Theory of Quantum Entanglement

Shao-Ming Fei

Capital Normal University, Beijing

Universität Bonn, Bonn

Richard Feynman 1980

Certain quantum mechanical effects cannot be simulated efficiently on a classical computer

Peter Shor 1994

polynomial time quantum algorithm for factoring integers   15=3 x 5

Number of order 10^{130}

limit of current classical method

42 days, number field sieve, 10^{12} operations/second

Number of order 10^{260}

classically intractable (million years)

quantum algorithm: 8 times longer

Classical bit: 0 or 1

Quantum bit (qubit): 2-d complex vector

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\alpha\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1$$

Quantum measurement:

$$\begin{cases} |1\rangle : & |\langle 1|\alpha\rangle|^2 = |a|^2 \\ |0\rangle : & |\langle 0|\alpha\rangle|^2 = |b|^2 \end{cases} \qquad \langle\alpha| = (|\alpha\rangle)^\dagger$$

Cryptography

♠ Cryptography with private key

A	B	C	D	E	...	...	X	Y	Z		?	,	.
00	01	02	03	04	...	...	23	24	25	26	27	28	29

S	H	A	K	E	N		N	O	T		S	T	I	R	R	E	D	
18	07	00	10	04	13	26	13	14	19	26	18	19	08	17	17	04	03	*Plaintext number* (P)
15	04	28	13	14	06	21	11	23	18	09	11	14	01	19	05	22	07	*Key numbers* (K)
03	11	28	23	18	19	17	24	07	07	05	29	03	09	06	22	26	10	*Code number* (C)

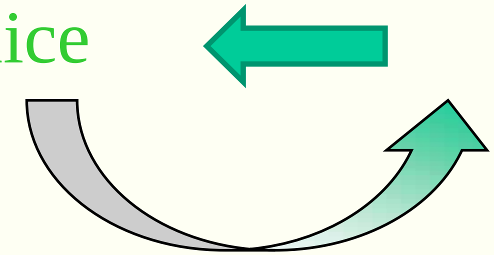
Key numbers: randomly selected from $0 \rightarrow 29$

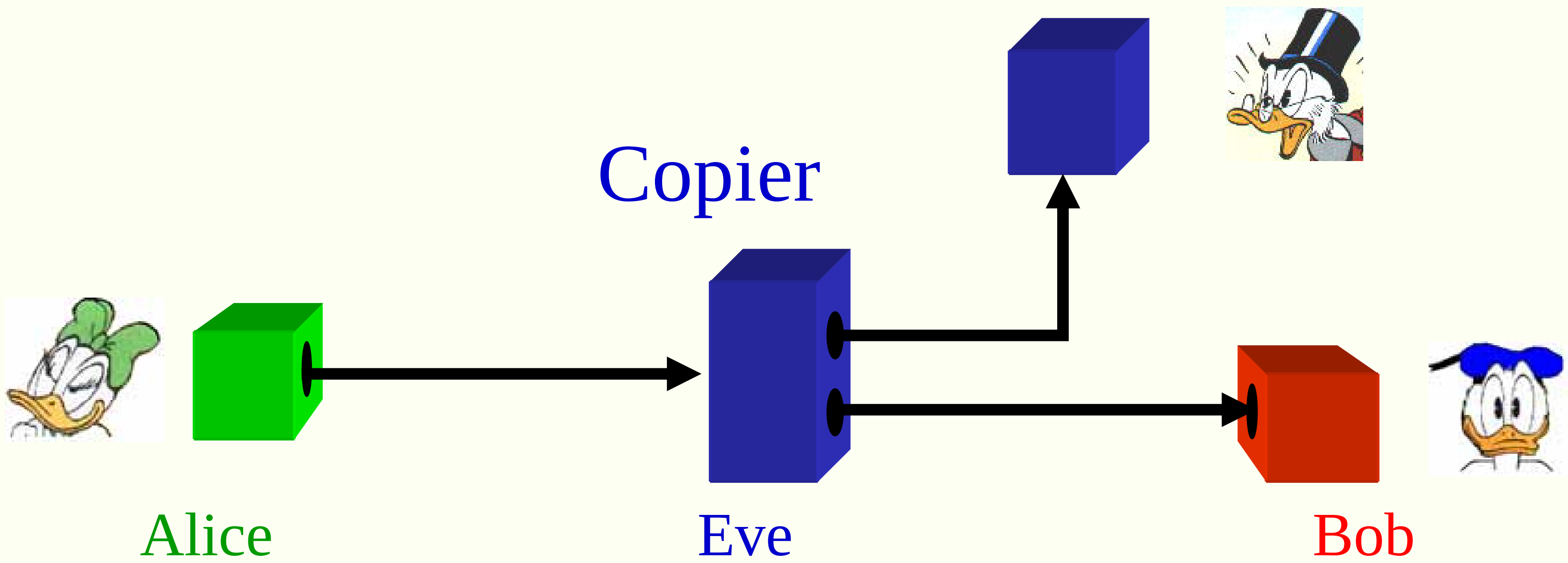
$$C = P + K \pmod{30}$$

Alice sends C to Bob

Bob: decription $C - K \pmod{30} = P$

No private key: Alice ← Bob $N = P\,Q$

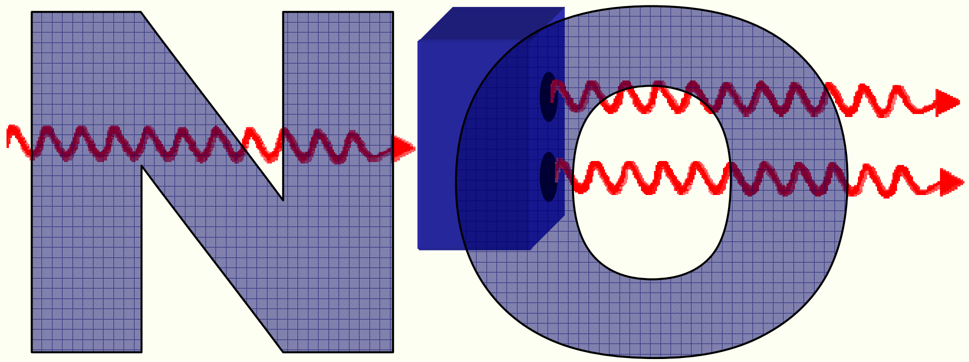




Impossible : Quantum Copier

No-Cloning

Copier:

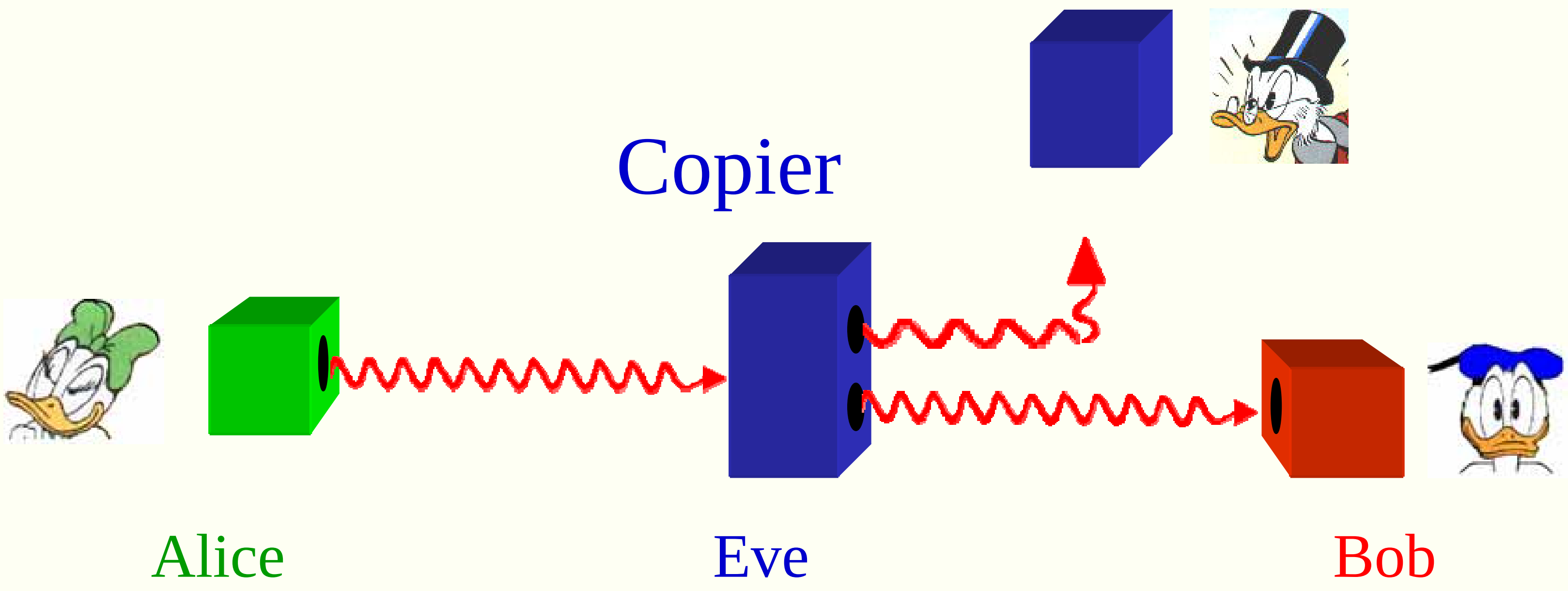


Quantum information is a **new** kind of information

Application:

Quantum Cryptography

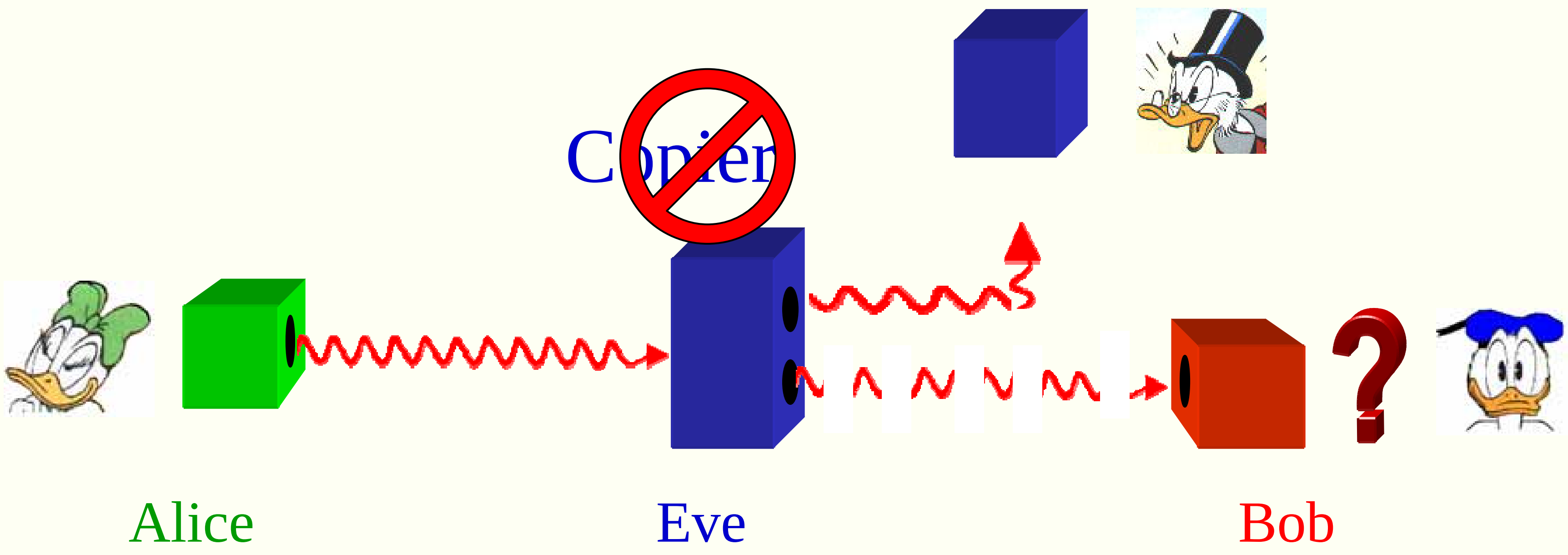
eavesdropping on quantum information ?



Application:

Quantum Cryptography

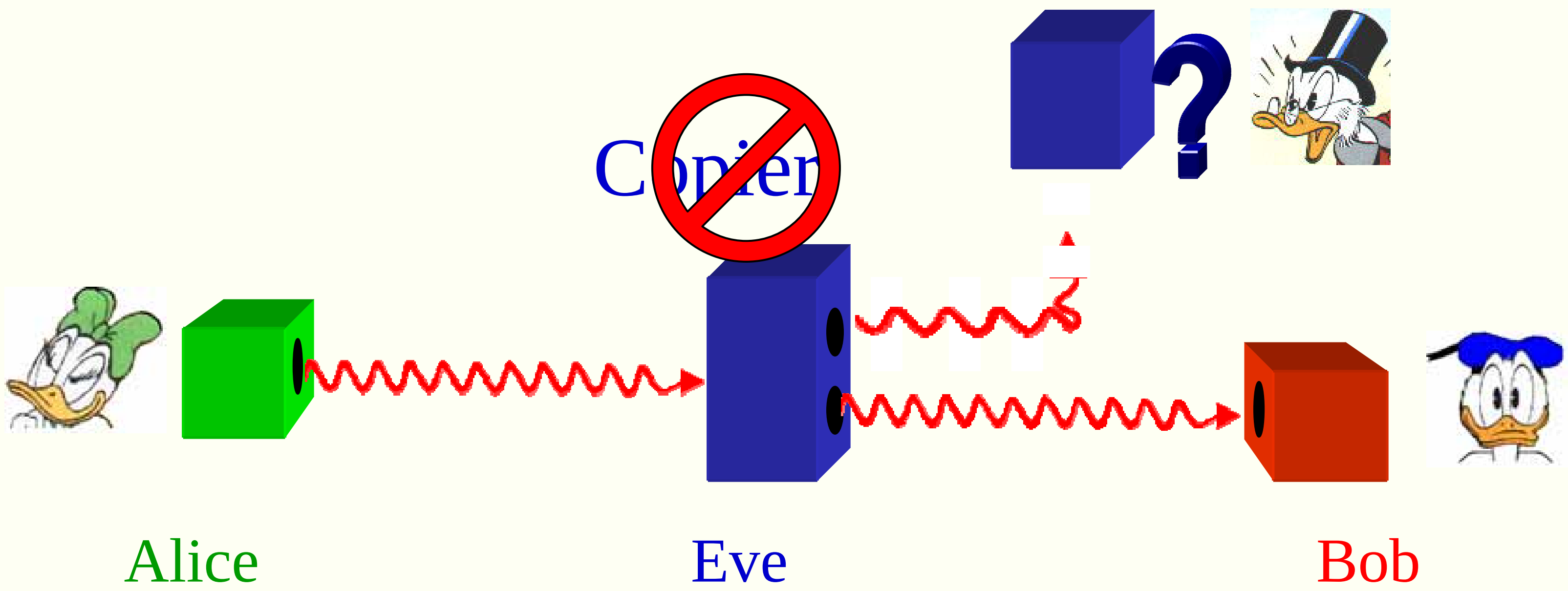
Detected eavesdropping on quantum information



Application:

Quantum Cryptography

Failed eavesdropping on quantum information

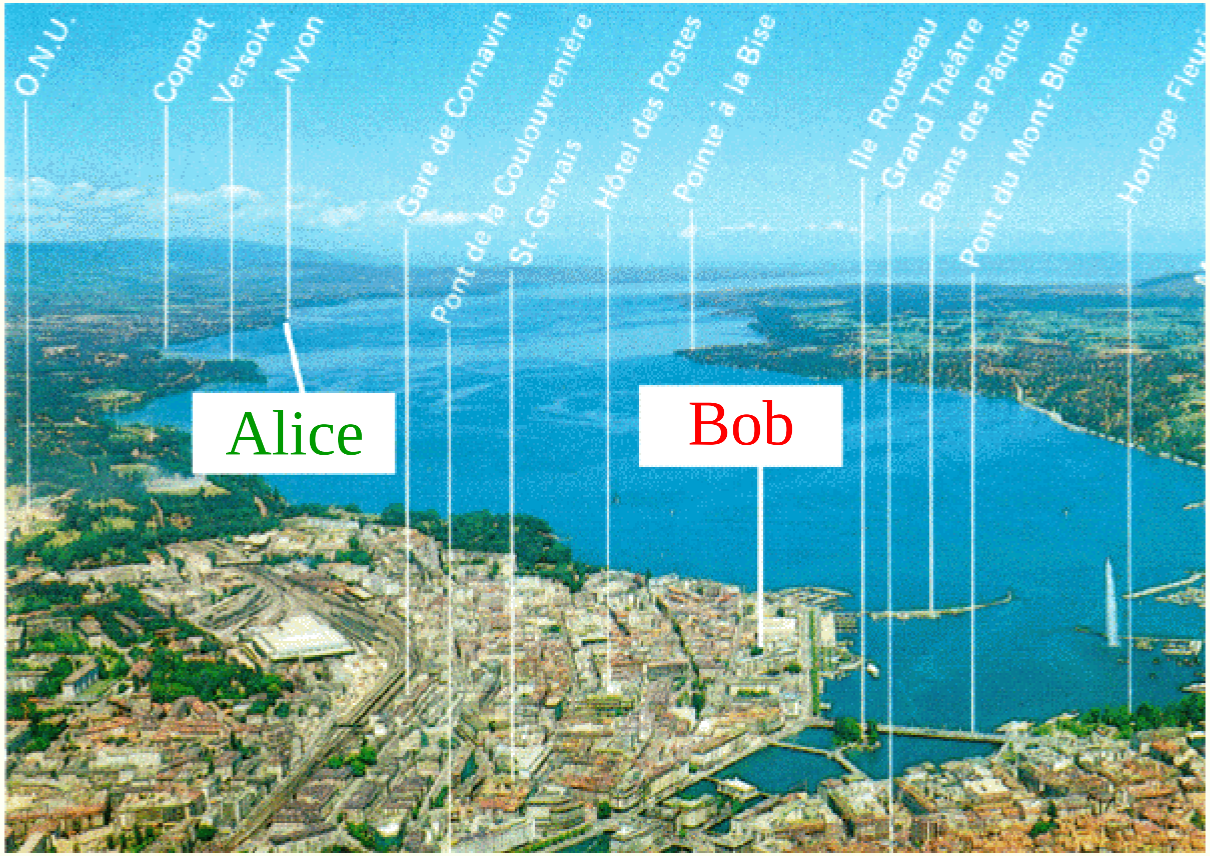


Suitable for key distribution

Experimental realization (among others)

by N. Gisin et al :

„A Plug and Play system for quantum cryptography“



23 km of standard optical fibre: supplied by

Multiqubits (qubit array):

$\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$

two qubits: basis vector $|0\rangle_1 \otimes |1\rangle_2 \equiv |01\rangle, |10\rangle, |00\rangle, |11\rangle$

n qubits: 2^n basis vectors

Entangled states

$$|\alpha> \neq (a_1|0> + b_1|1>) \otimes (a_2|0> + b_2|1>) \otimes \ldots \otimes (a_n|0> + b_n|1>)$$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ EPR (Einstein, Podolsky and Rosen) pair

$\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ Separable state

Quantum gates: unitary transformations U

Single-qubit:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \quad |0\rangle \to |0\rangle, \quad |1\rangle \to |1\rangle$$

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \quad |0\rangle \to |1\rangle, \quad |1\rangle \to |0\rangle$$

$$\mathbf{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \quad |0\rangle \to |1\rangle, \quad |1\rangle \to -|0\rangle$$

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \quad |0\rangle \to |0\rangle, \quad |1\rangle \to -|1\rangle$$

Hadamard Transformation

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Controlled-NOT gate, C_{not} (on two qubits)

$$C_{not} : \begin{array}{ccc} |00\rangle & \to & |00\rangle \\ |01\rangle & \to & |01\rangle \\ |10\rangle & \to & |11\rangle \\ |11\rangle & \to & |10\rangle \end{array} \qquad C_{not} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Impossible : Quantum Copier

