Entanglement Measure on 1 & 2 Dimensional Spin Systems

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Talk at QST 09

Based on work in progress with

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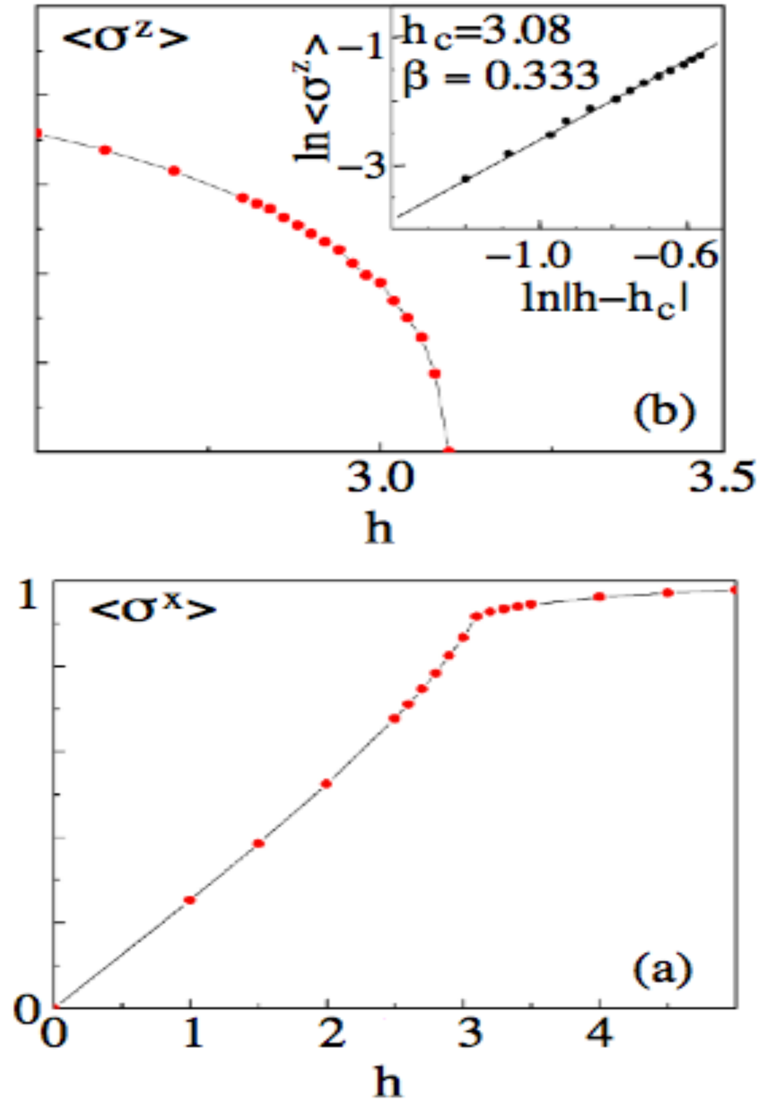
Outline

- Quantum Phase Transition
- Entanglement measure
- Tensor-entanglement RG
- Entanglement scaling
- Dynamics related issues

Quantum Phase transition

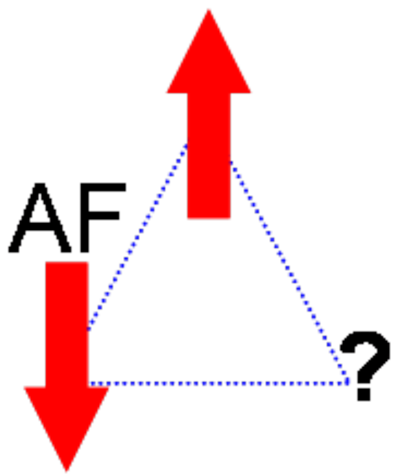
- Phase transition due to tuning of the coupling constant.
- For example, the Heisenberg model---spin systems.
- How to characterize quantum phase transition?

    a. Order parameters



Frustrated systems

- These systems usually have highly degenerate ground states.
- This implies that the entropy measure can be used to characterize the quantum phase transition.
- Especially, there are topological phases for some frustrated systems (no order parameters)



Entanglement measure

- The degeneracy of quantum state manifests as quantum entanglement.
- The entanglement measure can be used to characterize the topological phases.
- However, there is no universal entanglement measure of many body systems.
- Instead, the entanglement entropy like quantities are used.

