Quantum Dynamics of the Duffing Model for Qubit Readout

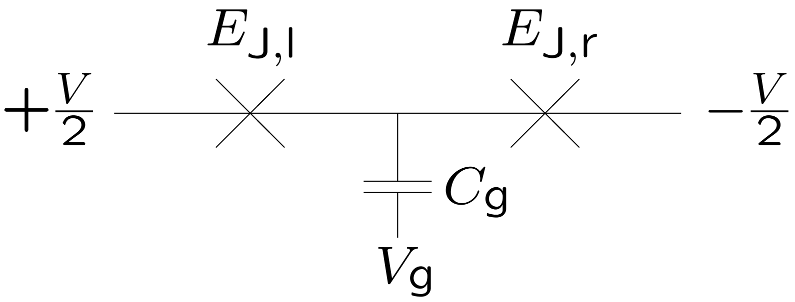
Alec Maassen van den Brink, Shu-Hao Yeh, and Dong-Bang Tsai
RCAS, Academia Sinica, Taipei, Taiwan

- Introduction: Numerical propagator method for driven systems
- Application to Duffing model of qubit readout
- Results
- Spectral analysis
- Conclusions

Quantum Technologies, TungHai Univ., 11 Sep 2009

Numerical propagator method for driven systems

- Example:



- Transport of Cooper pair through entire device only changes state of voltage source behind the scenes $\Rightarrow$
- Symmetry which does not conserve energy, but changes it by $\pm 2eV$
- Numerical diagonalization of H produces corresponding multiplets $E_k^{(n)} = E_k + 2eVn$: huge redundancy is drain on resources
- Equivalent: eliminate one d.o.f. at price of driving term with period $T \equiv 2\pi\hbar/2eV$. Makes connection to wide class of systems with AC-driving or more generally time-periodic Hamiltonian:

$$i\hbar\dot{\Psi}(t) = H(t)\Psi(t)\ , \qquad H(t+T) = H(t)$$

- Almost universal: Floquet method—Fourier expansion of $H(t)$, $\Psi(t)$. Determination of *quasi-energies* ϵ_k in $\Psi_k(t) = u_k(t)\,\exp(-i\epsilon_k t/\hbar)$, $u(t{+}T) = u(t)$ becomes eigenproblem in *extended Hilbert space*
- Fourier index is as extra d.o.f. Merely reversed previous elimination, so still problems with non-uniqueness and resource use
- Finally cracked with numerical propagator method: integrate *matrix* Schrödinger eqn

$$i\hbar\,\partial_t U(t) = H(t)U(t)\;, \qquad U(0) = \mathbf{1}$$

  and diagonalize $U(T)$
- All energies in one multiplet get mapped to *single* phase eigenvalue $\alpha_k = \exp(-i\epsilon_k^{(n)}T/\hbar)$ of *unitary* operator!
- Fully tested and confirmed on 1-qubit NMR problem (Rabi oscillations), where one can compare with analytic soln. N.B.: need to integrate only over short *driving* period, and still find oscillations on much longer *Rabi* period!

Duffing model

- Apply these ideas to *Duffing model*: oscillator with weak nonlinearity, damping, and near-resonant driving,

$$\ddot{x} + \omega_0^2 x = -\epsilon[\gamma\dot{x} + \alpha x^3 + f\cos(\omega t)]$$

  Classical bistability makes it interesting qubit detector
- Already relevant in experiments: superconducting implementation is called *Josephson bifurcation amplifier*. Since both final states are superconducting, one can e.g. try to perform repeated quantum non-demolition (QND) measurements on a qubit system [A. Lupașcu *et al.*, *Nature Physics* **3**, 119 (2007)]
- In quantum domain, bistability should be imperfect due to *tunneling* between the two limit cycles (limiting detector performance)
- Instead of Hamiltonian acting on wave function: *Liouville superoperator* acting on density matrix (quantum master equation)

$$
\begin{aligned}
i\hbar\dot{\rho} &= [H_S(t),\rho] + i\hbar\frac{\epsilon\gamma}{2}(\bar{n}{+}1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \\
&\quad + i\hbar\frac{\epsilon\gamma}{2}\bar{n}(2a^\dagger\rho a - aa^\dagger\rho - \rho a a^\dagger) \\
&\equiv \mathcal{L}(t)\{\rho\}\ ,
\end{aligned}
$$

$$
H_S(t) = \hbar\omega_0 a^\dagger a + \epsilon m\left[\frac{\alpha}{4}x^4 + f x \cos(\omega t)\right]\ ,
$$

$$
x = \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger)\ , \qquad \bar{n} = \frac{1}{e^{\hbar\omega_0/k_B T} - 1}
$$

