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Total coloring of planar graphs of maximum degree eight

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A B S T R A C T

The minimum number of colors needed to properly color the vertices and edges of a graph G is called the total chromatic number of G and denoted by $\chi''(G)$. It is known that if a planar graph G has maximum degree $\Delta \geqslant 9$, then $\chi''(G) = \Delta + 1$. Recently Hou et al. (Graphs and Combinatorics 24 (2008) 91–100) proved that if G is a planar graph with maximum degree 8 and with either no 5-cycles or no 6-cycles, then $\chi''(G) = 9$. In this Note, we strengthen this result and prove that if G is a planar graph with maximum degree 8, and for each vertex x, there is an integer $k_x \in \{3, 4, 5, 6, 7, 8\}$ such that there is no k_x-cycle which contains x, then $\chi''(G) = 9$.

1. Introduction

A total-k-coloring of a graph $G = (V, E)$ is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements get the same color. The total chromatic number of G, denoted by $\chi''(G)$, is the smallest integer k such that G has a total-k-coloring. It is clear that $\chi''(G) \geqslant \Delta + 1$ where Δ is the maximum degree of G. Behzad [1] and Vizing [6] conjectured independently that $\chi''(G) \leqslant \Delta + 2$ for every graph G. This conjecture was verified for general graphs with $\Delta \leqslant 5$. For planar graphs of large maximum degree, there is a stronger result. It is known that a planar graph G with $\Delta \geqslant 9$ has $\chi''(G) = \Delta + 1$ [2,7,4]. This stronger result does not hold for planar graphs of maximum degree at most 3. For $\Delta = 4, 5, 6, 7, 8$, it is unknown if every planar graph with maximum degree Δ is total-$(\Delta + 1)$-colorable. Shen et al. [5] proved that a planar graph G with maximum degree 8, and with no intersecting triangles has $\chi''(G) = 9$. Hou et al. [3] proved that a planar graph G with maximum degree 8, and with either no 5-cycle or no 6-cycle has $\chi''(G) = 9$. In this paper, we strengthen this result and prove the following result:

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Theorem 1. *Assume G is a planar graph with maximum degree 8. If for each vertex x, there is an integer $k_x \in \{3, 4, 5, 6, 7, 8\}$ such that G has no k_x-cycle which contains x, then $\chi''(G) = 9$.*

2. Proof of Theorem 1

Suppose the theorem is false. Let H be a planar graph with maximum degree 8, such that for each vertex $x \in V(H)$, there is an integer $k_x \in \{3, 4, 5, 6, 7, 8\}$ such that H has no k_x-cycle containing x. H is called a counterexample to the theorem if $\chi''(H) > 9$. Among all the counterexamples, take a graph G with minimum sum of its number of vertices and edges. G is called a minimum counterexample. For a vertex v of G, the degree $d(v)$ of v is the number of edges incident to v, and for a face f, the degree $d(f)$ of f is the number of edges on the boundary of f. A vertex of G of degree i (respectively, at most i or at least i) is called an i-*vertex* (respectively, i^--*vertex* or i^+-vertex). A face of degree at least i (respectively, at most i or at least i) is called an i-face (respectively, i^--*face or* i^+-*face*). The following remark is an easy observation:

Remark 2. The graph G is 2-connected, and hence has no vertices of degree 1.

The following lemma is adapted from the original proofs found in [3,2].

Lemma 3. *The graph G is 2-connected and has the following properties*:

(1) *If uv is an edge of G with $d(u) \leqslant 4$, then $d(u) + d(v) \geqslant \Delta + 2 = 10$ (and so 2-vertices of G are adjacent only to Δ-vertices).*
(2) *There is no cycle C in which half of its vertices are 2-vertices.*
(3) *The graph of all edges joining 2-vertices to Δ-vertices is a forest (and so there is an injective function Φ from the set of 2-vertices of G to the set of Δ-vertices of G such that $\Phi(v)$ is adjacent to v. The vertex $\Phi(v)$ is called the* master *of v).*
(4) *If $\{v, v_1, u_1\}$ induces a triangle and v_1 is a 2-vertex, then v is not adjacent to another 2-vertex v_2.*

