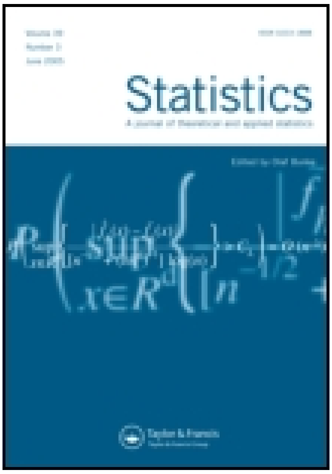

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A study of generalized skew-normal distribution

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A study of generalized skew-normal distribution

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Following the paper by Genton and Loperfido [*Generalized skew-elliptical distributions and their quadratic forms*, Ann. Inst. Statist. Math. 57 (2005), pp. 389–401], we say that $\mathbf{Z}$ has a generalized skew-normal distribution, if its probability density function (p.d.f.) is given by $f(z) = 2\phi_p(z; \boldsymbol{\xi}, \boldsymbol{\Omega})\pi(z - \boldsymbol{\xi})$, $z \in \mathbb{R}^p$, where $\phi_p(\cdot; \boldsymbol{\xi}, \boldsymbol{\Omega})$ is the p-dimensional normal p.d.f. with location vector $\boldsymbol{\xi}$ and scale matrix $\boldsymbol{\Omega}$, $\boldsymbol{\xi} \in \mathbb{R}^p$, $\boldsymbol{\Omega} > 0$, and π is a skewing function from $\mathbb{R}^p$ to $\mathbb{R}$, that is $0 \leq \pi(z) \leq 1$ and $\pi(-z) = 1 - \pi(z)$, $\forall z \in \mathbb{R}^p$. First the distribution of linear transformations of $\mathbf{Z}$ are studied, and some moments of $\mathbf{Z}$ and its quadratic forms are derived. Next we obtain the joint moment-generating functions (m.g.f.'s) of linear and quadratic forms of $\mathbf{Z}$ and then investigate conditions for their independence. Finally explicit forms for the above distributions, m.g.f.'s and moments are derived when $\pi(z) = \kappa(\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$ and κ is the normal, Laplace, logistic or uniform distribution function.

Keywords: elliptical distribution; independence; moment-generating function; multivariate skew-normal distribution; quadratic form

AMS 2000 Subject Classifications: Primary: 60E05; Secondary: 62H10

1. Introduction

There is a growing literature on classes of multivariate distributions which are more flexible than the normal ones. Azzalini and Dalla-Valle [1] introduced the skew-normal distribution which includes the normal distribution and has some properties like the normal and yet is skew. Its probability density function (p.d.f.) is

$$2\phi_p(z; \boldsymbol{\xi}, \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}'(z - \boldsymbol{\xi})), \quad z \in \mathbb{R}^p, \tag{1}$$

where $\phi_p(\cdot; \boldsymbol{\xi}, \boldsymbol{\Omega})$ is the p.d.f. of $N_p(\boldsymbol{\xi}, \boldsymbol{\Omega})$ distribution (the p-dimensional normal p.d.f. with location vector $\boldsymbol{\xi}$ and scale matrix $\boldsymbol{\Omega}$), $\boldsymbol{\xi}, \boldsymbol{\alpha} \in \mathbb{R}^p$, $\boldsymbol{\Omega} > 0$, and Φ is the cumulative distribution function (c.d.f.) of the univariate standard normal distribution. This class of distributions is useful in studying robustness and for modelling skewness. For their statistical properties and applications, we refer to Azzalini and Capitanio [2], Genton *et al.* [3], Loperfido [4], Gupta and Huang [5], Gupta and Kollo [6], and Lachos *et al.* [7] among others.

Generalizations of the skew-normal distribution with p.d.f. (1) have been proposed by many authors including Arnold and Beaver [8], Liseo and Loperfido [9], González-Farías *et al.* [10],

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Gupta and Chen [11], and Genton and Loperfido [12]. For useful reviews of developments in this field, see Genton [13], Azzalini [14] and Arellano-Valle and Azzalini [15]. Recently, Wang *et al.* [16] considered that $\boldsymbol{X}$ has p.d.f. (1) with $\boldsymbol{\xi} = \boldsymbol{0}$ and $\boldsymbol{\Omega}$ being the identity matrix $\boldsymbol{I}_p$, and then introduced the distribution of $\boldsymbol{Y} = \boldsymbol{a} + \boldsymbol{B}'\boldsymbol{X}$, where $\boldsymbol{a} \in \mathbb{R}^q$ and $\boldsymbol{B}$ is a $p \times q$ matrix, as an extension of the skew-normal distribution with p.d.f. (1). Indeed, if $\boldsymbol{B}$ has a full column rank, then the p.d.f. of $\boldsymbol{Y}$ exists and still has the form (1).

Following Genton and Loperfido [12] a p-dimensional random vector $\boldsymbol{Z}$ is said to have a generalized skew-normal distribution, denoted by $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$, if its p.d.f. is

$$2\phi_p(z; \boldsymbol{\xi}, \boldsymbol{\Omega})\pi(z - \boldsymbol{\xi}), \quad z \in \mathbb{R}^p, \tag{2}$$

where π is a skewing function from $\mathbb{R}^p$ to $\mathbb{R}$. That is $0 \le \pi(z) \le 1$ and $\pi(-z) = 1 - \pi(z)$, $\forall z \in \mathbb{R}^p$. It is worth noting that, for any bounded odd function φ such that $|\varphi(x)| \le l, x \in \mathbb{R}$, where $l > 0$ is fixed, $\varphi(\boldsymbol{\alpha}'z)/(2l) + 1/2$ can be taken to be a skewing function. Clearly, for $\pi(z) = 1/2$, $\boldsymbol{Z}$ simply has a $N_p(\boldsymbol{\xi}, \boldsymbol{\Omega})$ distribution. The skewing function π is flexible enough for the class of GSN distributions to include many known skew distributions. The case $\pi(z) = G(P_K(z))$, where G is the c.d.f. of a continuous random variable that is symmetric about 0 and $P_K(z)$ is an odd polynomial of order K defined on $\mathbb{R}^p$, has been introduced by Ma and Genton [17]. The special case by taking G being absolutely continuous and $P_K(z) = \boldsymbol{\alpha}'z$ was studied by Gupta and Chang [18] and Huang and Chen [19]. Gupta *et al.* [20] considered the case $\pi(z) = \Phi((1 + z'\boldsymbol{\Lambda}z)^{-1/2}\boldsymbol{\alpha}'z)$, where $\boldsymbol{\Lambda}$ is a non-negative diagonal matrix. Note that Azzalini and Capitanio [21] defined the class of distributions having p.d.f.

$$2\phi_p(z; \boldsymbol{\xi}, \boldsymbol{\Omega})G(w(z - \boldsymbol{\xi})), \quad z \in \mathbb{R}^p, \tag{3}$$

where w is an odd continuous function from $\mathbb{R}^p$ to $\mathbb{R}$, and then Wang *et al.* [22] showed that this class of distributions is the same as those having a p.d.f. of the form (2). It can be seen that $G(w(z))$ is a skewing function and conversely each skewing function $\pi(z)$ can be represented as $H(w(z))$, where H is a strictly increasing c.d.f. of a continuous random variable which is symmetric about 0 and $w(z) = H^{-1}(\pi(z))$. For simplicity, in this study, we present results by using GSN distributions with a p.d.f. of the form (2).