- For analogy with undamped time-periodic systems, can now study evolution superoperator $\mathcal{S}(t{=}2\pi/\omega)$
- But Markov treatment of damping suspect on *short time scale* ω^{-1}; only need dynamics on *long scale* $(\epsilon\gamma)^{-1}$

- Extract slow dynamics through *rotating-wave approximation* (RWA),

$$\tilde{\rho}(t) \equiv U(t)\rho(t)U^\dagger(t) \,, \qquad U(t) = e^{-i\omega N t} \,, \qquad N = a^\dagger a$$

$$i\frac{d\tilde{\rho}}{d\tau} = [\tilde{H}_S, \tilde{\rho}] + \frac{i}{2}(\bar{n}+1)(2a\tilde{\rho}a^\dagger - a^\dagger a\tilde{\rho} - \tilde{\rho}a^\dagger a) + \frac{i}{2}\bar{n}(2a^\dagger\tilde{\rho}a - aa^\dagger\tilde{\rho} - \tilde{\rho}aa^\dagger)$$
$$\equiv \tilde{\mathcal{L}}\{\tilde{\rho}\} \,,$$

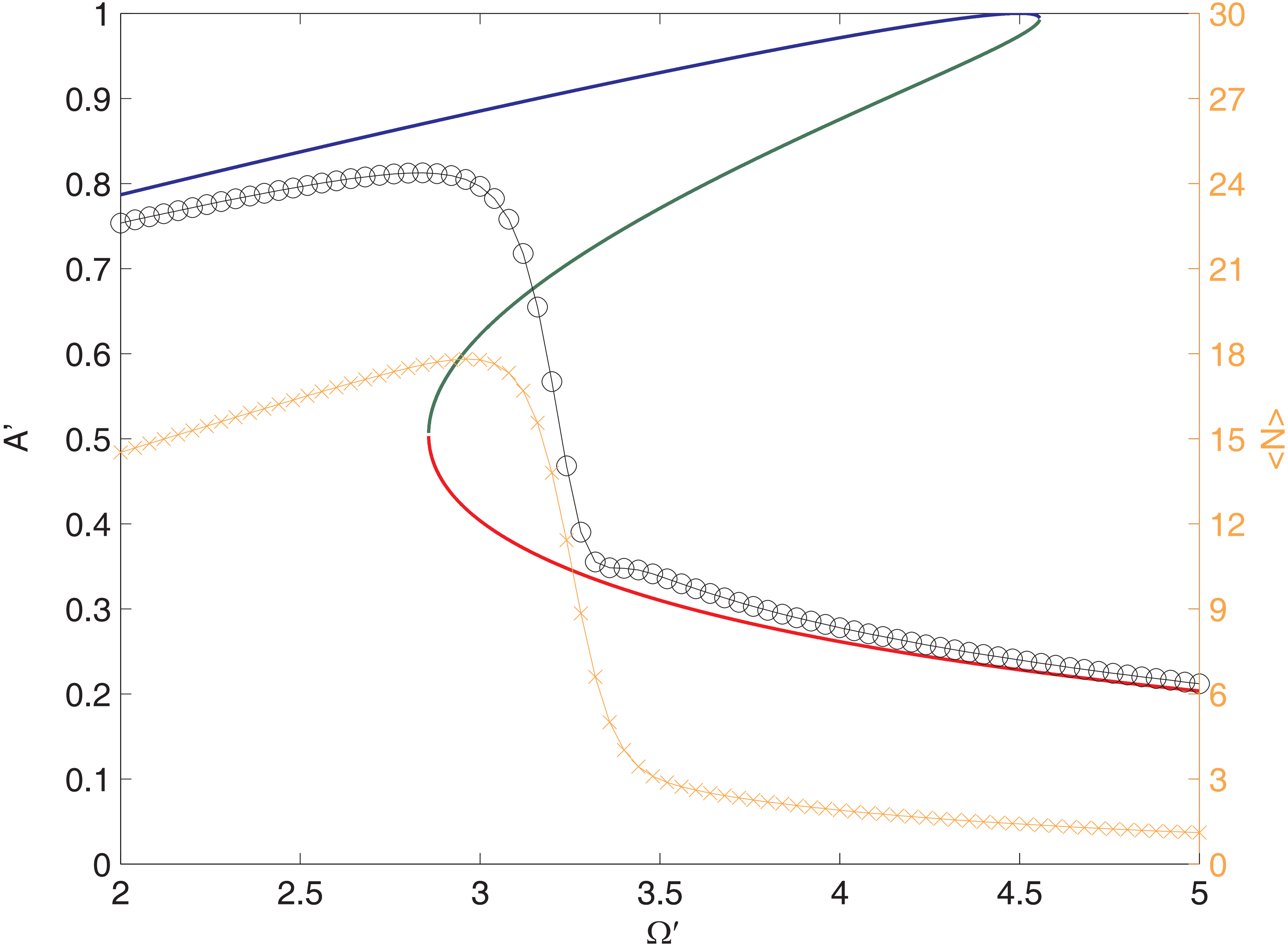
$$\tilde{H}_S = -\frac{\Omega'}{2}N + \frac{f'}{2\sqrt{2}}(a + a^\dagger) + \frac{3\alpha'}{8f'^2}(N^2 + N)$$