No-Cloning Theorem:

No U such that $\mathbf{U}(|\alpha 0\rangle) = |\alpha\alpha\rangle,\ \mathbf{U}(|\beta 0\rangle) = |\beta\beta\rangle$

[Proof] $|\gamma\rangle = (1/\sqrt{2})(|\alpha\rangle + |\beta\rangle)$

$$\mathbf{U}(|\gamma\mathbf{0}\rangle) = \mathbf{U}(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|\alpha\mathbf{0}\rangle + |\beta\mathbf{0}\rangle)) = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|\alpha\alpha\rangle + |\beta\beta\rangle)$$

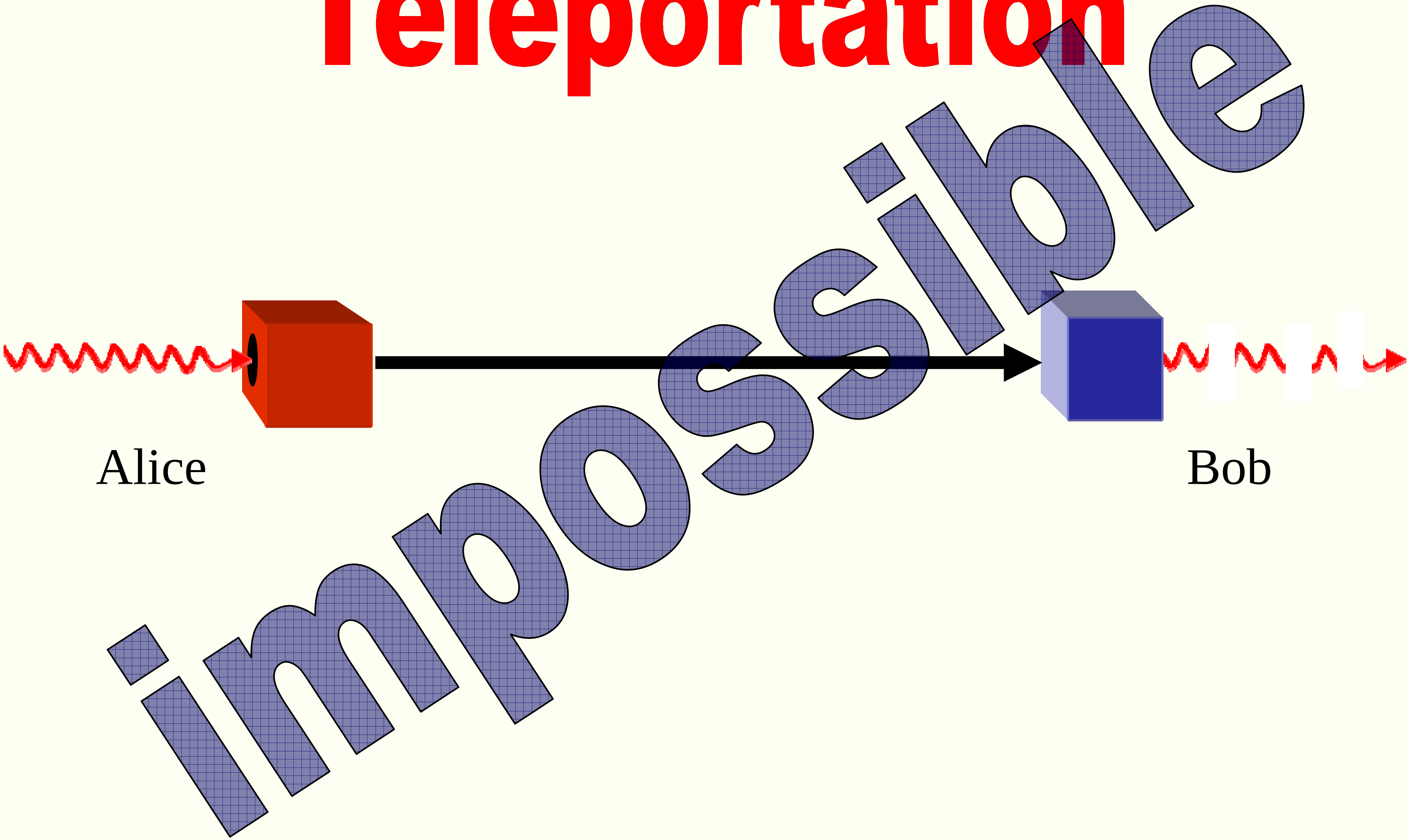
$$\mathbf{U}(|\gamma\mathbf{0}\rangle) = |\gamma\gamma\rangle = \frac{\mathbf{1}}{\mathbf{2}}(|\alpha\alpha\rangle + |\alpha\beta\rangle + |\beta\alpha\rangle + |\beta\beta\rangle)$$

Optimal cloning: Unitary trans. ⇨ best fidelity

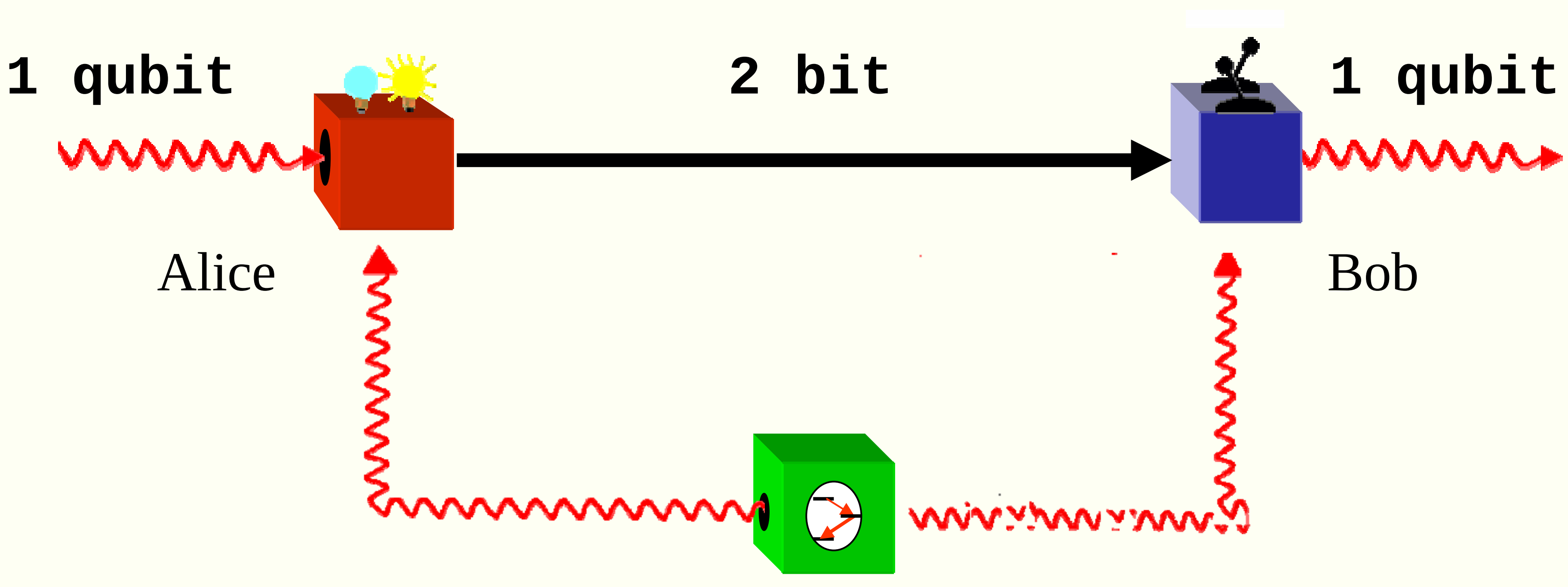
(general case: d-dim. ; N to M copies)

