GENERAL RELATIVITY (GR) FOR PEDESTRIANS

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OUTLINE OF TALK

- SOME REMARKS ON GR
- E&M VS GR
  $J_\mu \longleftrightarrow T_{\mu\nu}$
- FOUR TESTS OF GR
  * 5th TEST: PRECESSION OF GYROSCOPE IN SPACE NOT COVERED
- GR & GPS

EINSTEIN'S GR

- GRVITY AS SPACE-TIME GEOMETRY

- EQUIVALENCE PRINCIPLE

  * $\vec{F} = m_i \vec{a}$, m_i = INERTIAL MASS

GRAVIT. ATTRACTION:

  * $\vec{F} = -G m_g \frac{M}{r^2} \hat{r}$

    m_g = GRAVIT'NAL MASS

$$\boxed{m_i = m_g}$$ EQUIV. PRINCIPLE

$\vec{a} = -G \frac{M}{r^2} \hat{r}$

- UNIVERSAL FOR ALL BODIES IN A GRAVIT'L FIELD

- FOUR TESTS OF GR
  - * GRAVITATION REDSHIFT
  - * DEFLECTION OF LIGHT
  - ** TIME DELAY IN RADAR ECHOES
  - * PERIHELION PRECESSION OF MERCURY
- A SIMPLE APPROACH TO EXPLAIN THESE TESTS
  - J. SCHWINGER 1968

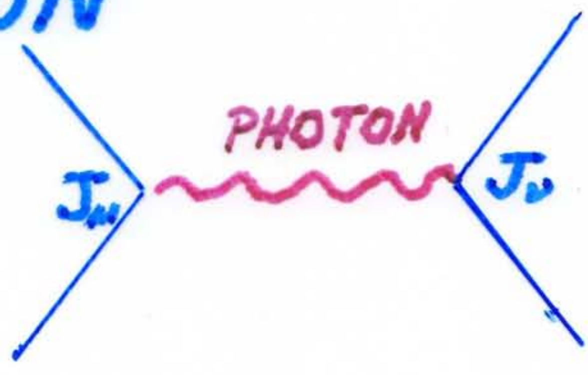
* CENTRAL OBJECT $T_{\mu\nu}$. THE STRESS ENERGY TENSOR

* GRAVIT'AL INTERACT'N BETWEEN TWO BODIES

E&M vs GR

E&M : FORCE CAN BE ATTRACTIVE OR REPULSIVE

IN QM LANGUAGE, EM FORCES ARE MEDIATED BY EXCHANGE OF A SPIN 1 MASSLESS PHOTON



$$\Delta S = \int dt\, \Delta L_I = \frac{1}{2}\int d^4x\, d^4x'\, J_\mu(x)\, D(x-x')\, J^\mu(x')$$

$$D(x) = \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2 + i\epsilon}\, e^{-ik\cdot x}$$

$$\int_{-\infty}^{\infty} dx^0\, D(x) = -\frac{1}{4\pi|\vec{x}|}$$

- QUASI-STATIC CASE

$$x^0 \approx x'^0$$

* THEN $\int dt\, \Delta L_I = -\int dt\, V(t)$

* $V(t) = \frac{1}{2}\int d^3x\, d^3x' \frac{J^0(\vec{x})J^0(\vec{x}') - \vec{J}(\vec{x})\cdot\vec{J}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$

- $J^0 J^0$: COULOMB POTENTIAL
- $\vec{J}\cdot\vec{J}$: AMPERE'S LAW

- IF EXCHANGED PHOTON IS REAL, IN MOMENTUM SPACE

$$k^2 = 0 \qquad \text{(MASSLESS)}$$

$$k^0 = |\vec{k}|$$

$$k^\mu = (k^0, 0, 0, k^3 = k)$$

- CURRENT CONSERVATION

$$k^\mu J_\mu(k) = 0 \Rightarrow k^0 J^0 - kJ^3 = 0 \Rightarrow J^0 = J^3$$

$$\therefore J^\mu(k) J_\mu(k) = -J_1 \cdot J_1 - J_2 \cdot J_2$$

- A PHOTON HAS ONLY 2 DEGREES OF F.