Global entanglement

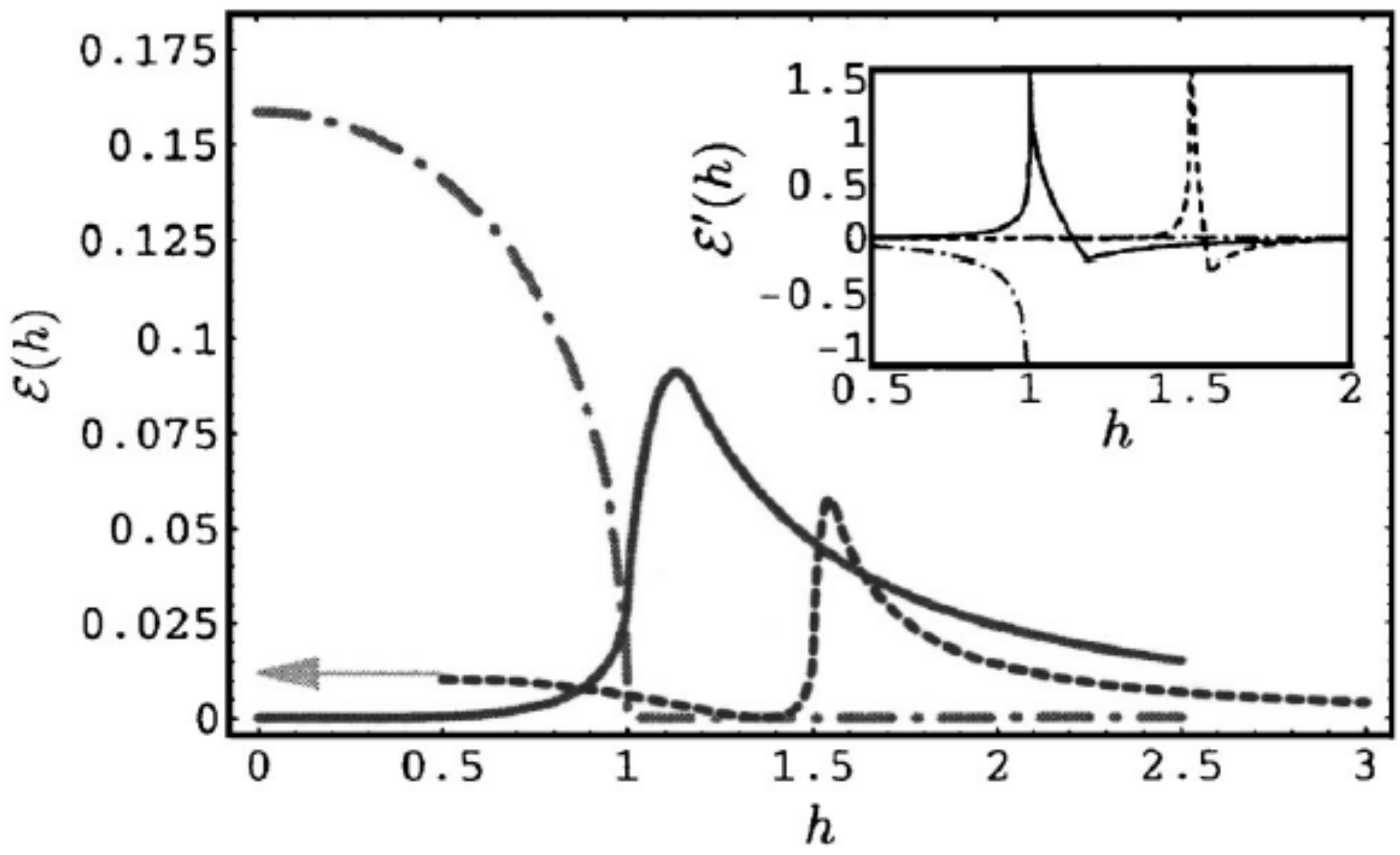
Global entanglement 【Wei et al. PRA 68, 042307(2003)】

- Consider pure states of N particles $|\psi\rangle$. The global multipartite entanglement $|\psi\rangle$ can be quantified by considering the maximum fidelity.

$$\Lambda_{\max} = \max\left|\langle\psi|\phi\rangle\right|$$

- The larger $\Lambda_{\max}$ indicates that the less entangled.
- A well-dfiened global measure of entanglement is

$$E(\psi) = -\log \Lambda_{\max}^2$$



Global entanglement density and its *h derivative for the ground state of three systems at N*. Ising*; anisotropic XY model; XX model.*

Our aim

- We are trying to check if the entanglement measure can characterize the quantum phase transition in 2d spin systems.
- Should rely on the numerical calculation
- Global entanglement can pass the test but is complicated for numerical implementation.
- One should look for more easier one.

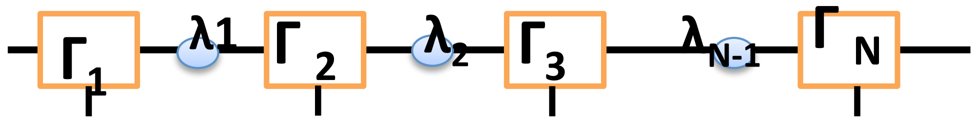
Matrix/Tensor Product States

- The numerical implementation for finding the ground states of 2d spin systems are based on the matrix/tensor product states.
- These states can be understood from a series of Schmidt (bi-partite) decomposition. It is QIS inspired.
- The ground state is approximated by the relevance of entanglement.

1D quantum state representation

Vidal 03

$$|\Psi\rangle = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} c_{i_1 \ldots i_n} \, | i_1 \ldots i_n \rangle$$



$$c_{i_1 \ldots i_n} = \sum_{\alpha_1 \ldots \alpha_{n-1}} \Gamma^{[1] i_1}_{\alpha_1} \; \lambda^{[1]}_{\alpha_1} \; \Gamma^{[2] i_2}_{\alpha_1 \alpha_2} \; \lambda^{[2]}_{\alpha_2} \; \Gamma^{[3] i_3}_{\alpha_2 \alpha_3} \ldots \; \Gamma^{[n] i_n}_{\alpha_{n-1}}$$

parameters » $nd\chi^2 << d^n$

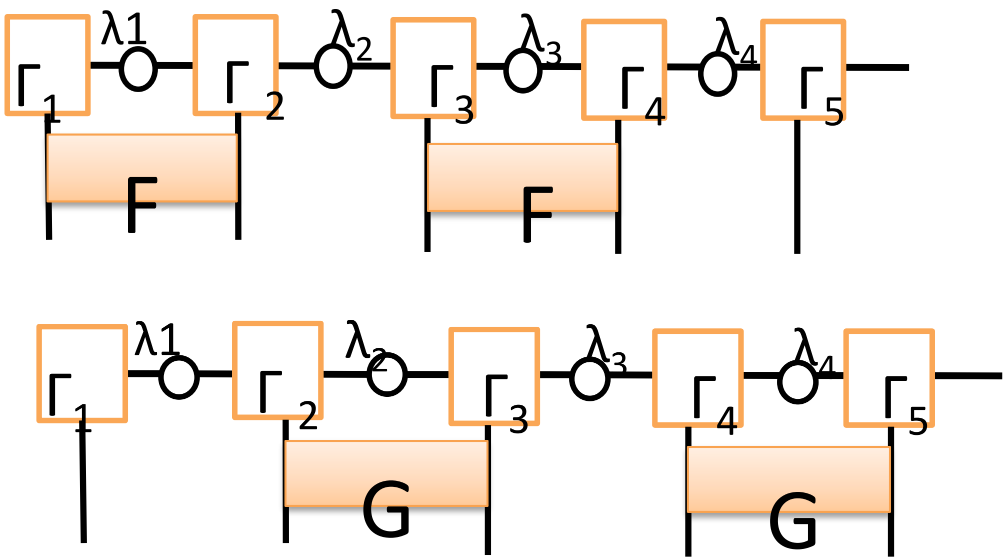
- Representation is efficient
- Single qubit gates involve only local update
- Two-qubit gates reduces to local updating

