Quotient Algebras of $\mathfrak{su}(N)$ and Their Applications

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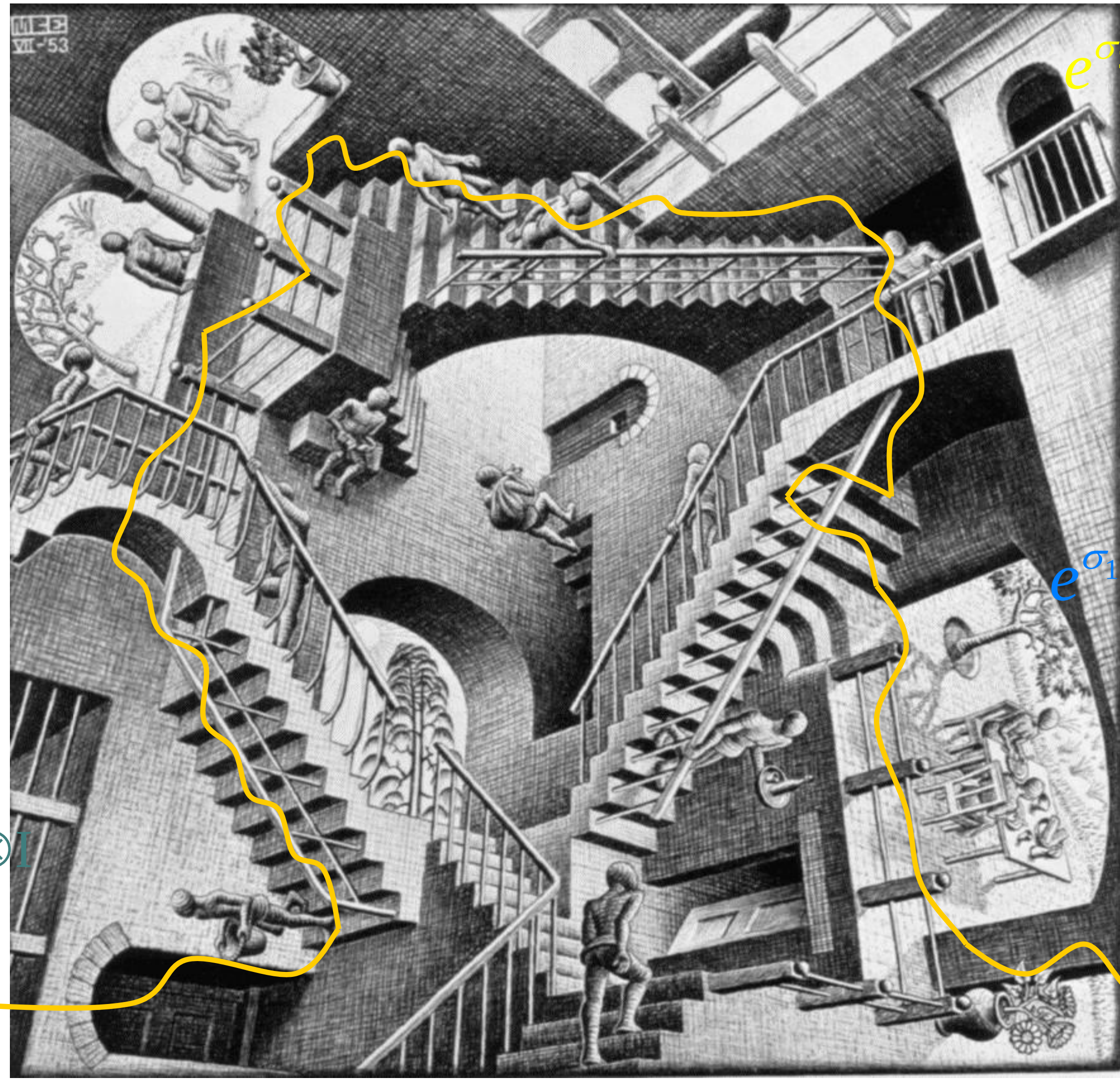
Decomposition of Unitary Transformations



$$e^{\sigma_3 \otimes \sigma_2 \otimes \sigma_1}$$

$$e^{I \otimes I \otimes I \otimes \sigma_2}$$

$$e^{\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_1}$$

$$e^{\sigma_1 \otimes I \otimes \sigma_1 \otimes \sigma_1}$$

$$e^{\sigma_1 \otimes \sigma_3 \otimes I}$$

$$e^{\sigma_1 \otimes \sigma_2 \otimes \sigma_3}$$

$$e^{\sigma_3 \otimes I \otimes \sigma_3 \otimes \sigma_2}$$

$$U$$

$$e^{I \otimes \sigma_3 \otimes \sigma_2}$$

$$e^{\sigma_2 \otimes I \otimes \sigma_1 \otimes I}$$

$$e^{\sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3}$$

$$e^{\sigma_2 \otimes \sigma_2 \otimes \sigma_3}$$

$$e^{\sigma_2 \otimes I \otimes I}$$

$$e^{I \otimes I \otimes \sigma_3 \otimes I}$$

$$e^{I \otimes \sigma_1 \otimes I}$$

$$e^{\sigma_1 \otimes I \otimes \sigma_3}$$

Schmidt Decomposition

$$|\psi\rangle = \sum_{i,j} c_{ij} \, |a_i\rangle \otimes |b_j\rangle$$

$$= \sum_{i} |a_i\rangle \otimes |\tilde{b}_i\rangle$$

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,k} |a_i\rangle \otimes |\tilde{b}_i\rangle\langle a_k| \otimes \langle\tilde{b}_k|$$

$$= \sum_{i,k} |a_i\rangle\langle a_k| \otimes |\tilde{b}_i\rangle\langle\tilde{b}_k|$$

$$\rho^A = \mathrm{Tr}_B \, \rho = \sum_{i,k,l} |a_i\rangle\langle a_k| \otimes \langle b_l|\tilde{b}_i\rangle\langle\tilde{b}_k|b_l\rangle$$

$$= \sum_{r} \lambda_r \, |a_r\rangle\langle a_r| \,, \text{ with } \sqrt{\lambda}_i \, |b_i\rangle = |\tilde{b}_i\rangle$$

$$|\psi\rangle = \sum_{l} \sqrt{\lambda}_l \, |a_l\rangle \otimes |b_l\rangle$$

Getting a *normal* representation for a pure state through partially tracing the corresponding density matrix

Singular Value Decomposition

Given a matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$, find the decomposition $A = K_1{}^t D K_2$,

$$A^t A = (K_2{}^t D K_1)(K_1{}^t D K_2) = K_2{}^t D^2 K_2 \qquad AA^t = \left(K_1{}^t D K_2\right)\left(K_2{}^t D K_1\right) = K_1{}^t D^2 K_1$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

the eigenvalues and eigenvectors of $A^t A$ and AA^t,

$$\lambda_1 = 1 \qquad \lambda_2 = 3$$

$$A^t A \qquad x_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \end{pmatrix}^t \qquad x_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}^t$$

$$AA^t \qquad y_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^t \qquad y_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}^t$$

$$K_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix} \qquad K_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Cartan Decomposition

For a semisimple Lie algebra $\mathfrak{g}$, there exists a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

s.t., $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $\mathrm{Tr}\{\mathfrak{k}\ \mathfrak{p}\} = 0$

where $\mathfrak{k}$ is a subalgebra and $\mathfrak{p}$ is a vector subspace of $\mathfrak{g}$.

There exists a maximal abelian subalgebra $\mathcal{A} \subset \mathfrak{p}$, such that the factorization is realizable, $g \in \mathfrak{g}$, $t_1, t_2 \in \mathfrak{k}$, and $a \in \mathcal{A}$,

$$e^{ig} = e^{it_1}\ e^{ia}\ e^{it_2} \quad (KAK \text{ Theorem})$$

e.g., $\mathfrak{su}(2) = \mathfrak{k} \oplus \mathfrak{p}$, take $\mathfrak{k} = span\{\sigma_1\}$, $\mathfrak{p} = span\{\sigma_2, \sigma_3\}$, $\mathcal{A} = span\{\sigma_3\}$,

for a $U \in \mathrm{SU}(2)$,

$$U = e^{ic_1\sigma_1} e^{ic_2\sigma_3} e^{ic_3\sigma_1}$$

A Conjugate Partition and Quotient Algebra of $\mathfrak{su}(8)$

$\mathcal{C}$

$\sigma_3 \otimes I \otimes I$

$I \otimes \sigma_3 \otimes I$

$I \otimes I \otimes \sigma_3$

$\sigma_3 \otimes \sigma_3 \otimes I$

$\sigma_3 \otimes I \otimes \sigma_3$

$I \otimes \sigma_3 \otimes \sigma_3$

$\sigma_3 \otimes \sigma_3 \otimes \sigma_3$

W_1	$I \otimes I \otimes \sigma_1$	$\sigma_3 \otimes I \otimes \sigma_1$	$I \otimes I \otimes \sigma_2$	$\sigma_3 \otimes I \otimes \sigma_2$	$\widehat{W}_1$
	$I \otimes \sigma_3 \otimes \sigma_1$	$\sigma_3 \otimes \sigma_3 \otimes \sigma_1$	$I \otimes \sigma_3 \otimes \sigma_2$	$\sigma_3 \otimes \sigma_3 \otimes \sigma_2$	
W_2	$I \otimes \sigma_1 \otimes I$	$\sigma_3 \otimes \sigma_1 \otimes I$	$I \otimes \sigma_2 \otimes I$	$\sigma_3 \otimes \sigma_2 \otimes I$	$\widehat{W}_2$
	$I \otimes \sigma_1 \otimes \sigma_3$	$\sigma_3 \otimes \sigma_1 \otimes \sigma_3$	$I \otimes \sigma_2 \otimes \sigma_3$	$\sigma_3 \otimes \sigma_2 \otimes \sigma_3$	
W_3	$I \otimes \sigma_1 \otimes \sigma_1$	$I \otimes \sigma_2 \otimes \sigma_2$	$I \otimes \sigma_1 \otimes \sigma_2$	$I \otimes \sigma_2 \otimes \sigma_1$	$\widehat{W}_3$
	$\sigma_3 \otimes \sigma_1 \otimes \sigma_1$	$\sigma_3 \otimes \sigma_2 \otimes \sigma_2$	$\sigma_3 \otimes \sigma_1 \otimes \sigma_2$	$\sigma_3 \otimes \sigma_2 \otimes \sigma_1$	
W_4	$\sigma_1 \otimes I \otimes I$	$\sigma_1 \otimes \sigma_3 \otimes I$	$\sigma_2 \otimes I \otimes I$	$\sigma_2 \otimes \sigma_3 \otimes I$	$\widehat{W}_4$
	$\sigma_1 \otimes I \otimes \sigma_3$	$\sigma_1 \otimes \sigma_3 \otimes \sigma_3$	$\sigma_2 \otimes I \otimes \sigma_3$	$\sigma_2 \otimes \sigma_3 \otimes \sigma_3$	
W_5	$\sigma_1 \otimes I \otimes \sigma_1$	$\sigma_2 \otimes I \otimes \sigma_2$	$\sigma_1 \otimes I \otimes \sigma_2$	$\sigma_2 \otimes I \otimes \sigma_1$	$\widehat{W}_5$
	$\sigma_1 \otimes \sigma_3 \otimes \sigma_1$	$\sigma_2 \otimes \sigma_3 \otimes \sigma_2$	$\sigma_1 \otimes \sigma_3 \otimes \sigma_2$	$\sigma_2 \otimes \sigma_3 \otimes \sigma_1$	
W_6	$\sigma_1 \otimes \sigma_1 \otimes I$	$\sigma_2 \otimes \sigma_2 \otimes I$	$\sigma_2 \otimes \sigma_1 \otimes I$	$\sigma_1 \otimes \sigma_2 \otimes I$	$\widehat{W}_6$
	$\sigma_1 \otimes \sigma_1 \otimes \sigma_3$	$\sigma_2 \otimes \sigma_2 \otimes \sigma_3$	$\sigma_2 \otimes \sigma_1 \otimes \sigma_3$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_3$	
W_7	$\sigma_1 \otimes \sigma_1 \otimes \sigma_1$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_2$	$\sigma_2 \otimes \sigma_2 \otimes \sigma_2$	$\sigma_1 \otimes \sigma_1 \otimes \sigma_2$	$\widehat{W}_7$
	$\sigma_2 \otimes \sigma_1 \otimes \sigma_2$	$\sigma_2 \otimes \sigma_2 \otimes \sigma_1$	$\sigma_2 \otimes \sigma_1 \otimes \sigma_1$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_1$	

