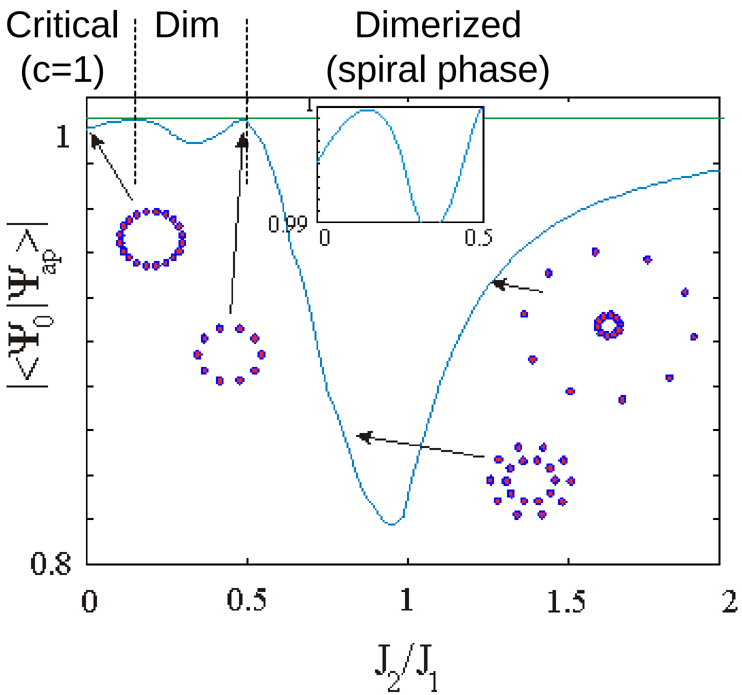




- Remarkable overlap with exact solution.
- For Δ=-1 and Δ=0 it is exact.

- Dimerized Heisenberg Model: z free, α=1/2

$$H=\sum_{1=1}^{N} J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2} \qquad (J_1=1)$$



- Remarkable overlap with exact solution.
- For J2=J1/2 it is exact (Majumdar-Gosh)

- **Su(2) invariance:** z free, α=1/2
  - The vertex operators are primary fields of the SU(2) WZW model with
    - spin ½
    - conformal weight h=1/4
    - Level k=1
    - With fusion rule: $\phi_{1/2} \times \phi_{1/2} = \phi_0$
  - The coefficients $c_{s_1..s_N}(\alpha = 1/2, z_1, \ldots, z_N) = \langle V_{s_1}(z_1) \ldots V_{s_N}(z_N) \rangle_{\mathrm{vac}}$ form a conformal block satisfying the Knizhnik-Zamolodchikov Equation:

$$\frac{k+2}{2} \frac{\partial}{\partial z_i} c(z_1, \ldots, z_N) = \sum_{j \neq i}^{N} \frac{\vec{S}_i \cdot \vec{S}_j}{z_i - z_j} c(z_1, \ldots, z_N)$$

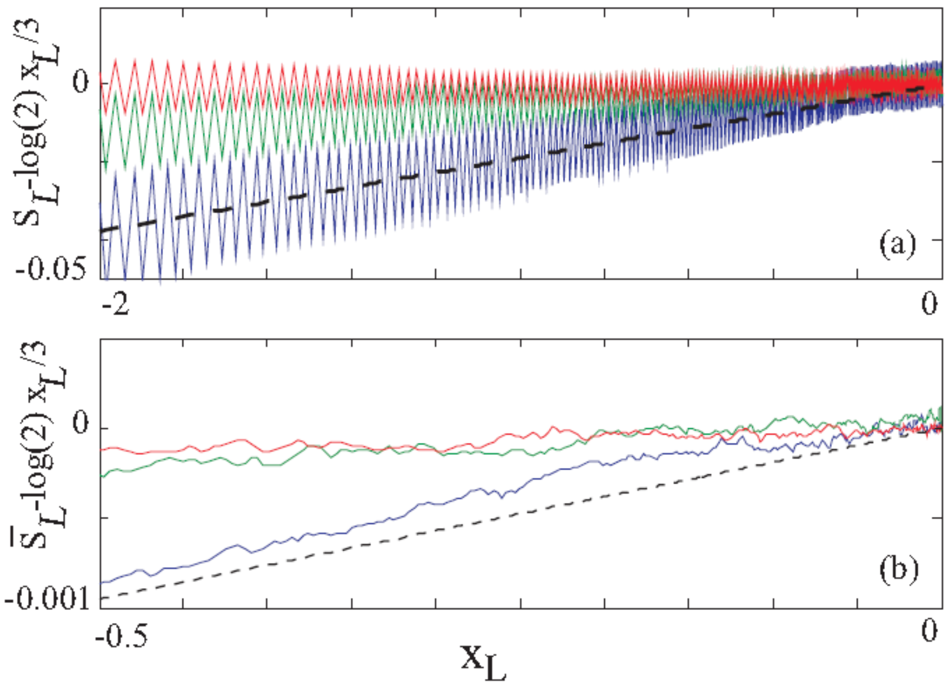
  - Using this equation it is easy to show that: $H | \Psi \rangle = E | \Psi \rangle$

$$H = -\sum_{n \neq m} \left(\frac{z_n z_m}{(z_n - z_m)^2} + \frac{1}{12} w_{n,m}(c_n - c_m) \right) \vec{S}_n \cdot \vec{S}_m$$

  - For the uniform case, we obtain the Haldane-Shastry Hamiltonian.

MATRIX PRODUCT STATES
EXACTLY SOLVABLE MODELS

- Su(2) invariance: z free, α=1/2
  - Taking random z's: (Monte Carlo, N=1000)



  - Scales like a critical theory (compare with Moore and Refael 05)

6. OTHER METHODS:
STRING BOND AND PLAQUETTES

OTHER METHODS

SPINS:

- MERA (Vidal 07)
- PEPS + MONTECARLO (Schuch, Wolf, Verstraete, IC, 08)
- TNS + MONTECARLO (Sandvik and Vidal, 08)
- EPS + MONTECARLO (Mezzacapo, Bonisegni, Schuch, IC, 09)
- iMPS + MONTECARLO (Sierra, IC, 09)

FERMIONS:

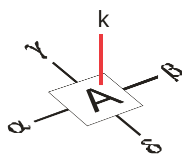
- fPEPS (Kraus, Schuch, Verstraete, IC, 09, Corboz, Orus, Bauer, Vidal, 09)
- fMERA (Corboz, Evently, Vidal, 09, Eisert et al 09)

STRING-BOND STATES

(Schuch, Wolf, Verstraete, IC, 2008)

- We first restrict ourselves to certain kind of PEPS:

$A^k_{\alpha\beta\gamma\delta}$ =

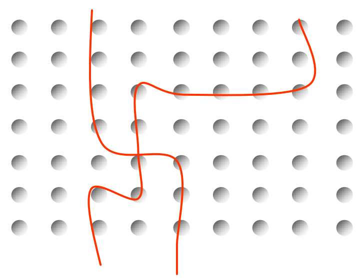


$A^i_{\alpha\beta\gamma\delta} = f_{\alpha\delta} g_{\beta\gamma}$ or $A^i_{\alpha\beta\gamma\delta} = f_{\alpha\beta} g_{\gamma\delta}$