``Solve" for MPS/TPS

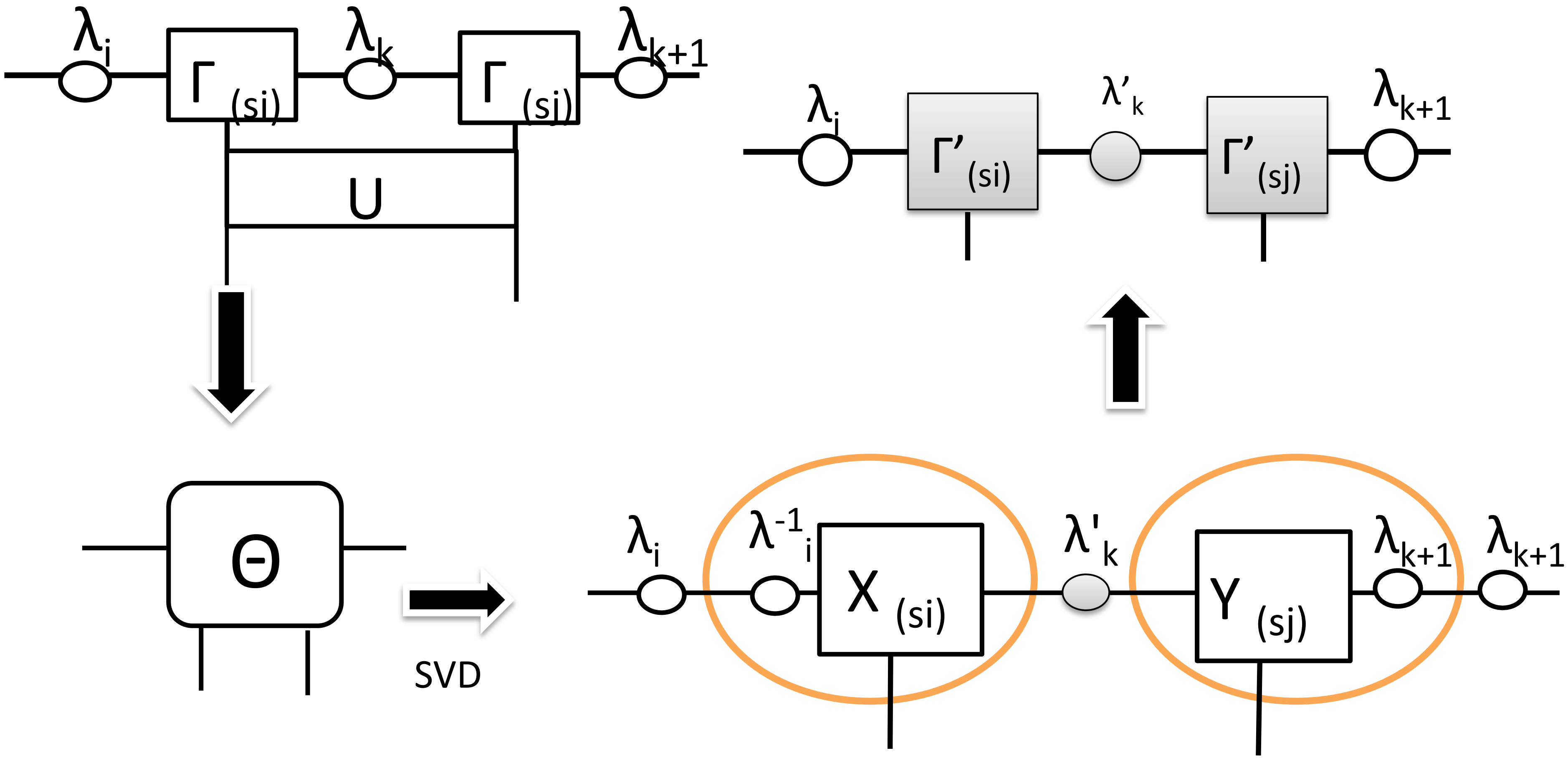
- In old days, the 1d spin system is numerically solved by the DMRG(density matrix RG).
- DMRG=Solving MPS based on von Neuman entropy
- For translationally invariant state, one can solve the MPS by variational methods.
- More efficiently by infinite time-evolving block decimation(iTEBD) method.

The dynamic of quantum state

- Real time $|\Psi_t\rangle = \exp(\text{-iHt})|\Psi_0\rangle$
- imaginary time $|\Psi_\tau\rangle = \dfrac{\exp(\text{-Ht})|\Psi_0\rangle}{\| \exp(\text{-Ht})|\Psi_0\rangle \|}$
- Trotter expansion $e^{-i(F+G)T} = [\, e^{-i(F+G)T\delta}\,]^{T/\delta}$



Evolution quantum state



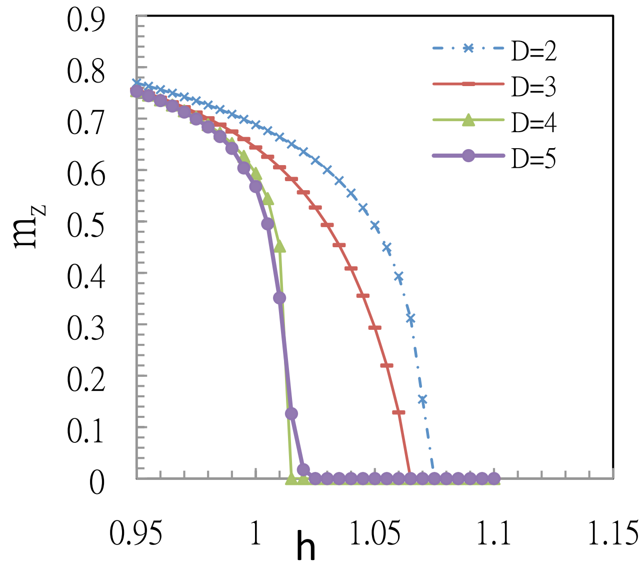
Matrix product state for 1D Ising model

- Ising model in a transverse magnetic field

$$H = -\sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z + h\ \sigma_i^x$$

- The ground state of H exhibits a second-order quantum phase transition as h is tuned across a critical value h = 1.