A Conjugate Partition and Quotient Algebra of $\mathfrak{su}(6)$

			$\mathcal{C}$			
W_1	$I\otimes\sigma_1$ $\mu_3\otimes\sigma_1$ $\mu_8\otimes\sigma_1$		$\mu_3\otimes I$	$I\otimes\sigma_2$ $\mu_3\otimes\sigma_2$ $\mu_8\otimes\sigma_2$		$\hat{W}_1$
			$\mu_8\otimes I$			
W_2	$\mu_1\otimes I$	$\mu_1\otimes\sigma_3$	$I\otimes\sigma_3$	$\mu_2\otimes I$	$\mu_2\otimes\sigma_3$	$\hat{W}_2$
			$\mu_3\otimes\sigma_3$			
W_3	$\mu_1\otimes\sigma_1$	$\mu_2\otimes\sigma_2$	$\mu_8\otimes\sigma_3$	$\mu_1\otimes\sigma_2$	$\mu_2\otimes\sigma_1$	$\hat{W}_3$
W_4	$\mu_4\otimes I$	$\mu_4\otimes\sigma_3$		$\mu_5\otimes I$	$\mu_5\otimes\sigma_3$	$\hat{W}_4$
W_5	$\mu_4\otimes\sigma_1$	$\mu_5\otimes\sigma_2$		$\mu_4\otimes\sigma_2$	$\mu_5\otimes\sigma_1$	$\hat{W}_5$
W_6	$\mu_6\otimes I$	$\mu_6\otimes\sigma_3$		$\mu_7\otimes I$	$\mu_7\otimes\sigma_3$	$\hat{W}_6$
W_7	$\mu_6\otimes\sigma_1$	$\mu_7\otimes\sigma_2$		$\mu_6\otimes\sigma_2$	$\mu_7\otimes\sigma_1$	$\hat{W}_7$

Conjugate Partition and Quotient Algebra

Definition With an abelian subspace $\mathcal{A}$ taken as the **center subalgebra**, the Lie algebra $\mathfrak{su}(N)$ admits the **conjugate partition** consisting of $\mathcal{A}$ and a finite number q of abelian **conjugate pairs** $\{W_i, \widehat{W}_i\}$, $1 \le i \le q$, namely,

$$su(N) = \mathcal{A} \oplus W_1 \oplus \widehat{W}_1 \oplus \cdots \oplus W_i \oplus \widehat{W}_i \oplus \cdots \oplus W_q \oplus \widehat{W}_q,$$

where the abelian subspaces W_i and $\widehat{W}_i$ hold the commutation relations, $\forall 1 \le i \le q,$

$$[W_i, \mathcal{A}] \subset \widehat{W}_i, \quad [\widehat{W}_i, \mathcal{A}] \subset W_i \text{ and } [W_i, \widehat{W}_i] \subset \mathcal{A};$$

furthermore, the subspaces in a conjugate partition form a **quotient algebra**, denoted as a multiplet of partition

$$\{\mathcal{Q}(\mathcal{A};q)\} \equiv \{\mathcal{A}; W_i, \widehat{W}_i, 1 \le i \le q\},$$

if the **condition of closure** is satisfied, $\forall\ 1 \le i, j \le q$ and $\exists\ 1 \le k \le q$,

$$\left[W_i, W_j\right] \subset \widehat{W}_k, \left[W_i, \widehat{W}_j\right] \subset W_k, \left[\widehat{W}_i, \widehat{W}_j\right] \subset \widehat{W}_k.$$

A Conjugate Partition and Qutient Algebra of $\mathfrak{su}(8)$ with Non-diagonal Center

			$\mathcal{A}$			
W_1	$I\otimes I\otimes\sigma_3$	$\sigma_1\otimes I\otimes\sigma_3$	$\sigma_1\otimes I\otimes I$	$I\otimes I\otimes\sigma_2$	$\sigma_1\otimes I\otimes\sigma_2$	$\widehat{W}_1$
	$I\otimes\sigma_1\otimes\sigma_3$	$\sigma_1\otimes\sigma_1\otimes\sigma_3$	$I\otimes\sigma_1\otimes I$	$I\otimes\sigma_1\otimes\sigma_2$	$\sigma_1\otimes\sigma_1\otimes\sigma_2$	
			$I\otimes I\otimes\sigma_1$			
W_2	$I\otimes\sigma_3\otimes I$	$\sigma_1\otimes\sigma_3\otimes I$	$\sigma_1\otimes\sigma_1\otimes I$	$I\otimes\sigma_2\otimes I$	$\sigma_1\otimes\sigma_2\otimes I$	$\widehat{W}_2$
	$I\otimes\sigma_3\otimes\sigma_1$	$\sigma_1\otimes\sigma_3\otimes\sigma_1$		$I\otimes\sigma_2\otimes\sigma_1$	$\sigma_1\otimes\sigma_2\otimes\sigma_1$	
			$\sigma_1\otimes I\otimes\sigma_1$			
W_3	$I\otimes\sigma_3\otimes\sigma_3$	$I\otimes\sigma_2\otimes\sigma_2$	$I\otimes\sigma_1\otimes\sigma_1$	$I\otimes\sigma_3\otimes\sigma_2$	$I\otimes\sigma_2\otimes\sigma_3$	$\widehat{W}_3$
	$\sigma_1\otimes\sigma_3\otimes\sigma_3$	$\sigma_1\otimes\sigma_2\otimes\sigma_2$	$\sigma_1\otimes\sigma_1\otimes\sigma_1$	$\sigma_1\otimes\sigma_3\otimes\sigma_2$	$\sigma_1\otimes\sigma_2\otimes\sigma_3$	
W_4	$\sigma_3\otimes I\otimes I$	$\sigma_3\otimes\sigma_1\otimes I$		$\sigma_2\otimes I\otimes I$	$\sigma_2\otimes\sigma_1\otimes I$	$\widehat{W}_4$
	$\sigma_3\otimes I\otimes\sigma_1$	$\sigma_3\otimes\sigma_1\otimes\sigma_1$		$\sigma_2\otimes I\otimes\sigma_1$	$\sigma_2\otimes\sigma_1\otimes\sigma_1$	
W_5	$\sigma_3\otimes I\otimes\sigma_3$	$\sigma_2\otimes I\otimes\sigma_2$		$\sigma_3\otimes I\otimes\sigma_2$	$\sigma_2\otimes I\otimes\sigma_3$	$\widehat{W}_5$
	$\sigma_3\otimes\sigma_1\otimes\sigma_3$	$\sigma_2\otimes\sigma_1\otimes\sigma_2$		$\sigma_3\otimes\sigma_1\otimes\sigma_2$	$\sigma_2\otimes\sigma_1\otimes\sigma_3$	
W_6	$\sigma_3\otimes\sigma_3\otimes I$	$\sigma_2\otimes\sigma_2\otimes I$		$\sigma_2\otimes\sigma_3\otimes I$	$\sigma_3\otimes\sigma_2\otimes I$	$\widehat{W}_6$
	$\sigma_3\otimes\sigma_3\otimes\sigma_1$	$\sigma_2\otimes\sigma_2\otimes\sigma_1$		$\sigma_2\otimes\sigma_3\otimes\sigma_1$	$\sigma_3\otimes\sigma_2\otimes\sigma_1$	
W_7	$\sigma_3\otimes\sigma_3\otimes\sigma_3$	$\sigma_3\otimes\sigma_2\otimes\sigma_2$		$\sigma_2\otimes\sigma_2\otimes\sigma_2$	$\sigma_3\otimes\sigma_3\otimes\sigma_2$	$\widehat{W}_7$
	$\sigma_2\otimes\sigma_3\otimes\sigma_2$	$\sigma_2\otimes\sigma_2\otimes\sigma_3$		$\sigma_2\otimes\sigma_3\otimes\sigma_3$	$\sigma_3\otimes\sigma_2\otimes\sigma_3$	

A Conjugate Partition and Qutient Algebra of $\mathfrak{su}(6)$ with Non-diagonal Center