- They can be efficiently contracted using MC $\approx N^2 d^2 D^3$
- This family can be easily extended.

$$|\Psi\rangle = \sum_{n_{11}\ldots n_{LL}} C_{\{n_{ij}\}} C_{\{n_{kl}\}} \ldots |n_{11},\ldots,n_{LL}\rangle$$

$\uparrow$ $\mathrm{tr}(A_n A_{n'} \ldots)$

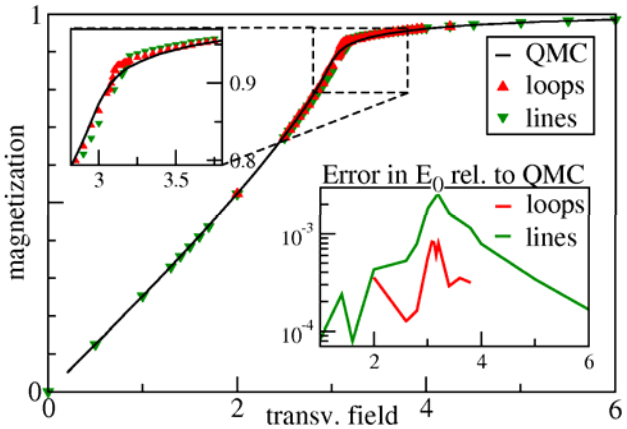


OTHER METHODS
STRING-BOND STATES

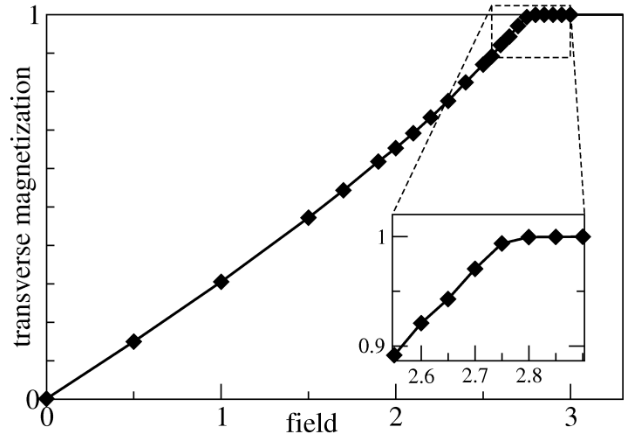
(Schuch, Verstraete, Wolf, and IC, 2008)
(Sfondrini, Cerrillo, Schuch, IC, submitted)

- 2D results (10x10 lattices):

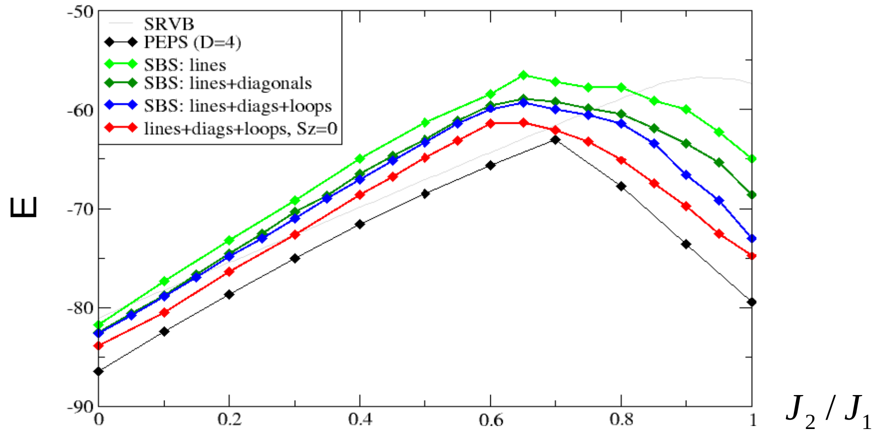
Ising model



XX frustrated model



$J_1 - J_2$ model

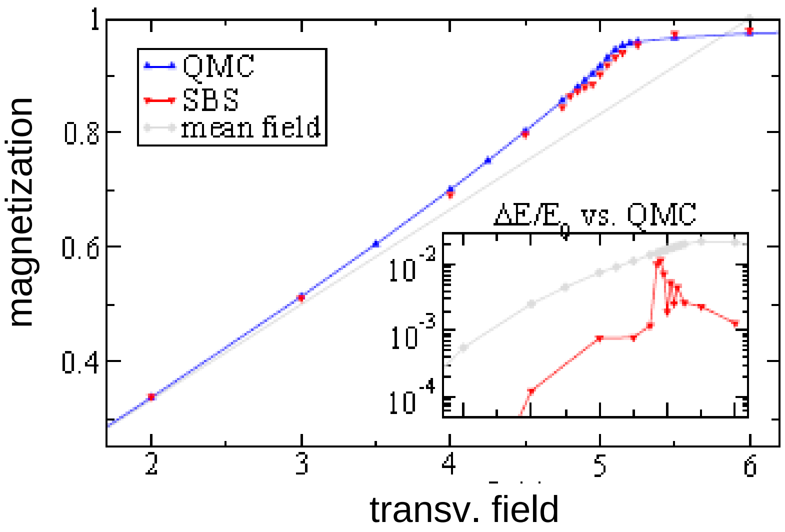


OTHER METHODS
STRING-BOND STATES

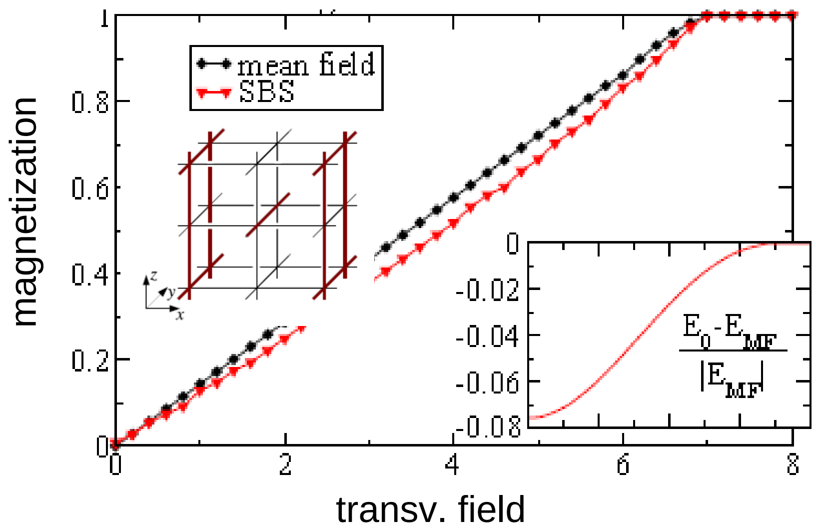
(Sfondrini, Cerrillo, Schuch, IC, submitted)

- 3D results:

Ising model (8x8x8)



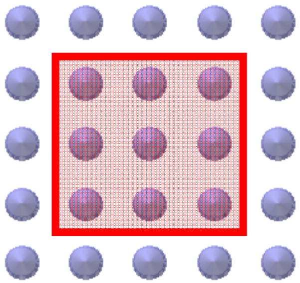
XX frustrated model (6x6x6)



ENTANGLED PLAQUETTE STATES
SPIN MODELS

(Mezzacapo, Schuch, Bonisegni, IC, 2009)