- Magnetization along the z direction .
  The derivative of magnetization
  has a singularity at h=1.01   (D = 5)



Global entanglement from matrix product state for 1D system

- Matrix product state (MPS) form

$$|\psi\rangle = \sum_{p_1,p_2\dots p_N=1}^{d} Tr(A^{p_1}A^{p_2}\dots A^{p_N})\,|p_1,p_2\dots p_N\rangle$$

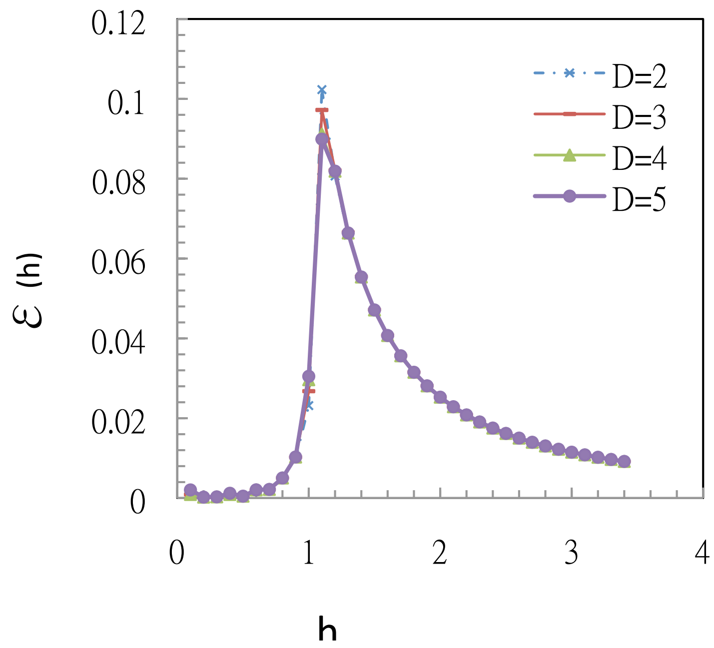
- For a system of N spin 1/2.
  the separable states $|\phi\rangle = \otimes_{i=1}^{n}(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle)$
  represented by a matrix $B^{[i]}(0)=\cos\theta_i\,,\; B^{[i]}(1)=\sin\theta_i$

- the fidelity can be written by transfer matrix
  $|\langle\psi|\phi\rangle| = |Tr\,(T_g^1 T_g^2 T_g^3 \dots T_g^n)|$ where $T_g^{[i]} = \sum_{s_i} A^{[i]}(s_i)\otimes B^{[i]}(s_i)$

【Q.-Q. Shi, R. Orus, J.-O. Fjaerestad, H.-Q. Zhou, arXiv:0901.2863 (2009)】

Global entanglement from matrix product state for 1D Ising model

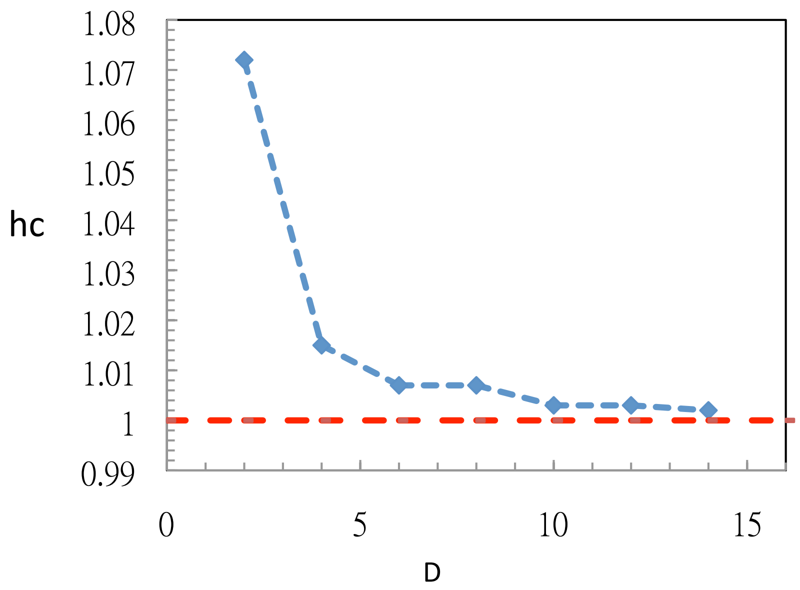
- The global entanglement density and its h, the entanglement has a nonsingular maximum at h=1.1.



- The von Neuman entanglement entropy also has similar feature.