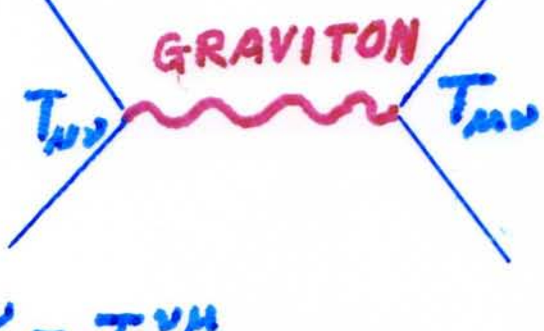
GRAVITATION

FORCES BETWEEN MASSES ARE MEDIATED BY A SPIN 2 MASSLESS GRAVITON

GRAVITON

- MASS = 0
- SPIN = 2, ALWAYS ATTRACTIVE
- ONLY 2 DEGREES OF F.

GRAVITON COUPLES TO STRESS ENERGY TENSOR $T_{\mu\nu}$



$$T^{\mu\nu} = T^{\nu\mu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\nu T^{\mu\nu} = 0$$

REAL GRAVITON EXCHANGE

- $k^\mu = (k, 0, 0, k^3 = k), \quad k^2 = 0$
- $T_{\mu\nu} T^{\mu\nu} \longrightarrow T'_{\mu\nu} T'^{\mu\nu}$

$$= \sum_{a,b=1,2} T'_{ab} T'^{ab}$$

- $T'^{ab} = T^{ab} - \frac{1}{2} \delta^{ab} \sum_c T^{cc}$
  - $\sum_a T'^{aa} = 0$

$\therefore$ ONLY 2 INDEPENDENT VARIABLES

- $T_{12} = T_{21}$
- $T_{11} = -T_{22}$

*GENERAL CASE $(k^2 \neq 0)$

- $\int dt\, \Delta L = -4\pi G \int d^4x\, d^4x'$

$$\left[T^{\mu\nu}(x) D(x-x') T_{\mu\nu}(x') - \frac{1}{2} T(x) D(x-x') T(x') \right], \quad T \equiv T^\mu{}_\mu$$

QUASI STATIC CASE

$x^0 \sim x^{0'}$

- $\int dx^{0'} D(x-x') = -\frac{1}{4\pi} \cdot \frac{1}{|\vec{x}-\vec{x}'|}$

$\int dt \, \Delta L = -\int dt \, E_{int}(x^0)$

- $E_{int} = -G \int d^3x \, d^3x' \left[T^{\mu\nu}(\vec{x}) \frac{1}{|\vec{x}-\vec{x}'|} T_{\mu\nu}(\vec{x}') \right.$

$\left. \qquad - T(\vec{x}) \frac{1}{|\vec{x}-\vec{x}'|} T(\vec{x}') \right]$

* $T^{\mu\nu}(\vec{x}) = T_1^{\mu\nu}(\vec{x}) + t^{\mu\nu}(\vec{x})$

STATIC M $\quad$ TEST BODY

$\vec{x} \sim 0$

- $T_1^{\mu\nu} : \int d^3x \, T_1^{00} = M$
  - $T_1^{0k} = T_1^{k\ell} = 0$
- $T_1 \equiv T_1{}^{\mu}{}_{\mu} = T_1^{00}$
- $t = t^{00} - t^{kk}$