Through an appropriate choice of the skewing function π in Equation (2), GSN distributions can systematically capture skewness and even multimodality. Note that, from Wang *et al.* [22], there is a stochastic representation of $\boldsymbol{Z}$ suitable for simulation, that is

$$\boldsymbol{Z} = \begin{cases} \boldsymbol{\xi} + \boldsymbol{U} & \text{if } V < \pi(\boldsymbol{U}), \\ \boldsymbol{\xi} - \boldsymbol{U} & \text{otherwise,} \end{cases}$$

where $\boldsymbol{U}$ is $N_p(\boldsymbol{0}, \boldsymbol{\Omega})$ distributed, V is uniformly distributed over $[0, 1]$, and $\boldsymbol{U}$ and V are independent. Additional properties of GSN distributions, which coincide or are close to the properties of the normal ones, have been discussed in Loperfido [23], Genton and Loperfido [12], Chang and Genton [24], and Lysenko *et al.* [25], etc.

The remainder of the article is structured as follows. In Section 2, the linear transformation of $\boldsymbol{Z}$ is investigated. Section 3 deals with some moments of $\boldsymbol{Z}$ and its quadratic forms. In Sections 4–6, we derive joint moment-generating functions (m.g.f.'s) of linear and quadratic forms of $\boldsymbol{Z}$ and then give conditions for independence of a linear form and a quadratic form, two quadratic forms, and two linear forms of $\boldsymbol{Z}$, respectively. In Section 7, explicit forms for the above distributions, moments and m.g.f's are derived when $\pi(z) = \kappa(\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$ and κ is the normal, Laplace, logistic or uniform distribution function.

2. Linear transformations

In this section, let $\boldsymbol{Z}$ have a $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$ distribution, $\boldsymbol{A}$ be a $p \times q$ matrix with rank q, and $\boldsymbol{b}$ be a $q \times 1$ vector, $1 \leq q \leq p$. We first show that the class of GSN distributions is closed under linear transformation, which is basic for developing other results and important for a family of distributions to have good properties when used to fit the data and make inferences. It is worth remarking that the marginal distribution of the first q coordinates of $\boldsymbol{Z}$ can be obtained from the following theorem by taking $\boldsymbol{A}' = [\boldsymbol{I}_q\ \boldsymbol{O}]$ and $\boldsymbol{b} = \boldsymbol{0}$.

THEOREM 2.1 *$\boldsymbol{A}'\boldsymbol{Z} + \boldsymbol{b}$ has a $\mathrm{GSN}_q(\boldsymbol{A}'\boldsymbol{\xi} + \boldsymbol{b}, \boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A}, \pi^*)$ distribution with*

$$\pi^*(z) = E_{\boldsymbol{U}_1}(\pi(\boldsymbol{U}_1 + \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1}z)), \tag{4}$$

where $\boldsymbol{U}_1$ is $N_p(\boldsymbol{0}, \boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Omega})$ distributed.

Proof For simplicity, we only prove the case for $\boldsymbol{\xi} = \boldsymbol{b} = \boldsymbol{0}$. The proof for the general situation is similar. First consider $p = q$. Obviously, $\boldsymbol{A}$ is non-singular. Hence, it is clear that $\boldsymbol{A}'\boldsymbol{Z}$ has a $\mathrm{GSN}_p(\boldsymbol{0}, \boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A}, \pi^*)$ distribution with $\pi^*(z) = \pi((\boldsymbol{A}')^{-1}z)$.

Next, consider $q < p$. Let $\boldsymbol{B}$ be a $p \times (p - q)$ matrix such that $\boldsymbol{D} = [\boldsymbol{A}\ \boldsymbol{B}]$ is a non-singular. Partition $\boldsymbol{D}^{-1}$ as

$$\boldsymbol{D}^{-1} = \begin{bmatrix} \boldsymbol{A}^{(-1)} \\ \boldsymbol{B}^{(-1)} \end{bmatrix},$$

where $\boldsymbol{A}^{(-1)}$ and $\boldsymbol{B}^{(-1)}$ are $q \times p$ and $(p - q) \times p$ matrices, respectively. Now we have the p.d.f. of $\boldsymbol{D}'\boldsymbol{Z}$ as follows

$$\begin{aligned} f_{\boldsymbol{D}'\boldsymbol{Z}}(z) &= 2\phi_p(z; \boldsymbol{0}, \boldsymbol{D}'\boldsymbol{\Omega}\boldsymbol{D})\pi((\boldsymbol{D}')^{-1}z) = f_{\boldsymbol{A}'\boldsymbol{Z},\boldsymbol{B}'\boldsymbol{Z}}(z_1, z_2) \\ &= \frac{2\exp\{-(1/2)(z_1'\boldsymbol{\Sigma}_{11}^{-1}z_1 + (z_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}z_1)'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(z_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}z_1))\}}{(2\pi)^{p/2}\sqrt{|\boldsymbol{\Sigma}_{11}||\boldsymbol{\Sigma}_{22\cdot 1}|}} \\ &\quad \cdot \pi(\boldsymbol{A}^{(-1)'}z_1 + \boldsymbol{B}^{(-1)'}z_2), z = [z_1'\ z_2']' \in \mathbb{R}^p, \end{aligned}$$

where $\boldsymbol{\Sigma}_{11} = \boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A}$, $\boldsymbol{\Sigma}_{21} = \boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{A}$ and $\boldsymbol{\Sigma}_{22\cdot 1} = \boldsymbol{B}'(\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Omega})\boldsymbol{B}$. Then the marginal p.d.f. of $\boldsymbol{A}'\boldsymbol{Z}$ can be derived as

$$\begin{aligned} f_{\boldsymbol{A}'\boldsymbol{Z}}(z_1) &= \int_{\mathbb{R}^{p-q}} f_{\boldsymbol{A}'\boldsymbol{Z},\boldsymbol{B}'\boldsymbol{Z}}(z_1, z_2)\,\mathrm{d}z_2 \\ &= 2\frac{1}{(2\pi)^{p/2}\sqrt{|\boldsymbol{\Sigma}_{11}||\boldsymbol{\Sigma}_{22\cdot 1}|}} \exp\left\{-\frac{1}{2}z_1'\boldsymbol{\Sigma}_{11}^{-1}z_1\right\} \\ &\quad \cdot \int_{\mathbb{R}^{p-q}} \exp\left\{-\frac{1}{2}\boldsymbol{v}'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{v}\right\} \pi(\boldsymbol{B}^{(-1)'}\boldsymbol{v} + (\boldsymbol{A}^{(-1)'} + \boldsymbol{B}^{(-1)'}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1})z_1)\,\mathrm{d}\boldsymbol{v} \\ &= 2\phi_q(z_1; \boldsymbol{0}, \boldsymbol{\Sigma}_{11})E_{\boldsymbol{V}}[\pi(\boldsymbol{B}^{(-1)'}\boldsymbol{V} + (\boldsymbol{A}^{(-1)'} + \boldsymbol{B}^{(-1)'}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1})z_1)], \quad z_1 \in \mathbb{R}^q, \end{aligned} \tag{5}$$

where $\boldsymbol{V}$ is $N_{p-q}(\boldsymbol{0}, \boldsymbol{\Sigma}_{22\cdot 1})$ distributed. Due to the fact $\boldsymbol{D}\boldsymbol{D}^{-1} = \boldsymbol{A}\boldsymbol{A}^{(-1)} + \boldsymbol{B}\boldsymbol{B}^{(-1)} = \boldsymbol{I}_p$, it yields that

$$\begin{aligned} \boldsymbol{B}^{(-1)'}\boldsymbol{\Sigma}_{22\cdot 1}\boldsymbol{B}^{(-1)} &= \boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Omega}, \\ \boldsymbol{A}^{(-1)'} + \boldsymbol{B}^{(-1)'}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} &= \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1}. \end{aligned}$$

