Rotated Einstein-Podolsky-Rosen States

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Outline

States with perfect correlation: (Einstein, Podolsky, Rosen, 1935)

On a bipartite system $\mathcal{AB}$

if the outcome of a measurement A on one subsystem $\mathcal{A}$ is known,

then the outcome of some measurement B on the other subsystem $\mathcal{B}$ can be predicted with certainty.

Outline

Examples of quantum states with perfect correlation:

- continuous system: EPR state (1935)

$$\Psi_{EPR}(x_1, x_2) = \int_{-\infty}^{\infty} e^{(2\pi i/h)(x_1 - x_2 + x_0)p} dp$$

- finite-dimensional system: Bohm state (1951)

$$\Phi_{Bohm} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Outline

Comparison of EPR state and Bohm state:

- Bohm state Φ_{Bohm} is well-defined in $\mathbb{C}^2 \otimes \mathbb{C}^2$;
  EPR state Ψ_{EPR} is not a well-defined vector $\mathscr{L}(\mathbb{R}^2)$!
- all states with perfect correlation on $\mathbb{C}^2 \otimes \mathbb{C}^2$ are unitarily equivalent to Bohm state Φ_{Bohm};
  for continuous system it is unknown.

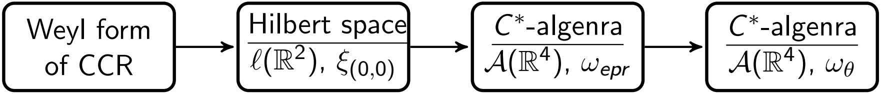
Outline

Goals:

- a well-defined formulation of EPR state
- rotated EPR states
- entanglement properties
- measurement of individual particles

Outline

Overview


Weyl form of CCR
Hilbert space
$\ell(\mathbb{R}^2)$, $\xi_{(0,0)}$
C^*-algenra
$\mathcal{A}(\mathbb{R}^4)$, ω_{epr}
C^*-algenra
$\mathcal{A}(\mathbb{R}^4)$, ω_{θ}

CCR

Wintner (1947): for $\hat{q}$ and $\hat{p}$ satisfying

$$[\hat{q}, \hat{p}] = i\,\mathbb{I}$$

there is no realization of $\hat{q}$ and $\hat{p}$ as bounded operators on Hilbert space.

Weyl form

- Instead of looking for realization of unbounded operators $\hat{q}$ and $\hat{p}$ we look for a realization of some bounded operators.
- Form the unitary operator $W(u)$, $u = (a, b)$

$$W(u) = e^{i(a\hat{q}+b\hat{p})}.$$

  The unitary operator $W(u)$ is clearly bounded.
- **Weyl form of canonical commutation relation (CCR)**:

$$W(u)W(v) = e^{-i\sigma(u,v)/2}W(u+v) \quad (1)$$

$$\sigma(u,v) = u_1v_2 - u_2v_1, \quad u, v \in \mathbb{R}^2. \quad (2)$$

Weyl form

For two particle systems we consider the following operator

$$W(a, b, c, d) = e^{i(a\hat{q}_1 + b\hat{p}_1 + c\hat{q}_2 + d\hat{p}_2)}.$$

Here (a, b) describes the first particle while (c, d) the second particle.

Representation: $(\mathcal{H}, \Omega)$
In order to represent a well-defined EPR state
we want to construct a Hilbert space $\mathcal{H}$
where $W(a, b, c, d)$ acts as a unitary operator on $\mathcal{H}$
and a vector state $\Omega \in \mathcal{H}$ with perfect correlation
which represents the EPR state.

EPR representations

Halvorson (2000):
first C^*-algebra formulation of $W(a,b,c,d)$, $a,b,c,d \in \mathbb{R}$ and then its GNS representation $\ell(\mathbb{R}^2)$.