• INTERACTION ENERGY E_{int}

• $$E_{int} = -GM\int d^3x \frac{1}{|\vec{x}|}[2t^{00}(\vec{x}) - t(\vec{x})]$$

$$= -GM\int d^3x \frac{1}{|\vec{x}|}[t^{00}(\vec{x}) + t^{kk}(\vec{x})]$$

• IF TEST BODY IS A MASS

$$t^{kk} = 0. \quad \int d^3x\, t^{00} = m$$

• $E_{int} = -G\frac{Mm}{R}$ NEWTON

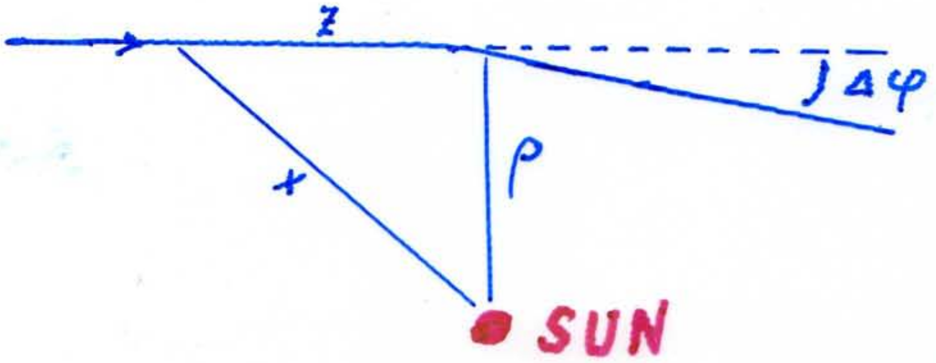
• GRAVTIONAL REDSHIFT

• ATOM TOTAL REST ENERGY

$$m_T = m + E_{int}$$
$$= m\left(1 - \frac{GM}{R}\right)$$

* m_T SETS THE SCALE OF ENERGIES OF ATOMIC RADIATION
* FREQUENCY IS REDUCEDBY $\left(1-\frac{GM}{R}\right)$

DEFLECTION OF LIGHT



TEST BODY : LIGHT BEAM

$$t^{\mu\nu} = \sigma p^{\mu} p^{\nu}$$

$$\therefore t = \sigma p^2 = 0 \quad \text{LIGHT}$$

$$\therefore E_{int} = -GM \int d^3x \, \frac{1}{|\vec{x}|} \left[2 t^{00}(\vec{x}) - \underset{0}{\underset{\parallel}{t(\vec{x})}} \right]$$

$$= - \frac{2GME}{(z^2 + \rho^2)^{1/2}}$$

BEFORE COMPUTING Δφ, NOTICE E_{int} IS TWICE THE NEWTONIAN VALUE

CACULATION OF $\Delta\varphi$

$$F_\perp = -\frac{\partial}{\partial\rho} E_{int} = \frac{\partial}{\partial\rho} \frac{2GMk}{(z^2+\rho^2)^{1/2}}$$

$$\Delta p_\perp = \int_{-\infty}^{\infty} dt\, F_\perp \quad , \qquad dz = c\,dt = dt \quad (c=1)$$

$$= 2GMk \frac{\partial}{\partial\rho} \int_{-\infty}^{\infty} dz \frac{1}{(z^2+\rho^2)^{1/2}}$$

$$\Delta\varphi = \frac{\Delta p_\perp}{p} = \frac{4MG}{\rho}$$

- SO THE ANGLE OF DEFLECTION FOR LIGHT IS TWICE THE NEWTONIAN RESULT — EINSTEIN'S PREDICTION
- 1919
  - EDDINGTON'S EXPEDIT'N TO WESTERN AFRICA
  - ECLIPSE DATA
  - 2ND GROUP WENT TO BRAZIL
  - SOME CONTROVERY
- 1979 REANALYSIS SURPORTED EINSTEIN
- PHYSICS TODAY MARCH 09

TIME DELAY IN RADAR ECHOES FROM INNER PLANETS

I.I. SHAPIRO 1964

THE SAME E_{int} REDUCES A PHOTON'S ENERGY BY A FACTOR OF $1-\frac{2GM}{R}$

$$E_T = k\left(1-\frac{2GM}{R}\right)$$

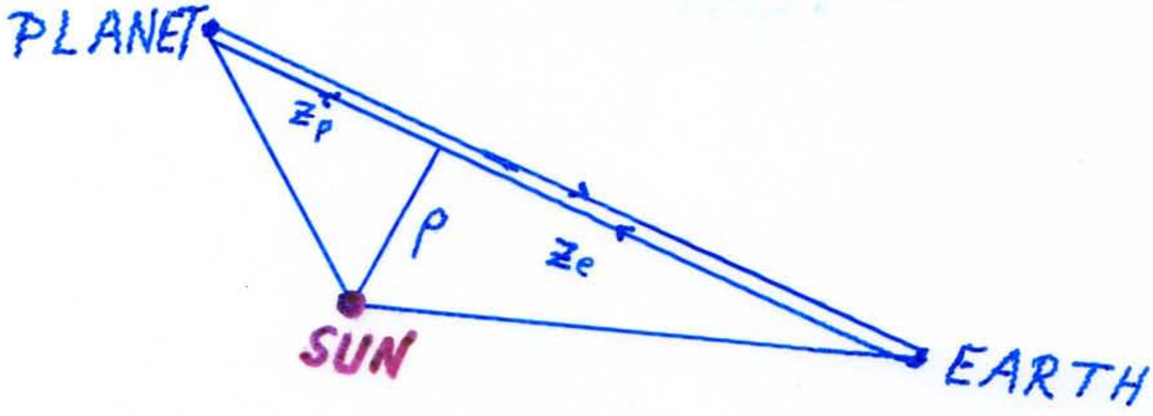
$$v = \frac{\partial E_T}{\partial k} = 1-\frac{2GM}{R} < 1$$