Hence Equation (5) can be rewritten as

$$f_{A'Z}(z_1) = 2\phi_q(z_1; \mathbf{0}, \boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A}) E_{U_1}(\pi(\boldsymbol{U}_1 + \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1} z_1)), \quad z_1 \in \mathbb{R}^q,$$

where $\boldsymbol{U}_1 = \boldsymbol{B}^{(-1)\prime}\boldsymbol{V}$ is $N_p(\mathbf{0}, \boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Omega})$ distributed. Finally, it is easy to check that $E_{U_1}(\pi(\boldsymbol{U}_1 + \boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})^{-1} z_1))$ is a skewing function, and the proof is completed. ■

The following corollary gives conditions such that linear functions of $\boldsymbol{Z}$ are still normally distributed. The proof can be obtained immediately by showing that $\pi^*(z) = \frac{1}{2}$, $z \in \mathbb{R}^q$, where π^* is given in Equation (4), hence is omitted.

COROLLARY 2.1 *Assume* $\pi(z) = \kappa(\boldsymbol{\alpha}' z)$. *If* $\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{A} = \mathbf{0}$, *then* $\boldsymbol{A}'\boldsymbol{Z}$ *has a* $N_q(\boldsymbol{A}'\boldsymbol{\xi}, \boldsymbol{A}'\boldsymbol{\Omega}\boldsymbol{A})$ *distribution.*

3. Moments

The first four moments of the skew-normal distribution with p.d.f. (1) and the first two moments of its quadratic forms were derived by Genton *et al.* [3]. They showed that the moments of the sample auto-covariance function and of the sample variogram estimator do not depend on the skewness parameter $\boldsymbol{\alpha}$, and gave some applications to time series and spatial statistics. Kim and Mallick [26] and Kim [27] subsequently extended these results to scale mixtures of the skew-normal distribution with p.d.f. (1). In this section, let $\boldsymbol{Z}$ have a $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$ distribution,

$$\boldsymbol{q} = 2\left.\frac{\partial E_U(\pi(\boldsymbol{U} + \boldsymbol{\Omega}\boldsymbol{t}))}{\partial \boldsymbol{t}}\right|_{\boldsymbol{t}=\mathbf{0}} \quad \text{and} \quad \boldsymbol{Q} = 2\left.\frac{\partial^3 E_U(\pi(\boldsymbol{U} + \boldsymbol{\Omega}\boldsymbol{t}))}{\partial \boldsymbol{t}\, \partial \boldsymbol{t}'\, \partial \boldsymbol{t}}\right|_{\boldsymbol{t}=\mathbf{0}}, \tag{6}$$

where $\boldsymbol{U}$ is $N_p(\mathbf{0}, \boldsymbol{\Omega})$ distributed. Now, we will obtain some parallel results for the GSN distribution. To this end, we need some of the operations involved in matrix algebra. That $\otimes$, tr and vec are the Kronecker product, trace and vectorizing operator, respectively. $\boldsymbol{K}_{mn}$ stands for the $mn \times mn$ commutation matrix consisting of $n \times m$ blocks, where the jith element in the ijth block equals 1 and all the other elements are zeros, $i = 1, \ldots, m, j = 1, \ldots, n$. For properties of these operators, the commutation matrix and the related matrix algebra, and details about calculations of the partial derivatives, the reader is referred to Schott [28] or Gupta and Nagar [29]. We also need a lemma by Genton and Loperfido [12].

LEMMA 3.1 *Let* τ *be any even function from* $\mathbb{R}^p$ *to* $\mathbb{R}$. *Assume* $\boldsymbol{\xi} = \mathbf{0}$. *Then the distribution of* $\tau(\boldsymbol{Z})$ *does not depend on* π.

The following theorem provides expressions for the first four moments of $\boldsymbol{Z}$ in terms of $\boldsymbol{\xi}$, $\boldsymbol{\Omega}$, $\boldsymbol{q}$ and $\boldsymbol{Q}$, which can be proved along the lines of Genton *et al.* [3].

THEOREM 3.1 *We have*

(i) $E(\boldsymbol{Z}) = \boldsymbol{q} + \boldsymbol{\xi}$,
(ii) $E(\boldsymbol{Z} \otimes \boldsymbol{Z}') = \boldsymbol{\Omega} + \boldsymbol{q}\boldsymbol{\xi}' + \boldsymbol{\xi}\boldsymbol{q}' + \boldsymbol{\xi}\boldsymbol{\xi}'$,
(iii)

$$\begin{aligned} E(\boldsymbol{Z} \otimes \boldsymbol{Z}' \otimes \boldsymbol{Z}) = {} & \boldsymbol{Q} + \boldsymbol{q} \otimes \boldsymbol{\Omega} + \boldsymbol{\Omega} \otimes \boldsymbol{q} + \mathrm{vec}(\boldsymbol{\Omega})\boldsymbol{q}' + \boldsymbol{q} \otimes \boldsymbol{\xi}' \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes \boldsymbol{q}' \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes \boldsymbol{\xi}' \otimes \boldsymbol{q} \\ & + \boldsymbol{\Omega} \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes \boldsymbol{\Omega} + \mathrm{vec}(\boldsymbol{\Omega}) \otimes \boldsymbol{\xi}' + \boldsymbol{\xi} \otimes \boldsymbol{\xi}' \otimes \boldsymbol{\xi}, \end{aligned}$$

(iv)