D-effectiveness for MPS of 1D Ising model

- Effective critical point hc* as a function of D



Tensor product state for 2D spin systems

- Tensor product state (TPS) form

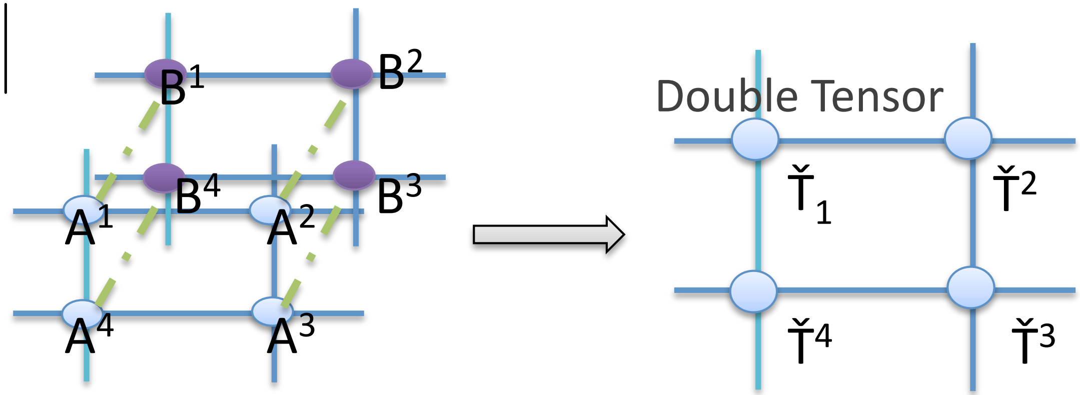
$$|\psi\rangle = \sum_{p_{1,1},p_{2,1}\dots p_{N,N}=1}^{d} Tr(A^{p_{1,1}}A^{p_{2,1}}\dots A^{p_{N,N}})\,|p_{1,1},p_{2,1}\dots p_{N,N}\rangle$$

- The fidelity can be written by transfer matrix

$$|\langle\psi|\phi\rangle| = |tTr\,(T_g^1 T_g^2 T_g^3 \dots T_g^n)|$$

where

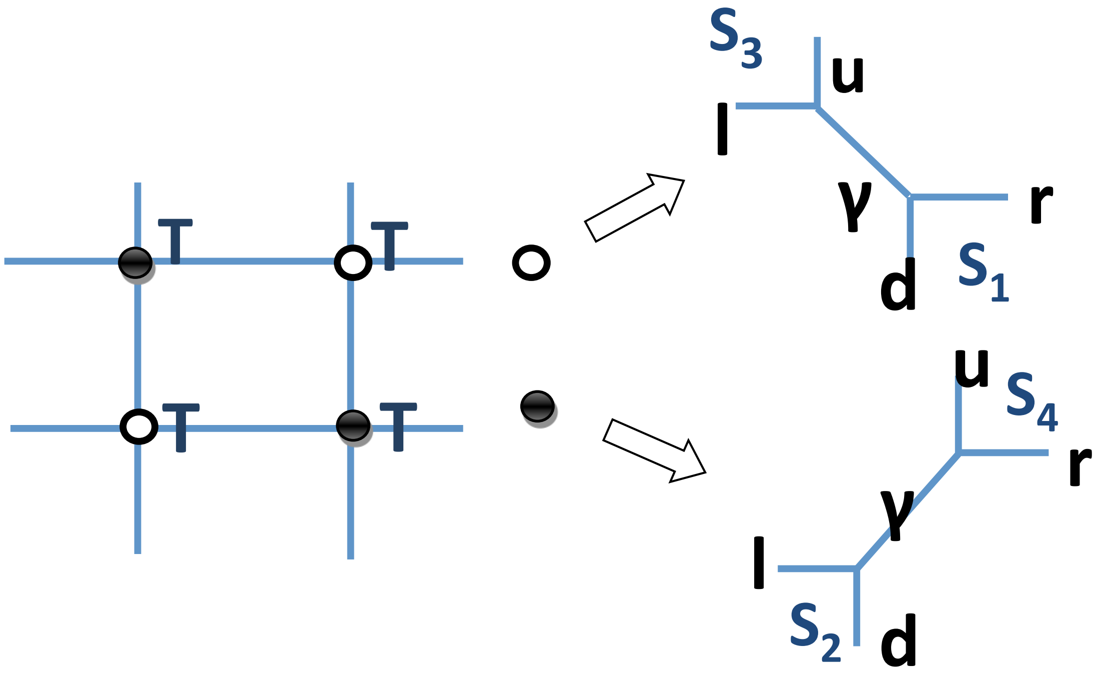
$$T_g^{[i]} = \sum_{s_i} A^{[i]}(s_i)\otimes B^{[i]}(s_i)$$



- It is difficult to calculate tensor trace (tTr), so we using the "TNRG" method to reduce the exponentially calculation to a polynomial calculation.

The TNRG method-I

- First, decomposing the rank-four tensor into two two rank-three tensors.



$$T_{rd;lu} = U \Lambda V^{+}$$

$$S^{1}_{rd\gamma} = \sqrt{\lambda_\gamma}\ U_{rd,\gamma}$$

$$S^{3}_{lu\gamma} = \sqrt{\lambda_\gamma}\ V_{lu,\gamma}$$

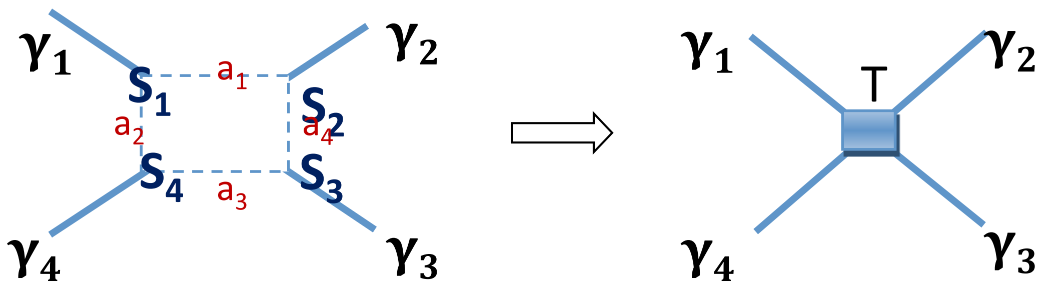
$$T_{ur,dl} = U \Lambda V^{+}$$

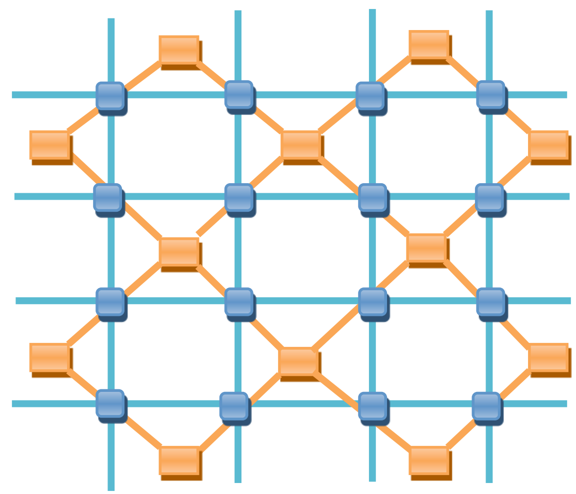
$$S^{4}_{ur\gamma} = \sqrt{\lambda_\gamma}\ U_{ur,\gamma}$$

$$S^{2}_{dl\gamma} = \sqrt{\lambda_\gamma}\ V_{dl,\gamma}$$

The TNRG method-II

- The second step is to build a new rank-four tensor. This introduces a coarse-grained square lattice.