Here:
first representation $\ell(\mathbb{R}^2)$ and then C^*-algebra formulation

$\ell(\mathbb{R}^2)$, the Hilbert space of square-summable functions from $\mathbb{R}^2$ to $\mathbb{C}$:

$$\begin{aligned} f &: \mathbb{R}^2 \to \mathbb{C}, \\ \langle f|g\rangle &= \sum \overline{f(\lambda,\mu)}g(\lambda,\mu), \\ \|f\| &= \left(\sum |f(\lambda,\mu)|^2\right)^{1/2}. \end{aligned}$$

EPR representations

$\xi_{(\lambda,\mu)}$: the characteristic function of the set $\{(\lambda,\mu)\}$:

$$\xi_{(\lambda,\mu)}(x,y) = \begin{cases} 1 & (x,y) = (\lambda,\mu) \\ 0 & (x,y) \neq (\lambda,\mu) \end{cases}$$

Define the following operator on $\ell(\mathbb{R}^2)$:

$$\begin{aligned} W(a,0,-a,0)\xi_{(\lambda,\mu)} &= e^{ia\lambda}\xi_{(\lambda,\mu)}, \quad (3)\\ W(0,b/2,0,-b/2)\xi_{(\lambda,\mu)} &= \xi_{(\lambda-b,\mu)}, \quad (4)\\ W(c/2,0,c/2,0)\xi_{(\lambda,\mu)} &= \xi_{(\lambda,\mu+c)}, \quad (5)\\ W(0,d,0,d)\xi_{(\lambda,\mu)} &= e^{id\mu}\xi_{(\lambda,\mu)}. \quad (6) \end{aligned}$$

EPR representation

$(\ell(\mathbb{R}^2), \xi_{(0,0)})$ has properties:

- $\xi_{(0,0)}$ is cyclic for $W(a,b,0,0)$, i.e.,

$$\overline{\{W(a,b,0,0)\xi_{(0,0)}\ ;\ a,b \in \mathbb{R}\}} = \ell(\mathbb{R}^2)$$

  and similarly for $W(0,0,c,d)$. Thus $\xi_{(0,0)}$ is entangled.

- $\xi_{(0,0)}$ is perfect correlated,i.e., EPR state:

$$(\xi_{(0,0)}, W(a,b,c,d)\xi_{(0,0)}) = \delta(a+c)\delta(b-d)e^{i(a\lambda_0+b\mu_0)}$$

- $\xi_{(0,0)}$ assigns
  - a dispersion-free value λ_0 to $\hat{q}_1 - \hat{q}_2$ and
  - a dispersion-free value μ_0 to $\hat{p}_1 + \hat{p}_2$.

C^*-algebra

C^*-**algebra formulation:**

- The one-particle system is described by $\mathcal{A}(\mathbb{R}^2)$:

$$\mathcal{A}(\mathbb{R}^2) = \overline{\left\{ \sum_{j=1}^{n} c_j W(u_j) \;\middle|\; u_j = (a_j, b_j) \in \mathbb{R}^2 \right\}}$$

- In $\mathcal{A}(\mathbb{R}^2)$ we have $*$-operation:

$$W(u)^* = W(-u).$$

- relation between the operator $*$ and the norm $\|\cdot\|$:

$$\|A^* A\| = \|A\|^2, \quad \forall A \in \mathcal{A}(\mathbb{R}^2).$$

C^*-algebra

Only observables and states are important!

- An **observable** associated with a measurement procedure corresponds to an element $A \in \mathcal{A}(\mathbb{R}^2)$ with
$$A = A^*.$$
- A **state** ω of a particle is given by a unital positive linear functional on $\mathcal{A}(\mathbb{R}^2)$:
$$\omega : \mathcal{A}(\mathbb{R}^2) \to \mathbb{C}.$$
$\omega(A) =$ the expectation value of the observable A in the state ω.