$$= 1-\Delta v$$

- EXAMPLES OF $\frac{\partial E}{\partial p} = v$
  - $E = \sqrt{p^2 + m^2}$
  - $\frac{\partial E}{\partial p} = \frac{p}{E} = v$
  - $E = \frac{p^2}{2m}$
  - $\frac{\partial E}{\partial p} = \frac{p}{m} = v$
- INVARIANT STATEMENT OF LIGHT

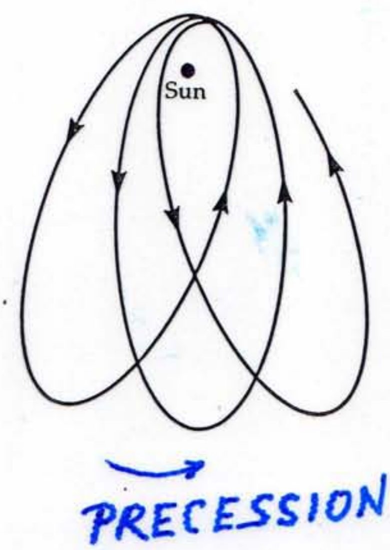
$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = 0$$

$$\not\Rightarrow \frac{d\vec{r}}{dt} = 1 \text{ ALWAYS}$$



- ADDITIONAL TIME DELAY
- $$\Delta t = 2\Delta U \int_E^P dz$$
  $$= 4GM \int_0^{z_e} dz \frac{1}{(z^2+\rho^2)^{1/2}}$$
  $$+ 4GM \int_0^{z_p} dz \frac{1}{(z^2+\rho^2)^{1/2}}$$
- $$\Delta t = 4GM \ln\left(\frac{2z_e}{\rho} \cdot \frac{2z_p}{\rho}\right)$$
- CONFIRMED BY VIKING MISSION TO MARS IN 1976 (I.I. SHAPIRO, et al.)

PERIHELION PRECESSION OF PLANETORY ORBITS



RECALL

$$E_{int} = -GM\int d^3x \, \frac{1}{|\vec{x}|}\left[t^{00}(\vec{x}) + t^{kk}(\vec{x})\right]$$

IF m MOVES SLOWLY, LOCALIZED AT $\vec{x} = \vec{R}$, THEN $t^{kk} = 0$

$$E_{int} = -G\frac{Mm}{R} \quad \text{NEWTON}$$

CORRECTION TO NEWTON

- POTENTIAL DUE TO PLANET'S MOTION

  * $t^{\mu\nu} \cong \frac{1}{m} p^\mu p^\nu \delta^3(x-R)$
  * $t^{kk} = \frac{1}{m} \vec{p}^{\,2} \delta^3(x-R)$

    $= \left(\frac{p}{m}\right)^2 t^{00} = \frac{2T}{m} t^{00}$

  - $T = \frac{p^2}{2m}$ KINETIC ENERGY

- $E_{int} = -GM \int d^3x \frac{1}{|\vec{x}|} [t^{00} + t^{kk}]$

  $= -\frac{GM}{R} \int d^3x \left(1 + \frac{2T}{m}\right) t^{00}$

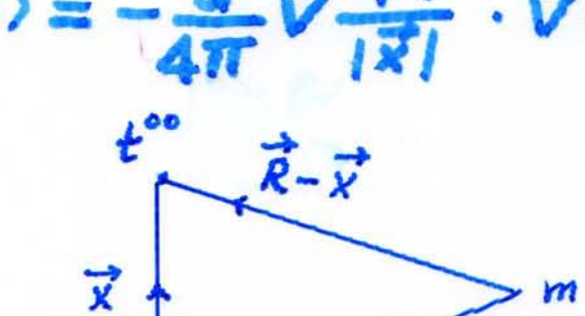
  - $\int d^3x\, t^{00} = m + T + \underline{\underline{\frac{1}{2}}} V = m\left(1 + \frac{T}{m} + \frac{1}{2}\frac{V}{m}\right)$