- Rescaled variables:

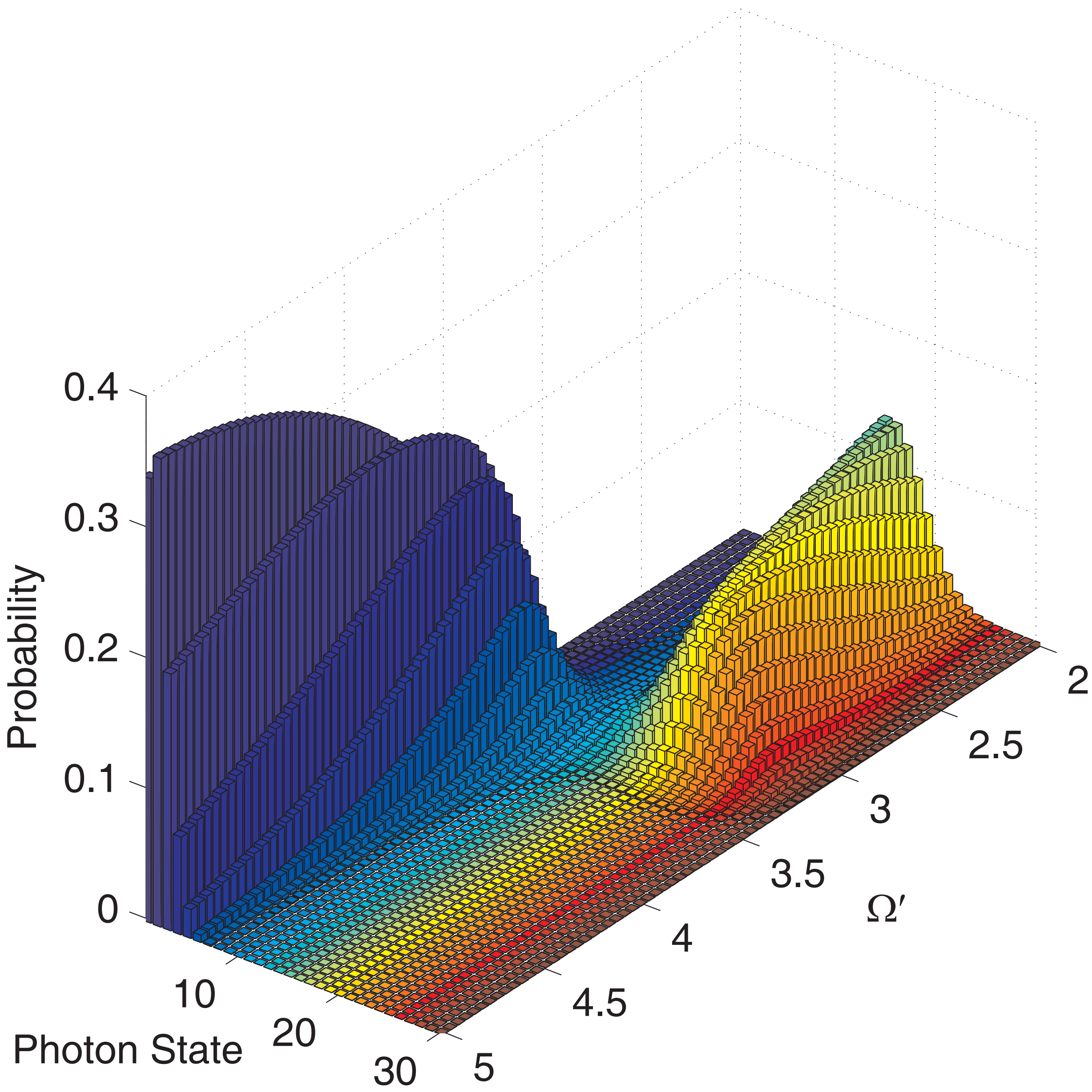
$$\Omega' = \frac{\Omega}{\omega_0\gamma} \,, \qquad \Omega = (\omega^2 - \omega_0^2)/\epsilon \,, \qquad \alpha' = \frac{\alpha f^2}{\omega_0^3\gamma^3} \,,$$