- We cover the spins with small overlapping plaquettes

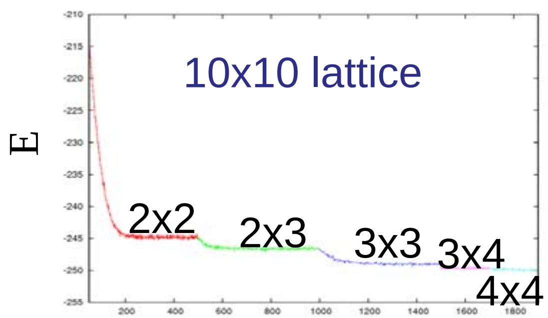


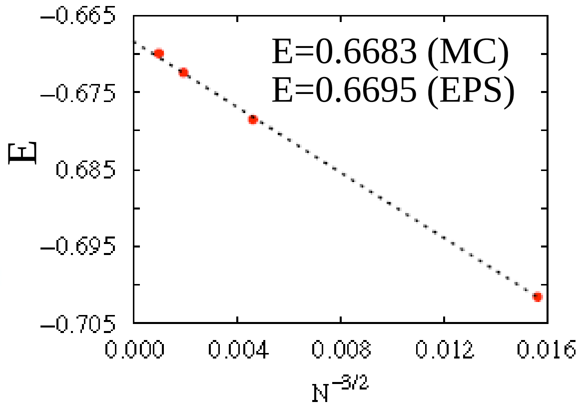
$$|\Psi\rangle = \sum_{n_{11}\ldots n_{LL}} C_{\{n_{ij}\}} C_{\{n_{kl}\}} \ldots |n_{11},\ldots,n_{LL}\rangle$$

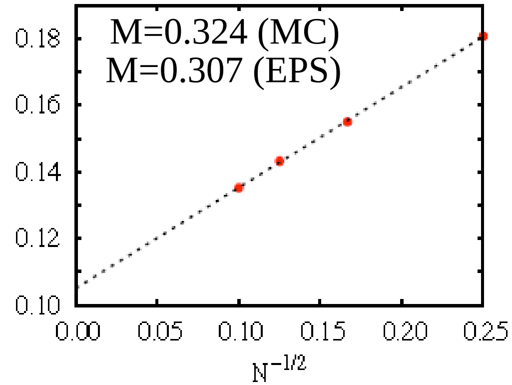
ENTANGLED PLAQUETTE STATES
SQUARE LATTICE

(Mezzacapo, Schuch, Bonisegni, IC, 2009)

- Heisenberg model:

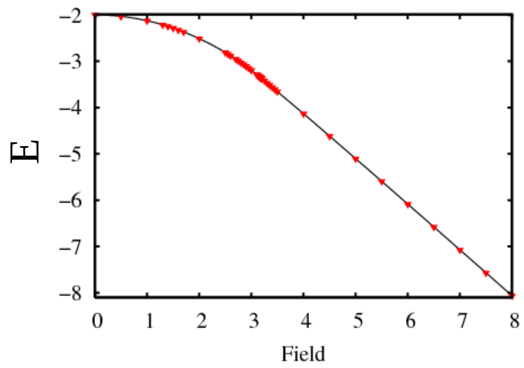




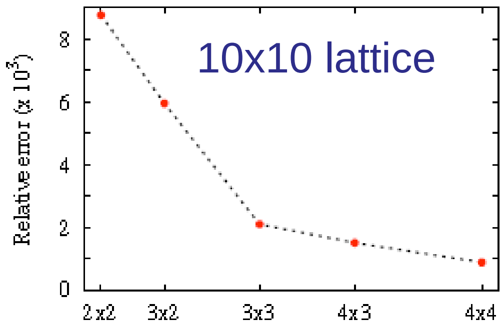


- Other models:

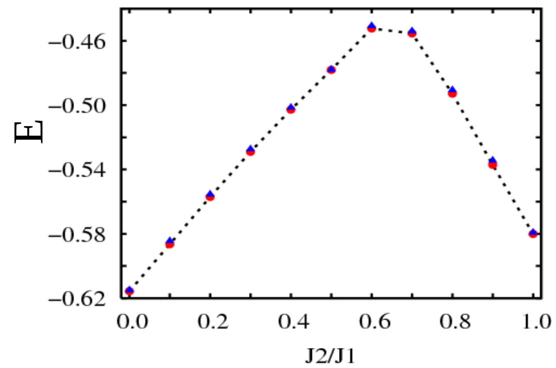
Ising model



XX model



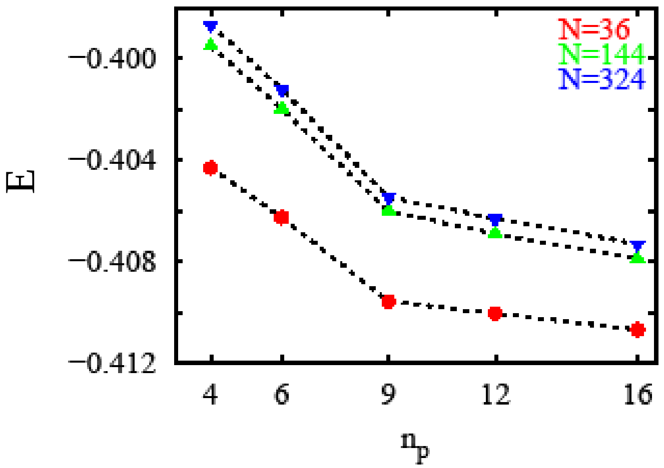
$J_1 - J_2$ model



ENTANGLED PLAQUETTE STATES
TRIANGULAR LATTICE (preliminary)

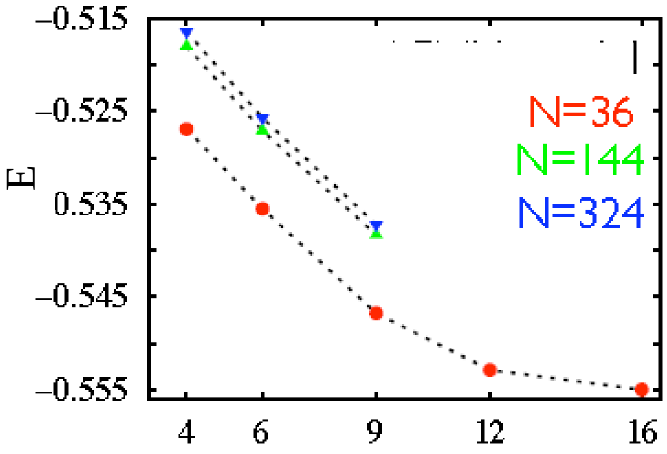
(Mezzacapo, et al, in preparation)

- AF XX model:



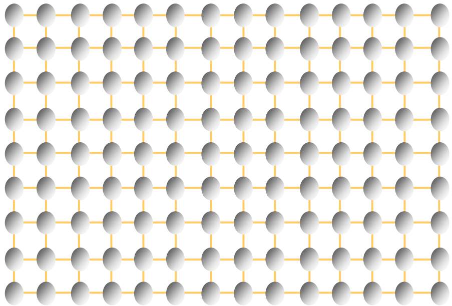
ΔE=0.5% (N=36)