				$\mathcal{C}$				
W_1	$I \otimes \sigma_3$	$\mu_1 \otimes \sigma_3$	$\mu_8 \otimes \sigma_3$	$\mu_1 \otimes I$	$I \otimes \sigma_2$	$\mu_1 \otimes \sigma_2$	$\mu_8 \otimes \sigma_2$	$\widehat{W}_1$
				$\mu_8 \otimes I$				
W_2	$\mu_3 \otimes I$	$\mu_3 \otimes \sigma_1$		$I \otimes \sigma_1$	$\mu_2 \otimes I$	$\mu_2 \otimes \sigma_1$		$\widehat{W}_2$
				$\mu_1 \otimes \sigma_1$				
W_3	$\mu_3 \otimes \sigma_3$	$\mu_2 \otimes \sigma_2$		$\mu_8 \otimes \sigma_1$	$\mu_3 \otimes \sigma_2$	$\mu_2 \otimes \sigma_3$		$\widehat{W}_3$
W_4	$\mu_4 \otimes I$	$\mu_4 \otimes \sigma_1$			$\mu_5 \otimes I$	$\mu_5 \otimes \sigma_1$		$\widehat{W}_4$
W_5	$\mu_4 \otimes \sigma_3$	$\mu_5 \otimes \sigma_2$			$\mu_4 \otimes \sigma_2$	$\mu_5 \otimes \sigma_3$		$\widehat{W}_5$
W_6	$\mu_6 \otimes I$	$\mu_6 \otimes \sigma_1$			$\mu_7 \otimes I$	$\mu_7 \otimes \sigma_1$		$\widehat{W}_6$
W_7	$\mu_6 \otimes \sigma_3$	$\mu_7 \otimes \sigma_2$			$\mu_6 \otimes \sigma_2$	$\mu_7 \otimes \sigma_3$		$\widehat{W}_7$

Abelian Subalgebras Extension

W_1	$I \otimes \sigma_1$	$\sigma_3 \otimes \sigma_1$	$\sigma_3 \otimes I$	$I \otimes \sigma_2$	$\sigma_3 \otimes \sigma_2$	$\widehat{W}_1$
W_2	$\sigma_1 \otimes I$	$\sigma_1 \otimes \sigma_3$	$I \otimes \sigma_3$	$\sigma_2 \otimes I$	$\sigma_2 \otimes \sigma_3$	$\widehat{W}_2$
W_3	$\sigma_1 \otimes \sigma_1$	$\sigma_2 \otimes \sigma_2$	$\sigma_3 \otimes \sigma_3$	$\sigma_2 \otimes \sigma_1$	$\sigma_1 \otimes \sigma_2$	$\widehat{W}_3$

$\mathcal{A}$

			$I \otimes \sigma_1$			
W_1	$I \otimes \sigma_3$	$\sigma_3 \otimes \sigma_3$	$\sigma_3 \otimes \sigma_1$	$I \otimes \sigma_2$	$\sigma_3 \otimes \sigma_2$	$\widehat{W}_1$
W_2	$\sigma_1 \otimes I$	$\sigma_1 \otimes \sigma_1$	$\sigma_3 \otimes I$	$\sigma_2 \otimes \sigma_1$	$\sigma_2 \otimes I$	$\widehat{W}_2$
W_3	$\sigma_1 \otimes \sigma_3$	$\sigma_2 \otimes \sigma_2$		$\sigma_1 \otimes \sigma_2$	$\sigma_2 \otimes \sigma_3$	$\widehat{W}_3$

All Cartan subalgebras of the Lie algebra $\mathfrak{su}(N)$, $2^{p-1} < N \le 2^p$, can be generated in the first p-shells through the process of the subalgebra extension.

All 4 Cartan decompositions,

$$\mathfrak{k}_1 = W_1 \oplus W_2 \oplus \widehat{W_3}$$

$$= span\{ \quad I \otimes \sigma_1,\ \sigma_1 \otimes I,\ \sigma_2 \otimes \sigma_1,\ \sigma_3 \otimes \sigma_1,\ \sigma_1 \otimes \sigma_3,\ \sigma_1 \otimes \sigma_2 \quad \}$$

$$\mathfrak{p}_1 = \mathcal{C} \oplus \widehat{W_1} \oplus \widehat{W_2} \oplus W_3$$

$$= span\{ \quad I \otimes \sigma_3,\ I \otimes \sigma_2,\ \sigma_2 \otimes I,\ \sigma_1 \otimes \sigma_1,\ \sigma_3 \otimes I,\ \sigma_3 \otimes \sigma_2,\ \sigma_2 \otimes \sigma_3,\ \sigma_2 \otimes \sigma_2,\ \sigma_3 \otimes \sigma_3 \quad \}$$

$$\mathfrak{k}_2 = W_1 \oplus \widehat{W_2} \oplus W_3 \qquad \mathfrak{p}_2 = \mathcal{C} \oplus \widehat{W_1} \oplus W_2 \oplus \widehat{W_3}$$

$$\mathfrak{k}_3 = \widehat{W_1} \oplus W_2 \oplus W_3 \qquad \mathfrak{p}_3 = \mathcal{C} \oplus W_1 \oplus \widehat{W_2} \oplus \widehat{W_3}$$

$$\mathfrak{k}_4 = \widehat{W_1} \oplus \widehat{W_2} \oplus \widehat{W_3} \qquad \mathfrak{p}_4 = \mathcal{C} \oplus W_1 \oplus W_2 \oplus W_3$$

A Predicision Rule of the Decomposition for $\mathfrak{su}(8)$

W_1	$I\otimes I\otimes\sigma_1$	$\sigma_3\otimes I\otimes\sigma_1$	$I\otimes I\otimes\sigma_2$	$\sigma_3\otimes I\otimes\sigma_2$	$\widehat{W}_1$
	$I\otimes\sigma_3\otimes\sigma_1$	$\sigma_3\otimes\sigma_3\otimes\sigma_1$	$I\otimes\sigma_3\otimes\sigma_2$	$\sigma_3\otimes\sigma_3\otimes\sigma_2$	
W_2	$I\otimes\sigma_1\otimes I$	$\sigma_3\otimes\sigma_1\otimes I$	$I\otimes\sigma_2\otimes I$	$\sigma_3\otimes\sigma_2\otimes I$	$\widehat{W}_2$
	$I\otimes\sigma_1\otimes\sigma_3$	$\sigma_3\otimes\sigma_1\otimes\sigma_3$	$I\otimes\sigma_2\otimes\sigma_3$	$\sigma_3\otimes\sigma_2\otimes\sigma_3$	
W_3	$I\otimes\sigma_1\otimes\sigma_1$	$I\otimes\sigma_2\otimes\sigma_2$	$I\otimes\sigma_1\otimes\sigma_2$	$I\otimes\sigma_2\otimes\sigma_1$	$\widehat{W}_3$
	$\sigma_3\otimes\sigma_1\otimes\sigma_1$	$\sigma_3\otimes\sigma_2\otimes\sigma_2$	$\sigma_3\otimes\sigma_1\otimes\sigma_2$	$\sigma_3\otimes\sigma_2\otimes\sigma_1$	
W_4	$\sigma_1\otimes I\otimes I$	$\sigma_1\otimes\sigma_3\otimes I$	$\sigma_2\otimes I\otimes I$	$\sigma_2\otimes\sigma_3\otimes I$	$\widehat{W}_4$
	$\sigma_1\otimes I\otimes\sigma_3$	$\sigma_1\otimes\sigma_3\otimes\sigma_3$	$\sigma_2\otimes I\otimes\sigma_3$	$\sigma_2\otimes\sigma_3\otimes\sigma_3$	
W_5	$\sigma_1\otimes I\otimes\sigma_1$	$\sigma_2\otimes I\otimes\sigma_2$	$\sigma_1\otimes I\otimes\sigma_2$	$\sigma_2\otimes I\otimes\sigma_1$	$\widehat{W}_5$
	$\sigma_1\otimes\sigma_3\otimes\sigma_1$	$\sigma_2\otimes\sigma_3\otimes\sigma_2$	$\sigma_1\otimes\sigma_3\otimes\sigma_2$	$\sigma_2\otimes\sigma_3\otimes\sigma_1$	
W_6	$\sigma_1\otimes\sigma_1\otimes I$	$\sigma_2\otimes\sigma_2\otimes I$	$\sigma_2\otimes\sigma_1\otimes I$	$\sigma_1\otimes\sigma_2\otimes I$	$\widehat{W}_6$
	$\sigma_1\otimes\sigma_1\otimes\sigma_3$	$\sigma_2\otimes\sigma_2\otimes\sigma_3$	$\sigma_2\otimes\sigma_1\otimes\sigma_3$	$\sigma_1\otimes\sigma_2\otimes\sigma_3$	
W_7	$\sigma_1\otimes\sigma_1\otimes\sigma_1$	$\sigma_1\otimes\sigma_2\otimes\sigma_2$	$\sigma_2\otimes\sigma_2\otimes\sigma_2$	$\sigma_1\otimes\sigma_1\otimes\sigma_2$	$\widehat{W}_7$
	$\sigma_2\otimes\sigma_1\otimes\sigma_2$	$\sigma_2\otimes\sigma_2\otimes\sigma_1$	$\sigma_2\otimes\sigma_1\otimes\sigma_1$	$\sigma_1\otimes\sigma_2\otimes\sigma_1$	

$\mathfrak{C}$

$\sigma_3\otimes I\otimes I$

$I\otimes\sigma_3\otimes I$

$I\otimes I\otimes\sigma_3$

$\sigma_3\otimes\sigma_3\otimes I$

$\sigma_3\otimes I\otimes\sigma_3$

$I\otimes\sigma_3\otimes\sigma_3$

$\sigma_3\otimes\sigma_3\otimes\sigma_3$

$\mathfrak{t}$

$\mathfrak{p}$

$\mathfrak{C}\subset\mathfrak{p}$

A Predicision Rule of the Decomposition for $\mathfrak{su}(6)$

			$\mathfrak{C}$			
W_1	$I\otimes\sigma_1$ $\mu_3\otimes\sigma_1$	$\mu_8\otimes\sigma_1$	$\mu_3\otimes I$	$I\otimes\sigma_2$ $\mu_3\otimes\sigma_2$	$\mu_8\otimes\sigma_2$	$\widehat{W}_1$
			$\mu_8\otimes I$			
W_2	$\mu_1\otimes I$	$\mu_1\otimes\sigma_3$	$I\otimes\sigma_3$	$\mu_2\otimes I$	$\mu_2\otimes\sigma_3$	$\widehat{W}_2$
			$\mu_3\otimes\sigma_3$			
W_3	$\mu_1\otimes\sigma_1$	$\mu_2\otimes\sigma_2$	$\mu_8\otimes\sigma_3$	$\mu_1\otimes\sigma_2$	$\mu_2\otimes\sigma_1$	$\widehat{W}_3$
W_4	$\mu_4\otimes I$	$\mu_4\otimes\sigma_3$		$\mu_5\otimes I$	$\mu_5\otimes\sigma_3$	$\widehat{W}_4$
			$\mathfrak{k}$			
W_5	$\mu_4\otimes\sigma_1$	$\mu_5\otimes\sigma_2$	$\mathfrak{p}$	$\mu_4\otimes\sigma_2$	$\mu_5\otimes\sigma_1$	$\widehat{W}_5$
W_6	$\mu_6\otimes I$	$\mu_6\otimes\sigma_3$	$\mathfrak{C}\subset\mathfrak{p}$	$\mu_7\otimes I$	$\mu_7\otimes\sigma_3$	$\widehat{W}_6$
W_7	$\mu_6\otimes\sigma_1$	$\mu_7\otimes\sigma_2$		$\mu_6\otimes\sigma_2$	$\mu_7\otimes\sigma_1$	$\widehat{W}_7$

In total, 2^p choices of Cartan decompositions for a given quotient algebra $\{\mathcal{Q}(\mathfrak{C};2^p-1)\}$.