(1) If G contains an edge uv with $d(u) \leqslant 4$ and $d(u) + d(v) \leqslant \Delta + 1$, then we can totally color $G \setminus \{uv\}$ with $\Delta + 1$ colors (by the minimality of G), erase the color on u, and then color uv and u in turn, since the number of colors that we may not use is at most $(\Delta - 1) + 1 = \Delta$ for uv and $4 + 4 = 8 < \Delta + 1$ for u. This contradicts the choice of G as a counterexample.
(2) This is an easy consequence of the previous statement.
(3) Suppose the statement is false then G contains a cycle $C = v_1 v_2 \ldots v_{2k} v_1$ of even length such that $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. Then, by the minimality of G, we can totally color $G \setminus \{v_1, v_3, \ldots, v_{2k-1}\}$ with $\Delta + 1$ colors. Each edge of C now has at most $(\Delta - 2) + 1 = \Delta - 1$ colors that may not be used on it, hence at least two that may, and so the problem of coloring the edges of C is equivalent to coloring the vertices of an even cycle, given a choice of two colors at each vertex; it is well known that this is possible. The 2-vertices of C are now easily colored, and that contradicts the choice of G as a counterexample to the theorem.
(4) On the contrary, suppose vv_1u_1 is a triangle with $d(v_1) = 2$ and v is adjacent to a 2-vertex v_2. The other neighbor of v_2 is u_2. Then, by minimality of G, we can totally color $G \setminus u_1 v_1$. Let ϕ be the coloring obtained, then erase the color on v_1, v_2. Assume that $\phi(vv_1) = 1$, $\phi(vu_1) = 2$ and $\phi(vv_2) = 3$. It is easy to verify that color 1 does not appear at u_1. Since otherwise, there must be a color α that does not appear at u_1. We color $v_1 u_1$ with α. The elements incident or adjacent to vertices v_1 and v_2 use at most four colors. So we can color v_1, v_2 properly. It follows that ϕ is extended to a total-$(\Delta + 1)$-coloring of G. If $\phi(v_2 u_2) \neq 2$, then recolor vv_2 with 2, vu_1 with 1, vv_1 with 3, and color $v_1 u_1$ with 2. Otherwise recolor vv_1 with 3, vv_2 with 1, and color $v_1 u_1$ with 1. Next color vertices v_1, v_2 properly. In any case, ϕ can be extended to a total-$(\Delta + 1)$-coloring of G. This contradicts the minimality of G.

Let V, E, F be the vertex set, edge set and face set of G, respectively. By Euler's formula,

$$\sum_{v \in V} \big(2d(v) - 6\big) + \sum_{f \in F} \big(d(f) - 6\big) = -6\big(|V| - |E| + |F|\big) = -12 < 0 \qquad (1)$$

We shall use discharging method to derive a contradiction. First, define the initial charge $c(x)$ for each $x \in V \cup F$ as follows: If v is a vertex, then $c(v) = 2d(v) - 6$; if f is a face then $c(f) = d(f) - 6$. The total initial charge is negative. We shall use a discharging procedure to lead to a nonnegative charge for every vertex and face.

A face is *small* if its length is less than 6, and is *large* if its length at least 7. We define the following two rounds of discharging. First round:

- If a 2-vertex is incident to a 3-face, then it receives 1 charge from each of its neighbors.
- Each large face f distributes its charge $d(f) - 6$ evenly among those 2-vertices that are incident to f.
- Each 4-vertex and 5-vertex sends 1/2 charge to each incident 4-face, 1/3 charge to each incident 5-face, and distribute the remaining extra charge evenly among all incident 3-faces.

We denote by $c'(x)$ the charge of a face or a vertex x after the first round. A small face or a 2-vertex that still has negative charge after the first round is called *deficient*. Second round:

- Each deficient small face f receives $|c'(f)|$ charge, evenly from its 6^+-boundary vertices.
- Each deficient 2-vertex x receives $|c'(x)|$ charge from its master.