$$\bar{T}'_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} = \sum_{a_1,a_2,a_3,a_4} S^1_{a_1 a_2 \gamma_1} S^2_{a_4 a_1 \gamma_2} S^3_{a_3 a_4 \gamma_3} S^4_{a_2 a_3 \gamma_4}$$

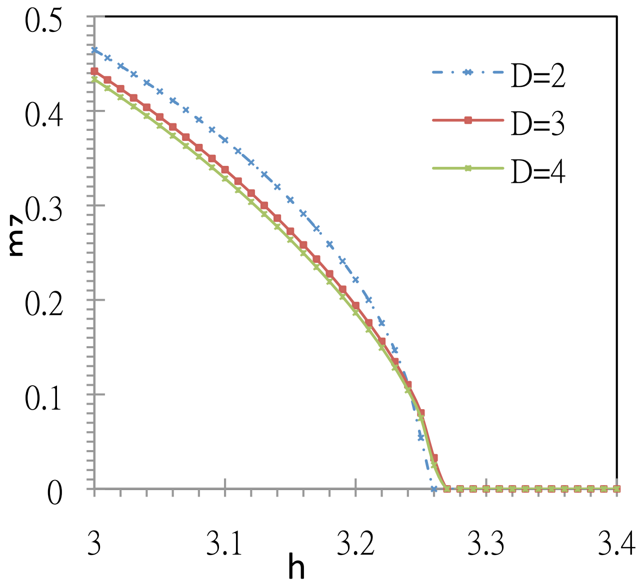
- Repeat the above two steps, until there are only four sites left. One can trace all bond indices to find the fidelity of the wave function.

Order parameter from MPS of transverse Ising

- Ising model in a transverse magnetic field

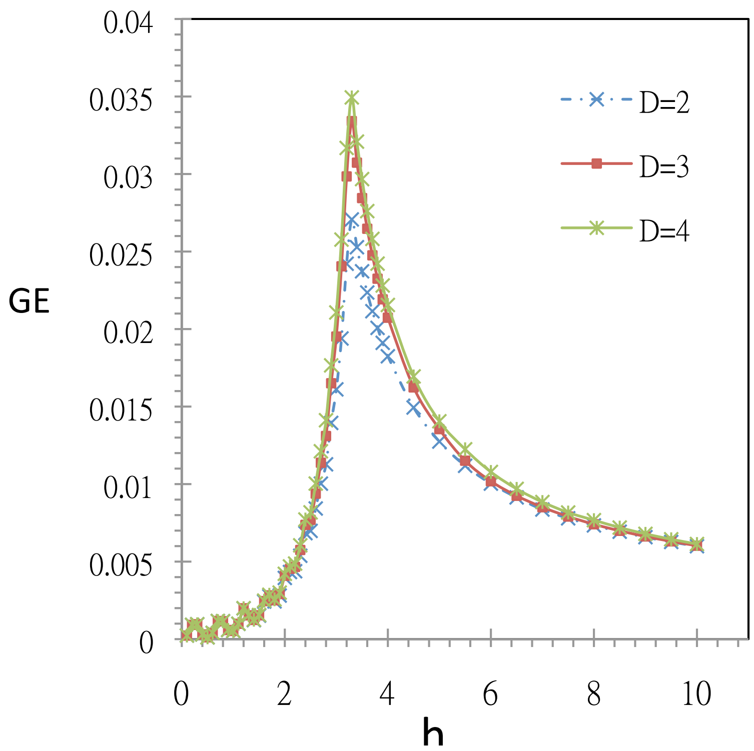
$$H = -\sum_{<i,j>} \sigma_i^z \sigma_j^z + h \ \sigma_i^x$$

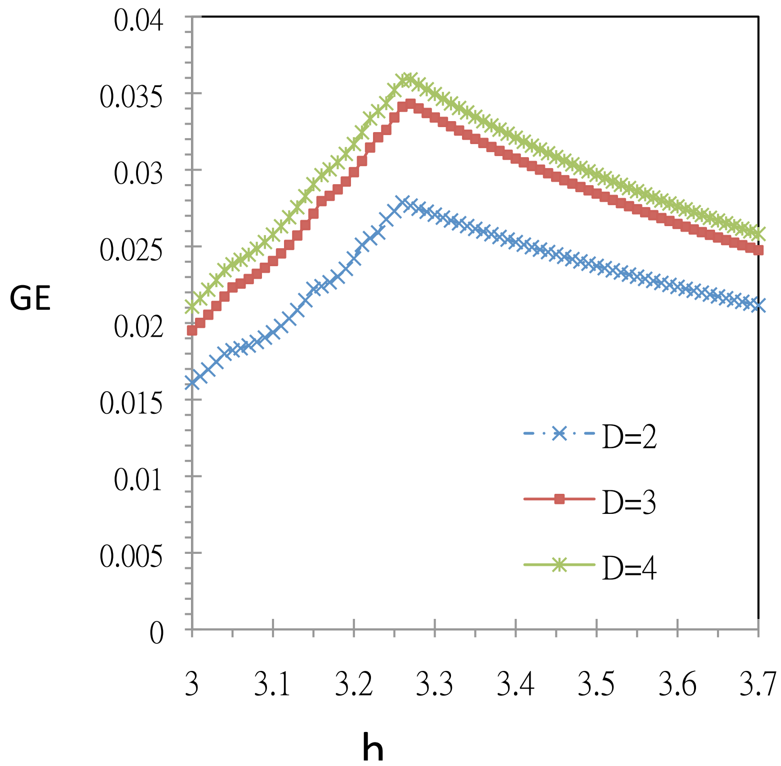
- Using the numerical method to compute the ground state wave function based on TPS representation.
- The transition point between the magnetic and paramagnetic states is h=3.04 unbiased quantum Monte and the mean-field value h=4.
- The derivative of magnetization has a singularity at h=3.25



Global entanglement for 2D transverse Ising

- The global entanglement density v.s. h, the entanglement has a nonsingular maximum at h=3.25.

