Symmetry properties of black holes
in higher dimensional general relativity

Akihiro Ishibashi

Cosmophysics, Theory Division, IPNS,
KEK (High Energy Accelerator Research Organization)

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Purpose of this talk

- attempt to give a brief overview of basic properties of higher dimensional black holes
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- present a proof of symmetry/rigidity theorems of higher dimensional black holes (Hollands, Al & Wald 07)
Outline

- Introduction
- Basic properties of $4D$ stationary BHs
- Basic properties of $D > 4$ BHs
- Symmetry properties of $D > 4$ BHs
- Remarks
Why Higher-dimensions?

- required in most attempts to unify the forces in Nature
  Kaluza-Klein, Supergravity, Superstring theories

- phenomenological ideas
  Braneworld / Large extra-dimensions

- help understand 4-dimensional gravity
  How special 4D gravity is!
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Higher Dimensional BH solutions play an important role
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*Higher Dimensional BH solutions play an important role*

**Focus:** Stationary Black Holes in $D > 4$ General Relativity
  – No compactified dimensions
Black holes in 4D general relativity

Asymptotically flat stationary BHs in 4-dimensions
Black holes in $4D$ general relativity

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- **Exact Solutions** — (Kerr-family metrics)
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- **Uniqueness** — (vacuum $\Rightarrow$ Kerr-metric)
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- **BH Mechanics** — (Thermodynamics)
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Which properties of $4D$ BHs are extended to $D > 4$?
Exact Solutions — much larger variety

Rotating Holes (Myers & Perry 86)  Rotating Rings (Emparan & Reall 02)
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Static vacuum ⇒ stable (AI & Kodama 03)

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  Some restrictions, e.g., (Galloway & Shoen 05)
Introduction

Overview of $D > 4$ black holes

Symmetry properties

Remarks

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  Hole and Rings w/ the same $(J, M)$
  
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  Uniqueness in $5D$ rotating holes/rings (Morisawa-Iida 04, Hollands & Yazadjiev 07)
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**Symmetry** — **This talk**

Rigidity Theorems (Hollands, Al & Wald 07)

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\[ \exists \text{ horizon} \iff 0 = g^{rr} = \Pi_i \left( 1 + \frac{(J_i/M)^2}{r^2} \right) - \frac{GM}{r^{D-3}} \]

as the last term dominates for small $r$ when $D \geq 6$
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- not uniquely specified by $(M, J_1, J_2)$

  *two ring-solutions w/ the same $(M, J_1, J_2 = 0)$*
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⇒ In $5D$, Uniqueness Theorem no longer holds as it stands
Exact solutions in $D \geq 4$

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  - Black-Saturn (“hole” + “ring”) (Elvang & Figueras 07)
Stability of static holes

Gravitational perturbations of $\forall D > 4$ static black holes
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  Tensor-mode w.r.t. $(D - 2)$-horizon manifold $\Sigma$
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  c.f. if $\Sigma$ is a highly clumpy Einstein-manifold,
  ⇒ tensor-mode instability (Gibbons & Hartnoll 02)
Stability of rotating holes: Some partial analysis

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  - looks like black-p-brane near the rotation axis
  - unstable due to Gregory-Laflamme modes? (Emparan & Myers 03)
Myers-Perry solution:

\[ ds^2 = -dt^2 + \frac{M}{\rho^2 r^{D-5}} (dt + a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 \\
+ \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\Omega^2_{(D-4)} \]

where

\[ \rho^2 = r^2 + a^2 \cos^2 \theta \quad \Delta = r^2 + a^2 - \frac{M}{r^{D-5}} \]

In the ultra-spinning limit: \( a \to \infty \) with \( \mu = M/a^2 \) kept finite, near the pole \( \theta = 0 \) \( (\sigma := a \sin \theta) \) the metric becomes

\[ ds^2 = -\left(1 - \frac{\mu}{r^{D-5}}\right) dt^2 + \left(1 - \frac{\mu}{r^{D-5}}\right)^{-1} dr^2 + r^2 d\Omega^2_{(D-4)} + d\sigma^2 + \sigma^2 d\phi^2 \]