S. Albeverio, S.M. Fei, Euro. Phys. J. B 14(2000)669

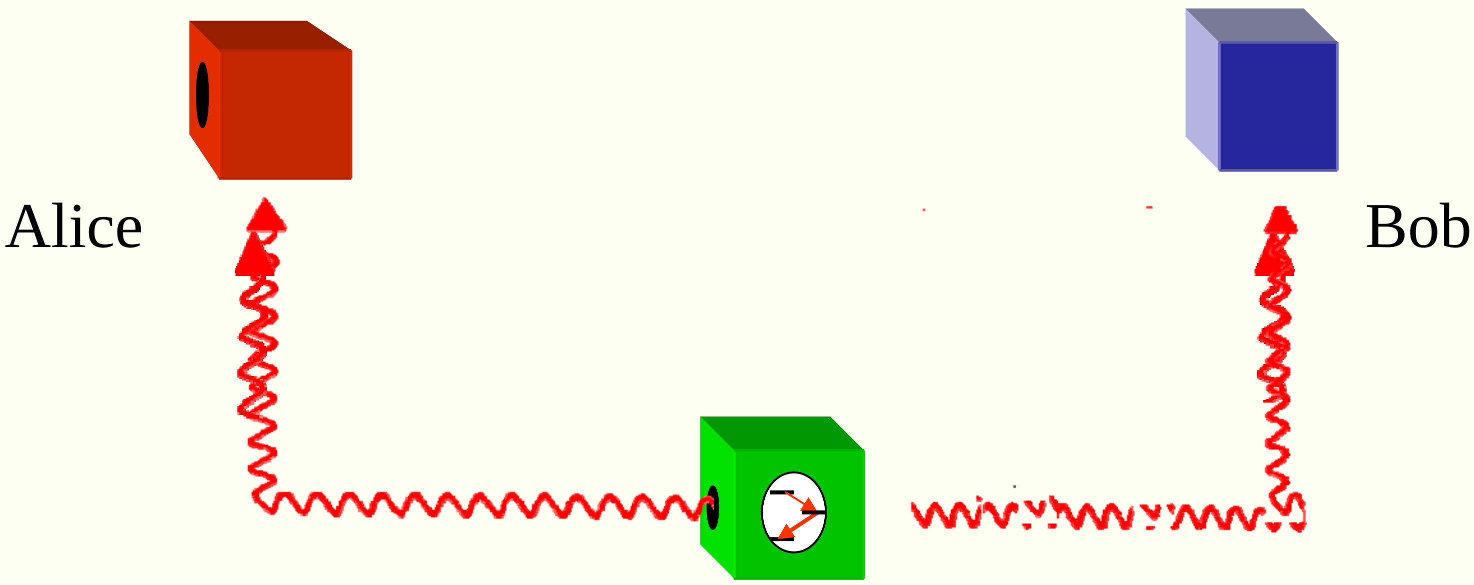
Teleportation



Entanglement enhanced Teleportation



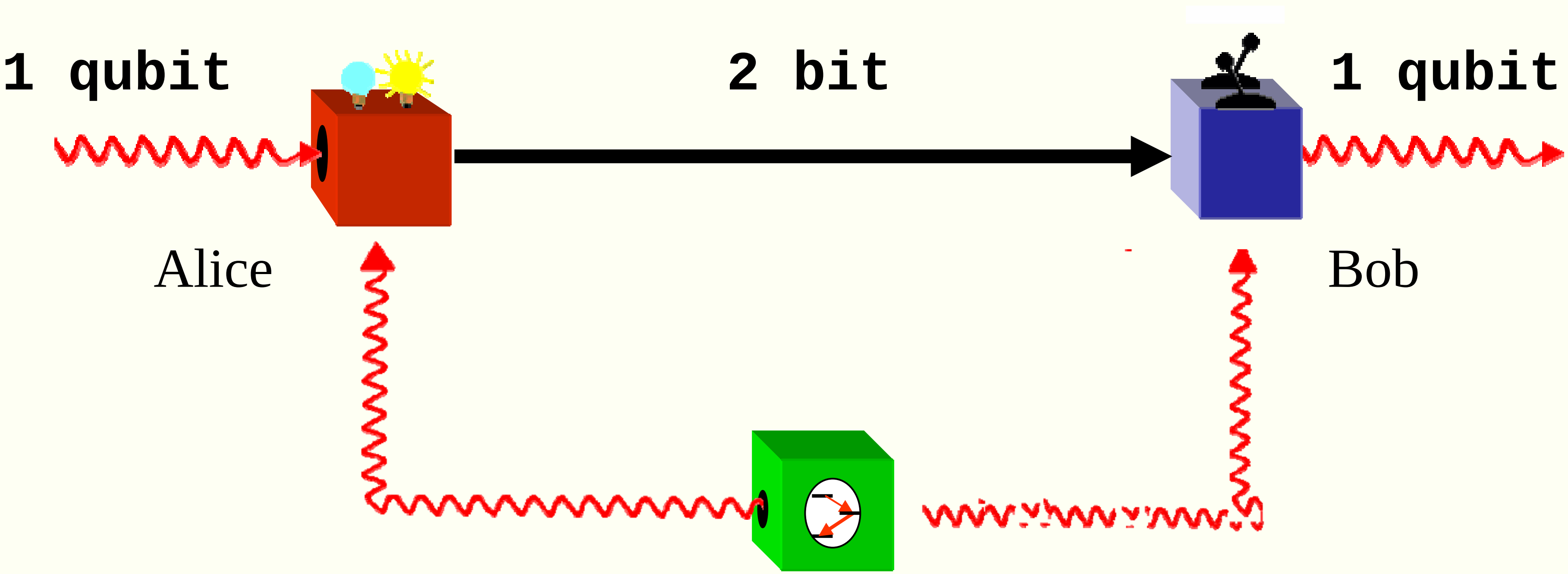
Entanglement enhanced Teleportation



$$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Entanglement enhanced Teleportation

Entanglement enhanced Teleportation



Alice:

$$\phi = \mathrm{a}|0\rangle + \mathrm{b}|1\rangle \qquad \psi_0 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\phi \otimes \psi_0 = \frac{1}{\sqrt{2}}(\mathrm{a}|000\rangle + \mathrm{a}|011\rangle + \mathrm{b}|100\rangle + \mathrm{b}|111\rangle)$$

$$(\mathbf{H} \otimes \mathbf{I} \otimes \mathbf{I})(\mathbf{C}_{\mathbf{not}} \otimes \mathbf{I})(\phi \otimes \psi_0) =$$

$$\tfrac{1}{2}(|00\rangle(\mathrm{a}|0\rangle + \mathrm{b}|1\rangle) + |01\rangle(\mathrm{a}|1\rangle + \mathrm{b}|0\rangle)$$

$$+|10\rangle(\mathrm{a}|0\rangle - \mathrm{b}|1\rangle) + |11\rangle(\mathrm{a}|1\rangle - \mathrm{b}|0\rangle))$$

Alice M

bits received	state	decoding
00	$\mathrm{a}\lvert 0\rangle + \mathrm{b}\lvert 1\rangle$	**I**
01	$\mathrm{a}\lvert 1\rangle + \mathrm{b}\lvert 0\rangle$	**X**
10	$\mathrm{a}\lvert 0\rangle - \mathrm{b}\lvert 1\rangle$	**Z**
11	$\mathrm{a}\lvert 1\rangle - \mathrm{b}\lvert 0\rangle$	**Y**

Bob U

General case: d-dim.; mixed states; optimal

Optimal fidelity

$$f_{\max}(\chi) = \frac{n\mathcal{F}(\chi)}{n+1} + \frac{1}{n+1}$$

S. Albeverio, S.M. Fei, Phys. Lett. A 276(2000)8

S. Albeverio, S.M. Fei and W.L. Yang, Commun. Theor. Phys. 38 (2002) 301-304; Phys. Rev. A 66(2002)012301.

M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. A 60, 1888 (1999).

Fully entangled fraction

$$\mathcal{F}(\chi) = \max\{\langle\Phi|(1\otimes U^{\dagger})\ \chi\ (1\otimes U)|\Phi\rangle\}$$

$$|\Phi\rangle = 1/\sqrt{n}\sum_{i=0}^{n-1}|ii\rangle$$ (Maximally entangled pure state)

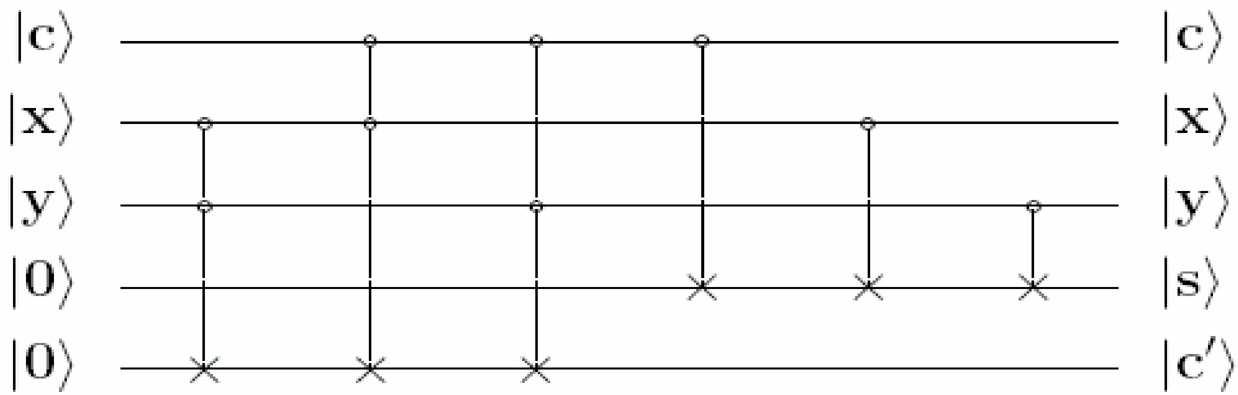
M. Li, S.M. Fei and Z.X. Wang, Phys. Rev. A,78(2008)032332.

Quantum computation

Toffoli gate (Controlled-Controlled-NOT)

$$T = |0\rangle\langle 0| \otimes I \otimes I + |1\rangle\langle 1| \otimes C_{not}$$

Quantum circuit: a 1-bit full adder



x, y data bits, s sum (modulo 2), c (c′) carry bit

Deutsch: possible to construct reversible quantum gates for any arbitrary classically computable function **f**

$$U_f|x,0\rangle \rightarrow |x,f(x)\rangle$$

Quantum Parallelism:

$$W:|00\ldots0\rangle \Rightarrow \frac{1}{\sqrt{2^n}}(|00\ldots0\rangle + |00\ldots1\rangle + \cdots + |11\ldots1\rangle) = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$$

$$U_f\left(\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x,0\rangle\right) = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}U_f(|x,0\rangle) = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x,f(x)\rangle$$

Quantum algorithm: Manipulating quantum parallelism ⇨ desired results with high probability

Shor's factorisation algorithm

Period finding and quantum Fourier transform

$\mathbf{f(x)}$, period $\mathbf{r}$: $\mathbf{f(x) = f(x + r)}$

$\mathbf{f(x)}$ can be efficiently computed from $\mathbf{x}$, $\mathbf{N/2 < r < N}$ for some $\mathbf{N}$

QC: $\mathbf{2n}$ qubits, $\mathbf{n = \lceil 2 \log N \rceil}$

Two 'registers', $\mathbf{X}$ and $\mathbf{Y}$, each $\mathbf{n}$ qubits

Initially prepared in the state $|\mathbf{0}\rangle|\mathbf{0}\rangle$

$\mathbf{H}$ applies to each qubit in the $\mathbf{X}$ register:

$$\frac{\mathbf{1}}{\sqrt{\mathbf{w}}}\sum_{\mathbf{x}=\mathbf{0}}^{\mathbf{w}-\mathbf{1}}|\mathbf{x}\rangle|\mathbf{0}\rangle, \qquad \mathbf{w}=\mathbf{2}^{\mathbf{n}}$$

$\mathbf{U_f}|\mathbf{x}\rangle|\mathbf{0}\rangle = |\mathbf{x}\rangle|\mathbf{f}(\mathbf{x})\rangle$

$$\frac{\mathbf{1}}{\sqrt{\mathbf{w}}}\sum_{\mathbf{x}=\mathbf{0}}^{\mathbf{w}-\mathbf{1}}|\mathbf{x}\rangle|\mathbf{f}(\mathbf{x})\rangle$$

