Choosability of Toroidal Graphs Without Short Cycles

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Abstract: Let G be a toroidal graph without cycles of a fixed length k, and $\chi_l(G)$ the list chromatic number of G. We establish tight upper bounds

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of $\chi_l(G)$ for the following values of k:

$$\chi_l(G) \leq \begin{cases} 4 & \text{if } k \in \{3,4,5\} \\ 5 & \text{if } k=6 \\ 6 & \text{if } k=7 \end{cases}$$

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1. INTRODUCTION

Motivated by the four color problem, coloring problems on graphs embedded on various surfaces have been studied extensively in the literature. In this article, we study list colorings of *toroidal graphs*, i.e., graphs that can be drawn on the torus without crossing edges. The concept of list coloring, a generalization of vertex coloring, was introduced by Vizing [23] and independently by Erdős et al. [6]. For each vertex of a graph G, let $L(v)$ denote a list of available colors for v. An *L-coloring* of G is a vertex coloring ϕ that colors each vertex v by a color $\phi(v) \in L(v)$ so that no adjacent vertices receive the same color. A graph G is *L-colorable* if it admits an L-coloring, and *k-choosable* (or *k-list-colorable*) if it is L-colorable for every assignment of k-element lists to the vertices. An L-coloring ϕ for G with every vertex v satisfying $|L(v)| = k$ is also called a *k-list-coloring*. The *list chromatic number* (or the *choice number*) $\chi_l(G)$ of a graph G is the smallest k for which G is k-choosable.

Thomassen [18] proved that every planar graph is 5-choosable, whereas Voigt [24] gave a planar graph that is not 4-choosable. Lam et al. [16, 17], independently Wang and Lih [26, 27], showed that for any $k \in \{3,4,5,6\}$, every planar graph G without k-cycles is 4-choosable. More recently, Farzad [7] showed that a planar graph without 7-cycles is 4-choosable. Several researchers [3, 4, 9, 12, 15, 22, 28] have also studied the choosability of graphs embedded on various surfaces.

In this article, we study list chromatic numbers of toroidal graphs without cycles of specific lengths. We note that there are quite a few papers in the literature [5, 10, 11, 13, 14, 19] dealing with vertex colorings of toroidal graphs. Let $k \geq 3$ be an integer and G a toroidal graph without k-cycles. In this article, we establish tight upper bounds of $\chi_l(G)$ for the following values of k:

$$\chi_l(G) \leq \begin{cases} 4 & \text{if } k \in \{3,4,5\} \\ 5 & \text{if } k=6 \\ 6 & \text{if } k=7 \end{cases}$$

Our main tool is the discharging method, which is used to obtain structural properties of toroidal graphs without k-cycles in Section 2. We use these results to obtain the upper bounds of $\chi_l(G)$ in Section 3, and discuss open problems in Section 4.

2. STRUCTURAL PROPERTIES

In this section, we consider structural properties of toroidal graphs without k-cycles, which will be used in the next section to establish upper bounds of list chromatic number of such toroidal graphs.

A graph G is *k-degenerate* if every subgraph H of G contains a vertex of degree at most k. Equivalently, a k-degenerate graph admits a linear ordering such that the forward degree of every vertex is at most k. Obviously, a k-degenerate graph is $(k+1)$-choosable. It is well-known that every planar graph is 5-degenerate, every toroidal graph is 6-degenerate, and every planar graph without 3-cycles is 3-degenerate. Wang and Lih [26] proved that every planar graph without 5-cycles is 3-degenerate, and Fijavž [8] et al. showed that every planar graph without 6-cycles is 3-degenerate. Here we will investigate degeneracies of toroidal graphs when some short cycles are excluded from the graphs.

An embedding of G is a 2-*cell embedding* if each face of the embedded graph is homeomorphic to the open unit disc. A cycle C of G is *noncontractible* if it separates the torus into two components with one being homeomorphic to a disc; otherwise, C is *noncontractible*. We assume that all embeddings considered in this article are 2-cell embeddings, and use $F(G)$ to denote the set of faces of G for a given embedding of G on the torus.