\( \Rightarrow \) Black-membrane metric \( \Rightarrow \) Gregory-Laflamme instability?
Stability of rotating holes: Some partial analysis

- $D(\text{odd}) \geq 7$: rotating holes \textit{(Kunduri-Lucietti-Reall 06)}

Special background: $J_1 = J_2 = \cdots J_{[(D-1)/2]}$
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⇒ **Stable** w.r.t. a subclass of tensor perturbations
  (tensor-modes w.r.t. $(D-3)$-base space)
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  - For \( \Lambda < 0 \) \( \Rightarrow \) superradiant instability is observed
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- For $\Lambda < 0$ ⇒ *superradiant instability* is observed

- Towards complete stability analysis of rotating holes:
  - decoupled master equations for zero-modes of vector and tensor fields in $5D$ Myers-Perry black holes
  with $J_1 = J_2$ enhanced symmetry  (Murata & Soda 07)
Topology of event horizon
Method 1: global analysis  (Chrusciel & Wald 94)

- Combine Topological Censorship and Cobordism of spacelike hypersurface $S$ with boundaries at horizon and infinity
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\text{Topological Censorship} \implies S \text{ is simply connected}
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\Sigma = \partial S \text{ is cobordant to } S^{D-2} \text{ via } S
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Topological Censorship $\Rightarrow S$ is simply connected

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In $4D$ $\Rightarrow \partial S$ must be $S^2$
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– powerful method in $4D$ but turns out to be not so in $D \geq 6$

   e.g., (Helfgott-Oz-Yanay 05)
Method 2: local analysis  \( (\text{Hawking 72}) \)

- Combine variational analysis \( \delta \theta / \delta \lambda \) and fact that outer-trapped surface must be inside BH, to show

\[
\int_{\Sigma} R > 0
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w/ \( \Sigma \) being a horizon cross-section and \( R \) scalar curvature of \( \Sigma \)
Topology of event horizon

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– generalizes to $D > 4$ \textit{(Galloway & Shoen 05)}
**Theorem:** (Galloway & Shoen 05, Galloway 07)

Consider a $\forall D \geq 4$ (stationary) black hole spacetime satisfying the dominant energy conditions. Then, the topology of (event) horizon cross-section $\Sigma$ must be such that $\Sigma$ admits metrics of positive scalar curvature.
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Remarks:

- $\Sigma$ can be topologically e.g., $S^{D-2}$, $S^m \times \cdots \times S^n$
  
  In 5D $\Rightarrow S^3$ or $S^1 \times S^2$
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- What if $\Lambda < 0$? $\Rightarrow$ more variety?
Symmetry property of black holes
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Assertion:

(1) The event horizon of a stationary, electro-vacuum BH is a Killing horizon

(2) If rotating, the BH spacetime must be axisymmetric

* Event Horizon: a boundary of causal past of distant observers
* Killing Horizon: a null hypersurface with a Killing symmetry vector field being normal to it
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\[ \cdots \text{ Black Hole Rigidity} \]
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- relates “global” (even horizon) to “local” (Killing horizon)
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- rotating hole $\Rightarrow$ extra-(axial) symmetry
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- foundation of BH Thermodynamics
  (Constancy of surface gravity $\Rightarrow$ Oth Law of Thermodynamics)
- rotating hole $\Rightarrow$ extra-(axial) symmetry
- a critical step toward proof of “Uniqueness” in $4D$ case
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  (Constancy of surface gravity ⇒ Oth Law of Thermodynamics)
- rotating hole ⇒ extra-(axial) symmetry
- a critical step toward proof of “Uniqueness” in \(4D\) case
- In \(D > 4\), Uniqueness no longer holds as it stands, and there seems to be a much larger variety of exact BH solutions
Why “rigidity” interesting?