$$
\begin{aligned}
&E(\boldsymbol{Z}\otimes\boldsymbol{Z}'\otimes\boldsymbol{Z}\otimes\boldsymbol{Z}')\\
&\quad=\boldsymbol{\Omega}\otimes\boldsymbol{\Omega}+\boldsymbol{K}_{pp}(\boldsymbol{\Omega}\otimes\boldsymbol{\Omega})+\operatorname{vec}(\boldsymbol{\Omega})(\operatorname{vec}(\boldsymbol{\Omega}))'+\boldsymbol{\Omega}\otimes\boldsymbol{\xi}\otimes\boldsymbol{\xi}'+\boldsymbol{\xi}\otimes\boldsymbol{\Omega}\otimes\boldsymbol{\xi}'\\
&\qquad+\boldsymbol{\xi}\otimes\boldsymbol{\xi}'\otimes\boldsymbol{\Omega}+\boldsymbol{\xi}'\otimes\boldsymbol{\Omega}\otimes\boldsymbol{\xi}+\operatorname{vec}(\boldsymbol{\Omega})\otimes\boldsymbol{\xi}'\otimes\boldsymbol{\xi}'+\boldsymbol{\xi}\otimes\boldsymbol{\xi}\otimes(\operatorname{vec}(\boldsymbol{\Omega}))'+\boldsymbol{\xi}\otimes\boldsymbol{\xi}'\otimes\boldsymbol{\xi}\otimes\boldsymbol{\xi}'\\
&\qquad+\boldsymbol{\Omega}\otimes\boldsymbol{q}\otimes\boldsymbol{\xi}'+\boldsymbol{\Omega}\otimes\boldsymbol{q}'\otimes\boldsymbol{\xi}+\boldsymbol{q}\otimes\boldsymbol{\Omega}\otimes\boldsymbol{\xi}'+\boldsymbol{q}'\otimes\boldsymbol{\Omega}\otimes\boldsymbol{\xi}+\boldsymbol{\xi}\otimes\boldsymbol{\Omega}\otimes\boldsymbol{q}'+\boldsymbol{\xi}'\otimes\boldsymbol{\Omega}\otimes\boldsymbol{q}\\
&\qquad+\boldsymbol{\xi}\otimes\boldsymbol{q}'\otimes\boldsymbol{\Omega}+\boldsymbol{\xi}'\otimes\boldsymbol{q}\otimes\boldsymbol{\Omega}+\boldsymbol{\xi}\otimes(\operatorname{vec}(\boldsymbol{\Omega})\boldsymbol{q}')'+(\operatorname{vec}(\boldsymbol{\Omega})\boldsymbol{q}')'\otimes\boldsymbol{\xi}+\boldsymbol{\xi}'\otimes(\operatorname{vec}(\boldsymbol{\Omega})\boldsymbol{q}')\\
&\qquad+(\operatorname{vec}(\boldsymbol{\Omega})\boldsymbol{q}')\otimes\boldsymbol{\xi}'+\boldsymbol{q}\otimes\boldsymbol{\xi}'\otimes\boldsymbol{\xi}\otimes\boldsymbol{\xi}'+\boldsymbol{\xi}\otimes\boldsymbol{q}'\otimes\boldsymbol{\xi}\otimes\boldsymbol{\xi}'+\boldsymbol{\xi}\otimes\boldsymbol{\xi}'\otimes\boldsymbol{q}\otimes\boldsymbol{\xi}'\\
&\qquad+\boldsymbol{\xi}\otimes\boldsymbol{\xi}'\otimes\boldsymbol{\xi}\otimes\boldsymbol{q}'+\boldsymbol{Q}\otimes\boldsymbol{\xi}'+\boldsymbol{Q}'\otimes\boldsymbol{\xi}+\boldsymbol{\xi}\otimes\boldsymbol{Q}'+\boldsymbol{\xi}'\otimes\boldsymbol{Q}.
\end{aligned}
$$

If $\boldsymbol{X}$ is a $p\times 1$ random vector, it can be found in Li [30] that

$$E(\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})=\operatorname{tr}(\boldsymbol{A}E(\boldsymbol{X}\otimes\boldsymbol{X}')), \tag{7}$$

$$E((\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})(\boldsymbol{X}'\boldsymbol{B}\boldsymbol{X}))=\operatorname{tr}((\boldsymbol{A}\otimes\boldsymbol{B})E(\boldsymbol{X}\otimes\boldsymbol{X}'\otimes\boldsymbol{X}\otimes\boldsymbol{X}')), \tag{8}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $p\times p$ symmetric matrices. Genton *et al.* [3] used Equations (7) and (8) to obtain expressions of $E(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z})$ and $\operatorname{Cov}(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z},\boldsymbol{Z}'\boldsymbol{B}\boldsymbol{Z})$ for $\pi(z)=\Phi(\boldsymbol{\alpha}'z)$. The case for $\pi(z)=\frac{1}{2}$, that is $\boldsymbol{Z}$ is $N_p(\boldsymbol{\xi},\boldsymbol{\Omega})$ distributed, can also be found in Schott [28, p. 395]. In order to obtain parallel results for a general skewing function π, first we give a preliminary lemma where due to Lemma 3.1 (see e.g. [28, p. 391, p. 413, and p. 394, respectively]), assertions (ii)–(iv) are the same as those in the normal case, and the proofs of assertions (i) and (v) are standard.

LEMMA 3.2 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $p\times p$ symmetric matrices, and $\boldsymbol{a}$ and $\boldsymbol{b}$ be $p\times 1$ vectors. Also assume $\boldsymbol{\xi}=\boldsymbol{0}$. Then*

(i) $E(\boldsymbol{a}'\boldsymbol{Z})=\boldsymbol{a}'\boldsymbol{q}$,
(ii) $E(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z})=\operatorname{tr}(\boldsymbol{A}\boldsymbol{\Omega})$,
(iii) $E(\boldsymbol{a}'\boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{b})=\boldsymbol{a}'\boldsymbol{\Omega}\boldsymbol{b}$,
(iv) $E(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{B}\boldsymbol{Z})=\operatorname{tr}(\boldsymbol{A}\boldsymbol{\Omega})\operatorname{tr}(\boldsymbol{B}\boldsymbol{\Omega})+2\operatorname{tr}(\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{B}\boldsymbol{\Omega})$,
(v) $E(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}\boldsymbol{a}'\boldsymbol{B}\boldsymbol{Z})=2\boldsymbol{a}'\boldsymbol{B}\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{q}+\boldsymbol{a}'\boldsymbol{B}\boldsymbol{q}\operatorname{tr}\boldsymbol{A}(\boldsymbol{\Omega})+\operatorname{tr}((\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{Q}\otimes\boldsymbol{a}'))$.

By noting that $\boldsymbol{Z}-\boldsymbol{\xi}$ is $\mathrm{GSN}_p(\boldsymbol{0},\boldsymbol{\Omega},\pi)$ distributed, we have the following immediate consequence of the above lemma.

THEOREM 3.2 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $p\times p$ symmetric matrices. Then*

(i) $E(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z})=\operatorname{tr}(\boldsymbol{A}\boldsymbol{\Omega})+2\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{q}+\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{\xi}$,
(ii) $\operatorname{Cov}(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z},\boldsymbol{Z}'\boldsymbol{B}\boldsymbol{Z})=2\operatorname{tr}(\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{B}\boldsymbol{\Omega})+4\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{B}\boldsymbol{\xi}+4\boldsymbol{\xi}'(\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{B}+\boldsymbol{B}\boldsymbol{\Omega}\boldsymbol{A})\boldsymbol{q}+2\operatorname{tr}((\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{Q}\otimes\boldsymbol{\xi}'))$ $+2\operatorname{tr}((\boldsymbol{B}\otimes\boldsymbol{A})(\boldsymbol{Q}\otimes\boldsymbol{\xi}'))-4\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{q}\boldsymbol{\xi}'\boldsymbol{B}\boldsymbol{q}$,
(iii) $\operatorname{Var}(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z})=2\operatorname{tr}((\boldsymbol{A}\boldsymbol{\Omega})^2)+4\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{\xi}+8\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{A}\boldsymbol{q}+4\operatorname{tr}((\boldsymbol{A}\otimes\boldsymbol{A})(\boldsymbol{Q}\otimes\boldsymbol{\xi}'))-4(\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{q})^2$.

4. Independence of a quadratic form and a linear form

Many statistics can be expressed as functions of linear and quadratic forms. In statistical inference, it is important to be able to determine whether linear and quadratic forms in the normal random vector are independently distributed. For instance, let $Y_1,Y_2,\ldots,Y_n$ be a random sample from the univariate normal distribution with mean μ and variance σ^2. It is known that the sample mean $\bar{Y}_n$

and the sample variance S_n^2 are independent and can be expressed as a linear form and a quadratic form of $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)'$, respectively, that is

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\mathbf{1}_n'\boldsymbol{Y},$$

and

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 = \frac{1}{n-1}\boldsymbol{Y}'\left(\boldsymbol{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'\right)\boldsymbol{Y},$$

where $\mathbf{1}_n = (1, 1, \ldots, 1)'$. Note that $\boldsymbol{Y}$ is $N_n(\mu\mathbf{1}_n, \sigma^2\boldsymbol{I}_n)$ distributed. Similar investigations for certain GSN random vectors were reported by Gupta and Huang [5], Huang and Chen [19], and Wang *et al*. [16], etc.