λ-Representation of the Conjugate Partition and Quotient Algebra of $\mathfrak{su}(N)$

A λ-generator, λ_{ij} or $\hat{\lambda}_{ij}$, is an off-diagonal $N \times N$ matrix and serves the role of σ_1 or σ_2 in the i-th and the j-th dimensions,

$$\lambda_{ij} = |i\rangle\langle j| + |j\rangle\langle i|, \qquad \hat{\lambda}_{ij} = -i|i\rangle\langle j| + i|j\rangle\langle i|.$$

A diagonal $N \times N$ matrix plays the role of σ_3,

$$d_{kl} = |k\rangle\langle k| - |l\rangle\langle l|.$$

$\frac{N(N-1)}{2}$ conjugate pairs of λ_{ij} and $\hat{\lambda}_{ij}$ with any $N-1$ independent d_{kl} form a complete set of independent generators of $\mathfrak{su}(N)$.

Orthogonality

$\tilde{\lambda}_{ij} = \lambda_{ij}$ or $\hat{\lambda}_{ij}$, $\mathcal{C}$ is the intrisic center subalgebra of $\mathfrak{su}(N)$,

$$\mathrm{Tr}\{\tilde{\lambda}_{ij}\tilde{\lambda}_{kl}\} = 0 \quad \text{and} \quad \mathrm{Tr}\{\tilde{\lambda}_{ij}\mathcal{C}\} = 0, \quad \tilde{\lambda}_{ij} \neq \tilde{\lambda}_{kl}.$$

The essential commutation relations to the construction of conjugate partitions and quotient algebras for $\mathfrak{su}(N)$, $1 \le i, j, k, l \le N$,

$$[\lambda_{ij}\ , d_{kl}] = i\hat{\lambda}_{ij}(-\delta_{ik} + \delta_{il} + \delta_{jk} - \delta_{jl})$$

$$[\hat{\lambda}_{ij}\ , d_{kl}] = i\lambda_{ij}(\delta_{ik} - \delta_{il} - \delta_{jk} + \delta_{jl})$$

$$[\lambda_{ij}\ , \hat{\lambda}_{ij}] = 2id_{ij}$$

$$[\lambda_{ij}\ , \lambda_{kl}] = i\hat{\lambda}_{ik}\delta_{jl} + i\hat{\lambda}_{il}\delta_{jk} + i\hat{\lambda}_{jk}\delta_{il} + i\hat{\lambda}_{jl}\delta_{ik}$$

$$[\lambda_{ij}\ , \hat{\lambda}_{kl}] = i\lambda_{ik}\delta_{jl} - i\lambda_{il}\delta_{jk} + i\lambda_{jk}\delta_{il} - i\lambda_{jl}\delta_{ik}$$

$$[\hat{\lambda}_{ij}\ , \hat{\lambda}_{kl}] = i\hat{\lambda}_{ik}\delta_{jl} - i\hat{\lambda}_{il}\delta_{jk} - i\hat{\lambda}_{jk}\delta_{il} + i\hat{\lambda}_{jl}\delta_{ik}$$

λ-Representation of a Quotient Algebra of $\mathfrak{su}(8)$

W_{001}

$\lambda_{12}+\lambda_{34}+\lambda_{56}+\lambda_{78}$
$\lambda_{12}+\lambda_{34}-\lambda_{56}-\lambda_{78}$
$\lambda_{12}-\lambda_{34}+\lambda_{56}-\lambda_{78}$
$\lambda_{12}-\lambda_{34}-\lambda_{56}+\lambda_{78}$

W_{010}

$\lambda_{13}+\lambda_{24}+\lambda_{57}+\lambda_{68}$
$\lambda_{13}+\lambda_{24}-\lambda_{57}-\lambda_{68}$
$\lambda_{13}-\lambda_{24}+\lambda_{57}-\lambda_{68}$
$\lambda_{13}-\lambda_{24}-\lambda_{57}+\lambda_{68}$

W_{011}

$\lambda_{14}+\lambda_{23}+\lambda_{58}+\lambda_{67}$
$\lambda_{14}+\lambda_{23}-\lambda_{58}-\lambda_{67}$
$\lambda_{14}-\lambda_{23}+\lambda_{58}-\lambda_{67}$
$\lambda_{14}-\lambda_{23}-\lambda_{58}+\lambda_{67}$

$diag\{1,1,1,1,-1,-1,-1,-1\}$
$diag\{1,1,-1,-1,1,1,-1,-1\}$
$diag\{1,-1,1,-1,1,-1,1,-1\}$
$diag\{1,1,-1,-1,-1,-1,1,1\}$
$diag\{1,-1,1,-1,-1,1,-1,1\}$
$diag\{1,-1,-1,1,1,-1,-1,1\}$
$diag\{1,-1,-1,1,-1,1,1,-1\}$

$\widehat{W}_{001}$

$\hat{\lambda}_{12}+\hat{\lambda}_{34}+\hat{\lambda}_{56}+\hat{\lambda}_{78}$
$\hat{\lambda}_{12}-\hat{\lambda}_{34}+\hat{\lambda}_{56}-\hat{\lambda}_{78}$
$\hat{\lambda}_{12}+\hat{\lambda}_{34}-\hat{\lambda}_{56}-\hat{\lambda}_{78}$
$\hat{\lambda}_{12}-\hat{\lambda}_{34}-\hat{\lambda}_{56}+\hat{\lambda}_{78}$

$\widehat{W}_{010}$

$\hat{\lambda}_{13}+\hat{\lambda}_{24}+\hat{\lambda}_{57}+\hat{\lambda}_{68}$
$\hat{\lambda}_{13}+\hat{\lambda}_{24}-\hat{\lambda}_{57}-\hat{\lambda}_{68}$
$\hat{\lambda}_{13}-\hat{\lambda}_{24}+\hat{\lambda}_{57}-\hat{\lambda}_{68}$
$\hat{\lambda}_{13}-\hat{\lambda}_{24}-\hat{\lambda}_{57}+\hat{\lambda}_{68}$

$\widehat{W}_{011}$

$\hat{\lambda}_{14}+\hat{\lambda}_{23}+\hat{\lambda}_{58}+\hat{\lambda}_{67}$
$\hat{\lambda}_{14}+\hat{\lambda}_{23}-\hat{\lambda}_{58}-\hat{\lambda}_{67}$
$\hat{\lambda}_{14}-\hat{\lambda}_{23}+\hat{\lambda}_{58}-\hat{\lambda}_{67}$
$\hat{\lambda}_{14}-\hat{\lambda}_{23}-\hat{\lambda}_{58}+\hat{\lambda}_{67}$

$$W_{100}\quad \begin{array}{l} \lambda_{15}+\lambda_{26}+\lambda_{37}+\lambda_{48} \\ \lambda_{15}+\lambda_{26}-\lambda_{37}-\lambda_{48} \\ \lambda_{15}-\lambda_{26}+\lambda_{37}-\lambda_{48} \\ \lambda_{15}-\lambda_{26}-\lambda_{37}+\lambda_{48} \end{array}$$

$$W_{101}\quad \begin{array}{l} \lambda_{16}+\lambda_{25}+\lambda_{38}+\lambda_{47} \\ \lambda_{16}+\lambda_{25}-\lambda_{38}-\lambda_{47} \\ \lambda_{16}-\lambda_{25}+\lambda_{38}-\lambda_{47} \\ \lambda_{16}-\lambda_{25}-\lambda_{38}+\lambda_{47} \end{array}$$

$$W_{110}\quad \begin{array}{l} \lambda_{17}+\lambda_{28}+\lambda_{35}+\lambda_{46} \\ \lambda_{17}+\lambda_{28}-\lambda_{35}-\lambda_{46} \\ \lambda_{17}-\lambda_{28}+\lambda_{35}-\lambda_{46} \\ \lambda_{17}-\lambda_{28}-\lambda_{35}+\lambda_{46} \end{array}$$

$$W_{111}\quad \begin{array}{l} \lambda_{18}+\lambda_{27}+\lambda_{36}+\lambda_{45} \\ \lambda_{18}+\lambda_{27}-\lambda_{36}-\lambda_{45} \\ \lambda_{18}-\lambda_{27}+\lambda_{36}-\lambda_{45} \\ \lambda_{18}-\lambda_{27}-\lambda_{36}+\lambda_{45} \end{array}$$

$$\begin{array}{l} diag\{1,1,1,1,-1,-1,-1,-1\} \\ diag\{1,1,-1,-1,1,1,-1,-1\} \\ diag\{1,-1,1,-1,1,-1,1,-1\} \\ diag\{1,1,-1,-1,-1,-1,1,1\} \\ diag\{1,-1,1,-1,-1,1,-1,1\} \\ diag\{1,-1,-1,1,1,-1,-1,1\} \\ diag\{1,-1,-1,1,-1,1,1,-1\} \end{array}$$

$$\begin{array}{l} \widehat{\lambda}_{15}+\widehat{\lambda}_{26}+\widehat{\lambda}_{37}+\widehat{\lambda}_{48} \\ \widehat{\lambda}_{15}+\widehat{\lambda}_{26}-\widehat{\lambda}_{37}-\widehat{\lambda}_{48} \\ \widehat{\lambda}_{15}-\widehat{\lambda}_{26}+\widehat{\lambda}_{37}-\widehat{\lambda}_{48} \\ \widehat{\lambda}_{15}-\widehat{\lambda}_{26}-\widehat{\lambda}_{37}+\widehat{\lambda}_{48} \end{array}\quad \widehat{W}_{100}$$

$$\begin{array}{l} \widehat{\lambda}_{16}+\widehat{\lambda}_{25}+\widehat{\lambda}_{38}+\widehat{\lambda}_{47} \\ \widehat{\lambda}_{16}+\widehat{\lambda}_{25}-\widehat{\lambda}_{38}-\widehat{\lambda}_{47} \\ \widehat{\lambda}_{16}-\widehat{\lambda}_{25}+\widehat{\lambda}_{38}-\widehat{\lambda}_{47} \\ \widehat{\lambda}_{16}-\widehat{\lambda}_{25}-\widehat{\lambda}_{38}+\widehat{\lambda}_{47} \end{array}\quad \widehat{W}_{101}$$

$$\begin{array}{l} \widehat{\lambda}_{17}+\widehat{\lambda}_{28}+\widehat{\lambda}_{35}+\widehat{\lambda}_{46} \\ \widehat{\lambda}_{17}+\widehat{\lambda}_{28}-\widehat{\lambda}_{35}-\widehat{\lambda}_{46} \\ \widehat{\lambda}_{17}-\widehat{\lambda}_{28}+\widehat{\lambda}_{35}-\widehat{\lambda}_{46} \\ \widehat{\lambda}_{17}-\widehat{\lambda}_{28}-\widehat{\lambda}_{35}+\widehat{\lambda}_{46} \end{array}\quad \widehat{W}_{110}$$

$$\begin{array}{l} \widehat{\lambda}_{18}+\widehat{\lambda}_{27}+\widehat{\lambda}_{36}+\widehat{\lambda}_{45} \\ \widehat{\lambda}_{18}+\widehat{\lambda}_{27}-\widehat{\lambda}_{36}-\widehat{\lambda}_{45} \\ \widehat{\lambda}_{18}-\widehat{\lambda}_{27}+\widehat{\lambda}_{36}-\widehat{\lambda}_{45} \\ \widehat{\lambda}_{18}-\widehat{\lambda}_{27}-\widehat{\lambda}_{36}+\widehat{\lambda}_{45} \end{array}\quad \widehat{W}_{111}$$