Measure $\mathbf{Y}$: $\mathbf{f}(\mathbf{x}) = \mathbf{u}$

$\mathbf{Y}$ register state collapses onto $|\mathbf{u}\rangle$

$$\frac{\mathbf{1}}{\sqrt{\mathbf{M}}}\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{M}-\mathbf{1}}|\mathbf{d_u}+\mathbf{jr}\rangle|\mathbf{u}\rangle$$

$\mathbf{d_u}+\mathbf{jr}$, $\mathbf{j}=\mathbf{0},\mathbf{1},\mathbf{2}\ldots\mathbf{M}-\mathbf{1}$, all $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x})=\mathbf{u}$, $\quad \mathbf{M}\simeq\mathbf{w}/\mathbf{r}$

Quantum Discrete Fourier transform:

$$\mathbf{U}_{\mathcal{FT}}|\mathbf{x}\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{w}}}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{w}-\mathbf{1}} \mathbf{e}^{\mathbf{i}\mathbf{2}\pi\mathbf{k}\mathbf{x}/\mathbf{w}}|\mathbf{k}\rangle$$

$$\mathbf{U}_{\mathcal{FT}}\frac{\mathbf{1}}{\sqrt{\mathbf{w}/\mathbf{r}}}\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{w}/\mathbf{r}-\mathbf{1}} |\mathbf{d}_{\mathbf{u}}+\mathbf{j}\mathbf{r}\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{r}}}\sum_{\mathbf{k}} \tilde{\mathbf{f}}(\mathbf{k})|\mathbf{k}\rangle$$

$$|\tilde{\mathbf{f}}(\mathbf{k})| = \begin{cases} \mathbf{1} & \text{if } \mathbf{k} \text{ is a multiple of } \mathbf{w}/\mathbf{r} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Measure $\Rightarrow \nu = \lambda\mathbf{w}/\mathbf{r}$, λ unknown

$$\frac{\nu}{\mathbf{w}} = \frac{\lambda}{\mathbf{r}} \qquad \mathbf{w} = 2^{\mathbf{n}}$$

If λ and $\mathbf{r}$ have no common factors, cancel $\nu/\mathbf{w}$ down to an irreducible fraction and thus obtain λ and $\mathbf{r}$

If λ and $\mathbf{r}$ have a common factor (unlikely for large $\mathbf{r}$), algorithm fails, repeat the algorithm

repetitions no greater than $\sim \log \mathbf{r}$ (usually much less) probability of success is arbitrarily close to $\mathbf{1}$

Take $\mathbf{f(x) = a^x}$ mod $\mathbf{N}$

$\mathbf{N}$ the number to be factorized, $\mathbf{a < N}$ is chosen randomly

Elementary number theory: for most choices of $\mathbf{a}$, $\mathbf{r}$ is even

$$a^{r+x} = a^x \ mod \ N$$

$$a^r = 1 \ mod \ N$$

$$(a^{r/2} + 1)(a^{r/2} - 1) = 0 \ mod \ N$$

$\mathbf{a^{r/2} \pm 1}$ shares a common factor with $\mathbf{N}$

L. Grover quantum searching: $\mathbf{O(N/2) \rightarrow O(\sqrt{N})}$

Quantum Information Processing

Initial State $|\psi>_o \Rightarrow$ Final State $|\psi>_t$

Unitary Transformations

+ Measurements

Quantum Entnglement:

- Quantum computation
- Quantum teleportation
- Dense coding
- Quantum cryptography
- Quantum error correction

Quantum Entanglement

$\mathcal{H}$: N-dim. complex Hilbert space, $|i\rangle$

Pure state (Vector) on $\mathcal{H}\otimes\mathcal{H}\otimes\ldots\otimes\mathcal{H}$

$$|\psi\rangle = \sum_{i,j,\ldots,k=1}^{N} a_{ij\ldots k}|ij\ldots k\rangle, \qquad a_{ij\ldots k}\in\mathbb{C} \qquad \in\mathcal{H}\otimes\mathcal{H}\otimes\ldots\otimes\mathcal{H}$$

$$\to \left(\sum_{i=1}^{N} a_i|i\rangle\right)\otimes\left(\sum_{j=1}^{N} b_j|j\rangle\right)\otimes\ldots\otimes\left(\sum_{k=1}^{N} c_k|k\rangle\right)$$

Separable!

$$\sum_{i,j,\ldots,k=1}^{N} a_{ij\ldots k}a^{*}_{ij\ldots k} = 1 \qquad |ij\ldots k\rangle \equiv |i\rangle\otimes|j\rangle\otimes\ldots\otimes|k\rangle$$

Pure State : $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$

Mean value of O : $\langle \mathrm{O} \rangle = \langle \psi | \mathrm{O} | \psi \rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|\mathrm{O}) = \mathrm{Tr}(\rho \mathrm{O})$

$$\langle \psi | = (|\psi\rangle)^{\dagger} \qquad \rho = |\psi\rangle\langle\psi|$$

Mixed state:

$$\rho = \sum \mathrm{p_i} |\psi_\mathrm{i}\rangle\langle\psi_\mathrm{i}|, \quad 0 < \mathrm{p_i} \leq 1, \quad \sum \mathrm{p_i} = 1$$

$\exists\ |\psi_\mathrm{i}\rangle$ such that $|\psi_\mathrm{i}\rangle$ are separable $\forall$ i: ρ Separable!

$\rho = \sum \mathrm{p_i} \rho_1^\mathrm{i} \otimes \rho_2^\mathrm{i} \otimes \ldots \otimes \rho_\mathrm{n}^\mathrm{i}$ Separable !

Measure： Bipartite $\mathcal{H}\otimes\mathcal{H}$

Multipartite $\mathcal{H}\otimes\mathcal{H}\otimes\ldots\otimes\mathcal{H}$?

Entanglement of Formation

Pure state
$$|\psi\rangle=\sum_{ij}a_{ij}|ij\rangle\in\mathcal{H}\otimes\mathcal{H}$$

$$E(|\psi\rangle)=-\mathrm{Tr}\left(\rho_1\log_2\rho_1\right)=-\mathrm{Tr}\left(\rho_2\log_2\rho_2\right)$$

$$\rho_1=AA^\dagger=\mathrm{Tr}_2|\psi\rangle\langle\psi|,\quad \rho_2=(A^\dagger A)^*=\mathrm{Tr}_1|\psi\rangle\langle\psi|,\quad (A)_{ij}=a_{ij}$$

Mixed state
$$\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|,\qquad E(\rho)=\min\sum_i p_iE(|\psi_i\rangle)$$

Two qubits (N=2)

$$E(|\psi\rangle) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right)$$

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

Concurrence C: $|\psi\rangle = a_{11}|00\rangle + a_{12}|01\rangle + a_{21}|10\rangle + a_{22}|11\rangle$

$$C = 2|a_{11}a_{22} - a_{12}a_{21}|, \quad |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 = 1$$

E ~ C: monotonically increasing

$$E(|\psi\rangle) \Longrightarrow E(C(|\psi\rangle)) \qquad E(\rho) \Longrightarrow E(C(\rho))$$

$$C(\rho) = \mathrm{Max}\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$: eigenvalue of $\sqrt{\rho\tilde{\rho}}$

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998)

High dimension N>2: no general solution

Isotropic states: $\rho \to \mathbf{U} \otimes \mathbf{U}^* \rho \left(\mathbf{U} \otimes \mathbf{U}^*\right)^\dagger$

B.M. Terhal, K.H. Vollbrecht, Phys. Rev. Lett. 85, 2625 (2000)

If $\mathbf{A}\mathbf{A}^\dagger$ has only two non-zero eigenvalues

generalized concurrence

S.M. Fei, J. Jost, X.Q. Li-Jost, G.F. Wang,
Phys. Lett. A 310 (2003) 333

High dimensional construction:

S.M. Fei, X.Q. Li-Jost, Rep. Math. Phys. 53(2004)195

More non-zero eigenvalues

S.M. Fei, Z.X. Wang, H. Zhao, Phys. Lett. A 329 (2004) 414-419

Theory of Quantum Entanglement

(II)

4th Winter School on Quantum Information Sciences
Feb. 14, 2009
Lanyang Campus, Tamkang University, Yilan

Measure: Bipartite $\mathcal{H} \otimes \mathcal{H}$

Entanglement of Formation

Pure state $|\psi\rangle = \sum_{ij} a_{ij}|ij\rangle \in \mathcal{H} \otimes \mathcal{H}$

$$E(|\psi\rangle) = -\mathrm{Tr}\left(\rho_1 \log_2 \rho_1\right) = -\mathrm{Tr}\left(\rho_2 \log_2 \rho_2\right)$$

$$\rho_1 = AA^\dagger = \mathrm{Tr}_2|\psi\rangle\langle\psi|, \quad \rho_2 = (A^\dagger A)^* = \mathrm{Tr}_1|\psi\rangle\langle\psi|, \quad (A)_{ij} = a_{ij}$$

Mixed state

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \qquad E(\rho) = \min \sum_i p_i E(|\psi_i\rangle)$$

Lower Bound for EoF

Partial transpose wrt subsystems

$$\rho \Longrightarrow \rho^{T_1} \qquad \langle \mathbf{ij}|\rho^{\mathbf{T_1}}|\mathbf{kl}\rangle = \langle \mathbf{kj}|\rho|\mathbf{il}\rangle$$

$$\rho = \left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{array}\right) \qquad \rho^{T_1} = \left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{31} & \rho_{32} \\ \rho_{21} & \rho_{22} & \rho_{41} & \rho_{42} \\ \hline \rho_{13} & \rho_{14} & \rho_{33} & \rho_{34} \\ \rho_{23} & \rho_{24} & \rho_{43} & \rho_{44} \end{array}\right)$$

Realignment:

Z: mxm block matrix with block size nxn

$$Z=\begin{pmatrix} Z_{11} & \cdots & Z_{1m} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \cdots & Z_{mm} \end{pmatrix} \Rightarrow \tilde{Z}=\begin{pmatrix} vec(Z_{11})^T \\ \vdots \\ vec(Z_{m1})^T \\ \vdots \\ vec(Z_{1m})^T \\ \vdots \\ vec(Z_{mm})^T \end{pmatrix}$$

$$A=[a_{ij}] \Rightarrow vec(A)=\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{1m} \\ \vdots \\ a_{mm} \end{pmatrix}$$

2x2 case

$$\rho=\left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{array}\right) \Rightarrow \tilde{\rho}=\left(\begin{array}{cccc} \rho_{11} & \rho_{21} & \rho_{12} & \rho_{22} \\ \hline \rho_{31} & \rho_{31} & \rho_{32} & \rho_{42} \\ \hline \rho_{13} & \rho_{23} & \rho_{14} & \rho_{24} \\ \hline \rho_{33} & \rho_{43} & \rho_{34} & \rho_{44} \end{array}\right)$$

Pure state

$$|\psi\rangle = \sum_i \sqrt{\mu_i}|e_i f_i\rangle \qquad \rho = |\psi\rangle\langle\psi|$$

$$\|\rho^{T_1}\| = \|\tilde{\rho}\| = (\sum_{k=1}^{m} \sqrt{\mu_k})^2 = \lambda \qquad \lambda \in [1, m]$$

$$R(\lambda) = \min_{\vec{\mu}}\left\{E(\psi) \mid \left(\sum_{k=1}^{m} \sqrt{\mu_k}\right)^2 = \lambda\right\} = H_2[\gamma(\lambda)] + [1-\gamma(\lambda)]\log_2(m-1)$$

$$H_2(x) = -x\log_2 x - (1-x)\log_2(1-x) \qquad \gamma(\lambda) = \frac{1}{m^2}[\sqrt{\lambda} + \sqrt{(m-1)(m-\lambda)}]^2$$

Let $\mathcal{E} \leq E$ be a convex, monotonically increasing function

$$E(\rho) = \sum_i p_i E(\rho^i) \geq \sum_i p_i \mathcal{E}(\lambda^i) \geq \mathcal{E}\left(\sum_i p_i \lambda^i\right) \geq \begin{cases} \mathcal{E}(\|\rho^{T_1}\|) \\ \mathcal{E}(\|\tilde{\rho}\|) \end{cases}$$

$$\|\rho^{T_1}\| \leq \sum_i p_i \|(\rho^i)^{T_1}\| \qquad \|\tilde{\rho}\| \leq \sum_i p_i \|\widetilde{\rho^i}\|$$

$$\mathcal{E}(\lambda) = \mathrm{co}[R(\lambda)]$$

“co” means the convex hull, which is the largest convex function that is bounded above by a given function

Theorem.— For any $m \otimes n$ $(m \leq n)$ mixed quantum state ρ, the entanglement of formation $E(\rho)$ satisfies

$$E(\rho) \geq \begin{cases} 0, & \Lambda = 1, \\ H_2[\gamma(\Lambda)] + [1-\gamma(\Lambda)]\log_2(m-1), & \Lambda \in [1, \frac{4(m-1)}{m}], \\ \frac{\log_2(m-1)}{m-2}(\Lambda - m) + \log_2 m, & \Lambda \in [\frac{4(m-1)}{m}, m], \end{cases}$$

$$R(\Lambda) = H_2[\gamma(\Lambda)] + [1-\gamma(\Lambda)]\log_2(m-1)$$

$$\gamma(\Lambda) = \frac{1}{m^2}[\sqrt{\Lambda} + \sqrt{(m-1)(m-\Lambda)}]^2$$

$$\Lambda = \max(\| \rho^{T_1} \|, \| \tilde{\rho} \|)$$

$$H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

K. Chen, S. Albeverio, S.M. Fei, Phys. Rev. Lett. 95(2005)210501

S.M. Fei, X. Li-Jost, Phys. Rev. A 73(2006)024302

Lower Bound for Concurrence

$$C(|\psi\rangle) = \sqrt{2(1 - \mathrm{Tr}\rho_1^2)}$$

Uhlmann 2000, Rungta et al, Albeverio and Fei 2001

$$\rho_1 = Tr_2(|\psi\rangle\langle\psi|)$$

$$C(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$$

Theorem: For any $m \otimes n$ $(m \leq n)$ mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq \sqrt{\frac{2}{m(m-1)}}\left(\max(\|\rho^{T_1}\|, \|\tilde{\rho}\|) - 1\right)$$

K. Chen, S. Albeverio, S.M. Fei, Phys. Rev. Lett. 95(2005)040504

Example

3x3 Bound Entangled State

$$
\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle), \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle \\
|\psi_4\rangle &= \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)
\end{aligned}
$$

$$\rho = \frac{1}{4}\left(Id - \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|\right)$$

$$\|\rho^{T_1}\| = 1$$

$$\|\mathcal{R}(\rho)\| = 1.087$$

$$C(\rho) \geq 0.05$$

the state is entangled !

Example

Isotropic states

$$\rho_F = \frac{1-F}{d^2-1}(Id - |\Psi^+\rangle\langle\Psi^+|) + F(|\Psi^+\rangle\langle\Psi^+|)$$

$$|\Psi^+\rangle \equiv \sqrt{1/d}\sum_{i=1}^{d} |ii\rangle$$

F>1/d: entangled

$$\| \rho_F^{T_1} \| = \| \tilde{\rho}_F \| = dF$$

Rudolph, quant-ph/0202121;

Vidal and Werner, PRA 65, 032314 (2002).

Concurrence

$$C(\rho_F) = \sqrt{\frac{2}{d(d-1)}}(dF-1)$$

Rungta and Caves, PRA 67, 012307 (2003)

EOF

$$E(\rho_F) = \mathrm{co}[R(dF)]$$

Terhal and Vollbrecht, PRL 85, 2625 (2000)

The lower bounds are exact for both concurrence and EOF !

Lower Bound for Concurrence of Tripartite States

$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$$ dim. m, n, p

$$C(|\psi\rangle) = \sqrt{3 - \mathrm{Tr}(\rho_A^2 + \rho_B^2 + \rho_C^2)}$$

$$C(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$$

X.H. Gao, S.M. Fei and K. Wu, Phys. Rev. A 74 (Rapid Comm.)(2006)050303

Lower bound: covariance matrix approach ρ_{AB} $\mathcal{H}_d^A \otimes \mathcal{H}_d^B$.

A_k (resp. B_k) be d^2 observables on $\mathcal{H}_d^A$ (resp. $\mathcal{H}_d^B$)

--- orthonormal normalized basis of the observable space

$$\{M_k\} = \{A_k \otimes I, I \otimes B_k\}$$

Covariance matrix

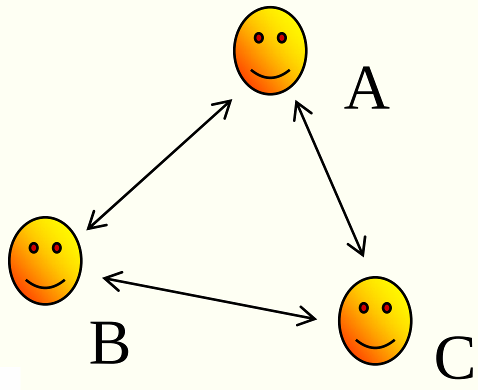
$$\gamma_{ij}(\rho_{AB}, \{M_k\}) = \frac{\langle M_i M_j \rangle + \langle M_j M_i \rangle}{2} - \langle M_i \rangle \langle M_j \rangle \Rightarrow \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

$$C_{ij} = \langle A_i \otimes B_j \rangle_{\rho_{AB}} - \langle A_i \rangle_{\rho_A} \langle B_j \rangle_{\rho_B}$$

$$C(\rho) \geq \frac{2||C||_{KF} - (1 - Tr\rho_A^2) - (1 - Tr\rho_B^2)}{\sqrt{2M(M-1)}}$$ ⇒ Multipartite

M. Li, S.M. Fei and Z.X. Wang, J. Phys. A 41 (Fast track commun.) (2008)202002

Monogamy relations



Pure three qubit state $|\phi\rangle_{ABC}$

Concurrence $C_{AB}^2 + C_{AC}^2 \leq C_{A(BC)}^2$

$\rho_{AB} = Tr_C(|\phi\rangle_{ABC}\langle\phi|)$ $\rho_{AC} = Tr_B(|\phi\rangle_{ABC}\langle\phi|)$

Negativity $\mathcal{N} = \|\rho^{T_A}\| - 1$

$$\mathcal{N}_{AB}^2 + \mathcal{N}_{AC}^2 \leq \mathcal{N}_{A(BC)}^2$$

High dimensional case

Y.C. Ou, H. Fan and S.M. Fei, Phys. Rev. A, 78(2008)012311.