In this section, let $\boldsymbol{Z}$ have a $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$ distribution. Also let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $p \times p$ symmetric and $p \times q$ matrices, respectively. We now give without proof the joint m.g.f. of $\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}$ and $\boldsymbol{B}'\boldsymbol{Z}$, which are useful for obtaining not only moments but also their independence.

THEOREM 4.1 *The joint m.g.f. of $\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}$ and $\boldsymbol{B}'\boldsymbol{Z}$ is given by*

$$\begin{aligned} M_1(s, \boldsymbol{t}) = {} & \frac{2\exp\{s\boldsymbol{\xi}'\boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{t}'\boldsymbol{B}'\boldsymbol{\xi} + (1/2)(\boldsymbol{B}\boldsymbol{t} + 2s\boldsymbol{A}\boldsymbol{\xi})'(\boldsymbol{\Omega}^{-1} - 2s\boldsymbol{A})^{-1}(\boldsymbol{B}\boldsymbol{t} + 2s\boldsymbol{A}\boldsymbol{\xi})\}}{|\boldsymbol{I}_p - 2s\boldsymbol{A}\boldsymbol{\Omega}|^{1/2}} \\ & \cdot E_{\boldsymbol{U}_s}[\pi(\boldsymbol{U}_s + (\boldsymbol{\Omega}^{-1} - 2s\boldsymbol{A})^{-1}(\boldsymbol{B}\boldsymbol{t} + 2s\boldsymbol{A}\boldsymbol{\xi}))], \quad s \in \mathbb{S}_1, \boldsymbol{t} \in \mathbb{R}^q, \end{aligned} \tag{9}$$

where $\boldsymbol{U}_s$ is $N_p(\boldsymbol{0}, (\boldsymbol{\Omega}^{-1} - 2s\boldsymbol{A})^{-1})$ distributed and $\mathbb{S}_1 = \{x|\boldsymbol{\Omega}^{-1} - 2x\boldsymbol{A} > 0, x \in \mathbb{R}\}$.

When $\boldsymbol{\xi} = \boldsymbol{0}$ and $\pi(z) = \Phi(\boldsymbol{\alpha}'z)$, the necessary and sufficient conditions $\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{B} = \boldsymbol{O}$ and $\boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$ for the independence of $\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}$ and $\boldsymbol{B}'\boldsymbol{Z}$ can be obtained by using Equation (9) and along the lines of the proof of Theorem 3 of Gupta and Huang [5], where $\boldsymbol{B}$ is assumed to be a vector. For the general case, using $M_1(s, \boldsymbol{t}) = M_1(s, \boldsymbol{0})M_1(0, \boldsymbol{t})$, $s \in \mathbb{S}_1, \boldsymbol{t} \in \mathbb{R}^q$, can yield the necessary and sufficient condition for the independence of $\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}$ and $\boldsymbol{B}'\boldsymbol{Z}$. We present some examples in the following.

Example 4.1 Assume $\boldsymbol{\xi} = \boldsymbol{0}$ and $\pi(z) = \kappa(h(z'\boldsymbol{\Lambda}z)\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$, $\boldsymbol{\Lambda}$ is a non-negative diagonal matrix and h is any function defined on non-negative real numbers. If $\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{B} = \boldsymbol{O}, \boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$ and $\boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{\Lambda} = \boldsymbol{O}$, then $\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}$ and $\boldsymbol{B}'\boldsymbol{Z}$ are independent.

Proof From Equation (9) and

$$(\boldsymbol{\Omega}^{-1} - 2s\boldsymbol{A})^{-1} = \boldsymbol{\Omega}\sum_{j=0}^{\infty}(2s)^j(\boldsymbol{A}\boldsymbol{\Omega})^j, \quad ||2s\boldsymbol{A}\boldsymbol{\Omega}|| < 1,$$

where as usual $||\cdot||$ is the matrix norm, if $\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{B} = \boldsymbol{O}, \boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$ and $\boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{\Lambda} = \boldsymbol{O}$, it yields

$$M_1(s, \boldsymbol{t}) = \frac{\exp\{(1/2)\boldsymbol{t}'\boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{B}\boldsymbol{t}\}}{|\boldsymbol{I}_p - 2s\boldsymbol{A}\boldsymbol{\Omega}|^{1/2}}, \quad s \in \mathbb{S}_1, \quad \boldsymbol{t} \in \mathbb{R}^q,$$

which in turn implies $\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}$ and $\boldsymbol{B}'\boldsymbol{Z}$ are independent. ■

Example 4.1 is an extension of Theorem 3.1 of Huang and Chen [19], where $h(x) \equiv 1$ and κ is assumed to be an absolutely continuous c.d.f. such that the p.d.f. of κ is symmetric about 0. Note that if h is a constant function, then the condition $\boldsymbol{B}'\boldsymbol{\Omega}\boldsymbol{\Lambda} = \boldsymbol{O}$ is unnecessary. It can also be seen that if $h(x) = 1/\sqrt{1+x}$ and $\kappa(x) = \Phi(x)$, the corresponding GSN distribution is that studied in Gupta *et al.* [20].

The next example concerns the case $\boldsymbol{\xi} \neq \mathbf{0}$. The proof is similar to Example 4.1 hence is omitted.

Example 4.2 Assume $\boldsymbol{\xi} \neq \mathbf{0}$ and $\pi(z) = \kappa(\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$. If $\boldsymbol{A\Omega B} = \boldsymbol{O}$ and either $\boldsymbol{A\Omega\alpha} = \mathbf{0}$ or $\boldsymbol{B}'\boldsymbol{\Omega\alpha} = \mathbf{0}$, then $\boldsymbol{Z}'\boldsymbol{AZ}$ and $\boldsymbol{B}'\boldsymbol{Z}$ are independent.

5. Independence of two quadratic forms

Quadratic forms of normal random vectors are of great importance in many branches of statistics, such as the least-squares methods, the analysis of variance and regression analysis. Among the related results, Cochran's theorem is useful in proving that certain quadratic forms of a normal vector are independently distributed as central or non-central chi-squared distributions. In this section, again let $\boldsymbol{Z}$ have a $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$ distribution and $\boldsymbol{A}_1, \boldsymbol{A}_2$ be $p \times p$ symmetric matrices. We first give the joint m.g.f. of two quadratic forms of $\boldsymbol{Z}$, and then investigate their independence.