We denote by $c^*(x)$ the charge of a face or a vertex x after the second round. We shall prove that all the faces and all the vertices have nonnegative charge $c^*(x)$ after the second round.

If f is a small face and is incident to a 6^+-vertex, then either it becomes nonnegative in the first round of discharging, or is deficient after the first round and hence becomes nonnegative after the second round of discharging.

Assume f is a small face not incident to any 6^+-vertex. Then f is not incident to any 4^--vertex by Lemma 3(1) and hence all the vertices incident to f are 5-vertices. If f is a 4-face, then each incident vertex sends 1/2 charge to f in the first round, and hence $c'(f) = c^*(f) = 0$. If f is a 5-face, then each incident vertex sends 1/3 charge to f in the first round, and hence $c'(f) = c^*(f) > 0$.

Assume $f = v_1 v_2 v_3$ is a 3-face. Recall $d(v_1) = d(v_2) = d(v_3) = 5$.

- First we consider the case that each of v_1, v_2, v_3 is incident to at most three 3-faces. Since $c(v_i) = 4$ and each 4^+-face incident to v_i receives at most 1/2 charge from v_i, we conclude that each v_i sends at least 1 charge to f, and hence $c'(f) = c^*(f) \geqslant 0$.
- Next we consider the case that v_1 is incident to at least four 3-faces, and hence both v_1, v_2, v_3 are contained in cycles of length 3, 4, 5, 6. If v_2 is also incident to at least four 3-faces, then it is easy to ver-

ify that v_2 is contained in cycles of length l for $l = 3, 4, \dots, 8$. Thus v_2 is incident to at most three 3-faces. Similarly v_3 is incident to at most three 3-faces. Moreover, if v_2 (resp. v_3) is incident to three 3-faces, then the two other faces incident to v_2 (resp. v_3) cannot be 7^--faces, since then v_2 (resp. v_3) would be contained in 7- and 8-cycles as well, contrary to our choice of G. Therefore v_2 (resp. V_3) is also incident to two 8^+-faces. Therefore, each of v_2 and v_3 sends charge at least 5/4 to f. Since v_1 sends at least 4/5 charge to f, we have $c'(f) = c^*(f) \geqslant 0$.

A 6-face f does not send out any charge and hence $c^*(f) = c(f) = 0$. A 7^+-face distributes its charge $d(f) - 6$ to incident 2-vertices and sends out no other charge, and hence $c^*(f) \geqslant 0$.

It remains to show that every vertex v has $c^*(v) \geqslant 0$. If v is a 2-vertex, then it is clear that v receives at least 2 charge from incident faces and adjacent vertices. So $c^*(v) = 0$. If v is a 3-vertex, then v does not send out any charge and hence $c^*(v) = c(v) = 0$. If v is a 4-, 5-vertex, then the only charge send out by v is by the last rule of the first round of discharging, and it is obvious that $c'(v) = c^*(v) \geqslant 0$. In the following, we consider 6^+-vertices.

Lemma 4. *Suppose x is a 6^+-vertex and f is a small face incident to x.*

(1) *If x is a 6-vertex and f is a 3-face, then f receives at most 5/4 charge from x.*
(2) *If x is a 7^+-vertex and f is a 3-face, then f receives at most 3/2 charge from x.*
(3) *If x is a 6-vertex and f is a 4-face then f receives at most 2/3 charge from x.*
(4) *If x is a 7^+-vertex and f is a 4-face, then f receives at most 1 charge from x.*
(5) *If f is a 5-face then f receives at most 1/3 charge from x.*

Proof. (1) Assume x is a 6-vertex and f is a 3-face. If the three boundary vertices of f are 6^+-vertices, then each of them sends 1 charge to f. If f has two 6^+-boundary vertices, then the other boundary vertex is a 4-vertex or a 5-vertex, and hence sends at least 1/2 charge to f, and each of the two 6^+-vertices sends at most 5/4 charge to f. If x is the only 6^+-boundary vertex of f, then the other two vertices, say y, y', are 5-vertices. If one of y, y', say y, is incident to a 6^+-face or incident to two 4^+-faces, then y sends at least 1 charge to f, and y' sends at least 4/5 charge to f, hence x sends at most 6/5 charge to f. Otherwise for each of y, y', four of its incident faces are 3-faces and the other is a 5^--face. It is easy to verify that in this case, x is contained in cycles of length l for each $l = 3, 4, \dots, 8$.