Suppose that G is a graph embedded on the torus. For $x\in V(G)\cup F(G)$, let $d_G(x)$ (simply $d(x)$) denote the degree of x in G. A vertex (or a face) of degree k is called a *k-vertex* (or *k-face*). Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of the vertices of G, respectively. For a face $f\in F(G)$, we use $b(f)$ to denote the closed boundary walk of f and write $f=[u_1u_2\dots u_n]$ if $u_1,u_2,\dots,u_n$ are the vertices on the boundary walk in the clockwise order. The set of boundary vertices of f is denoted by $V(f)$. A face f of G is called a *simple* face if $b(f)$ forms a cycle. Obviously, when $\delta(G)\geq 2$, for $k\leq 5$, each k-face is a simple face. For $v\in V(G)$, let $N(v)$ denote the set of neighbors of v in G. Furthermore, let $T(v)$ and $Q(v)$ denote, respectively, the set of 3-faces and 4-faces incident with v. If $S\subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and simply write $G-S=G[V(G)\setminus S]$.

Lemma 1. *Let G be a toroidal graph. If G contains no 4-cycle and $\delta(G)\geq 4$, then it contains either*

(a) *an even cycle where each vertex is a 4-vertex, or*
(b) *a 6-cycle $x_1x_2\dots x_6x_1$ with a chord x_1x_3 such that $d(x_1)\leq 5$ and $d(x_i)=4$ for every remaining vertex x_i.*

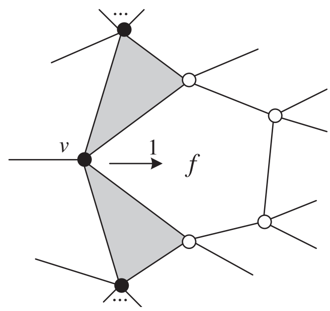

FIGURE 1. Special vertex v, where black vertices are major vertices and other vertices are 4-vertices.

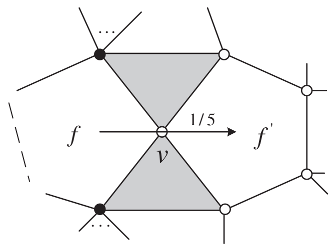

FIGURE 2. Source f and sink f' with vertex v being a joint-vertex from f to f'. Black vertices are major vertices and other vertices are 4-vertices.

Proof. By contradiction. Let G be a connected counterexample to the lemma, i.e., G is a connected toroidal graph such that $\delta(G) \geq 4$ and G contains neither 4-cycles nor configurations in (a) or (b). Without loss of generality, we may assume that G is embedded on the torus. We will show by the discharging method that such G does not exist. Note that G contains no 4-faces, and thus also no two 3-faces with a common edge.

We first define a few terms. A vertex v is a *major vertex* if $d(v) \geq 5$, and a 5-vertex v on a 5-face f is a *special vertex for f* if we have the configuration shown in Figure 1. Note that a 5-vertex can be a special vertex for at most one face.

A face f with $d(f) \geq 5$ is a *source* of a 5-face f', and then f' is a *sink* of f, if they have exactly the connection shown in Figure 2. The vertex v in the figure is referred to as the *joint-vertex* from f to f'. Note that f can be multiple sources of f' through different joint-vertices.

We define an initial weight w for vertices and faces as follows:

$$w(v) = 2d(v) - 6 \text{ for each vertex } v \text{ and } w(f) = d(f) - 6 \text{ for each face } f.$$

It follows from Euler's formula that $\sum_{x\in V(G)\cup F(G)} w(x)=0$. We now use the following discharging rules **(R1)**, **(R2)**, and **(R3)** in this order to obtain a new weight w'.

(R1) Each vertex v sends 1 to each incident 3-face, and a special vertex for a 5-face f sends 1 to f.

(R2) Each vertex sends its remaining weight evenly to the remaining faces incident with it.

(R3) For each joint-vertex from a source to a sink, the source transfers $\frac{1}{5}$ to the sink.

We now show that after discharging, the new weight w' satisfies

$$w'(x)\geq 0 \text{ for all } x\in V(G)\cup F(G), \text{ and there is a face } f \text{ with } w'(f)>0.$$

This is a contradiction since the total weight, which is 0, is unchanged during the discharging, and hence the lemma holds.

Consider the weight $w'(v)$ of an arbitrary vertex v. Since no 3-faces in G share edges, the number of 3-faces incident with v is at most $d(v)/2$, and thus v sends at most $d(v)/2$ to 3-faces incident with it. Therefore the remaining weight of v after applying rule **(R1)** is 1 if v is a special vertex and at least $(2d(v)-6)-d(v)/2\geq 0$ as $d(v)\geq 4$. It follows that $w'(v)=0$.

We now consider the weight $w'(f)$ of an arbitrary face f. First we note the following three simple facts that will be used frequently in our discussions.