- V IS SHARED BETWEEN THE SUN & PLANET, SO ATTRIBUTE $\frac{1}{2}V$ TO EACH

DERIVATION

CONTRIBUTION TO t^{00} DUE TO GRAVITATIONAL INTERACTION BETWEEN PLANET AND SUN

$$t^{00}_{int}(\vec{x}) = -\frac{G}{4\pi}\vec{\nabla}\frac{M}{|\vec{x}|}\cdot\vec{\nabla}\frac{m}{|\vec{R}-\vec{x}|}$$



• CHECK

$$\int d^3x\, t^{00}_{int} = -\frac{G}{4\pi}\int d^3x\, \vec{\nabla}\frac{M}{|x|}\cdot\vec{\nabla}\frac{m}{|\vec{x}-R|}$$

$$= \frac{G}{4\pi}\int d^3x\, \frac{M}{|x|}\,\nabla^2\frac{m}{|\vec{x}-\vec{R}|}$$

$$= -\,G\frac{Mm}{R}$$

$$* \quad = V \qquad \text{AS REQUIRED}$$

• THIS t^{00}_{int} GIVES EXTRA TERM TO CORRECTION TO V:

$$-GM\int d^3x\, \frac{1}{|\vec{x}|}\, t^{00}_{int}(\vec{x})$$

$$= -GM\int d^3x\, \frac{1}{|\vec{x}|}\left(-\frac{GMm}{4\pi}\right)\vec{\nabla}\frac{1}{|\vec{x}|}\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{R}|}$$

$$= \frac{G^2M^2m}{4\pi}\int d^3x\, \frac{1}{2}\,\vec{\nabla}\frac{1}{|\vec{x}|^2}\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{R}|}$$

$$= \frac{1}{2}\left(\frac{G^2M^2m}{R^2}\right)$$

$$* \quad = V\cdot\left(\frac{1}{2}\frac{V}{m}\right)$$

- THESE EFFECTS CORRECT THE NEWTON POTENTIAL TO

- $V\left(1+\frac{2T}{m}\right)\left(1+\frac{T}{m}+\frac{1}{2}\frac{V}{m}\right)$

  $\approx V + V\left(\frac{3T}{m}\right) + \frac{1}{2}\frac{V^2}{m}$

- ONE MORE CORRECTION DUE TO SPECIAL RELATIVITY

  - $(p^2+m^2)^{\frac{1}{2}} - m = T - \frac{T^2}{2m}$

- $\therefore\quad V \rightarrow V + \Delta V$

- $\Delta V = V\left(\frac{3T}{m}\right) + \frac{V^2}{2m} - \frac{T^2}{2m}$

- $E = T + V$
- $\Delta V = -\frac{E^2}{2m} + \frac{4E}{m}V - \frac{3}{m}V^2$
- $-\frac{E^2}{2m}$: CONST., NO EFFECT
- $\frac{4E}{m}V$: STILL $\sim\frac{1}{R}$. NO PREHELION PRECESSION
- $V_{eff} = V - \frac{3}{m}V^2$. $\quad V = -\frac{GMm}{R}$

- GOLDSTEIN: CLASSICAL MECHANICS
- $u \equiv \frac{1}{R}$
- $\frac{d^2u}{d\varphi^2} + u = -\frac{1}{L_1^2}\frac{d}{du}\frac{V_{eff}}{m}$
- $L_1 = \frac{l}{m}$ ANGULAR MOMENTUM PER UNIT MASS
- $V_{eff} = -GMmu - 3G^2M^2mu^2$
- $\therefore \frac{d}{du}\frac{V_{eff}}{m} = -GM - 6G^2M^2u$