λ-Representation of a Conjugate Partition and Quotient Algebra of $\mathfrak{su}(4)$

			$\mathcal{C}$			
W_{01}	$\lambda_{12}+\lambda_{34}$	$\lambda_{12}-\lambda_{34}$	$diag\{1,1,-1,-1\}$	$\hat{\lambda}_{12}+\hat{\lambda}_{34}$	$\hat{\lambda}_{12}-\hat{\lambda}_{34}$	$\widehat{W}_{01}$
W_{10}	$\lambda_{13}+\lambda_{24}$	$\lambda_{13}-\lambda_{24}$	$diag\{1,-1,1,-1\}$	$\hat{\lambda}_{13}+\hat{\lambda}_{24}$	$\hat{\lambda}_{13}-\hat{\lambda}_{24}$	$\widehat{W}_{10}$
W_{11}	$\lambda_{14}+\lambda_{23}$	$\lambda_{14}-\lambda_{23}$	$diag\{1,-1,-1,1\}$	$\hat{\lambda}_{14}+\hat{\lambda}_{23}$	$\hat{\lambda}_{14}-\hat{\lambda}_{23}$	$\widehat{W}_{11}$

An alternative,

			$\mathcal{C}$			
W_{01}	λ_{12}	$\hat{\lambda}_{34}$	$diag\{1,1,-1,-1\}$	$\hat{\lambda}_{12}$	λ_{34}	$\widehat{W}_{01}$
W_{10}	λ_{13}	$\hat{\lambda}_{24}$	$diag\{1,-1,1,-1\}$	$\hat{\lambda}_{13}$	λ_{24}	$\widehat{W}_{10}$
W_{11}	λ_{14}	$\hat{\lambda}_{23}$	$diag\{1,-1,-1,1\}$	$\hat{\lambda}_{14}$	λ_{23}	$\widehat{W}_{11}$

W_{001} or $\widehat{W}_{001}$

W_{010} or $\widehat{W}_{010}$

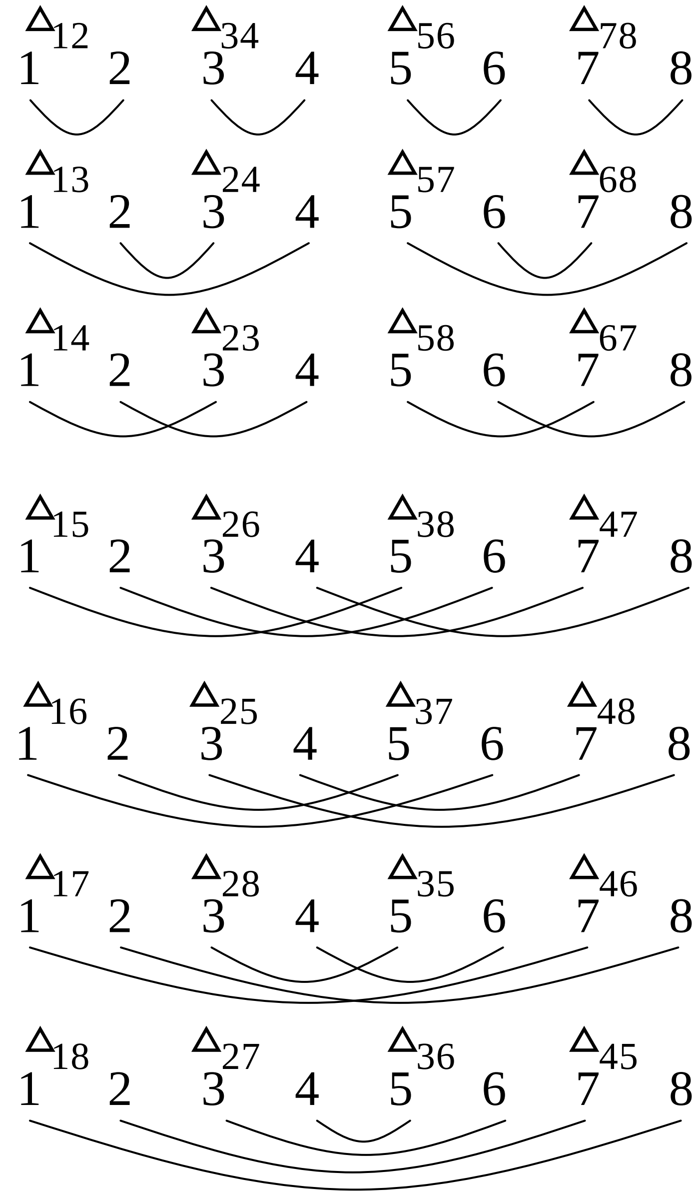
W_{011} or $\widehat{W}_{011}$

W_{100} or $\widehat{W}_{100}$

W_{101} or $\widehat{W}_{101}$

W_{110} or $\widehat{W}_{110}$

W_{111} or $\widehat{W}_{111}$



$$[W_{\zeta}, W_{\eta}] \subset \widehat{W}_{\zeta+\eta}, \; [W_{\zeta}, \widehat{W}_{\eta}] \subset W_{\zeta+\eta}, \text{ and } [\widehat{W}_{\zeta}, \widehat{W}_{\eta}] \subset \widehat{W}_{\zeta+\eta}$$

conjugate partit ≠ Q.A.   binary pariti & its permut   binary partit discerned only in eigenspace

λ-Representation of a Conjugate Partition and Quotient Algebra of $\mathfrak{su}(6)$

W_{001}	$\lambda_{12}-\lambda_{34}$ $\lambda_{12}+\lambda_{34}-\lambda_{56}$ $\lambda_{12}+\lambda_{34}-2\lambda_{56}$		$diag\{1,-1,1,-1,1,-1\}$ $diag\{1,1,-1,-1,0,0\}$ $diag\{1,1,1,1,-2,-2\}$ $diag\{1,-1,-1,1,0,0\}$ $diag\{1,-1,1,-1,-2,2\}$	$\hat{\lambda}_{12}-\hat{\lambda}_{34}$ $\hat{\lambda}_{12}+\hat{\lambda}_{34}+\hat{\lambda}_{56}$ $\hat{\lambda}_{12}+\hat{\lambda}_{34}-2\hat{\lambda}_{56}$		$\widehat{W}_{001}$
W_{010}	$\lambda_{13}+\lambda_{24}$	$\lambda_{13}-\lambda_{24}$		$\hat{\lambda}_{13}+\hat{\lambda}_{24}$	$\hat{\lambda}_{13}-\hat{\lambda}_{24}$	$\widehat{W}_{010}$
W_{011}	$\lambda_{14}+\lambda_{23}$	$\lambda_{14}-\lambda_{23}$		$\hat{\lambda}_{14}+\hat{\lambda}_{23}$	$\hat{\lambda}_{14}-\hat{\lambda}_{23}$	$\widehat{W}_{011}$
W_{100}	$\lambda_{15}+\lambda_{26}$	$\lambda_{15}-\lambda_{26}$		$\hat{\lambda}_{15}+\hat{\lambda}_{26}$	$\hat{\lambda}_{15}-\hat{\lambda}_{26}$	$\widehat{W}_{100}$
W_{101}	$\lambda_{16}+\lambda_{25}$	$\lambda_{16}-\lambda_{25}$		$\hat{\lambda}_{16}+\hat{\lambda}_{25}$	$\hat{\lambda}_{16}-\hat{\lambda}_{25}$	$\widehat{W}_{101}$
W_{110}	$\lambda_{35}+\lambda_{46}$	$\lambda_{35}-\lambda_{46}$		$\hat{\lambda}_{35}+\hat{\lambda}_{46}$	$\hat{\lambda}_{35}-\hat{\lambda}_{46}$	$\widehat{W}_{110}$
W_{111}	$\lambda_{36}+\lambda_{45}$	$\lambda_{36}-\lambda_{45}$		$\hat{\lambda}_{36}+\hat{\lambda}_{45}$	$\hat{\lambda}_{36}-\hat{\lambda}_{45}$	$\widehat{W}_{111}$

The removing process

$$\mathfrak{su}(8) \qquad\qquad\qquad\qquad\qquad\qquad \mathfrak{su}(6)$$

$$W_{001}\quad \begin{matrix} \lambda_{12}+\lambda_{34}+\lambda_{56}+\lambda_{78} \\ \lambda_{12}+\lambda_{34}-\lambda_{56}-\lambda_{78} \\ \lambda_{12}-\lambda_{34}+\lambda_{56}-\lambda_{78} \\ \lambda_{12}-\lambda_{34}-\lambda_{56}+\lambda_{78} \end{matrix} \longrightarrow \begin{matrix} \lambda_{12}+\lambda_{34}+\lambda_{56} \\ \lambda_{12}+\lambda_{34}-\lambda_{56} \\ \lambda_{12}-\lambda_{34}+\lambda_{56} \\ \lambda_{12}-\lambda_{34}-\lambda_{56} \end{matrix} \twoheadrightarrow \begin{matrix} \lambda_{12}-\lambda_{34} \\ \lambda_{12}+\lambda_{34}-\lambda_{56} \\ \lambda_{12}+\lambda_{34}-2\lambda_{56} \end{matrix} \quad W_{001}$$

Algebra Isomorphism

The quotient algebra of the Lie algebra $\mathfrak{su}(N)$, where $2^{p-1} < N < 2^p$ and $p \in \mathbb{Z}^+$, is isomorphic to that of $\mathfrak{su}(2^p)$.