(2) Assume x is a 7^+-vertex and f is a 3-face. If the other two boundary vertices of f are 5-vertices, then x sends 7/5 charge to f. If one of the other two boundary vertices of f is a 6-vertex, then the third boundary vertex of f is a 4^+-vertex, and hence x sends at most 5/4 charge to f. Otherwise f has at least two 7^+-boundary vertices, and x sends at most 3/2 charge to f.

(3) Assume x is a 6-vertex and f is a 4-face. If f has a 3^--boundary vertex u, then the two neighbors of u in the boundary of f are 7^+-vertices. So the boundary of f has three 6^+-vertices, and hence each of them sends 2/3 charge to f. Otherwise each boundary vertex of f sends 1/2 charge to f.

(4) Assume x is a 7^+-vertex and f is a 4-face. If f has two 6^+-boundary vertices, then each 6^+-vertex sends at most 1 charge to f. Assume x is the only 6^+-boundary vertex of f. By Lemma 3(1), the other three boundary vertices of f are 5-vertices. Each of them sends 1/2 charge to f and hence x also sends 1/2 charge to f.

(5) If f is a 5-face, then f has at least three 5^+-boundary vertices. Hence each of them sends at most 1/3 charge to f. □

Final charge of 6-vertices. Assume x is a 6-vertex, and let x_i be its neighbors for $i = 1, 2, \dots, 6$. By Lemma 3(1), each x_i has degree at least 4. This is important since this will allow us to avoid faces that share more than one edge. The only charges sent out by x are to the small faces incident to x.

If x is incident to five or six 3-faces, then x is contained in cycles of lengths 3, 4, 5, 6, 7. Thus x is not contained in any 8-cycle. This implies that each of the 3-faces incident to x is adjacent to a 8^+-face. So each x_i is incident to at least two 8^+-faces. If x_i has degree 4 or 5, then x_i sends at least 1 charge to each of its incident 3-faces. Therefore x sends at most 1 charge to its incident faces, and hence $c^*(x) \geqslant 0$.

If x is incident to four 3-faces, then x is not incident to a 4-face, for otherwise x is contained in cycles of length l for $l = 3, 4, \dots, 8$. As each 3-face receives at most 5/4 charge from x by Lemma 4 and each 5-face incident to x receives at most 1/3 charge from x, we conclude that $c^*(x) \geqslant 6 - 4 \times 5/4 - 2 \times 1/3 \geqslant 0$.

If x is adjacent to at most three 3-faces, then $c^*(x) \geqslant 6 - 3 \times 5/4 - 3 \times 2/3 \geqslant 0$.

Final charge of 7-vertices. Assume x is a 7-vertex. Recall that the neighbors of x have degree at least 3. Again this is important to prevent some faces from sharing more than one edge. If among the faces incident to x, one is a 6^+-face and another is a 5^+-face, then x sends at most 1/3 charge to these two faces, and at most 3/2 charge to any other face. Hence $c^*(x) \geqslant 8 - 15/2 - 1/3 \geqslant 0$. If x is incident to at least three 5-faces, then x sends at most 1 charge to these three faces, and at most 3/2 charge to any other face. Hence $c^*(x) \geqslant 8 - 4 \times 3/2 - 1 \geqslant 0$. If x is incident to at most two 3-faces, then x sends at most 1 charge to any other face, so $c^*(x) \geqslant 8 - 2 \times 3/2 - 5 \geqslant 0$. So among the seven faces incident to x, at least six are 5^--faces, at least five are 4^--faces and at least three are 3-faces. A case by case analysis shows that x is contained in cycles of length l for $l = 3, 4, \dots, 8$ in that case.