- Heisenberg

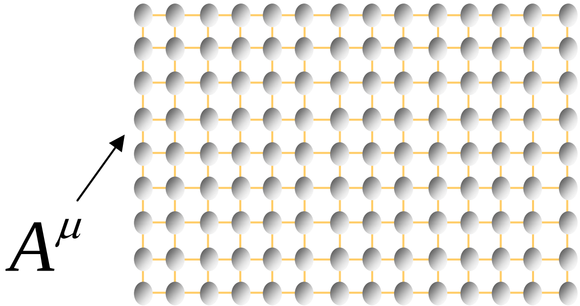


ΔE=1% (N=36)

OUTLOOK



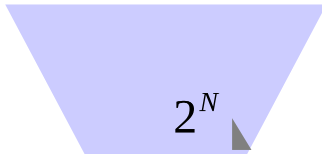
$$|\Psi\rangle = \sum c_{i_1,i_2,\ldots} |i_1, i_2, \ldots\rangle$$



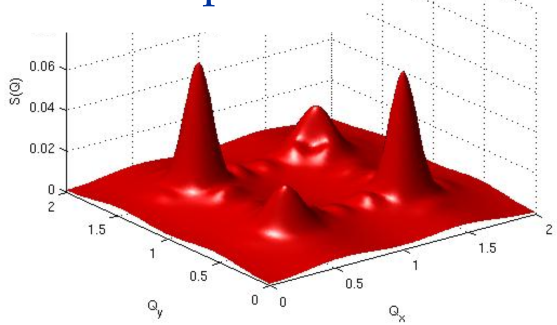
$$|\Psi(A^\mu)\rangle$$

- „Quantum Mechanics“ of PEPS:
  - Symmetries
  - Topological order
  - Excitations
  - Gaps
  - Criticality

- Efficient descripitons:
  - Proofs
  - New families
  - Different problems



- New computational methods:



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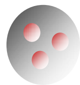
LATTICE PROBLEMS
PEPS

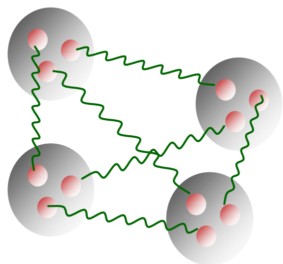
Verstraete and IC 04

Let us consider our particles as composite objects



Spin ½ neutron is composed of quarks.

- Auxiliary particles have different dimensions: D
- We only „see“ the macroscopic objects:
  We project onto a 2-dimensional subspace.
- The auxiliary particles are in a very simple state: maximally entangled.

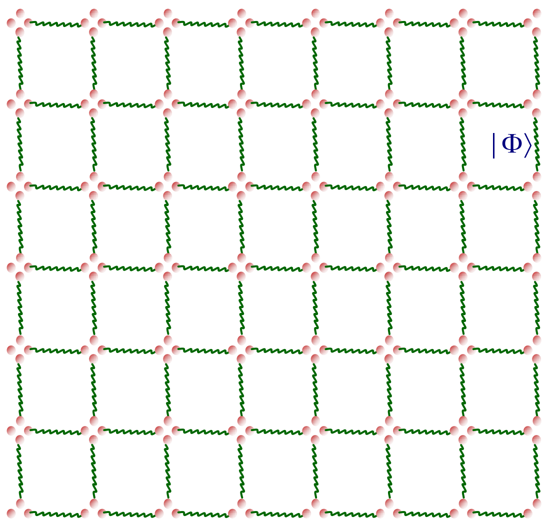


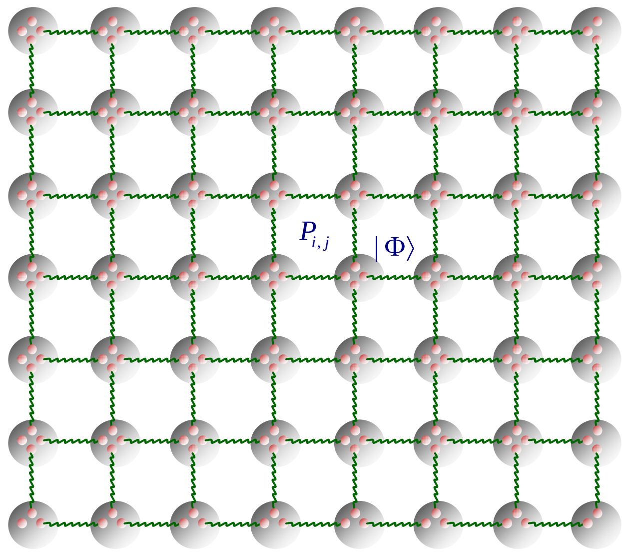
- The state is determined by the way we project onto the 2d space.
  The state is completely characterized by N projectors.

PEPS
DEFINITION

Verstraete and IC 04

Example: 2-dimensional system



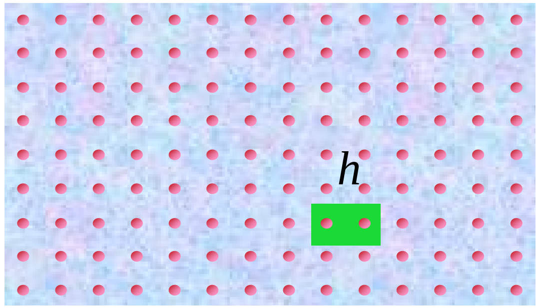


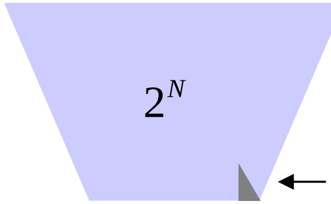
$$P = \sum A^{i}_{\alpha\beta\gamma\delta} \, |i\rangle\langle\alpha\beta\gamma\delta|$$

- All the information is contained in the P‘s
- There are $2D^4N$ parameters

QUANTUM MANY-BODY SYSTEMS

EFFICIENT DESCRIPTIONS: LATTICES





- Example: spin 1/2

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

- General 2-body interactions:

$$H = \sum_{n,m=1}^{N} \sum_{\alpha,\beta=0}^{3} \lambda_{\alpha,\beta}^{n,m} \sigma_\alpha^n \otimes \sigma_\beta^m$$

there are $16N^2$ coefficients

$$\Psi_0(\lambda_{n,m}^{\alpha,\beta})$$

- Short-range, homogeneous

$$H = \sum_{\|n-m\|<1} \sum_{\alpha,\beta=0}^{3} \lambda_{\alpha,\beta}^{n-m} \sigma_\alpha^n \otimes \sigma_\beta^m$$

there are $16R^d$ coefficients

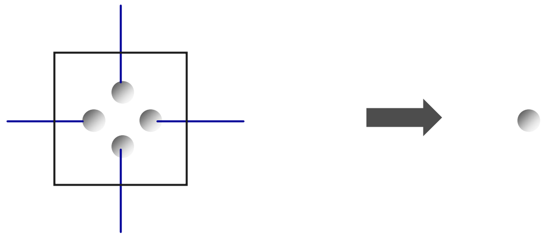
OTHER METHODS:
STRING-BOND STATES

OTHER METHODS
STRING-BOND STATES

(Schuch, Wolf, Verstraete, IC, 2008)

- We first restrict ourselves to certain kind of PEPS:

P:

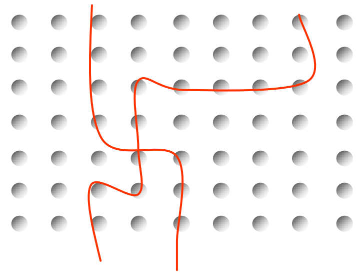


$$P = |0\rangle\langle\Psi_0| + |1\rangle\langle\Psi_1| \quad \text{with} \quad |\Psi_{0,1}\rangle = |\varphi_{0,1}\rangle|\varphi_{0,1}\rangle$$

- They can be efficiently contracted using MC $\approx N^2 d^2 D^3$
- This family can be easily extended.

$$|\Psi\rangle = \sum_{n_{11}\dots n_{LL}} C_{\{n_{ij}\}} C_{\{n_{kl}\}} \dots |n_{11},\dots,n_{LL}\rangle$$

↑ $\mathrm{tr}(A_n A_{n'}\dots)$

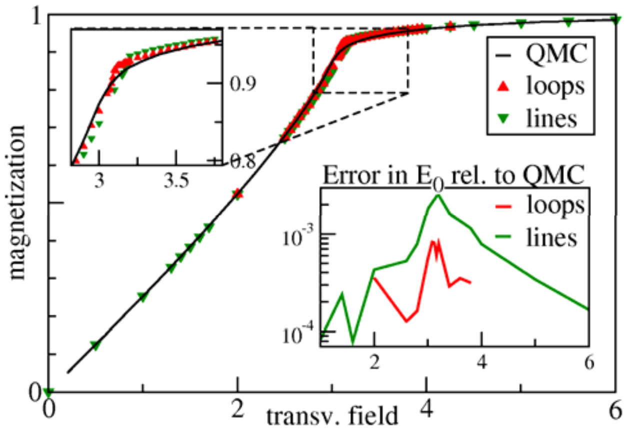


OTHER METHODS
STRING-BOND STATES

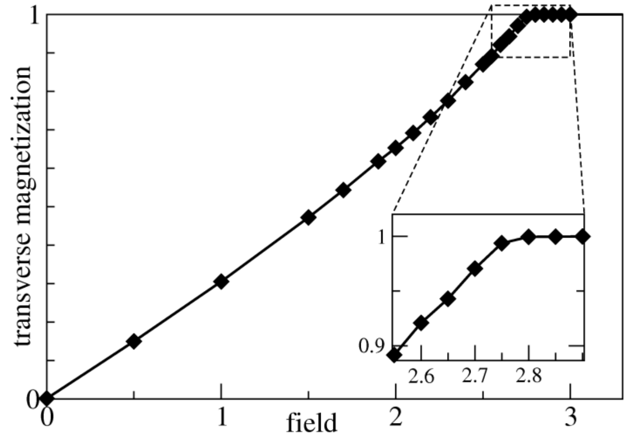
(Schuch, Verstraete, Wolf, and IC, 2008)
(Sfondrini, Cerrillo, Schuch, IC, submitted)

- 2D results (10x10 lattices):

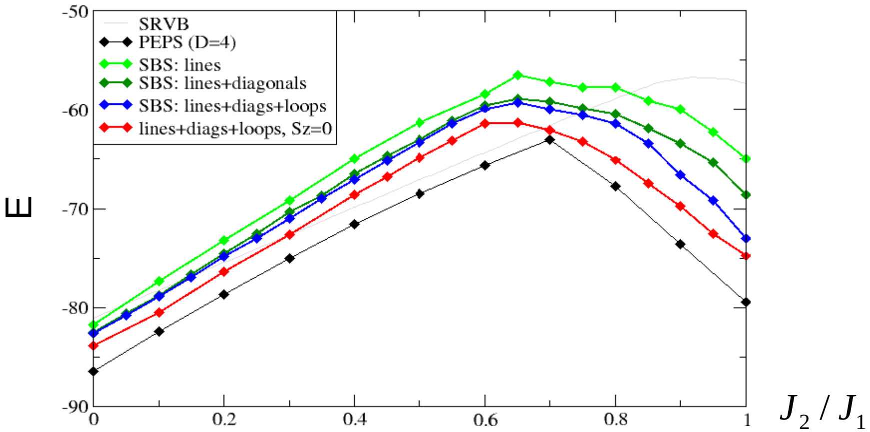
Ising model



XX frustrated model



$J_1 - J_2$ model

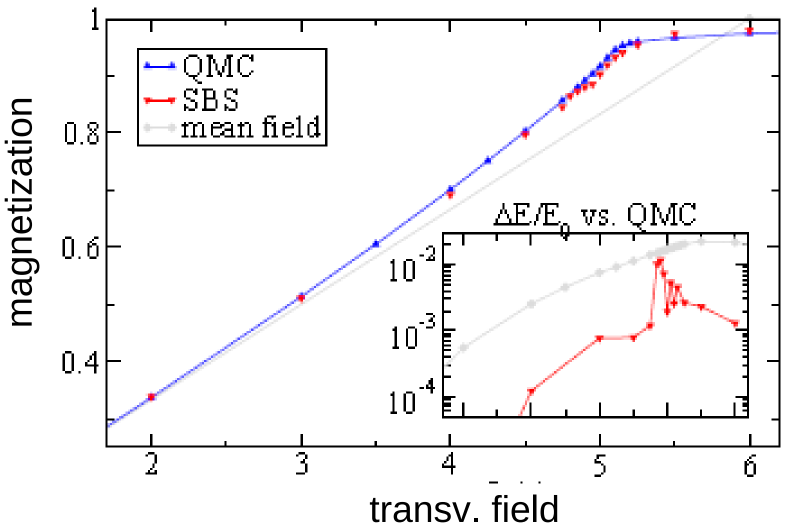


OTHER METHODS
STRING-BOND STATES

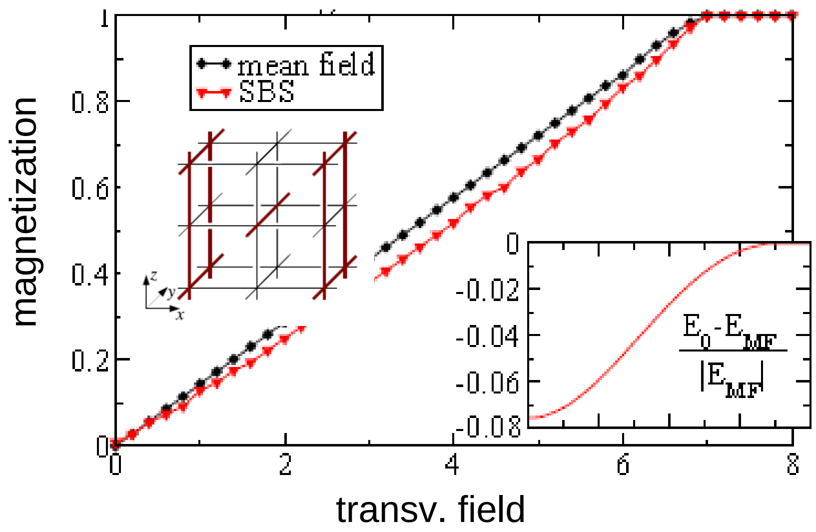
(Sfondrini, Cerrillo, Schuch, IC, submitted)