Theorem Every Cartan subalgebra $\mathfrak{C}$ of the Lie algebra $\mathfrak{su}(N)$, $2^{p-1} < N \le 2^p$, can generate a quotient algebra of rank zero $\{Q(\mathfrak{C}; 2^p - 1)\}$.

Path, a Recursive Decomposition

A bifurcation tree by the *KAK* theorem,

1-st level $\mathcal{A}_{[1]}$

2-nd level $\mathcal{A}_{[2]}$ $\mathcal{A}_{[2]}$

3-th level $\mathcal{A}_{[3]}$ $\mathcal{A}_{[3]}$ $\mathcal{A}_{[3]}$ $\mathcal{A}_{[3]}$

$\vdots$ $\vdots$ $\ddots$ $\vdots$

p-th level $\mathcal{A}_{[p]}$ $\mathcal{A}_{[p]}$ $\mathcal{A}_{[p]}$ $\cdots$ $\cdots$ $\mathcal{A}_{[p]}$ $\mathcal{A}_{[p]}$ $\mathcal{A}_{[p]}$

$\mathfrak{t}_{[p]}$ $\mathfrak{t}_{[p]}$ $\mathfrak{t}_{[p]}$ $\mathfrak{t}_{[p]}$ $\cdots$ $\mathfrak{t}_{[p]}$ $\mathfrak{t}_{[p]}$ $\mathfrak{t}_{[p]}$ $\mathfrak{t}_{[p]}$

For an arbitrary $U \in \mathrm{SU}(2^{p-1} < N \leq 2^{p})$, $t_{[k],i} \in \mathfrak{t}_{[k]}$ and $a_{[k],i} \in \mathcal{A}_{[k]}$,

$$U = e^{it_{[\mathfrak{p}],1}} e^{ia_{[\mathfrak{p}],1}} e^{it_{[\mathfrak{p}],1}} \cdots e^{it_{[\mathfrak{p}],2^{\mathfrak{p}-1}}} e^{ia_{[1]}} e^{it_{[\mathfrak{p}],2^{\mathfrak{p}-1}}} \cdots e^{it_{[\mathfrak{p}],2^{\mathfrak{p}}}} e^{ia_{[\mathfrak{p}],2^{\mathfrak{p}-1}}} e^{it_{[\mathfrak{p}],2^{\mathfrak{p}}}}$$

$$e^{\{\mathcal{A}_{[k]};\, k=1,2,\cdots,p,p+1\}} = Q_p Q_{p-1} \cdots Q_1 e^{\{\mathcal{C}_{[k]};\, k=1,2,\cdots,p,p+1\}} (Q_1)^{+} \cdots (Q_{p-1})^{+} (Q_p)^{+}$$

$$\mathcal{A}_{[k]} = Q_k^{+} \mathcal{C}_{[k]} Q_k$$

Primitive Gate Decompositions

Every unitary transformation can be fully decomposed into a product of local and nonlocal gates.

Computation Universality

A set of any one single bipartite nonlocal gate with local gates is computationally universal.

But, is this the unique type of decomposition that we can have?
Are there alternatives or any more refined choices? Yes!

The Three Different Types of Decompositions of $\mathfrak{su}(N)$

p.518, Helgason

Type-**AI**

$$\mathrm{SU}(N) \to \mathrm{SO}(N) \to \mathrm{SU}(N) \to \mathrm{SO}(N) \to \cdots$$

Type-**AII**

$$\mathrm{SU(2N)} \to \mathrm{Sp(N)} \to \mathrm{SU(2N)} \to \mathrm{Sp(N)} \to \cdots$$

Type-**AIII**

$$\mathrm{SU}(p+q) \to \mathrm{SU}(p) \otimes \mathrm{SU(q)} \otimes \mathrm{U} \to \mathrm{SU}(p+q) \to \mathrm{SU}(p) \otimes \mathrm{SU(q)} \otimes \mathrm{U} \to \cdots$$

Criteria of a Center Subalgebra

Lemma For a partition of $\mathfrak{su}(2^p) = \mathcal{A} \oplus W_1 \oplus \widehat{W_1} \oplus \cdots \oplus W_m \oplus \widehat{W_m} \oplus \cdots \oplus W_q \oplus \widehat{W_q}$ consisting of the center subspace $\mathcal{A}$ and a number q of conjugate pairs $\{W_m, \widehat{W_m}\}$, $1 \le m \le q$, $\mathcal{A}$ is **abelian** if the property of the **conjugate partition** is satisfied,

$$[W_m, \mathcal{A}] \subset \widehat{W_m}, \quad [\widehat{W_m}, \mathcal{A}] \subset W_m \quad \text{and} \quad [W_m, \widehat{W_m}] \subset \mathcal{A}.$$

Lemma For two arbitrary conjugate pairs $\{W_m, \widehat{W_m}\}$ and $\{W_n, \widehat{W_n}\}$, there exists a unique conjugate pair $\{W_s, \widehat{W_s}\}$, $1 \le m, n, s \le q$, $\mathcal{A}$ is a **bi-subalgebra** or a **coset of a bi-subalgebra** in a Cartan subalgebra $\mathfrak{C} \subset \mathfrak{su}(2^p)$ if the **condition of closure** is satisfied,

$$[W_m, W_n] \subset \widehat{W_s}, \quad [W_m, \widehat{W_n}] \subset W_s, \quad [\widehat{W_m}, W_n,] \subset W_s \quad \text{and} \quad [\widehat{W_m}, \widehat{W_n},] \subset \widehat{W_s}.$$

Isomorphism Relations

As the bi-addition for the spinor generators in a Cartan subalgebra $\mathfrak{C}$, the operation $\sqcap$ is allowed for the set $\{\mathfrak{B} : \mathfrak{B} \subset \mathfrak{C}\} \cup \mathfrak{C}$ consisting of $\mathfrak{C}$ and all maximal bi-subalgebras $\mathfrak{B}$ in $\mathfrak{C}$ such that a third one is always derivable from two arbitrary maximal bi-subalgebras $\mathfrak{B}_1$ and $\mathfrak{B}_2 \subset \mathfrak{C}$,

$$\mathfrak{B}_1 \sqcap \mathfrak{B}_2 \equiv (\mathfrak{B}_1 \cap \mathfrak{B}_2) \cup (\mathfrak{B}_1^c \cap \mathfrak{B}_2^c) \subset \mathfrak{C},$$

where $\mathfrak{B}_1^c = \mathfrak{C} - \mathfrak{B}_1$ and $\mathfrak{B}_2^c = \mathfrak{C} - \mathfrak{B}_2$.

$$\mathfrak{B}_{\alpha+\beta} = \mathfrak{B}_{\alpha} \sqcap \mathfrak{B}_{\beta},$$

in an appropriate coordinate.

The isomorphism relations hold,

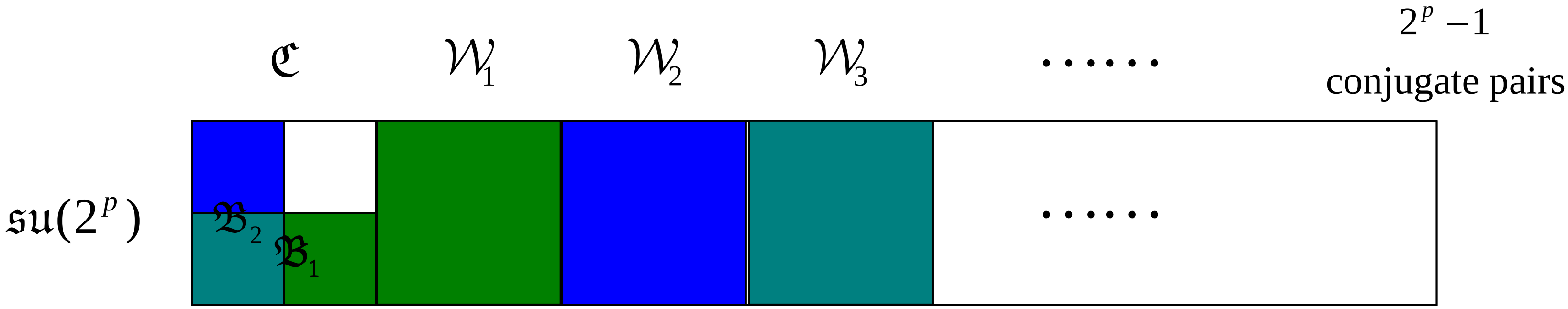
hypercube topology

$$\{\mathfrak{B} : \mathfrak{B} \subset \mathfrak{C}\} \cup \mathfrak{C} \simeq \mathfrak{C} \simeq \mathbb{Z}_2^p$$

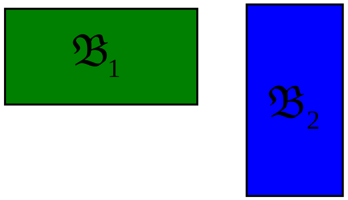
$$\simeq \{\mathcal{W} : \mathcal{W} \subset \mathfrak{su}(2^p) - \mathfrak{C} \text{ and } [\mathcal{W}, \mathfrak{B}] = 0, \mathfrak{B} \subset \mathfrak{C}\} \cup \mathfrak{C}.$$

A Cartan subalgebra $\mathfrak{C} \subset \mathfrak{su}(2^p)$ has a total number $2^p - 1$ of maximal bi-subalgebras, each of which determines a **conjugate - pair subspace** by the commutator rule $[\mathcal{W}, \mathfrak{B}] = 0$.

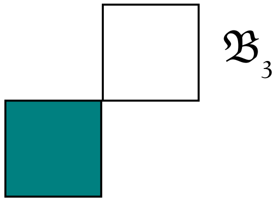
Partition of $\mathfrak{su}(2^p)$ by a Cartan Subalgebra $\mathfrak{C}$



A **conjugate-pair subspace** $\mathcal{W}$ is determined from a maximal bi-subalgebra $\mathfrak{B} \subset \mathfrak{C}$ according to the commutator rule $[\mathcal{W}, \mathfrak{B}] = 0$.



The 3rd maximal bi-subalgebra $\mathfrak{B}_3 = \mathfrak{B}_1 \sqcap \mathfrak{B}_2$ and the subspace $\mathcal{W}_3 = [\mathcal{W}_1, \mathcal{W}_2]$ satisfy the relation $[\mathcal{W}_3, \mathfrak{B}_3] = 0$.