Final charge of 8-vertices. Assume x is an 8-vertex. If x is not the master of any 2-vertex, then an argument similar to that in the previous paragraph shows that $c^*(x) \geqslant 0$. Assume x is a master of a 2-vertex y and $c^*(x) < 0$.

If x is incident to at least three 6^+-faces, then these three faces do not receive any charge from x, hence $c^*(x) \geqslant 10-2-5\times 3/2 \geqslant 0$. Thus we assume that among the eight faces incident to x, at most two are 6^+-faces.

Case 1. x is incident to exactly two 6^+-faces.

If the 2-vertex y receives at most 1 charge from x, then $c^*(x) \geqslant 10-1-6\times 3/2 \geqslant 0$. Thus we assume that y receives more than 1 charge from x. In particular, y is not incident to any triangles.

If x is incident to a 5-face, then $c^*(x) \geqslant 10-2-5\times 3/2-1/3 \geqslant 0$. If x is incident to at least two 4-faces, then $c^*(x) \geqslant 10-2-4\times 3/2-2 \geqslant 0$. Thus x is incident to no 5-faces and incident to at most one 4-face.

If the two faces incident to y are 6^+-faces, then either all the other faces are 3-faces, or one of them is a 4-face and the others are 3-faces. In any case, it is easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$. Assume the two faces incident to y are a 4-face and a 6^+-face. The other faces incident to x are all 3-faces, except one more 6^+-face. If one of the 6^+-faces is a 7^--face, then it is again easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$. Assume both 6^+-faces are 8^+-faces. Then y receives at least $2/3$ charge from an incident 8^+-face, as the surplus charge of that face is evenly distributed among its incident 2-vertices. But then $c^*(x) \geqslant 10-4/3-5\times 3/2-1 \geqslant 0$.

Case 2. x is incident to exactly one 6^+-face.

If among the other faces incident to x, at least five are 3-faces, then it is easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$. If there are at most two 3-faces, then $c^*(x) \geqslant 10-2-2\times 3/2-5 \geqslant 0$. If there are three 3-faces and at least one 5-face, $c^*(x) \geqslant 10-2-3\times 3/2-3-1/3 \geqslant 0$. If there are three 3-faces and no 5-face, then it is again easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$. Thus we may assume that there are exactly four 3-faces incident to x. If the other three faces are 5-faces, then $c^*(x) \geqslant 10-2-4\times 3/2-3\times 1/3 \geqslant 0$. Thus at least one of the other faces is a 4-face. But then it is again easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$.

Case 3. x is incident to no 6^+-face.

If x is incident to four 5-faces, then $c^*(x) \geqslant 10-2-4\times 3/2-4\times 1/3 \geqslant 0$. If x is incident to three 5-faces and at least one 4-face, then $c^*(x) \geqslant 10-2-4\times 3/2-3\times 1/3-1 \geqslant 0$. If x is incident to three 5-faces and no 4-face then it is again easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$. If x is incident to two 5-faces and at least four 4-faces, then $c^*(x) \geqslant 10-2-2\times 3/2-2\times 1/3-4 \geqslant 0$. If x is incident to two 5-faces and at most three 4-faces, then x is incident to at least three 3-faces. It is again easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$. If x is incident to one 5-face and at least six 4-faces, then $c^*(x) \geqslant 10-2-3/2-1/3-6 \geqslant 0$. If x is incident to one 5-face and at most five 4-faces, then x is incident to at least two 3-faces. It is again easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$. If x is incident to no 5-face, and no 3-face, then $c^*(x) \geqslant 10-2-8 \geqslant 0$. If x is incident to no 5-face and at least one 3-face, then it is again easy to verify that x is contained in cycles of length l for $l=3,4,\dots,8$.

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