Two-particle systems

- Let $\mathcal{A} = \mathcal{B} = \mathcal{A}(\mathbb{R}^2)$. The two particle system is then given by
$$\mathcal{A} \otimes \mathcal{B} \cong \mathcal{A}(\mathbb{R}^4)$$
where the elements $W(u) \in \mathcal{A}(\mathbb{R}^4)$ satisfy CCR similarly as (1) with
$$\sigma_2 = \sigma \oplus \sigma.$$
- A state ω on $\mathcal{A}(\mathbb{R}^4)$ is a positive linear functional on $\mathcal{A}(\mathbb{R}^4)$ with norm one:
$$\omega : \mathcal{A}(\mathbb{R}^4) \to \mathbb{C}$$
- Basically,
$$\omega : W(a,b,c,d) \mapsto \omega(W(a,b,c,d)) \in \mathbb{C}.$$

EPR states

Halvorson (2000):
The EPR state ω_{epr} is as a unital positive linear functional on $\mathcal{A}(\mathbb{R}^4)$:

$$\omega_{epr}(W(a,b,c,d)) = \delta(a+c)\delta(b-d)e^{i(a\lambda_0+b\mu_0)}. \tag{7}$$

ω_{epr} assigns

- a dispersion-free value λ_0 to $\hat{q}_1 - \hat{q}_2$ and
- a dispersion-free value μ_0 to $\hat{p}_1 + \hat{p}_2$:

states and representations

Each state ω on a C^*-algebra $\mathcal{A}$ gives a representation $\mathcal{H}$ of $\mathcal{A}$ and ω is represented as a vector Ω in $\mathcal{H}$,

$$(\mathcal{A}, \omega) \Longleftrightarrow (\mathcal{H}, \Omega)$$

Here:

$$\begin{array}{rcl} \mathcal{A} & = & \mathcal{A}(\mathbb{R}^4) \\ \omega = \omega_{epr} & \Longleftrightarrow & \Omega = \xi_{(0,0)} \\ \mathcal{H} & = & \ell(\mathbb{R}^2) \\ \omega_{epr}(W(a,b,c,d)) & = & (\xi_{(0,0)}, W(a,b,c,d)\xi_{(0,0)}) \end{array}$$

$W(a,b,c,d)$ acts on $\ell(\mathbb{R}^2)$ according to Equations (3)–(6) and

$$\overline{\{W(a,b,c,d)\xi_{(0,0)}\ ;\ a,b,c,d \in \mathbb{R}\}} = \ell(\mathbb{R}^2)$$

essential points of ω_{epr}

- ω_{epr} can be viewed as a mapping from the phase space $\mathbb{R}^4$ to $\mathbb{C}$ with some additional conditions:

$$\begin{array}{rcl} \mathbb{R}^4 & \longrightarrow & \mathbb{C} \\ (a,b,c,d) & \longmapsto & \delta(a+c)\delta(b-d)e^{i(a\lambda_0+b\mu_0)} = \omega_{epr}(W(a,b,c,d)) \end{array}$$

- rotation in phase space $\Longleftrightarrow$ rotation of ω_{epr}.

Rotated EPR states

Bohr (1935):
in EPR states $(\hat{q}, \hat{p})$ cab be replaced by $(\hat{Q}, \hat{P})$

$$\hat{Q}_1 = \hat{q}_1 \cos\theta + \hat{q}_2 \sin\theta, \qquad \hat{Q}_2 = -\hat{q}_1 \sin\theta + \hat{q}_2 \cos\theta,$$
$$\hat{P}_1 = \hat{p}_1 \cos\theta + \hat{p}_2 \sin\theta, \qquad \hat{P}_2 = -\hat{p}_1 \sin\theta + \hat{p}_2 \cos\theta.$$

1. This corresponds to a rotation R_θ in phase space.
2. The commutation relation remains, i.e.,

$$[\hat{Q}_j, \hat{P}_j] = i, \quad j = 1, 2.$$

Rotated EPR states

Remark of Bohr (1935) + ω_{epr} by Halvorson (2000)