Separability

Entanglement is invariant under local unitary tran.

$$|\Psi_2\rangle = \sum_{i,j=1}^{N} a_{ij}\ |ij\rangle \qquad U_1 \otimes U_2$$

Invariants $I_\alpha = \mathrm{Tr}(AA^\dagger)^\alpha, \quad \alpha = 1, ..., N$

$$C_N^2 = \sqrt{\frac{N}{N-1}(I_1^2 - I_2)}$$

$$N = 2: \quad C_2^2 = C = 2|a_{11}a_{22} - a_{12}a_{21}|$$

$C_N^2 = 0\ (1) \ \Leftrightarrow\ |\Psi_2\rangle$ separable（max.entangled）

Multipartite

$$|\Psi_M\rangle = \sum_{i_1,\ldots,i_M=1}^{N} a_{i_1,\ldots,i_M} |i_1 \ldots i_M\rangle$$

Invariants:

$$I_1 = \sum_{i_1,\ldots,i_M=1}^{N} a_{i_1,\ldots,i_M} a^*_{i_1,\ldots,i_M} \equiv 1$$

$$I_{\alpha\beta} = \sum a_{\alpha\beta} a^*_{\alpha\beta'} a_{\alpha'\beta'} a^*_{\alpha'\beta}$$

$$C_N^M = \sqrt{\frac{N}{d(N-1)}\left(d\, I_1^2 - I_2 - \ldots - I_d\right)} \qquad \left(d = 2^{M-1} - 1\right)$$

$$C_N^M = 0 \;\Leftrightarrow\; |\Psi_M\rangle$$ separable

S. Albeverio and S.M Fei, J. Opt. B: 3(2001)1

Separability of mixed states: no general criteria

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|, \quad 0 < p_i \leq 1, \quad \sum p_i = 1$$

$\exists\ |\psi_i\rangle$ such that $|\psi_i\rangle$ are separable $\forall$ i: ρ Sep.

$$\rho = \sum \mathbf{p_i} \rho_{\mathbf{1}}^{\mathbf{i}} \otimes \rho_{\mathbf{2}}^{\mathbf{i}} \otimes \ldots \otimes \rho_{\mathbf{n}}^{\mathbf{i}}$$ Sep.

a) Peres (PPT) criterion:

$$\rho \text{ separable} \implies \text{partial transpose } \rho^{T_1} \geq 0$$

Peres PRL **77**, 1413 (1996)

$$\textit{Positive partial transpose (PPT):} \quad \langle \mathbf{ij}|\rho^{T_1}|\mathbf{kl}\rangle = \langle \mathbf{kj}|\rho|\mathbf{il}\rangle$$

$$\rho = \sum \mathbf{p_i}\rho_\mathbf{1}^\mathbf{i} \otimes \rho_\mathbf{2}^\mathbf{i} \otimes \ldots \otimes \rho_\mathbf{n}^\mathbf{i}$$

2x2, 2x3: PPT ⟺ Separable

Horodeckis, Phys. Lett. A **223**,1 (1996)

b) Realignment:

Separable $\longrightarrow$ $$\|\tilde{\rho}\| \leq 1$$

$||M||$: sum of all the singular values of M

Chen and Wu, Quant. Inf. Comp. 3, 193 (2003)
Rudolph, Quant. Inf. Proc. 4, 219 (2005)
Albeverio, Chen, Fei, Phys. Rev. A 68(2003)062313

c) Reduction: $\rho_1 = \mathrm{Tr}_2\rho,\ \rho_2 = \mathrm{Tr}_1\rho$

$$\rho \text{ separable} \Rightarrow \rho_1 \otimes I - \rho \geq 0,\ I \otimes \rho_2 - \rho \geq 0$$

d) Majorization:

vector $x = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_d^{\downarrow}),\ x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_d^{\downarrow}$

x is majorized by y: $x \prec y$ if

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$$ = (k=d)

$x \prec y$ if and only $x = Dy$ D:stochastic matrix

ρ_{AB} separable ⇨ $\lambda_{AB} \prec \lambda_A \quad \lambda_{AB} \prec \lambda_B$

$\lambda_{AB},\ \lambda_A,\ \lambda_B$ eigenvalues of $\rho_{AB},\ \rho_A,\ \rho_B$

e) Rank 2 (Necessary and sufficient condition)

ρ: *a rank two state in* $\mathcal{H} \otimes \mathcal{H}$,

$|\mathbf{E_1}\rangle,\ |\mathbf{E_2}\rangle$ Eigenvectors (non-zero)

S. Albeverio, S.M. Fei and D. Goswami, Phys. Lett. A286(2001)91

$$\mathcal{H}_\mathbf{1} \otimes \mathcal{H}_\mathbf{2} \otimes \ldots \otimes \mathcal{H}_\mathbf{M}$$

S.M. Fei, X.H. Gao, X.H. Wang, Z.X. Wang and K. Wu,
Phys. Lett. A300(2002)559;
Int. J. Quant. Inform. 1(2003)37

Multipartite Schmidt-correlated State

$$\rho = \sum_{m,n=0}^{N-1} a_{mn} |m \cdots m\rangle \langle n \cdots n|, \qquad \sum_{m=0}^{N-1} a_{mm} = 1.$$

Fully separable $\Longleftrightarrow$ PPT

Fully separable (maximally entangled)

$\Updownarrow$

$$\|\tilde{\rho}\| = 1 \; (N \;)$$

M.J. Zhao, S.M. Fei and Z.X. Wang, Phys. Lett. A 372(2008)2552

Bell Inequalities

Separable ➡ $|\langle A_1 A_2 + A_1 B_2 + B_1 A_2 - B_1 B_2 \rangle| \leq 2$

All generalized GHZ states $|\psi\rangle = \cos\alpha|0,...,0\rangle + \sin\alpha|1,...,1\rangle$ of many qubits violate the Bell inequality maximally

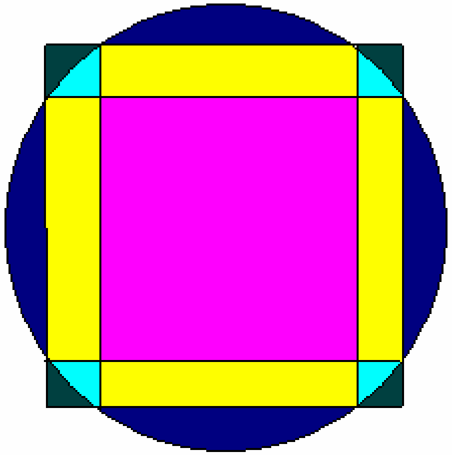
$$\frac{1}{2}|\langle \mathcal{B}_{N-1}(A_N + A'_N) + (A_N - A'_N)\rangle_{\text{LHV}}| \leq 1$$

$\mathcal{B}_{N-1}$ quantum mechanical Bell operator of WWZB inequalities for N-1 particles

Only two measurement settings of each party

K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. A (R)74(2006)050101

Tripartite B.Z. Sun and S.M. Fei, Phys. Rev. A 74 (2006) 032335



For states ρ in S_{1-23}, S_{2-13} and S_{12-3}, respectively

$$|\langle \mathcal{D}_3^{(1)} \rangle_\rho| \le \sqrt{2}, \quad |\langle \mathcal{D}_3^{(2)} \rangle_\rho| \le 1, \quad |\langle \mathcal{D}_3^{(3)} \rangle_\rho| \le 1$$

$$|\langle \mathcal{D}_3^{(1)} \rangle_\rho| \le 1, \quad |\langle \mathcal{D}_3^{(2)} \rangle_\rho| \le \sqrt{2}, \quad |\langle \mathcal{D}_3^{(3)} \rangle_\rho| \le 1$$

$$|\langle \mathcal{D}_3^{(1)} \rangle_\rho| \le 1, \quad |\langle \mathcal{D}_3^{(2)} \rangle_\rho| \le 1, \quad |\langle \mathcal{D}_3^{(3)} \rangle_\rho| \le \sqrt{2}$$

$$\langle \mathcal{D}_3^{(1)} \rangle_\rho^2 + \langle \mathcal{D}_3^{(2)} \rangle_\rho^2 + \langle \mathcal{D}_3^{(3)} \rangle_\rho^2 \le 3$$

Multi-mode states

Z.G. Li, S.M. Fei, Z.X.Wang and K. Wu, Phys. Rev. A 75(2007)012311

Classification under Local Unitary Transformations

$$\rho \Longrightarrow (U_1 \otimes \ldots \otimes U_M)\rho(U_1 \otimes \ldots \otimes U_M)^{-1} \qquad U_i U_i^\dagger = U_i^\dagger U_i = 1$$

Orbits (dimension， topology， geometry)

Equivalence criterion

$$\rho' = (U_1 \otimes U_2 \otimes \ldots \otimes U_M)\,\rho\,(U_1 \otimes U_2 \otimes \ldots \otimes U_M)^\dagger$$

Pure state:

$$|\Psi\rangle = \sum_{i,j,\ldots,k} a_{ij\ldots k}|ij\ldots k\rangle \qquad |\Psi'\rangle = \sum_{i,j,\ldots,k} a'_{ij\ldots k}|ij\ldots k\rangle$$

$$|\Psi'\rangle = u_1 \otimes u_2 \otimes \ldots \otimes u_M|\Psi\rangle$$?

Bipartite

$$|\Psi\rangle = \sum_{i,j} a_{ij}|ij\rangle \qquad A' = u_1^t A u_2$$

Invariants:

$$I_\alpha = \mathrm{Tr}(AA^\dagger)^\alpha \qquad \alpha = 1, 2, \ldots, N$$

Mutipartite:

S. Albeverio, L. Cattaneo, S.M. Fei, X.H. Wang, Rep. Math. Phys. 56 (2005)341-350; Int. J. Quant. Inform. 3 (2005) 603-609.

Z.H. Yu, X. Jost-Li, Q.Z. Li, J.T. Lv and S.M. Fei, Differential Geometry of Bipartite Quantum States, Rep. Math. Phys. 60(2007)125-133

X.H. Wang, S.M. Fei and K. Wu, J. Phys. A 41 (2008) 025305

Mixed state (bipartite): Equivalence criterion

I. Invariants under local unitary transformations

$$\rho = \sum_{i=1}^{n} \lambda_i |\nu_i\rangle\langle\nu_i|$$

$$|\nu_i\rangle = \sum_{k,l=1}^{N} a^i_{kl} |k\rangle \otimes |l\rangle, \quad a^i_{kl} \in \mathbb{C}, \quad \sum_{k,l=1}^{N} a^i_{kl} a^{i*}_{kl} = 1 \qquad (A_i)_{kl} = a^i_{kl}$$

$$\rho_i = \mathrm{Tr}_2 |\nu_i\rangle\langle\nu_i| = A_i A_i^\dagger, \quad \theta_i = (\mathrm{Tr}_1 |\nu_i\rangle\langle\nu_i|)^* = A_i^\dagger A_i$$

$$\Omega(\rho)_{ij} = \mathrm{Tr}(\rho_i \rho_j), \quad \Theta(\rho)_{ij} = \mathrm{Tr}(\theta_i \theta_j)$$

Generic $\quad \det(\Omega(\rho)) \neq 0 \qquad \det(\Theta(\rho)) \neq 0$

$$X(\rho)_{ijk} = \mathrm{Tr}(\rho_i \rho_j \rho_k) \qquad Y(\rho)_{ijk} = \mathrm{Tr}(\theta_i \theta_j \theta_k)$$