THEOREM 5.1 *The joint m.g.f. of $\boldsymbol{Z}'\boldsymbol{A}_1\boldsymbol{Z}$ and $\boldsymbol{Z}'\boldsymbol{A}_2\boldsymbol{Z}$ is given by*

$$\begin{aligned} M_2(s_1, s_2) = {} & \frac{2\exp\{\boldsymbol{\xi}'(s_1\boldsymbol{A}_1 + s_2\boldsymbol{A}_2 + 2(s_1\boldsymbol{A}_1 + s_2\boldsymbol{A}_2)\boldsymbol{\Sigma}_{s_1,s_2}^{-1}(s_1\boldsymbol{A}_1 + s_2\boldsymbol{A}_2))\boldsymbol{\xi}\}}{|\boldsymbol{I}_p - 2s_1\boldsymbol{A}_1\boldsymbol{\Omega} - 2s_2\boldsymbol{A}_2\boldsymbol{\Omega}|^{1/2}} \\ & \cdot E_{\boldsymbol{U}_{s_1,s_2}}[\pi(\boldsymbol{U}_{s_1,s_2} + 2\boldsymbol{\Sigma}_{s_1,s_2}^{-1}(s_1\boldsymbol{A}_1 + s_2\boldsymbol{A}_2)\boldsymbol{\xi}], (s_1, s_2) \in \mathbb{S}_2, \end{aligned} \tag{10}$$

where $\boldsymbol{U}_{s_1,s_2} \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma}_{s_1,s_2}^{-1})$, $\boldsymbol{\Sigma}_{s_1,s_2} = \boldsymbol{\Omega}^{-1} - 2s_1\boldsymbol{A}_1 - 2s_2\boldsymbol{A}_2$ and $\mathbb{S}_2 = \{(x_1, x_2) | \boldsymbol{\Omega}^{-1} - 2x_1\boldsymbol{A}_1 - 2x_2\boldsymbol{A}_2 > 0, x_1, x_2 \in \mathbb{R}\}$.

Using $M_2(s_1, s_2) = M_2(0, s_2)M_2(s_1, 0)$, $(s_1, s_2) \in \mathbb{S}_2$, the necessary and sufficient condition for the independence of the two quadratic forms can be obtained. When $\boldsymbol{\xi} = \mathbf{0}$, according to Lemma 3.1, we have that the distributions of quadratic forms of $\boldsymbol{Z}$ do not depend on π. Then by Theorem 3.2 of Huang and Chen [19] the following special case of Theorem 5.1 can be obtained immediately.

Example 5.1 Assume $\boldsymbol{\xi} = \mathbf{0}$. The two quadratic forms $\boldsymbol{Z}'\boldsymbol{A}_1\boldsymbol{Z}$ and $\boldsymbol{Z}'\boldsymbol{A}_2\boldsymbol{Z}$ are independent if and only if $\boldsymbol{A}_1\boldsymbol{\Omega}\boldsymbol{A}_2 = \boldsymbol{O}$.

When $\boldsymbol{\xi} \neq \mathbf{0}$, by using Equation (10) and

$$(\boldsymbol{\Omega}^{-1} - 2s_1\boldsymbol{A}_1 - 2s_2\boldsymbol{A}_2)^{-1} = \boldsymbol{\Omega}\sum_{j=0}^{\infty}(2s_1\boldsymbol{A}_1\boldsymbol{\Omega} + 2s_2\boldsymbol{A}_2\boldsymbol{\Omega})^j, \quad ||2s_1\boldsymbol{A}_1\boldsymbol{\Omega} + 2s_2\boldsymbol{A}_2\boldsymbol{\Omega}|| < 1,$$

we have

Example 5.2 Assume $\boldsymbol{\xi} \neq \mathbf{0}$ and $\pi(z) = \kappa(\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$. If $\boldsymbol{A}_1\boldsymbol{\Omega}\boldsymbol{A}_2 = \boldsymbol{O}$ and either $\boldsymbol{A}_1\boldsymbol{\Omega\alpha} = \mathbf{0}$ or $\boldsymbol{A}_2\boldsymbol{\Omega\alpha} = \mathbf{0}$, then the two quadratic forms $\boldsymbol{Z}'\boldsymbol{A}_1\boldsymbol{Z}$ and $\boldsymbol{Z}'\boldsymbol{A}_2\boldsymbol{Z}$ are independent.

6. Independence of two linear forms

The literature on characterizing the normal law by some natural assumptions of linear forms, including their independence, is rather extensive. Some excellent reviews can be found in Patel and Read [31], Kagan and Wesołowski [32], and Kotz *et al.* [33]. Again let $\boldsymbol{Z}$ have a $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$ distribution. Also let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be $p \times q$ and $p \times r$ matrices, respectively. As before, we give the joint m.g.f. of two linear forms of $\boldsymbol{Z}$ to explore their independence.

THEOREM 6.1 *The joint m.g.f. of $\boldsymbol{B}_1'\boldsymbol{Z}$ and $\boldsymbol{B}_2'\boldsymbol{Z}$ is given by*

$$\begin{aligned} M_3(\boldsymbol{t}_1, \boldsymbol{t}_2) = {} & 2\exp\{\tfrac{1}{2}(\boldsymbol{B}_1\boldsymbol{t}_1 + \boldsymbol{B}_2\boldsymbol{t}_2)'\boldsymbol{\Omega}(\boldsymbol{B}_1\boldsymbol{t}_1 + \boldsymbol{B}_2\boldsymbol{t}_2) + (\boldsymbol{B}_1\boldsymbol{t}_1 + \boldsymbol{B}_2\boldsymbol{t}_2)'\boldsymbol{\xi}\} \\ & \cdot E_{\boldsymbol{U}}[\pi(\boldsymbol{U} + \boldsymbol{\Omega}(\boldsymbol{B}_1\boldsymbol{t}_1 + \boldsymbol{B}_2\boldsymbol{t}_2))], \quad \boldsymbol{t}_1 \in \mathbb{R}^q, \quad \boldsymbol{t}_2 \in \mathbb{R}^r, \end{aligned} \tag{11}$$

where $\boldsymbol{U}$ is $N_p(\boldsymbol{0}, \boldsymbol{\Omega})$ distributed.

As in Sections 4 and 5, we only study the independence of two linear forms for certain GSN distributions.

Example 6.1 Assume $\pi(z) = \kappa(\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$, and $\kappa(x)$ is absolutely continuous and satisfies $\lim_{x\to\infty} \kappa(x) = l$, $l \neq 0, \frac{1}{2}$. The two linear forms $\boldsymbol{B}_1'\boldsymbol{Z}$ and $\boldsymbol{B}_2'\boldsymbol{Z}$ are independent if and only if $\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{B}_2 = \boldsymbol{O}$ and either $\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$ or $\boldsymbol{B}_2'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$.

Proof Using $M_3(\boldsymbol{t}_1, \boldsymbol{t}_2) = M_3(\boldsymbol{t}_1, \boldsymbol{0})M_3(\boldsymbol{0}, \boldsymbol{t}_2)$, $\boldsymbol{t}_1 \in \mathbb{R}^q$, $\boldsymbol{t}_2 \in \mathbb{R}^r$, we obtain that the necessary and sufficient condition for the independence of $\boldsymbol{B}_1'\boldsymbol{Z}$ and $\boldsymbol{B}_2'\boldsymbol{Z}$ is

$$\frac{2E_{\boldsymbol{U}}[\kappa(\boldsymbol{\alpha}'\boldsymbol{U} + \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_1\boldsymbol{t}_1)]E_{\boldsymbol{U}}[\kappa(\boldsymbol{\alpha}'\boldsymbol{U} + \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_2\boldsymbol{t}_2)]}{E_{\boldsymbol{U}}[\kappa(\boldsymbol{\alpha}'\boldsymbol{U} + \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_1\boldsymbol{t}_1 + \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_2\boldsymbol{t}_2)]} = \exp\{\boldsymbol{t}_1'\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{B}_2\boldsymbol{t}_2\}, \tag{12}$$

for all $\boldsymbol{t}_1 \in \mathbb{R}^q$, $\boldsymbol{t}_2 \in \mathbb{R}^r$. Obviously, if (a) $\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{B}_2 = \boldsymbol{O}$ and (b) either $\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$ or $\boldsymbol{B}_2'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$, then $\boldsymbol{B}_1'\boldsymbol{Z}$ and $\boldsymbol{B}_2'\boldsymbol{Z}$ are independent.