$\Rightarrow$ rotated EPR states (Huang, 2007)

Rotated EPR states

Introduce a new basis $E_\theta = [u_1, v_1, u_2, v_2]$ for $\mathbb{R}^4$,

$$u_1 = (\cos\theta, 0, \sin\theta, 0) \qquad v_1 = (0, \cos\theta, 0, \sin\theta),$$
$$u_2 = (-\sin\theta, 0, \cos\theta, 0), \qquad v_2 = (0, -\sin\theta, 0, \cos\theta).$$

$$W(u_1) \Leftrightarrow \hat{Q}_1 \qquad W(v_1) \Leftrightarrow \hat{P}_1$$
$$W(u_2) \Leftrightarrow \hat{Q}_2, \qquad W(v_2) \Leftrightarrow \hat{P}_2.$$

Rotated EPR states

Idea:
A state which assigns sharp values to $W(u_1)$ and $W(v_2)$ has the same perfect correlation as the EPR states.

Definition:
A **rotated EPR state** ω_θ is a unital positive linear functional on $\mathcal{A}(\mathbb{R}^4)$ such that $\hat{Q}_1$ and $\hat{P}_2$ have the sharp value 0 can then be defined as

$$\omega_\theta(W(a,b,c,d)_\theta) = \delta_{b,0}\delta_{c,0} \tag{8}$$

$(a,b,c,d)_\theta$ denote the coordinates with respect to the basis E_θ.
(Here: for simply $\lambda_0 = 0$, $\mu_0 = 0$ for ω_θ and ω_{epr})

Royated states and EPR state

Some Remarks:

1. ω_θ with $\theta = n\pi$ or $(n \pm 1/2)\pi$ is a product state.
2. ω_{epr} corresponds to ω_θ with $\theta = -\pi/4$.
3. Let R_θ denotes the rotation $R_\theta(a,b,c,d)_\theta = (a,b,c,d)_{\theta=-\pi/4}$ Then we have

$$\begin{aligned}\omega_\theta(W(a,b,c,d)_\theta) &= \omega_{epr}((W(a,b,c,d)_{-\pi/4})) \\ &= \omega_{epr}(\tau_\theta(W(a,b,c,d)_\theta))\end{aligned}$$

where τ_θ is the *-automorphism of $\mathcal{A}(\mathbb{R}^4)$ corresponding to R_θ.

Rotated EPR states and their representations

Define $W(a,b,c,d)$ on $\mathcal{B}(\ell(\mathbb{R}^2))$:

$$\begin{aligned} W(a,0,0,0)\xi_{(\lambda,\mu)} &= e^{ia\cos\theta\lambda}\xi_{(\lambda,\mu-a\sin\theta)}, \\ W(0,b,0,0)\xi_{(\lambda,\mu)} &= e^{-ib\sin\theta\mu}\xi_{(\lambda-b\cos\theta,\mu)}, \\ W(0,0,c,0)\xi_{(\lambda,\mu)} &= e^{ic\sin\theta\lambda}\xi_{(\lambda,\mu+c\cos\theta)}, \\ W(0,0,0,d)\xi_{(\lambda,\mu)} &= e^{id\cos\theta\mu}\xi_{(\lambda-d\sin\theta,\mu)}. \end{aligned}$$

$\xi_{(0,0)}$ has the following properties:

$$\overline{\{W(a,b,c,d)\xi_{(0,0)}\ ;\ a,b,c,d\in\mathbb{R}\}} = \ell(\mathbb{R}^2)$$

$$\omega_\theta(A) = (\xi_{(0,0)}, \pi_\theta(A)\xi_{(0,0)})$$

Rotated EPR representation

Property I:

$\xi_{(0,0)}$ has the entanglement property:

$$\overline{\{W(a,b,0,0)\xi_{(0,0)}\ ;\ a,b\in\mathbb{R}\}} = \overline{\{W(0,0,c,d)\xi_{(0,0)}\ ;\ c,d\in\mathbb{R}\}} = \ell(\mathbb{R}^2).$$

Rotated EPR representation

Property (II):

Halvorson (2000):
ω_{epr} maximally violate Bell's inequalities.