Theorem: ρ Equiv. L.U. ⟺

$$\mathbf{J}^s(\rho) = \mathbf{Tr}_2(\mathbf{Tr}_1 \rho^s), \quad s = 1, \ldots, N^2$$

$$\Omega(\rho), \quad \Theta(\rho), \quad X(\rho), \quad Y(\rho)$$

S. Albeverio, S.M. Fei, P. Parashar, W.L. Yang,
Phys. Rev. A 68 (Rapid Comm.) (2003) 010303

Not full rank B.Z. Sun, S.M. Fei, X.Q. Li-Jost, Z.X. Wang, J. Phys. A39 (2006) L43

If $\rho_1, \ \theta_1$ each of its eigenvalues has multiplicity one

S. Albeverio, S.M. Fei, D. Goswami, Phys. Lett. A 340(2005)37;
J. Phys. A 40(2007)11113

II. Matrix tensor product decomposition approach

$\rho = \mathbf{X}\Lambda\mathbf{X}^\dagger, \qquad \rho' = \mathbf{Y}\Lambda\mathbf{Y}^\dagger \qquad \Lambda = \mathbf{diag}(\lambda_1, \lambda_2, ..., \lambda_{\mathbf{MN}})$

[Theorem]. If ρ and ρ' are not degenerate, they are equivalent under local unitary transformations if and only if $V = XDY^\dagger$, $D = diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_{MN}})$, contains a unitary tensor decomposable element for some $\theta_i \in \mathbb{R}$, $rank(\tilde{V}) = 1$.

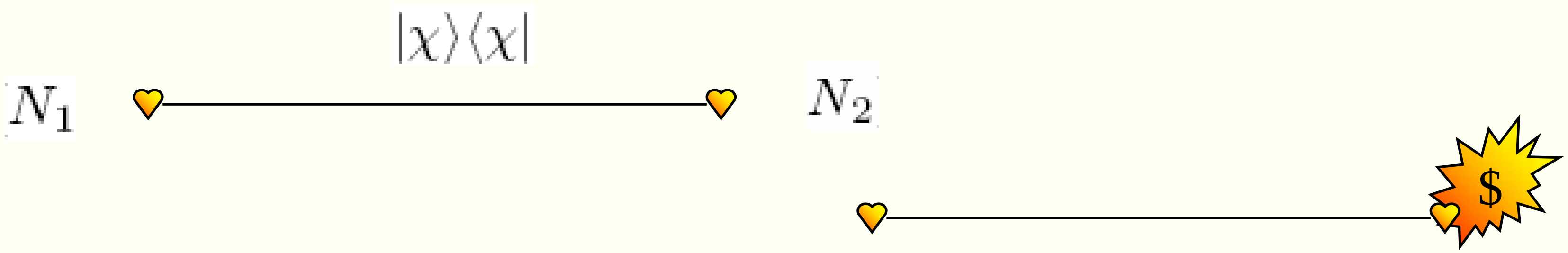
($V = V_1 \otimes V_2$ iff $rank(\tilde{V}) = 1$)

S.M. Fei, N.H. Jing, Phys. Lett. A 342(2005)77
X.H. Gao, S. Albeverio, S.M. Fei, Z.X. Wang, Commun. Theor. Phys. 45 (2006) 267-270.

SLOCC $\rho_2 = (P \otimes Q)\rho_1(P \otimes Q)^\dagger$

Evolution of quantum entanglement:

Bipartite system with one subsystem undergoes a noisy channel



$$\rho' = (\mathbb{1} \otimes \$)|\chi\rangle\langle\chi|$$

$C(\rho')$ $\sim$ $C[|\chi\rangle]$?

$$C[|\chi\rangle] = \sqrt{\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} |C_{\alpha\beta}|^2}$$

$$C_{\alpha\beta} = \langle\chi|(L_\alpha \otimes L_\beta)|\chi^*\rangle$$

$L_\alpha,\ \alpha = 1, \cdots, N_1(N_1-1)/2$
$L_\beta,\ \beta = 1, \cdots, N_2(N_2-1)/2$

generators of $SO(N_1)$ and $SO(N_2)$ resp.

$$|\chi\rangle = (M_\chi \otimes \mathbb{1})|\phi\rangle$$

$$M_\chi = \sqrt{N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} A_{ij}|i\rangle\langle j|$$

$$|\phi\rangle = \sum_{n=1}^{N_2} |n\rangle \otimes |n\rangle/\sqrt{N_2}$$

$$\rho' = (M_\chi \otimes \mathbb{1})\rho_\$ (M_\chi^\dagger \otimes \mathbb{1})$$

$$\rho_\$ = (\mathbb{1} \otimes \$)|\phi\rangle\langle\phi|$$

If $\rho_\$$ is a pure state (e.g. channel \$ is unitary or stochastic quantum operation given by a local filter)

$$C[\rho'] = \sqrt{\sum_{\alpha=1}^{N_1(N_1-1)/2} \sum_{\beta=1}^{N_2(N_2-1)/2} |C'_{\alpha\beta}|^2}$$

$$C'_{\alpha\beta} = \frac{N_2}{2}\sum_{\gamma=1}^{N_2(N_2-1)/2} C_{\alpha\gamma}[|\chi\rangle] C_{\gamma\beta}[\rho_{\$'}]$$

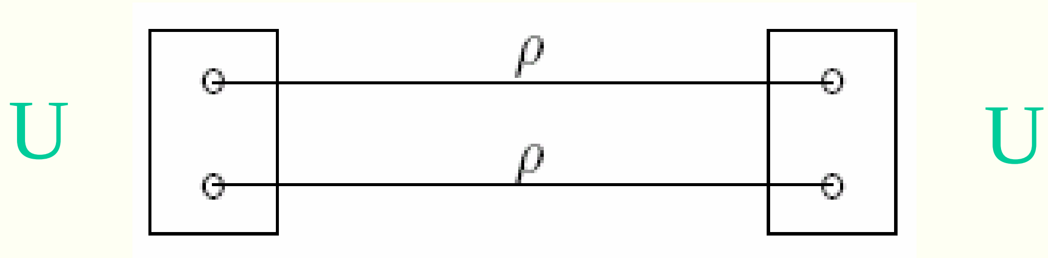
For $N_1 \otimes 2$ system: $C[\rho'] = C[|\chi\rangle]C[\rho_\$]$

(2x2: Nature Physics 4, 99 (2008))

For general mixed initial state: $C\left[(\mathbb{1} \otimes \$)\rho_0\right] \leq \frac{N_2}{2} C(\rho_0) C[\rho_\$]$

Z.G. Li, S.M. Fei, Z.D. Wang and W.M. Liu, Phys. Rev. A,79 (2009) 024303

Distillation:



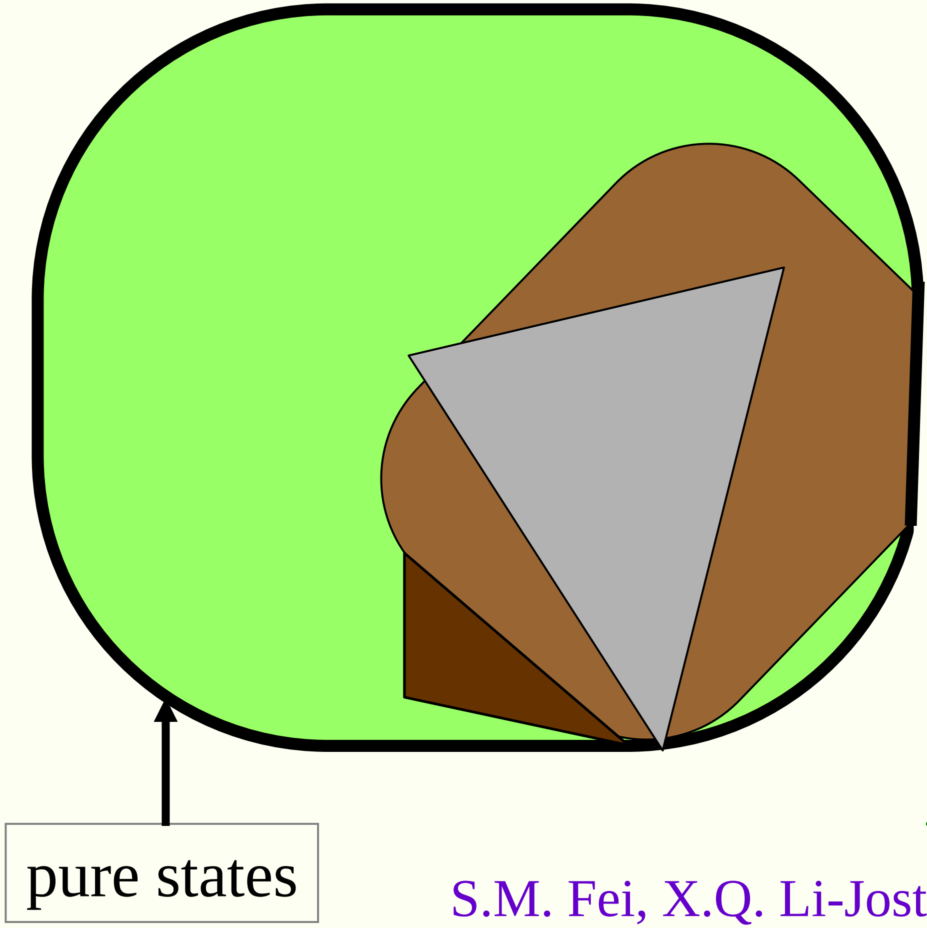
Measurement

Entangled, but not distillable: Bound entangled

All PPT entangled states are bound entangled!

Entanglement

No Classical Counterpart



All states

unentangled states:
mixtures of products

bound entangled (PPT)
states: not distillable

bound entangled
(NPPT) states ?

Much remains to be clarified!

S.M. Fei, X.Q. Li-Jost, B.Z. Sun, Phys. Lett. A 352 (2006) 321

Thanks !