- relates “global” (even horizon) to “local” (Killing horizon)
- foundation of BH Thermodynamics
  (Constancy of surface gravity ⇒ Oth Law of Thermodynamics)
- rotating hole ⇒ extra-(axial) symmetry
- a critical step toward proof of “Uniqueness” in 4D case
- In $D > 4$, Uniqueness no longer holds as it stands, and there seems to be a much larger variety of exact BH solutions

⇒ “Rigidity”–if holds also in $D > 4$—places important restrictions on possible exact BH solutions
Rigidity theorem in $D = 4$

Proof in $4D$: Hawking (1972), Hawking & Ellis (1973)
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However

Hawking’s proof for $4D$ case relies heavily on the fact that event horizon cross-section $\Sigma$ is topologically 2-sphere

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Goal: Prove BH Rigidity Theorem in $D \geq 4$

No Assumption on Topology of Event Horizon
Let \((M, g)\) be a \(D \geq 4\), analytic, asymptotically flat, stationary vacuum BH solution to Einstein’s equation. Assume event horizon \(\mathcal{H}\) is analytic, non-degenerate, and topologically \(\mathbb{R} \times \Sigma\) with cross-sections \(\Sigma\) being compact, connected.
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Let $(M, g)$ be a $D \geq 4$, analytic, asymptotically flat, stationary vacuum BH solution to Einstein’s equation. Assume event horizon $\mathcal{H}$ is analytic, non-degenerate, and topologically $\mathbb{R} \times \Sigma$ with cross-sections $\Sigma$ being compact, connected.

**Theorem 1:** There exists a Killing field $K^a$ in the entire exterior of the BH such that $K^a$ is normal to $\mathcal{H}$ and commutes with the stationary Killing vector filed $t^a$ $\Rightarrow$ “Killing horizon”
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**Theorem 1:** There exits a Killing field $K^a$ in the entire exterior of the BH such that $K^a$ is normal to $\mathcal{H}$ and commutes with the stationary Killing vector filed $t^a$ $\Rightarrow$ “Killing horizon”

**Theorem 2:** If $t^a$ is not normal to $\mathcal{H}$, i.e., $t^a \neq K^a$, then there exist mutually commuting Killing vector fields $\varphi^a_{(1)}, \ldots, \varphi^a_{(j)}$ $(j \geq 1)$ with period $2\pi$ and $t^a = K^a + \Omega_{(1)} \varphi^a_{(1)} + \cdots + \Omega_{(j)} \varphi^a_{(j)}$, where $\Omega_{(j)}$’s constants. $\Rightarrow$ “Axisymmetry”
Brief sketch of proof of Theorem 1

“Trial foliation” $\Sigma$ &
“candidate” vector $K^a$

$K^a$ depends on $\Sigma$
Brief sketch of proof of Theorem 1

"Trial foliation" $\Sigma$ &
"candidate" vector $K^a$

Step 1
Construct a "candidate" Killing field $K^a$
on $H$ which satisfies

\[ K^a K_a = 0 \text{ and } \mathcal{L}_t K^a = 0 \text{ on } H \]
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However, there is \textbf{No reason why }$\alpha$\textbf{ need be constant}
However, there is **No reason why** $\alpha$ **need be constant**

— wish to find “**correct**” $\tilde{K}^a$ with $\tilde{\alpha} = \text{const.} =: \kappa$ on $\mathcal{H}$ by choosing a **new “correct” foliation** $\tilde{\Sigma}$
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When one solves this equation, the spacetime dimensionality comes to play a role
Find correct foliation $\tilde{\Sigma}$: $4D$ case

Hawking 73
Find correct foliation $\tilde{\Sigma}$: $4D$ case    

In $4D$, horizon cross-section $\Sigma$ is 2-sphere, and therefore the orbits of $s^a$ must be closed.
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Hawking 73

**fixed point**

In $4D$, horizon cross-section $\Sigma$ is $2$-sphere, and therefore the orbits of $s^a$ must be closed.