MODIFIED ORBIT EQ:

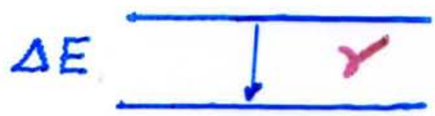
- $\frac{d^2u}{d\varphi^2} + \left(1 - \frac{6G^2M^2}{L_1^2}\right)u = \frac{GM}{L_1^2}$
- $\varphi' = \left(1 - \frac{3G^2M^2}{L_1^2}\right)\varphi$, REDUCTION OF SCALE FOR φ
- $\varphi' = 2\pi$
- $\varphi = 2\pi + \Delta\varphi$
- $\Delta\varphi = \frac{6\pi G^2M^2}{L_1^2} = 43''$/CENTURY MERCURY

GLOBAL POSITIONING SYSTEM (GPS)

- PURE SCIENCE + TECHNOLOGY
- WIDELY USED, RARELY UNDERSTOOD
- ATOMIC CLOCKS
- SATELLITES
- SPECIAL & GENERAL RELATIVITY CORRECTIONS
- CONSTANT MONITORING
- 1973 CONCEPT OF GPS
- 1991 FIRST USED IN GULF WAR
- US $ 2 B NOW → $ 30 B IN 10 YEARS

ATOMIC CLOCKS

- PURPOSE : PRECISE MEAS'NT OF SMALL ENERGY DIFF.



$\Delta E = hf$

- 1945 I.I. RABI PROPOSAL
- 1952 NBS FIRST Cs^{133} ATOMIC CLOCK
- 1958 COMMERCIAL Cs CLOCK
  $20K EACH
- 1967 1 SEC ≡ 9192631770 CYCLES IN Cs^{133}'s HYPERFINE TRANSITIONS
- 1999 NIST-F1
  $\Delta t \sim$ 1.7 PARTS IN 10^{15}
  <1SEC IN 20M YEARS

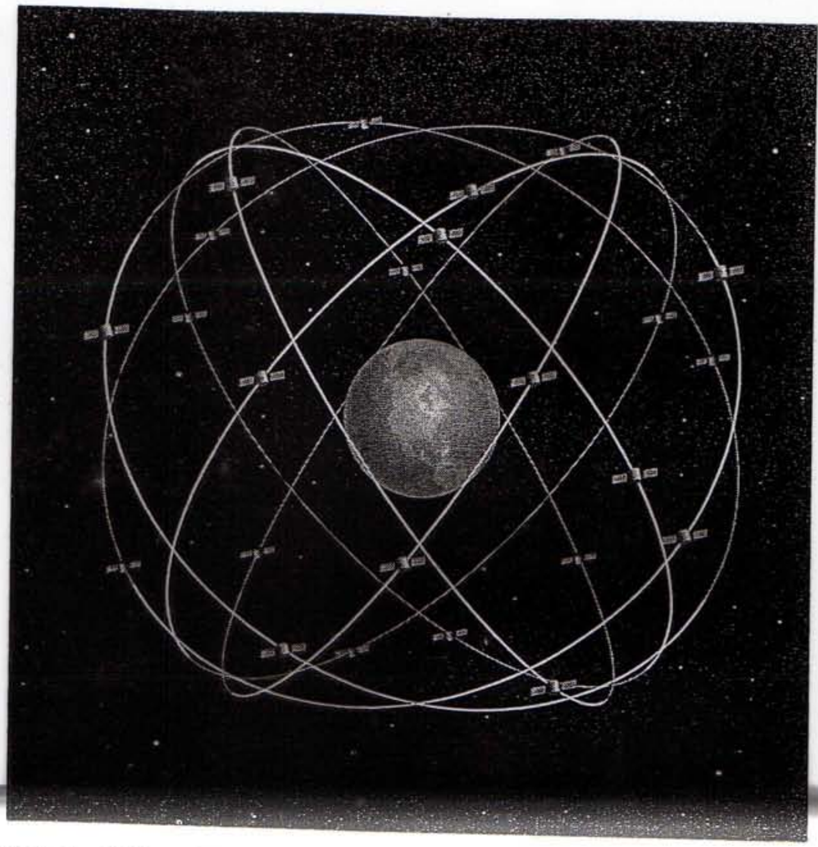


FIGURE 6.9 GPS satellite constellation: The GPS constellation of 24 satellites are arranged in 6 equally spaced orbit planes.