$$\tau = \epsilon\gamma t \,, \qquad f' = \frac{f}{\gamma}\sqrt{\frac{m}{\hbar\omega_0}}$$

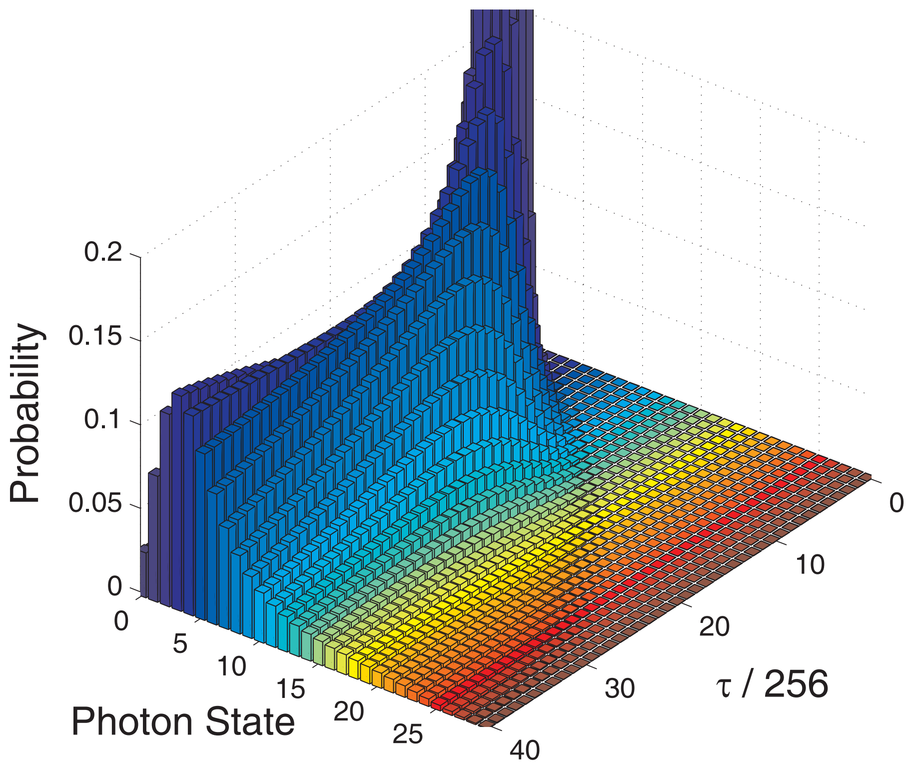
- Parallels canonical transform to “Van der Pol coordinates” in classical case; f' only “quantum” parameter
- Fast scale $\sim \omega^{-1}$ eliminated from problem