There is a discrete isometry "$\Gamma$" which maps each null generator into itself.

Akihiro Ishibashi  
IPNS, KEK
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Find correct foliation $\tilde{\Sigma}$: $D > 4$ case

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e.g., $5D$ Myers-Perry BH w/ 2-rotations $\Omega_{(1)}$, $\Omega_{(2)}$:

$\Sigma \approx S^3$, \quad t^a = K^a + s^a$

$s^a = \Omega_{(1)} \varphi_{(1)}^a + \Omega_{(2)} \varphi_{(2)}^a$
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$$

$$
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$$

Each rotation Killing vector $\varphi^a$ has closed orbits but $s^a$ does not if $\Omega(1)$ and $\Omega(2)$ are incommensurable.
Solution to $D > 4$ case:

(i) When $s^a$ has closed orbits on $\Sigma \Rightarrow$ we are done!

$$\kappa = \frac{1}{P} \int_0^P \alpha[\phi_s(x)] ds \quad P : \text{period} \quad \phi_s : \text{isom. on } \Sigma \text{ by } s^a$$
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"time-average"  "space-average"
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$$f(x) = \kappa \int_0^\infty P(x,T)dT, \quad P(x,T) = \exp \left(- \int_T^\infty \alpha[\phi_s(x)]ds\right)$$
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— since $\forall \epsilon > 0, P(x, T) < e^{(\epsilon - \kappa)T}$, for sufficiently large $T$,
  $f(x)$ above is well-defined
Brief sketch of proof of Theorem 2

— wish to show \( t^a = K^a + \Omega_{(1)} \varphi_{(1)}^a + \cdots + \Omega_{(j)} \varphi_{(j)}^a \)
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— Get horizon Killing vector field \( K^a \) by Theorem 1

\( \Rightarrow \) Then \( S^a \equiv t^a - K^a \) generates Abelian group, \( \mathcal{G} \), of isometries on horizon cross-sections \( \Sigma \)
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— Extend $U(1)^N$ into the entire spacetime by analyticity
Immediate generalizations:

- can apply to \textit{Einstein-Λ-Maxwell} system
e.g., \textit{charged-AdS-BHs}
Remarks

Immediate generalizations:

- can apply to Einstein-$\Lambda$-Maxwell system
e.g., charged-AdS-BHs

- combined together with Staticity Theorems

\[ d = 4 \quad \text{Sudarsky & Wald (92)} \quad d > 4 \quad \text{Rogatko (05)} \]

⇒ The assertion is rephrased as

Stationary, non-extremal BHs in $D \geq 4$ Einstein-Maxwell system are either static or axisymmetric
Remarks

— can apply to any “horizon” defined as the “boundary” of causal past of a complete timelike orbit $\gamma$ of $t^a$
eq e.g., cosmological horizon
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- can remove analyticity assumption for the BH interior
  by using initial value formulation w/ initial data for $K^a$ on the bifurcate horizon
Remarks

It would not appear to be straightforward to generalize to:

- Theories w/ higher curvature terms and/or exotic source

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1. **Theories** w/ higher curvature terms and/or exotic source
   - Present proof relies on Einstein’s equations
2. **Non-trivial topology** at infinity / BH exterior
   - Horizon Killing field $K^a$ may **not** have a single-valued analytic extension
3. **Extremal BHs** (i.e., BHs w/ degenerate horizon $\kappa = 0$)
Interesting questions:

— Does there exist a $D > 4$ BH solution with only two commuting Killing fields (i.e., w/ isom. $\mathbb{R} \times U(1)$)? (Reall 03)
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$\Rightarrow$ Hunt new (less-symmetric) black objects!
Summary

4D Black holes:
Summary

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- $4D$ Black holes: Restricted by Uniqueness Theorems
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- $D > 4$ Black holes: More varieties
  ⇒ More “surprises” await us!