Here: similar arguments as Halvorson (2000),
ω_θ maximally violate Bell's inequalities.

Rotated EPR states

Property (III):

ω_θ has the perfect correlation:

If the outcome of one measurement on one subsystem is obtained, then the outcome of some measurement on the other subsystem can be predicted with certain.

Rotated EPR states

Property (IV):

No information about individual particle can be obtained!

- The operator $\hat{q}_1$ does not exist.
  Due to the weak discontinuity of $\omega_\theta(W(a,0,0,0))$

$$\begin{aligned} \omega_\theta(W(a,0,0,0)) &= \langle \xi_{(0,0)}, \pi(W(a,0,0,0))\xi_{(0,0)} \rangle \\ &= \begin{cases} 1, & a = 0 \\ 0, & a \neq 0 \end{cases}, \end{aligned}$$

  the limit does not exist!

$$\lim_{a \to 0} \frac{W(a,0,0,0) - \mathbb{I}}{a} \xi_{(0,0)} =: i\hat{q}\, \xi_{(0,0)}$$

- Similarly, $\hat{q}_2$, $\hat{p}_1$ and $\hat{p}_2$ does not exist.

Rotated EPR states

- Keyl, Schlingemann, and Werner (2003):
  In an EPR states the probability of finding a particle at infinity is one!
- Halvorson (2004):
  In any representation where the position operator has eigenstates, there is no momentum operator, and vice versa.

Rotated EPR states

Property (V):

The uncertainty principle implies that two representations ω_θ and $\omega_{\theta'}$ are not unitarily equivalent if $\theta \neq \theta' + n\pi$ or $\theta \neq \theta' + (n + 1/2)\pi$, i.e., there is no unitary operator U on $\ell(\mathbb{R}^2)$ such that

$$\begin{aligned} W_\theta(x) &= U^\dagger W_{\theta'}(x) U \\ \xi_{(0,0)} &= U \xi_{(0,0)} \end{aligned}$$

where W_θ denotes operations according θ on $\ell(\mathbb{R}^2)$.

Physical meaning: external unitary operator outside the observable algebra $\mathcal{A}(\mathbb{R}^4)$ is necessary to change ω_θ into $\omega_{\theta'}$

Rotated EPR states

- For finite systems $M_n \otimes M_n$:
  states with perfect correlation:

$$\Phi = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |e_j f_j\rangle, \quad \{e_j\}, \{f_j\} \text{ ONB for } \mathbb{C}^n$$

  All states satisfying perfect correlation can transfered into each other by a unitary operator on the same vector space $\mathbb{C}^n \otimes \mathbb{C}^n$.

- For infinite systems $\mathcal{A}(\mathbb{R}^n)$:
  there exists non-unitarily equivalent states satisfying perfect correlation.
  $\Rightarrow$ New entanglement phenomenon.

Summary

1. Rotated EPR states ω_θ is constructed!

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2. No information of individual particle from ω_θ can be obtained!

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1. Rotated EPR states ω_θ is constructed!
2. No information of individual particle from ω_θ can be obtained!
3. ω_θ non-unitarily equivalent to $\omega_{\theta'}$
4. Another construction ω_ϕ:

$$\omega_\phi(W(a,b,c,d)_\phi) = \delta_{b,0}\delta_{c,0}e^{i(a\lambda_0 + d\mu_0)}$$

$$\begin{aligned} s_1 &= (\cos\phi, 0, 0, \sin\phi) & t_1 &= (0, \cos\phi, -\sin\phi, 0), \\ s_2 &= (-\sin\phi, 0, 0, \cos\phi), & t_2 &= (0, -\sin\phi, -\cos\phi, 0). \end{aligned}$$

Thank you for your attention!