- WHY SUCH ACCURATE CLOCKS?
  - $C = 3 \times 10^{10}$ CM/S
  - LIGHT TRAVELS 30 cm IN 1 NS (10^{-9} sec) !!
- SATELLITES
  - A SYSTEM OF 24 SATELLITES
  - EACH IN A 12-HOUR ORBIT
  - 6 ORBITAL PLANES ABOUT EARTH
  - AT LEAST 4 SATELLITES ABOVE HORIZON AT ANY POINT ON EARTH
  - SIGNALS FROM THESE FOUR DETERMINE X, Y, Z & t.

A HYPOTHETICAL WORLD

- EARTH CENTER AT REST IN AN INERTIAL FRAME
- EACH SATELLITE SENDS PERIODICALLY SIGNALS ENCODED WITH TIME & LOCATION OF EMISSION
- AN OBSERVER THAT RECEIVES SIGNAL AN INTERVAL LATER CAN CALCULATE THE DISTANCE FROM THE SATELLITES
- SIGNALS FROM 4 SATELLITES $\Rightarrow$ OBSERVER'S X, Y, Z & t.

CORRECTIONS TO PROPER TIME ON SATELLITE CLOCKS TO GIVE TIME OF THE INERTIAL FRAME:

- TIME DILATION: SATELLITES MOVE W.R.T. OBSERVER.
- SIGNALS RECEIVED AT SLOWER RATE THAN EMITTED

SATELLITE SPEED V_s

$$\frac{V_s^2}{R_s} = \frac{GM}{R_s^2}$$

$R_s \sim 2.7 \times 10^4$ Km

$V_s \sim 3.9$ Km/s

$$\frac{V_s}{c} \simeq 1.3 \times 10^{-5}$$

$$\Delta t' \sim \gamma \Delta t, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \simeq 1 + \frac{1}{2}\beta^2$$

FRACTIONAL CORRECT'N $= \frac{1}{2}\beta^2 = 0.84 \times 10^{-10}$

GRAVITTAT'L REDSHIFT-BIG SURPRISE

- IN OUR HYPOTHETICAL WORLD
  - A DISTANT OBSERVER AT REST & FAR AWAY FROM ANY GRAVIT. SOURCE
  - GRAVIT. POTENTIAL AT SATELLITE $= -\frac{GM}{R_S} = -V_S^2$
- FRACTIONAL CORRECTION TO EMISSION RATE DUE TO GRAVIT. REDSHIFT $= \left(\frac{V_S}{c}\right)^2 = 1.6 \times 10^{-10}$
- TWICE AS BIG AS TIME DILATION!
- IN LESS THAN 5 HOURS ACCUMULATED ERROR ~ 1 KM!

- ACTUAL GPS: NOT IN AN INERTIAL FRAME WITH ITS TIME DEFINED BY CLOCKS AT ∞
- RATHER, A FRAME ROTATING WITH EARTH & CLOCKS ON EARTH'S SURFACE
- TIME DILATION

  SATELLITE SPEED $v \sim 4\,km/s$

  SATELLITE CLOCKS SLOWER

  *BY $\frac{\Delta t}{t} \sim 10^{-10}$
- GRAVITATIONAL BLUE (NOT RED) SHIFT $\frac{\Delta t}{t} \sim 5 \times 10^{-10}$

  SATELLITE CLOCKS FASTER
- MANY OTHER CORRECTIONS

CONCLUSIONS

- INTERACTIONS BETWEEN "MASSES" ($T_{\mu\nu}$) VIA ONE GRAVITON EXCHANGE
- SIMPLE EXPLANATION FOR
  - GRAVITATIONAL REDSHIFT
  - LIGHT BENDING
  - TIME DELAY
- SUPPLEMENTED BY HIGHER ORDER (IN $\frac{1}{c^2}$) TERMS IN t_{oo} ⇒ PREHELION PRECESS'N
- GPS = WONDERFUL APPLICATION OF PURE SCIENCE (GR) TO TECHNOLOGY