Conversely, assume that $\boldsymbol{B}_1'\boldsymbol{Z}$ and $\boldsymbol{B}_2'\boldsymbol{Z}$ are independent. Suppose that (b) is not true, that is both $\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{\alpha} \neq \boldsymbol{0}$ and $\boldsymbol{B}_2'\boldsymbol{\Omega}\boldsymbol{\alpha} \neq \boldsymbol{0}$. Then we can choose fixed vectors $\boldsymbol{s}_1 \in \mathbb{R}^q$, $\boldsymbol{s}_2 \in \mathbb{R}^r$ such that $\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_1\boldsymbol{s}_1 \neq 0$ and $\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_2\boldsymbol{s}_2 \neq 0$. Let $\boldsymbol{t}_1 = c_1\boldsymbol{s}_1$ and $\boldsymbol{t}_2 = c_2\boldsymbol{s}_2$, where $c_1, c_2 \in \mathbb{R}$. For $i = 1, 2$, if $\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_i\boldsymbol{s}_i > 0$, then we let $c_i \to \infty$, otherwise let $c_i \to -\infty$, in each case, $\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_i\boldsymbol{t}_i$ tends to ∞. Now the limit of the left-hand side of Equation (12) is $2l$ and the limit of the right-hand side of Equation (12) equals 0, or 1, or ∞. The contradiction implies (b) is true. Applying Lemma 3.2, it can be obtained that

$$\mathrm{Cov}(\boldsymbol{B}_1'\boldsymbol{Z}, \boldsymbol{B}_2'\boldsymbol{Z}) = \boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{B}_2 - 4(E_{\boldsymbol{U}}(g(\boldsymbol{\alpha}'\boldsymbol{U})))^2\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{\alpha}\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{B}_2,$$

where $\boldsymbol{U}$ is $N_p(\boldsymbol{0}, \boldsymbol{\Omega})$ distributed and $g(x)$ is the derivative of $\kappa(x)$. In view of (b) and the independence of $\boldsymbol{B}_1'\boldsymbol{Z}$ and $\boldsymbol{B}_2'\boldsymbol{Z}$ implies $\mathrm{Cov}(\boldsymbol{B}_1'\boldsymbol{Z}, \boldsymbol{B}_2'\boldsymbol{Z}) = \boldsymbol{O}$, it turns out $\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{B}_2 = \boldsymbol{O}$, which completes the proof. ■

The proof of the following example is similar to Example 4.1. Note that if h is a constant function, then the condition $\boldsymbol{B}_2'\boldsymbol{\Omega}\boldsymbol{\Lambda} = \boldsymbol{O}$ is unnecessary.

Example 6.2 Assume $\pi(z) = \kappa(h(z'\boldsymbol{\Lambda}z)\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$, $\boldsymbol{\Lambda}$ is a non-negative diagonal matrix and h is any function defined on non-negative real numbers. If $\boldsymbol{B}_1'\boldsymbol{\Omega}\boldsymbol{B}_2 = \boldsymbol{O}$, $\boldsymbol{B}_2'\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$ and $\boldsymbol{B}_2'\boldsymbol{\Omega}\boldsymbol{\Lambda} = \boldsymbol{O}$, then $\boldsymbol{B}_1'\boldsymbol{Z}$ and $\boldsymbol{B}_2'\boldsymbol{Z}$ are independent.

7. Some special GSN models

From Theorems 2.1, 4.1, 5.1, and 6.1, we see that marginal distributions or each m.g.f. of linear and quadratic forms for the $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$ distribution has a factor of the form as follows:

$$E_{\boldsymbol{V}}(\pi(\boldsymbol{V} + \boldsymbol{b})), \tag{13}$$

where $\boldsymbol{b} \in \mathbb{R}^p$ and $\boldsymbol{V}$ is $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ distributed. Furthermore, Equation (13) can be used to derive $\boldsymbol{q}$ and $\boldsymbol{Q}$ given in Equation (6), which consist in the first four moments of the $\mathrm{GSN}_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \pi)$ distribution and the first two moments of its quadratic forms as mentioned in Theorems 3.1–3.2. In this section, we present explicit forms of Equation (13), $\boldsymbol{q}$ and $\boldsymbol{Q}$ for some examples of $\pi(z)$.

Example 7.1 Suppose $\pi(z) = \Phi(\boldsymbol{\alpha}' z)$. From Zacks [34, pp. 53–59], we have

$$E_{\boldsymbol{V}}(\Phi(\boldsymbol{\alpha}'\boldsymbol{V} + \boldsymbol{\alpha}'\boldsymbol{b})) = \Phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{b}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} + 1)^{1/2}}\right). \tag{14}$$

Letting $\boldsymbol{b} = \boldsymbol{\Omega t}$, $\boldsymbol{\Sigma} = \boldsymbol{\Omega}$ and then taking derivatives with respect to $\boldsymbol{t}$ in Equation (14) yields that

$$\boldsymbol{q} = \sqrt{\frac{2}{\pi}} \frac{1}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha} + 1)^{1/2}} \cdot \boldsymbol{\Omega}\boldsymbol{\alpha},$$

$$\boldsymbol{Q} = -\sqrt{\frac{2}{\pi}} \frac{1}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha} + 1)^{3/2}} \cdot (\boldsymbol{\Omega}\boldsymbol{\alpha}) \otimes (\boldsymbol{\Omega}\boldsymbol{\alpha}) \otimes (\boldsymbol{\Omega}\boldsymbol{\alpha})'.$$

Example 7.2 Suppose $\pi(z) = G_1(\boldsymbol{\alpha}' z)$, where G_1 is the c.d.f. of the Laplace distribution, namely

$$G_1(x) = \begin{cases} 1 - \dfrac{1}{2}\exp\left\{-\dfrac{x}{\gamma}\right\}, & x \geq 0, \\ \dfrac{1}{2}\exp\left\{\dfrac{x}{\gamma}\right\}, & x < 0, \end{cases}$$

where $\gamma > 0$. By routine computations, we have

$$\begin{aligned} E_{\boldsymbol{V}}(G_1(\boldsymbol{\alpha}'\boldsymbol{V} + \boldsymbol{\alpha}'\boldsymbol{b})) &= \frac{1}{2}\exp\left\{\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} + 2\gamma\boldsymbol{\alpha}'\boldsymbol{b}}{2\gamma^2}\right\}\Phi\left(\frac{-\gamma\boldsymbol{\alpha}'\boldsymbol{b} - \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}{\gamma(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right) \\ &\quad + \Phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{b}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right) - \frac{1}{2}\exp\left\{\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} - 2\gamma\boldsymbol{\alpha}'\boldsymbol{b}}{2\gamma^2}\right\}\Phi\left(\frac{\gamma\boldsymbol{\alpha}'\boldsymbol{b} - \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}{\gamma(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right), \end{aligned} \tag{15}$$