- 3D results:

Ising model (8x8x8)



XX frustrated model (6x6x6)

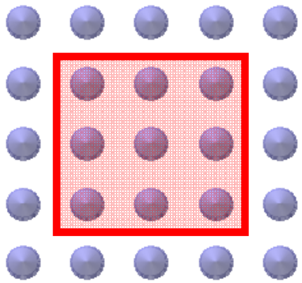


OTHER METHODS:
ENTANGLED PLAQUETTE STATES

ENTANGLED PLAQUETTE STATES
SPIN MODELS

(Mezzacapo, Schuch, Bonisegni, IC, 2009)

- We cover the spins with small overlapping plaquettes

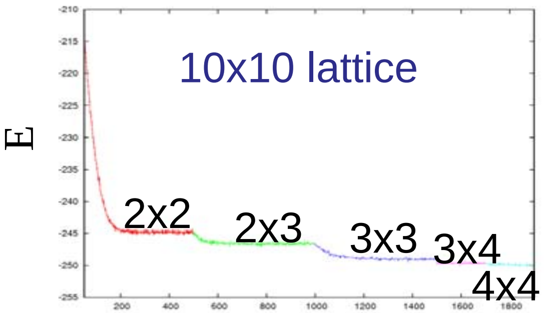


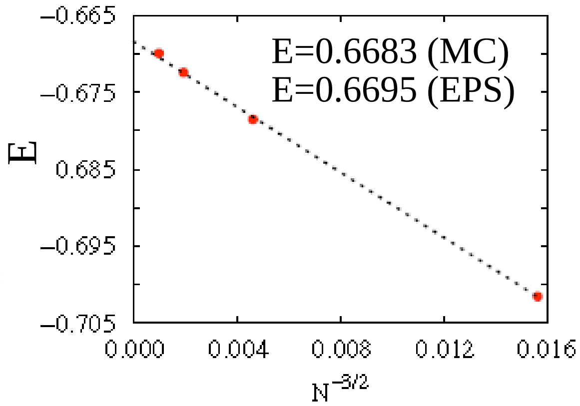
$$|\Psi\rangle = \sum_{n_{11}\ldots n_{LL}} C_{\{n_{ij}\}} C_{\{n_{kl}\}} \ldots |n_{11},\ldots,n_{LL}\rangle$$

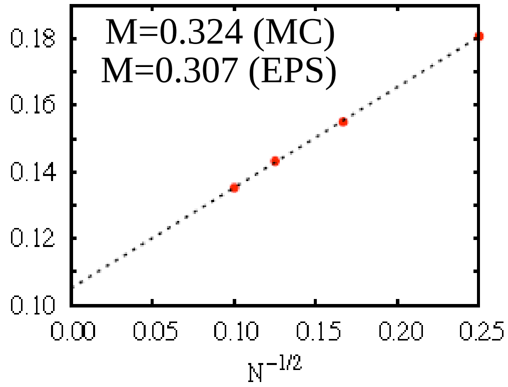
ENTANGLED PLAQUETTE STATES
SQUARE LATTICE

(Mezzacapo, Schuch, Bonisegni, IC, 2009)

- Heisenberg model:

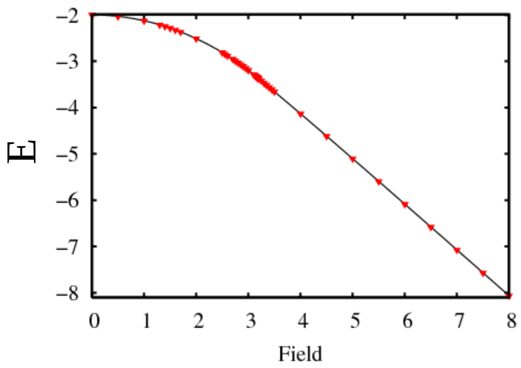




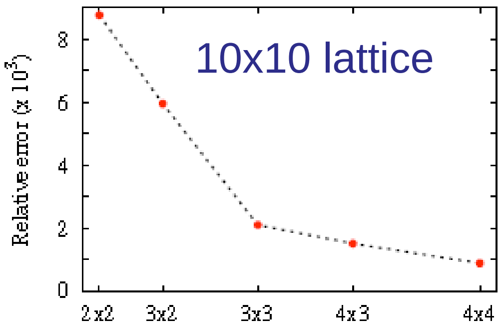


- Other models:

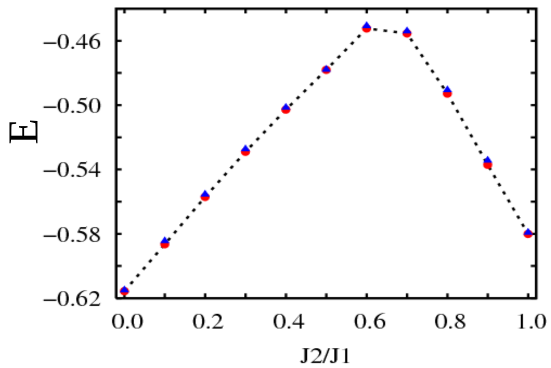
Ising model



XX model



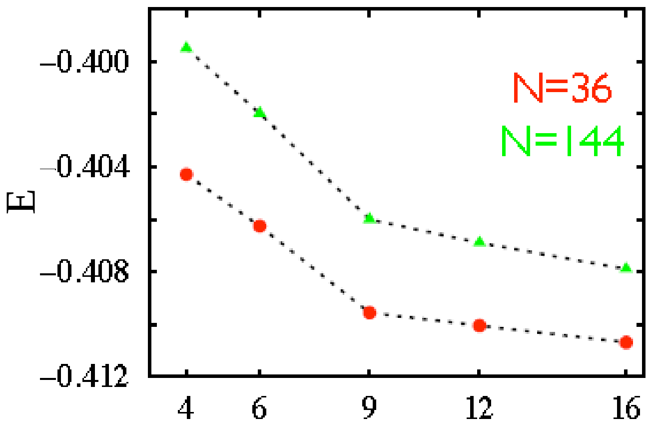
$J_1 - J_2$ model



ENTANGLED PLAQUETTE STATES
TRIANGULAR LATTICE (preliminary)

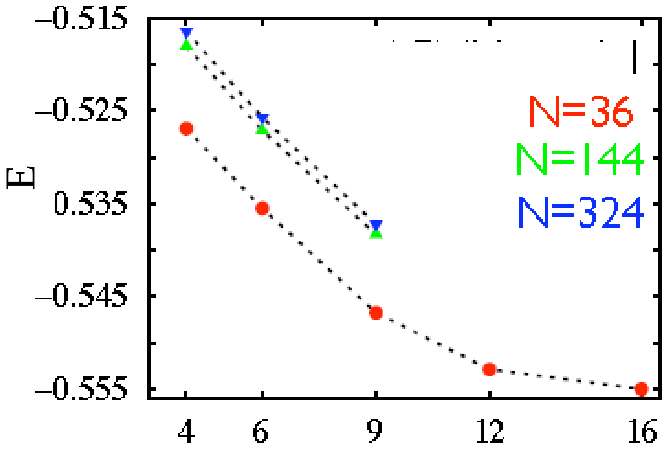
(Mezzacapo, et al, in preparation)

- AF XX model:



ΔE=0.5% (N=36)

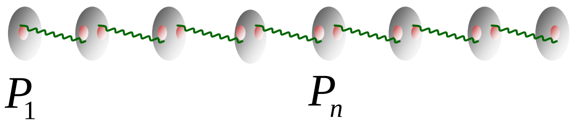
- Heisenberg



ΔE=1% (N=36)

MATRIX PRODUCT STATES
1 DIMENSION

- 1D systems:



$$P_n : C^D \otimes C^D \to C^2$$

- In a basis: $P_n = \sum_{x,y=1}^{D} \sum_{s=-1}^{1} [A_n]^{s}_{x,y} \, | s \rangle \langle x, y |$
- Matrix Product States:

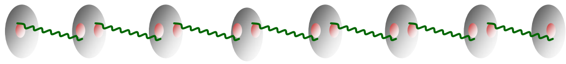
  (Fannes, Narchtengaele, Werner 08)

$$| \Psi \rangle = \sum_{s_1 \ldots s_N = \pm 1} \mathrm{tr}(A_1^{s_1} \ldots A_N^{s_N}) \, | s_1, \ldots, s_N \rangle$$

- For finite D, all correlation functions decay exponentially.
- In order to describe exactly a critical system, we have to take $D \to \infty$
- We have taken a QFT for the auxiliary particles.

MATRIX PRODUCT STATES
CFT

- 1D systems:



$$P_n : C^D \otimes C^D \to C^2 \qquad P_n = \sum_{x,y=1}^{D} \sum_{s=-1}^{1} [A_n]^s_{x,y} \, | s \rangle \langle x, y |$$

$$V_s(\alpha, z_n) =: e^{i\sqrt{\alpha} s \phi(z_n)} :$$

vertex operator

$\phi(z)$ chiral free boson field

z: complex number

- State: $|\Psi\rangle = \sum_{s_1 \ldots s_N = \pm 1} c_{s_1 .. s_N}(\alpha, z_1, \ldots, z_N) \, | s_1 \ldots s_N \rangle$

Variational parameters

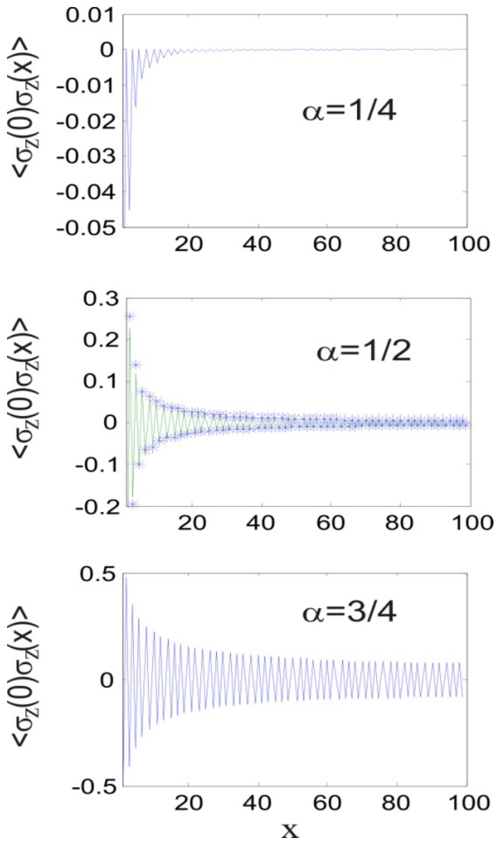
$$c_{s_1 .. s_N}(\alpha, z_1, \ldots, z_N) = \langle V_{s_1}(\alpha, z_1) \ldots V_{s_N}(\alpha, z_N) \rangle_{\text{vac}}$$

- The auxiliary particles correspond to a critical theory (CFT c=1).
- The spin theory, for some values of α, corresponds to a critical theory (c=1)
- We can use the „technology“ of CFT.

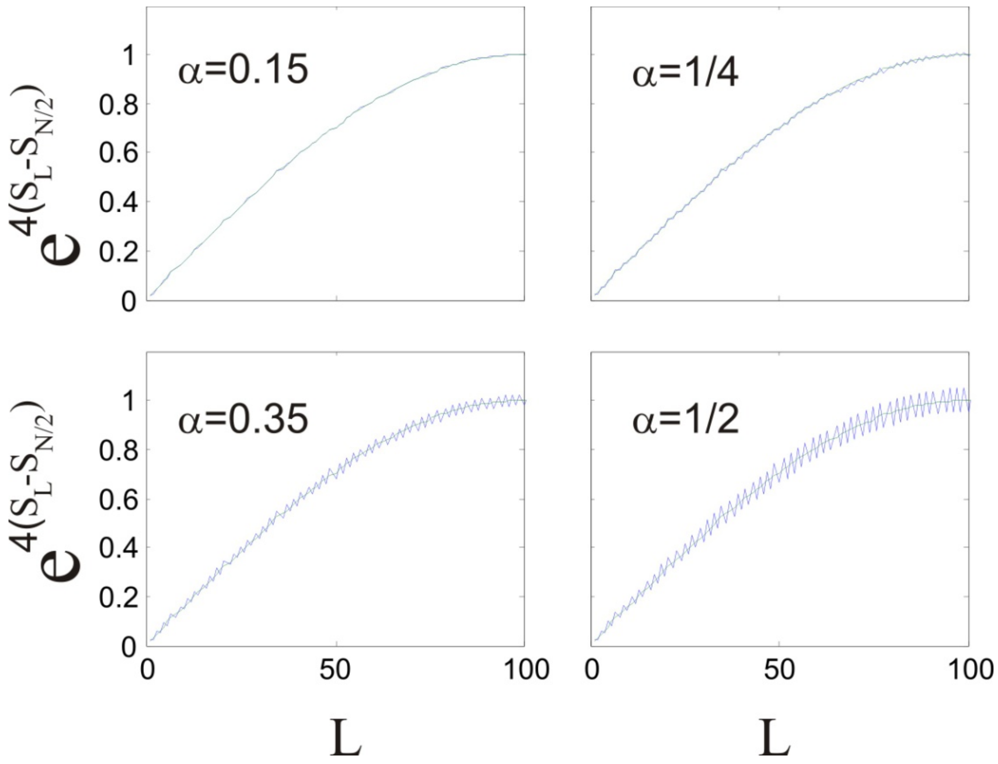
MATRIX PRODUCT STATES
CRITICALITY

- Translationally invariant: z equidistant in the unit circle.