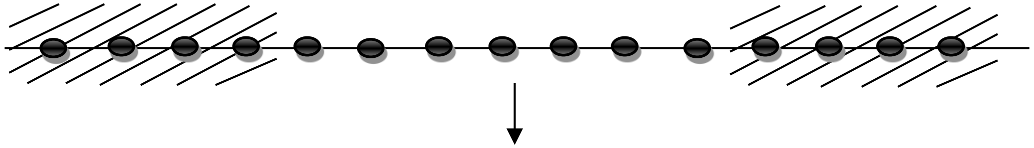




Scaling of entanglement

Entanglement entropy for a reduced block in spin chains

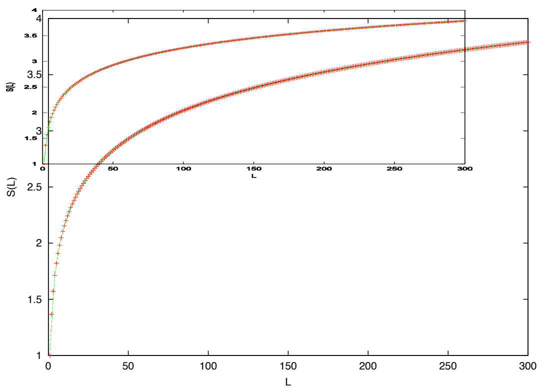
L spins



$\rho_L \longrightarrow S_L$

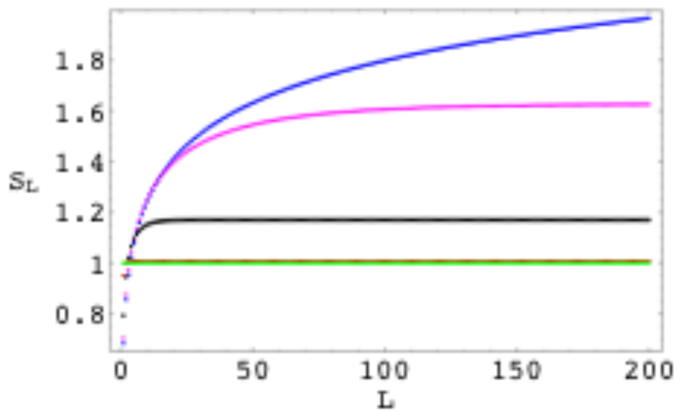
【 Vidal et al. PRL,90,227902 (2003) 】

At Quantum Phase Transition



$$S_L \xrightarrow[L\to\infty]{} \frac{c}{3}\log_2 L$$

Away from Quantum Phase Transition



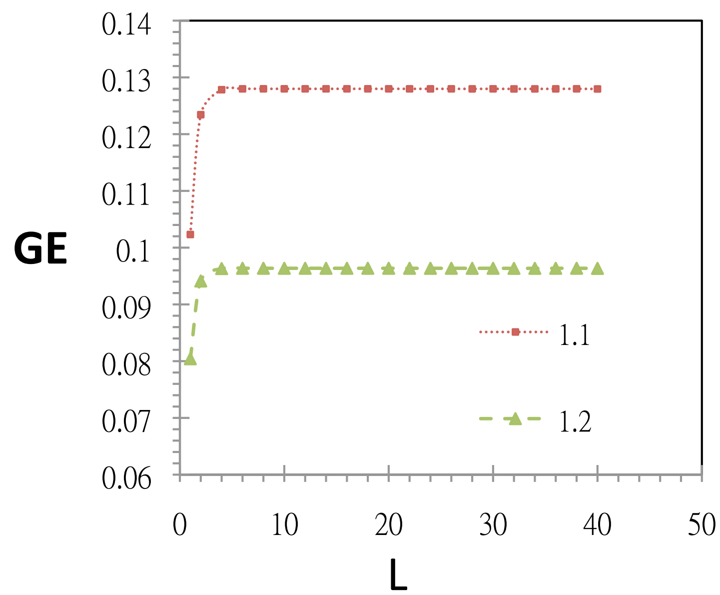
$$S_{L=N/2\to\infty} = \frac{c}{6}\log_2 |1-h|$$

Scaling of entanglement

【Latorre et al. PRB,78,024410 (2008)】

- For matrix product state representation for a fixed value of D , this entropy saturates at a distance $L \sim D^{\kappa}$ *(kind of correlation length)*
- Behaviors of 1D global entanglement off critical point

Could we attain the entropy scaling at critical point from MPS?



Scaling of entanglement

- In 2D system, we want to calculate the block entropy and block global entanglement.

Could we find the scaling behavior like 1D system?

$$S_{LxL} \xrightarrow[L \to \infty]{} L(1 + c \ln L)$$

Area law

Block GE v.s. Quantum state RG

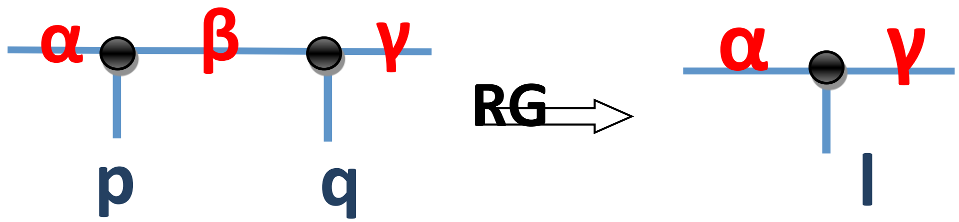
- Numerically, it is involved to calculate the block entanglement because the block trial state is complicated.
- Instead (Wei '08): entanglement per block of size L = entanglement per site of the L-th time quantum state RG(merging of sites).
- We can the use quantum state RG to check the scaling law of GE.