which is a slight generalization of Lemma 5.1 of Huang and Chen [19]. Again letting $\boldsymbol{b} = \boldsymbol{\Omega t}$, $\boldsymbol{\Sigma} = \boldsymbol{\Omega}$ and then taking derivatives with respect to $\boldsymbol{t}$ in Equation (15) yields

$$\boldsymbol{q} = \left\{\frac{2}{\gamma}\exp\left\{\frac{\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha}}{2\gamma^2}\right\}\left[\Phi\left(\frac{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\gamma}\right) - \frac{\gamma}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\phi\left(\frac{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\gamma}\right)\right] + \frac{(2/\pi)^{1/2}}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\right\} \cdot \boldsymbol{\Omega}\boldsymbol{\alpha},$$

$$\begin{aligned} \boldsymbol{Q} &= \left\{\frac{2}{\gamma^3}\exp\left\{\frac{\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha}}{2\gamma^2}\right\}\left[\Phi\left(-\frac{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\gamma}\right) + \frac{\gamma^3 - \gamma\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha}}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{3/2}}\phi\left(\frac{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\gamma}\right)\right] - \frac{(2/\pi)^{1/2}}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{3/2}}\right\} \\ &\quad \cdot (\boldsymbol{\Omega}\boldsymbol{\alpha}) \otimes (\boldsymbol{\Omega}\boldsymbol{\alpha}) \otimes (\boldsymbol{\Omega}\boldsymbol{\alpha})'. \end{aligned}$$

Example 7.3 Suppose $\pi(z) = G_2(\boldsymbol{\alpha}'\boldsymbol{z})$, where G_2 is the c.d.f. of the logistic distribution, namely

$$G_2(x) = \frac{1}{1+\exp\{-x/\delta\}}, \quad -\infty < x < \infty, \tag{16}$$

where $\delta > 0$. Using the Taylor series expansion for $(1+z)^{-1}$, we obtain

$$\begin{aligned} E_{\boldsymbol{V}}(G_2(\boldsymbol{\alpha}'\boldsymbol{V}+\boldsymbol{\alpha}'\boldsymbol{b})) = \sum_{j=0}^{\infty}\binom{-1}{j}\Bigg[&\exp\left\{-\frac{j\boldsymbol{\alpha}'\boldsymbol{b}}{\delta}+\frac{j^2\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})}{2\delta^2}\right\}\Phi\left(\frac{\delta\boldsymbol{\alpha}'\boldsymbol{b}-j\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}{\delta(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right) \\ &+\exp\left\{\frac{(j+1)\boldsymbol{\alpha}'\boldsymbol{b}}{\delta}+\frac{(j+1)^2\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})}{2\delta^2}\right\}\Phi\left(-\frac{\delta\boldsymbol{\alpha}'\boldsymbol{b}+(j+1)\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}{\delta(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right)\Bigg]. \end{aligned}$$

By a similar argument as in Example 7.2, it yields that

$$\begin{aligned} \boldsymbol{q} = \Bigg\{&4\sum_{j=1}^{\infty}\binom{-1}{j-1}\exp\left\{\frac{j^2\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha}}{2\beta^2}\right\}\left[\frac{j}{\delta}\Phi\left(-\frac{j(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\delta}\right)-\frac{1}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\phi\left(\frac{j(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\delta}\right)\right] \\ &+\sqrt{\frac{2}{\pi}}\frac{1}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\Bigg\}\cdot\boldsymbol{\Omega}\boldsymbol{\alpha}, \\ \boldsymbol{Q} = \Bigg\{&4\sum_{j=1}^{\infty}\binom{-1}{j-1}\exp\left\{\frac{j^2\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha}}{2\delta^2}\right\}\left[\left(\frac{1}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{3/2}}-\frac{j^2}{\delta^2(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\right)\phi\left(\frac{j(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\delta}\right)\right. \\ &\left.+\frac{j^3}{\delta^3}\Phi\left(-\frac{j(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}{\delta}\right)\right]-\sqrt{\frac{2}{\pi}}\frac{1}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{3/2}}\Bigg\}\cdot(\boldsymbol{\Omega}\boldsymbol{\alpha})\otimes(\boldsymbol{\Omega}\boldsymbol{\alpha})\otimes(\boldsymbol{\Omega}\boldsymbol{\alpha})'. \end{aligned}$$

Example 7.4 Suppose $\pi(z) = G_3(\boldsymbol{\alpha}'\boldsymbol{z})$, where G_3 is the c.d.f. of the uniform distribution, namely

$$G_3(x) = \begin{cases} 1, & x \geq \theta, \\ \dfrac{(\theta+x)}{(2\theta)}, & -\theta \leq x < \theta, \\ 0, & x < -\theta, \end{cases} \tag{17}$$

where $\theta > 0$. It can be obtained that

$$\begin{aligned} E_{\boldsymbol{V}}(G_3(\boldsymbol{\alpha}'\boldsymbol{V}+\boldsymbol{\alpha}'\boldsymbol{b})) = &\frac{\boldsymbol{\alpha}'\boldsymbol{b}+\theta}{2\theta}\Phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{b}+\theta}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right)-\frac{\boldsymbol{\alpha}'\boldsymbol{b}-\theta}{2\theta}\Phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{b}-\theta}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right) \\ &+\frac{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}{2\theta}\left(\phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{b}+\theta}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right)-\phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{b}-\theta}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha})^{1/2}}\right)\right). \end{aligned}$$

Again we have that

$$\begin{aligned} \boldsymbol{q} &= \frac{1}{\theta}\left\{2\Phi\left(\frac{\theta}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\right)-1\right\}\cdot\boldsymbol{\Omega}\boldsymbol{\alpha}, \\ \boldsymbol{Q} &= -\frac{2}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{3/2}}\phi\left(\frac{\theta}{(\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\right)\cdot(\boldsymbol{\Omega}\boldsymbol{\alpha})\otimes(\boldsymbol{\Omega}\boldsymbol{\alpha})\otimes(\boldsymbol{\Omega}\boldsymbol{\alpha})'. \end{aligned}$$

Remark 7.1 It can be seen that for each of the above four examples $\boldsymbol{Q}$ has the following form

$$C \cdot (\boldsymbol{\Omega\alpha}) \otimes (\boldsymbol{\Omega\alpha}) \otimes (\boldsymbol{\Omega\alpha})',$$

where C is a scalar. Hence the term $\mathrm{tr}((\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{Q} \otimes \boldsymbol{\xi}'))$ in assertions of Lemma 3.2 and Theorem 3.2 with $\pi(z) = \kappa(\boldsymbol{\alpha}'z)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$ and κ is a c.d.f. which comes from one of the normal, Laplace, logistic or uniform distribution, will have an elegant explicit form by showing that

$$\mathrm{tr}((\boldsymbol{A} \otimes \boldsymbol{B})((\boldsymbol{\Omega\alpha}) \otimes (\boldsymbol{\Omega\alpha}) \otimes (\boldsymbol{\Omega\alpha})' \otimes \boldsymbol{\xi}')) = \boldsymbol{\alpha}'\boldsymbol{\Omega A \Omega \alpha \xi}'\boldsymbol{B\Omega\alpha}.$$

Finally, the reason that we do not consider the case of $G_4(\boldsymbol{\alpha}'z)$, where G_4 is the c.d.f. of the Student's t distribution, which is also studied in Nadarajah and Kotz [35] and Arellano-Valle *et al.* [36], is that the closed form of Equation (13) cannot be obtained in our situation.

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