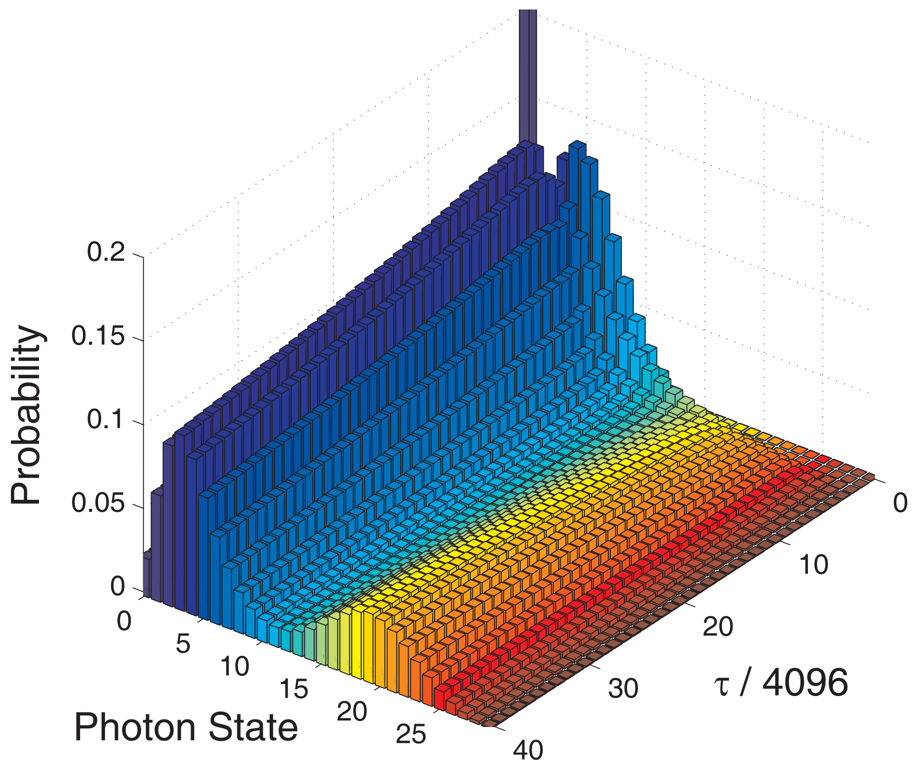


A'
Ω′
<N>
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
2.5
3
3.5
4
4.5
5
0
3
6
9
12
15
18
21
24
27
30



Probability
0
0.1
0.2
0.3
0.4
Photon State
10
20
30
Ω′
2
2.5
3
3.5
4
4.5
5


Probability
0.2
0.15
0.1
0.05
0
Photon State
0
5
10
15
20
25
τ / 256
0
10
20
30
40


Probability
0.2
0.15
0.1
0.05
0
Photon State
0
5
10
15
20
25
τ / 4096
0
10
20
30
40

Spectral analysis

- In coarse-grained RWA approach, revisit idea from time-periodic case: diagonalize $\tilde{\mathcal{L}}$!
- Unique eigenvalue $\tilde{\lambda}_1 = 0$, with hermitian, normalizable eigen-$\tilde{\rho}_1$: stationary state
- Unique 2nd-smallest $\tilde{\lambda}_2$, with hermitian, traceless $\tilde{\rho}_2$
- $\tilde{\rho}_2$ has same population peaks as $\tilde{\rho}_1$, but with *opposite signs* causing equilibration
- $\mathrm{Re}\,\tilde{\lambda}_2 = 0 \Rightarrow$ *incoherent* tunneling
- $\mathrm{Im}\,\tilde{\lambda}_2 < 0 \Rightarrow$ stability
- $|\tilde{\lambda}_2| \ll |\mathrm{Im}\,\tilde{\lambda}_{k\geq 3}| \Rightarrow$ separation of time scales: $\tilde{\rho}_{1,2}$ suffice for late times, many $\tilde{\rho}$'s needed for initial state

Conclusions

- For same parameters as the classical problem, bistability disappears in quantum case due to *tunneling* . . .
- . . . opening up *third*, ultra-long, time scale $(\epsilon\gamma|\tilde{\lambda}_2|)^{-1}$ with no classical counterpart
- Error process for qubit readout: if final state independent of initial conditions, no detection took place
- Thus, to observe counterpart to classical bistability, must go beyond stationary averages—either *full distributions* or *dynamic evolution*
- Spectral approach cleanly isolates tunneling from intermediate-time dynamics; enables study of classical limit $f' \uparrow$
- Try to make analytical sense, e.g. in coherent-state representation