1D Quantum state RG

Quantum state RG 【Verstraete et al. PRL 94, 140601 (2005)】

- Map two neighboring spins to one new block spin

$$\widetilde{A}^{(pq)}_{(\alpha\gamma)} := \sum_{\beta=1}^{D} A^{p}_{\alpha\beta} A^{q}_{\beta\gamma}$$

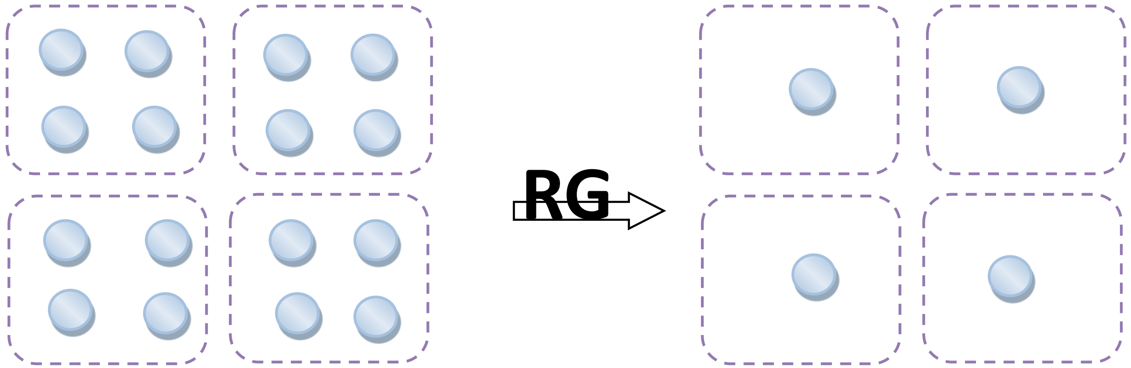


- Identify states which are equivalent under local unitary operations.

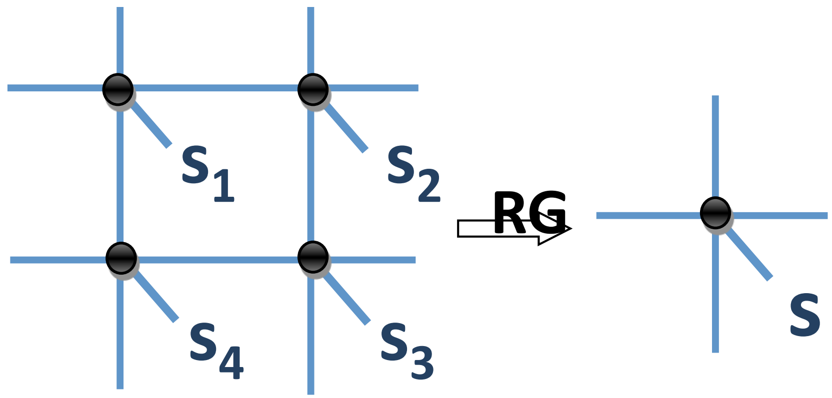
By SVD $$\widetilde{A}^{(pq)}_{(\alpha\gamma)} = \sum_{l=1}^{\min(d^2,D^2)} U^{+(pq)}_{l} \lambda^{l} \mathrm{V}^{1}_{(\alpha\gamma)}$$

$$\therefore A^{p} \xrightarrow{RG} A'^{l} = \lambda^{l} V^{l}$$

2D Quantum state RG



- Map one block of four neighboring spins to one new block spin.
- By TRG and SVD method



Dynamics related issues

- Use real time iTEBD, one can study some dynamical issue related to quantum entanglement.
- One is to study evolution of the global entanglement, such as its creation and death.
- The other is to see if entanglement will help the quantum state transfer or not.

Quantum state transfer

- Teleportation

We need maxima entangle state and local operators between sender and receiver.

C. H. Bennett et al., Phys. Rev. Lett. 70, 1895 (1993).

- Swap

It needs a series of swap gates

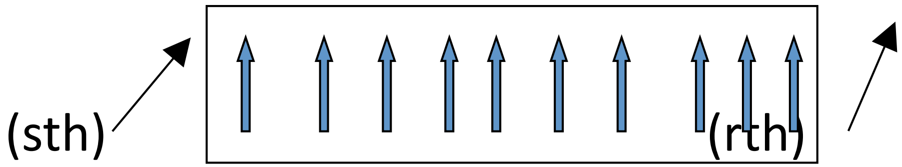
- The spin chain

S. Bose, Phys. Rev. Lett. ***91****, 207901 2003*

The protocol

Step1.

The initial quantum state between sender(sth) and receiver(rth) is ground state.



$$|0000...00000\rangle$$

Step2.

The sender prepares a encoded spin (s=1/2).

$$|\psi\rangle_{in} = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

- Step3.

Both sender and receiver open the interaction with quantum channel, and the state evolves with Heisenberg interaction.

$$H = -J \sum_{<i,j>} \vec{\sigma^i} \bullet \vec{\sigma^j} - \sum_{i=1}^{N} B_i \sigma_z^i$$

- Step4.

The average fidelity in the Bloch sphere.

$$F_{av} = \frac{1}{4\pi} \int {}_{in}\langle \psi | \rho_{rth}(t) | \psi \rangle_{in} \, d\Omega$$

ρ_{rth} is reduced density matrix for rth site.

The channel entanglement

- What will happen if we change the initial unentangle quantum channel to entangled state?
- Is it faster to reach the optimal fidelity?
- will the fidelity become better when the channel is entangled at beginning?