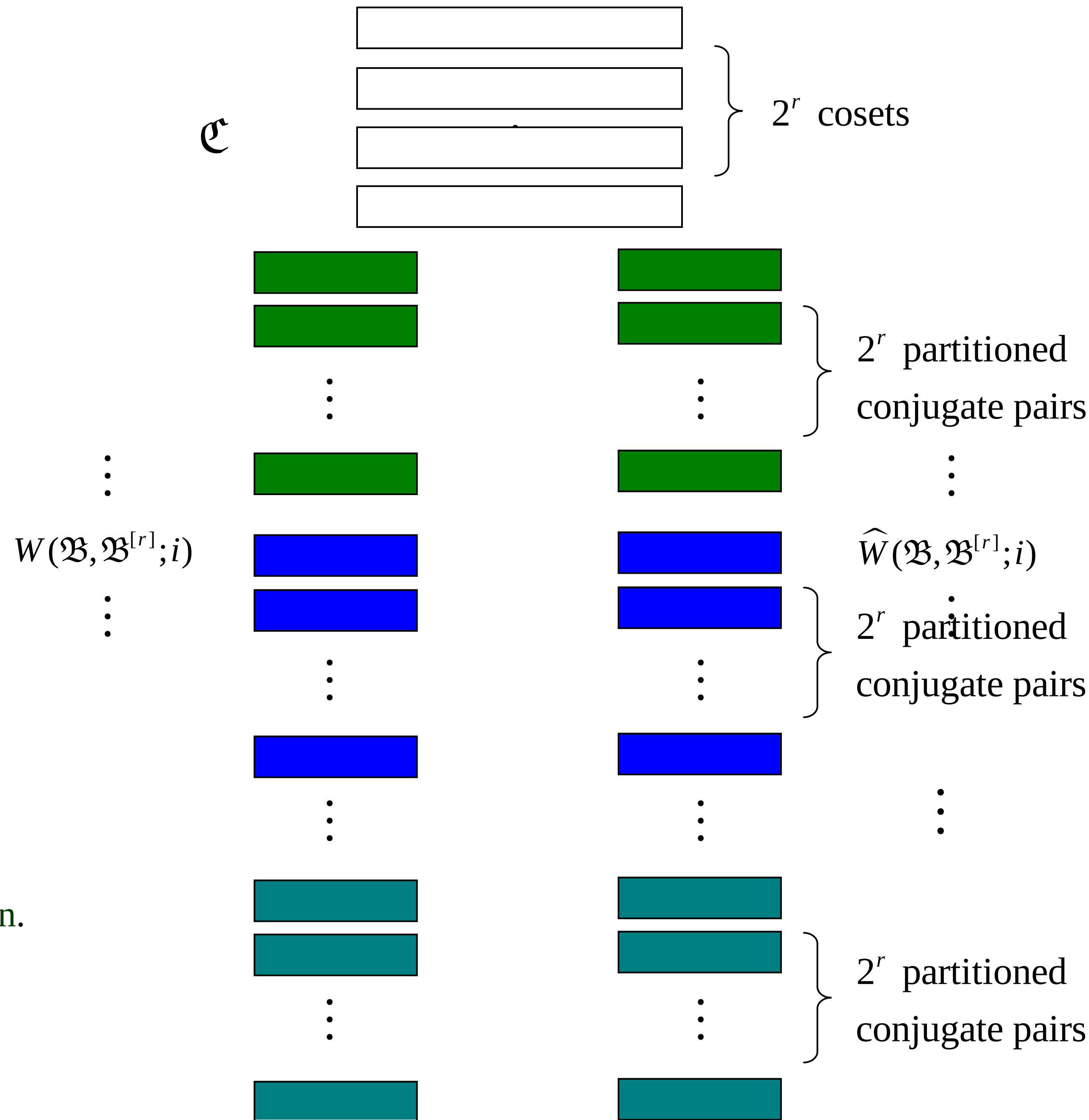


$\mathfrak{su}(2^p)$

$\mathfrak{C}$ is partitioned into 2^r cosets by an r-th maximal bi-subalgebra $\mathfrak{B}^{[r]} \subset \mathfrak{C}$.

Every conjugate-pair subspace is partitioned into 2^r partitioned conjugate pairs by the coset rule of partition.

Every partitioned conjugate pair is further bisected into two conditioned subspaces by the coset rule of bisection.



Theorem In the **quotient - algebra partition of rank *r*** $\{\mathcal{P}_{\mathcal{Q}}(\mathfrak{B}^{[r]})\}$ given by an r-th maximal bi-subalgebra $\mathfrak{B}^{[r]}$ of a Cartan subalgebra $\mathfrak{C} \subset \mathfrak{su}(N)$, $2^{p-1} < N \le 2^p$, there determine a **quotient algebra of rank *r*** given by $\mathfrak{B}^{[r]}$, denoted as $\{\mathcal{Q}(\mathfrak{B}^{[r]})\}$, when $\mathfrak{B}^{[r]}$ is taken as the center subalgebra, or a **co - quotient algebra of rank r** given by a non-null conditioned subspace $W^{\varepsilon}(\mathfrak{B}, \mathfrak{B}^{[r]}; l) \in \{\mathcal{P}_{\mathcal{Q}}(\mathfrak{B}^{[r]})\}$ else from $\mathfrak{B}^{[r]}$, denoted as $\{\mathcal{Q}(W^{\varepsilon}(\mathfrak{B}, \mathfrak{B}^{[r]}; l))\}$, when $W^{\varepsilon}(\mathfrak{B}, \mathfrak{B}^{[r]}; l)$ as the center subalgebra, here $\mathfrak{B} \subseteq \mathfrak{C}$, $\varepsilon \in \mathbb{Z}_2$ and $l \in \mathbb{Z}_2^r$.

Lemma A **quotient - algebra partition of rank *r*** for the Lie algebra $\mathfrak{su}(N)$ is **isomorphic** to that of $\mathfrak{su}(2^p)$, where $2^{p-1} < N \le 2^p$, $0 \le r \le r_0 < p$, and the dimension has the factorization $N = 2^{r_0} N'$ **with** N' **being odd**.

Main Theorem **Every Lie algebra $\mathfrak{su}(N)$ admits structures of quotient and co - quotient algebras up to rank r_0 and its quotient and co - quotient algebras of rank *r* are isomorphic to those of $\mathfrak{su}(2^p)$, where $2^{p-1} < N \le 2^p, 0 \le r \le r_0 < p$, and the dimension has the factorization $N = 2^{r_0} N'$ for N' being odd.**

Determination of Cartan Decompositions

Lemma **The subalgebra $\mathfrak{k}$ of a Cartan decomposition $\mathfrak{su}(N) = \mathfrak{k} \oplus \mathfrak{p}$ is a proper maximal subgroup of a quotient - algebra partition $\{\mathcal{P}_{\mathcal{Q}}(\mathfrak{B}^{[r]})\}$ under the operation of tri - addition.**

Theorem The quotient-algebra partition of rank r $\{\mathcal{P}_{\mathcal{Q}}(\mathfrak{B}^{[r]})\}$ given by an r-th maximal bi-subalgebra $\mathfrak{B}^{[r]}$ of a Cartan subalgebra $\mathfrak{C} \subset \mathfrak{su}(N)$ admits **Cartan decompositions of types AI**, **AII** and **AIII** as $1 \le r < p$ and only **types AI** and **AIII** as $r = 0$.

Types of a decomposition $\mathfrak{k} \oplus \mathfrak{p}$: the **maximal abelian subalgebra** of $\mathfrak{p}$ is

a Cartan subalgebra $\mathfrak{C}$ ⟶ **type AI**

a maxiaml bi-subalgebra $\mathfrak{B} \subset \mathfrak{C}$ ⟶ **type AII**

the complement $\mathfrak{B}^c = \mathfrak{C} - \mathfrak{B}$ ⟶ **type AIII**

Corollary A quotient algebra of rank r $\{\mathcal{Q}(\mathfrak{B}^{[r]};2^{p+r}-1)\}$ admits Cartan decompositions of types **AI** and **AII** as $1\le r<p$ and only type **AI** as $r=0$.

Corollary The co-quotient algebra of rank r $\{\mathcal{Q}(W^{\varepsilon}(\mathfrak{B},\mathfrak{B}^{[r]};l);2^{p+r}-2^{2r-2})\}$ given by a non-null conditioned subspace $W^{\varepsilon}(\mathfrak{B},\mathfrak{B}^{[r]};l)\in\{\mathcal{Q}(\mathfrak{B}^{[r]})\}$ else from $\mathfrak{B}^{[r]}\subseteq\mathfrak{B}$ admits Cartan decompositions of types **AI**, **AII,** and **AIII** as $1<r<p$ and only the types **AI** and **AIII** as $r=1$.

Corollary The co-quotient algebra of rank r $\{\mathcal{Q}(W^{\sigma}(\mathfrak{B},\mathfrak{B}^{[r]};s);2^{p+r}-1)\}$ given by a conditioned subspace $W^{\varepsilon}(\mathfrak{B},\mathfrak{B}^{[r]};l)\in\{\mathcal{Q}(\mathfrak{B}^{[r]})\}$ with $\mathfrak{B}^{[r]}\not\subset\mathfrak{B}$ admits Cartan decompositions of types **AI**, **AII,** and **AIII** as $1\le r<p$ and only the types **AI** and **AIII** as $r=0$.

Let $\mathfrak{D}_{\Omega}\{\mathfrak{B}^{[r]}\}$ denote a collection of all Cartan decompositions of type **AΩ** determined from quotient-algebra partitions generated by the set of all r-th maixmal bi-subalgebras $\{\mathfrak{B}^{[r]}\}$ in Cartan subalgebras $\{\mathfrak{C}\} \subset \mathfrak{su}(2^p)$, where the type **AΩ** is referring to the type **AI**, **AII** or **AIII**.

There exist the identities for the sets of decompositions for $1 \leq r, r'' < p$ and $1 < r' < p$,

$$\mathfrak{D}_{\mathbf{I}}\{\mathfrak{C}\} = \mathfrak{D}_{\mathbf{I}}\{\mathfrak{B}^{[r]}\},\ \mathfrak{D}_{\mathbf{II}}\{\mathfrak{B}^{[1]}\} = \mathfrak{D}_{\mathbf{II}}\{\mathfrak{B}^{[r']}\}, \text{ and } \mathfrak{D}_{\mathbf{III}}\{\mathfrak{C}\} = \mathfrak{D}_{\mathbf{III}}\{\mathfrak{B}^{[r'']}\}.$$

These sets are respectively the complete sets of decompositions of types **AI**, **AII** and **AIII**.

Decomposition Sequences and the *l*-th Level Decomposition

A **decomposition sequence** seq_{dec} is prepared from a quotient algebra $\{\mathcal{Q}(\mathcal{A};q)\}$ given by a center subalgebra $\mathcal{A}$,

$$seq_{dec} = \{\mathcal{A}_{[l]} : l = 0,1,\cdots M,\ \mathcal{A}_{[0]} = \mathcal{A} \text{ and } \mathcal{A}_{[M]} = \mathfrak{t}_{[M]}\}.$$

The l-th member subspace $\mathcal{A}_{[l]}$ is a maximal abelian subalgebra in the vector subspace $\mathfrak{p}_{[k]}$ of the **$(l-1)$ - th level decomposition** $\mathfrak{t}_{[l]} \oplus \mathfrak{p}_{[l]},\ 1 \le l \le M$.