Correlation functions



Area law (Renyi entropy)

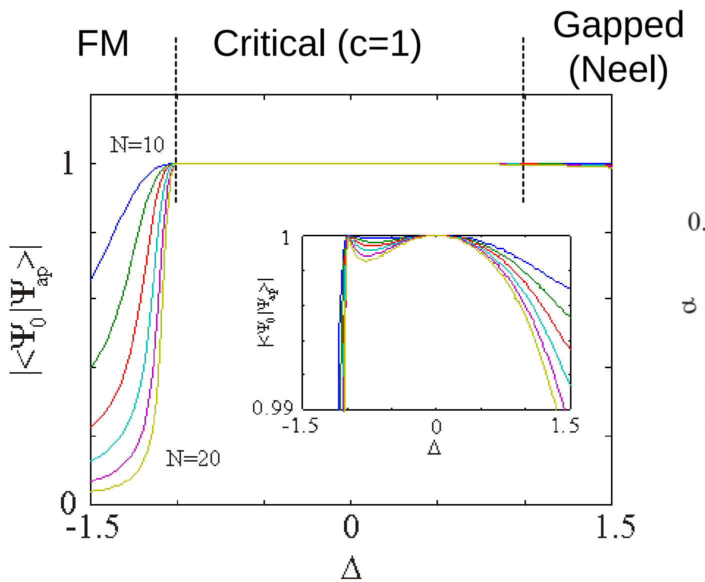


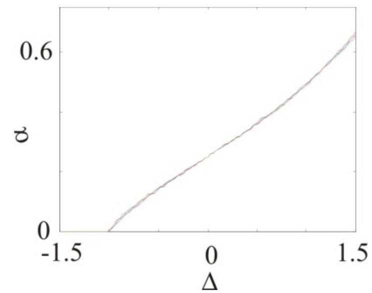
- $\alpha \in (0, \frac{1}{2}]$ critical with c=1
- Non-critical otherwise.

MATRIX PRODUCT STATES
VARIATIONAL CALCULATIONS

- Anisotropic Heisenberg Model: z equidistant in the unit circle.

$$H = \sum_{1=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$



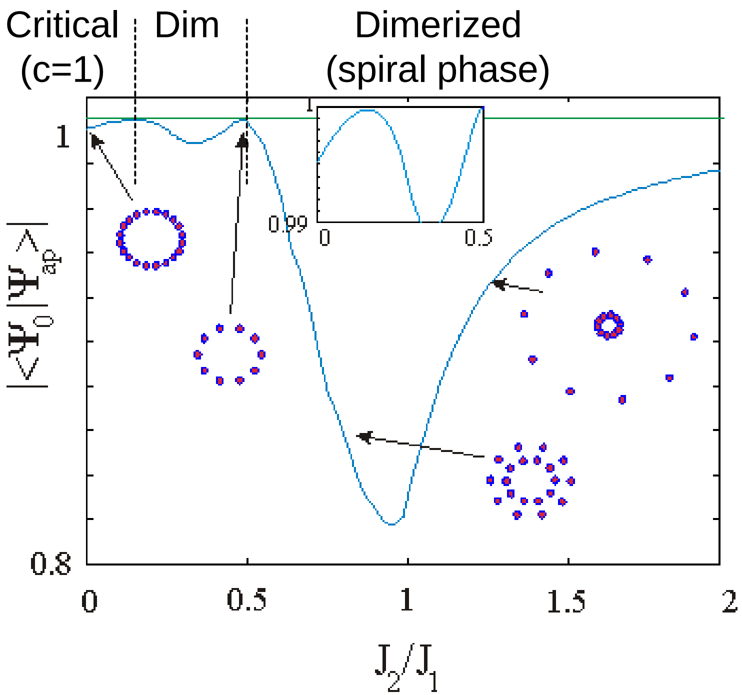


- Remarkable overlap with exact solution.
- For Δ=-1 and Δ=0 it is exact.

MATRIX PRODUCT STATES
VARIATIONAL CALCULATIONS

- Dimerized Heisenberg Model: z free, α=1/2

$$H=\sum_{1=1}^{N} J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2} \qquad (J_1=1)$$



- Remarkable overlap with exact solution.
- For J2=J1/2 it is exact (Majumdar-Gosh)

MATRIX PRODUCT STATES
EXACTLY SOLVABLE MODELS

- Su(2) invariance: z free, α=1/2
  - The vertex operators are primary fields of the SU(2) WZW model with
    - spin ½
    - conformal weight h=1/4
    - Level k=1
    - With fusion rule: $\phi_{1/2} \times \phi_{1/2} = \phi_0$
  - The coefficients $c_{s_1..s_N}(\alpha = 1/2, z_1, ..., z_N) = \langle V_{s_1}(z_1)...V_{s_N}(z_N)\rangle_{\text{vac}}$ form a conformal block satisfying the Knizhnik-Zamolodchikov Equation:

$$\frac{k+2}{2}\frac{\partial}{\partial z_i}c(z_1,\dots,z_N) = \sum_{j\neq i}^{N}\frac{\vec{S}_i \cdot \vec{S}_j}{z_i - z_j}c(z_1,\dots,z_N)$$

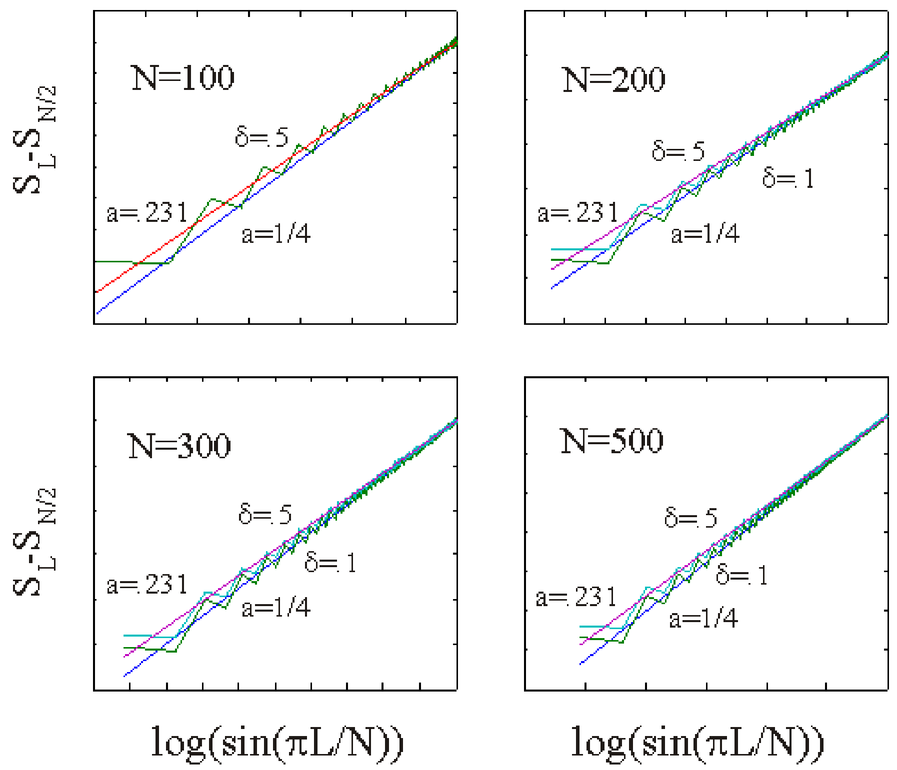
  - Using this equation it is easy to show that: $H|\Psi\rangle = E|\Psi\rangle$

$$H = -\sum_{n\neq m}\left(\frac{z_n z_m}{(z_n - z_m)^2} + \frac{1}{12}w_{n,m}(c_n - c_m)\right)\vec{S}_n \cdot \vec{S}_m$$

  - For the uniform case, we obtain the Haldane-Shastry Hamiltonian.

MATRIX PRODUCT STATES
EXACTLY SOLVABLE MODELS

- Su(2) invariance: z free, α=1/2
  - Taking random z's:



  - Scales like a critical theory (compare with Moore and Refael 05)

7. Numerical methods