The subalgebra $\mathfrak{t}_{[l-1]} = \mathfrak{t}_{[l]} \oplus \mathfrak{p}_{[l]}$, *i.e.*, an **$(l-1)$-th level decomposition,** is an **$(l-1)$ - th maximal subgroup** of the quotient-algebra partition of rank r $\{\mathcal{P}_{\mathcal{Q}}(\mathfrak{B}^{[r]})\}$ under the operation of **tri - addition**, where $\mathfrak{t}_{[0]} = \{\mathcal{P}_{\mathcal{Q}}(\mathfrak{B}^{[r]})\}$, $\mathfrak{p}_{[0]} = \{0\}$, $1 \le l \le M \le p + r$ and $0 \le r < p$.

Types of the l-th Level Decomposition

The l-th level decomposition $\mathfrak{k}_{[l]} \oplus \mathfrak{p}_{[l]}$ in a given decomposition sequence is of type **AΩ** if there exists a Cartan decomposition of type **AΩ** $\mathfrak{k}_{\Omega} \oplus \mathfrak{p}_{\Omega}$ such that

$$\mathfrak{k}_{[l]} \subset \mathfrak{k}_{\Omega} \text{ and } \mathfrak{p}_{[l]} \subset \mathfrak{p}_{\Omega},$$

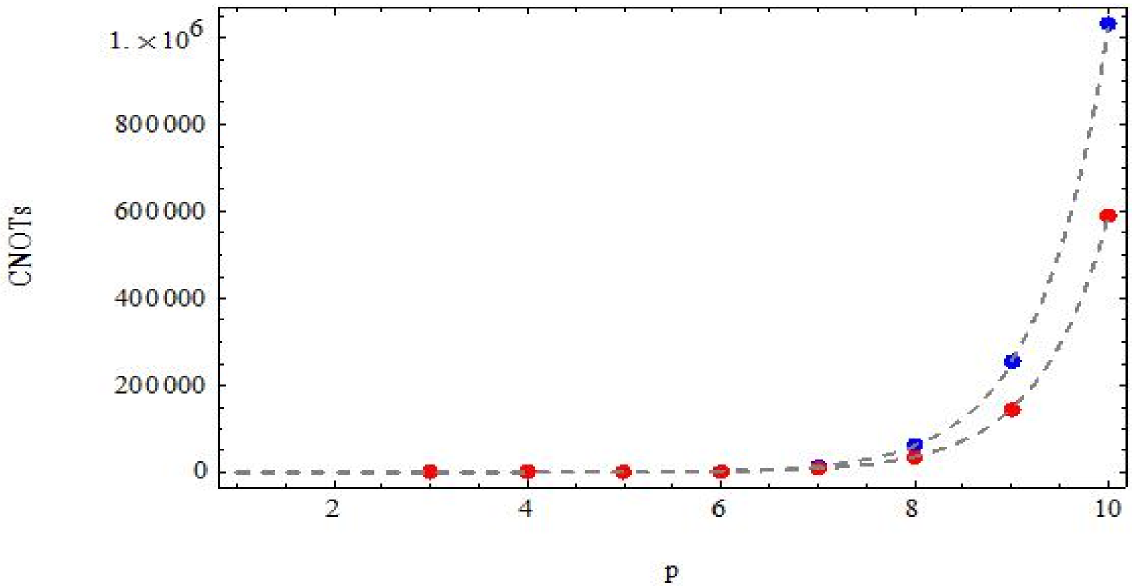
where the type **AΩ** is referring to type **AI**, **AII** or **AIII**.

Every level of the decomposition in a sequence allows all the three types **AI**, **AII** and **AIII**.

Optimal Quantum Circuits

The number of CNOTs required in the selected decomposition sequence is

$$3\cdot 2^{2p-4}+3\sum_{n=2}^{p-1}\sum_{m=1}^{n}\frac{n!(2m-1)!}{(n-m)!m!}2^{2p-2n-2} \xrightarrow[\text{gate identities}]{\text{Reduction by}} \frac{9}{16}2^{2p}-\frac{3}{2}2^{p}$$



Up to specific mathematical motivations or physical constraints, the problems of determining the optimal quantum evolutions can be termed into the concept of searching appropriate decompoistion sequences, namely Quantum Control.

Error-Correction Codes

The process of error-correction:

$$\underbrace{|\phi\rangle}_{k\text{-qubit information}} \xrightarrow[\text{by } U_E]{\text{encoded}} \underbrace{|\psi\rangle = \sum_{i=1}^{M} a_i |\psi_i\rangle}_{n\text{-qubit state written in a basis } \{|\psi_0\rangle,\ |\psi_1\rangle,\ \cdots,\ |\psi_{M-1}\rangle\}} \xrightarrow[\text{corruptions by a set of errors } E=\{E_1,\, E_2,\, \cdots,\, E_N\}]{\text{transmission}} |\psi_m{}'\rangle = E_m |\psi\rangle = \sum_{i=1}^{M} a_i E_m |\psi_i\rangle$$

$$\xrightarrow[\text{by } U_D]{\text{decoded}} |\phi_m{}'\rangle \xrightarrow{\text{measure and correction}} |\phi\rangle$$

A quantum code satisfies the following condition relating its basis states to the error operators

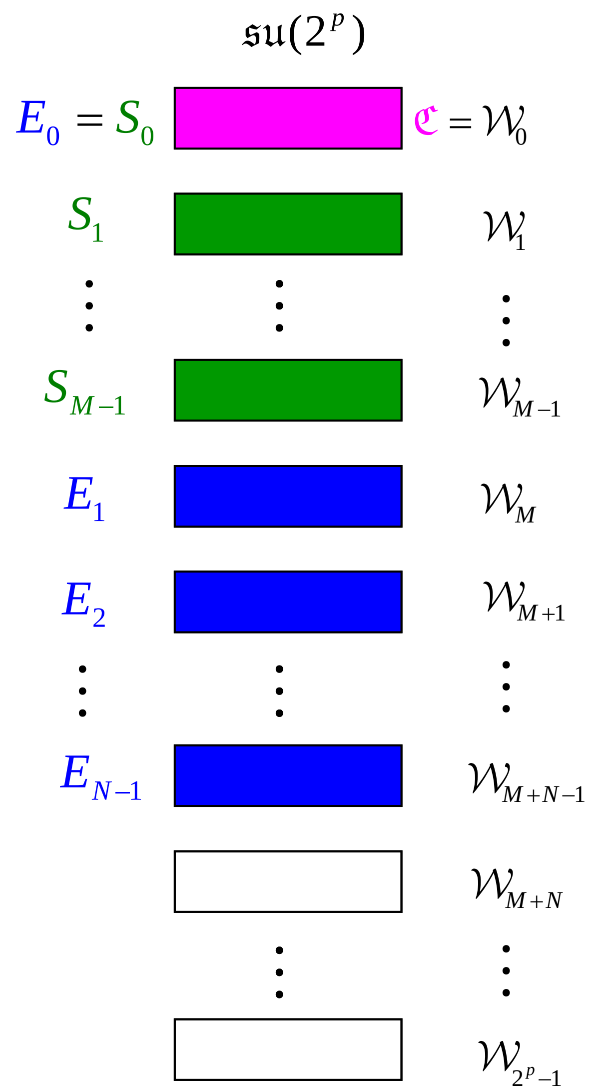
$$\langle \psi_i | E_m^\dagger E_n | \psi_j \rangle = \begin{cases} \delta_{ij}\delta_{mn} & \longrightarrow \text{orthogonality for a non-degenerate code} \\ \delta_{ij} C_{mn} & \longrightarrow \text{non-orthogonality for a degenerate code} \end{cases}$$

Constructing Quantum Error-Correction Codes

Every Cartan subalgebra of the Lie algebra $\mathfrak{su}(2^p)$ can decide the codes $[[p, M]]$ with the code length p and code-subspace dimension M.

An initial state $|\psi_0\rangle$ is prepared by the spinors of an arbitrary Cartan subalgebra $\mathfrak{C} \subset \mathfrak{su}(2^p)$ and the set $\{S_0|\psi_0\rangle, S_1|\psi_0\rangle, \cdots, S_{M-1}|\psi_0\rangle\}$ forms a basis of a code $[[p, M]]$ for a set of spinors $\mathcal{B}=\{S_0, S_1, \cdots, S_{M-1}\}$.

Given a set of N spinor errors $E=\{E_0, E_1, \cdots, E_{N-1}\}$, the code $[[p, M]]$ is nondegenerate for E if the N errors of E are distributed in N different conjugate pairs respectively, or the code $[[p, M]]$ is degenerate for E if at least two errors of E are included in the same conjugate pair.



Classifications of Quantum Codes

For the quantum code $[[p, M]]$, the code space is spanned by M independent codewords

$$|\psi_0\rangle = S_0 |\psi_0\rangle, \;\; |\psi_1\rangle = S_1 |\psi_0\rangle, \;\; \cdots, \;\; |\psi_{M-1}\rangle = S_{M-1} |\psi_0\rangle.$$

$$\underbrace{\mathcal{B} = \{S_0,\ S_1, \cdots,\ S_{M-1}\}}_{\text{codeword spinors}} \qquad \underbrace{|\psi_0\rangle = \frac{1}{\sqrt{2^m}} \sum_{l=1}^{2^m} (-1)^{\varepsilon_l} |\alpha_l\rangle}_{\text{initial state}}$$

	$\mathcal{B}$ a bi-subalgebra or coset of $\mathfrak{su}(2^p)$	$\{\alpha_l\}$ a subgroup or coset of $\mathbb{Z}_2^p$	
I	Yes	Yes	additive
II	No	Yes	non-additive
III	Yes	No	non-additive
IV	No	No	non-additive

Holy Grails of QIS Theory

Decoherence **Entanglement**

The **sufficient** and **necessary** conditions to determine the **separability** of a quantum state of an arbitrary dimension.

$$\rho^{AB} \overset{?}{=} \sum_j p_j \rho_j^A \otimes \rho_j^B, \qquad \sum_j p_j = 1,\ p_j \geq 0.$$

Found such conditions for a large class of states.

Summary

A quotient-algebra partition in brief is a partition over a unitary Lie algebra where each subspace of the partition is abelian and all the subspaces obey the quaternion condition of closure. The scheme of the quotient algebra has a very wide scope of applications. The subjects are ranging from optimal (scalable) quantum gates, quantum error correction codes, quantum entanglement, coherence and many others of interest currently in rapid progress.