Fact 1. If a 4-vertex v is incident with at most one 3-face, then it sends at least $\frac{1}{3}$ to every face incident with it.

Fact 2. A 5-vertex v sends every face incident with it at least $\frac{1}{2}$, and at least $\frac{2}{3}$ if v is not a special vertex.

Fact 3. A d-vertex with $d\geq 6$ sends at least 1 to each face incident with it.

If $d(f)=3$, then $w'(f)=-3+1+1+1=0$ (rule **(R1)**). Otherwise $d(f)\geq 5$, and we note that the number of major vertices in $V(f)$ is no less than the number k of sinks of f. It follows that $w'(f)\geq(d(f)-6)+(\frac{1}{2}-\frac{1}{5})k>0$ (rule **(R3)** and Facts 2 and 3) unless $d(f)=6$ and f has no sink. For this case, because G contains no configuration in (a), $V(f)$ contains at least one major vertex and thus $w'(f)\geq\frac{1}{2}>0$ (Facts 2 and 3).

If f is a 5-face, then $w(f)=-1$ and we consider three cases depending on the number of major vertices in $V(f)$.

Case 1. $V(f)$ contains no major vertex, i.e., every vertex of $V(f)$ is a 4-vertex.

In this case, f is not a source. Consider an arbitrary vertex $v\in V(f)$. If v is incident with at most one 3-face, then v sends at least $\frac{1}{3}$ to f (Fact 1). Otherwise since G contains no configuration in (b), there is a source f' with v being a joint-vertex from f' to f, and thus f' sends $\frac{1}{5}$ to f through v (rule (R3)). It follows that at least $\frac{1}{5}$ will be sent to f through v, and thus $w'(f)\geq -1+5\cdot\frac{1}{5}=0$. Furthermore, $w'(f)=0$ if and only if f has the configuration shown in Figure 3.

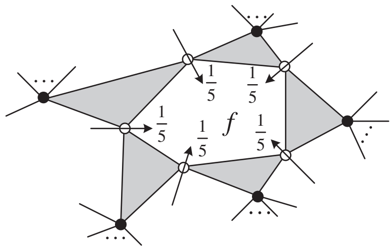

FIGURE 3. The configuration with $w'(f)=0$ for Case 1.

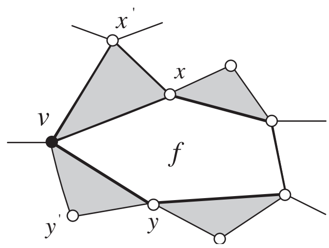

FIGURE 4. The situation for Case 2 when v is not a special vertex and both x and y are incident with two 3-faces. Thick lines indicate configuration (b).

Case 2. $V(f)$ contains exactly one major vertex v.

In this case, f is not a source either. If $d(v)\geq 6$ or v is a special vertex for f, then v sends at least 1 to f, implying $w'(f)\geq 0$. Otherwise, v cannot be a special vertex for other 5-faces and thus it sends $\frac{2}{3}$ to f (Fact 2). If both neighbors x and y of v in $V(f)$ are incident with two 3-faces, then we have the situation in Figure 4. Since v is not a special vertex for f, one of the two vertices x' and y', say x', must be a 4-vertex; but this yields a configuration in (b), a contradiction. Therefore one of x and y is incident with at most one 3-face and hence sends at least $\frac{1}{3}$ to f (Fact 1). Therefore $w'(f)\geq -1+\frac{2}{3}+\frac{1}{3}\geq 0$.

Case 3. $V(f)$ contains at least two major vertices.

First note that f can be a source of at most two sinks. If $V(f)$ has more than two major vertices, then $w'(f)\geq -1+3\cdot\frac{1}{2}-2\cdot\frac{1}{5}>0$ (Facts 2 and 3). Otherwise $V(f)$ has exactly two major vertices. If neither is a special vertex for a 5-face other than f, then each major vertex sends at least $\frac{2}{3}$ to f and thus $w'(f)\geq -1+2\cdot\frac{2}{3}-\frac{1}{5}>0$ as f has at most one sink. Otherwise one major vertex v is a special vertex for a 5-face other than f, and thus the other major vertex is adjacent with v and f has no sink (see Figures 1 and 2). Let u be the 4-vertex in $V(f)$ adjacent with v. Then edge uv in not in the boundary of any 3-face and therefore u is incident with at most one 3-face. It follows

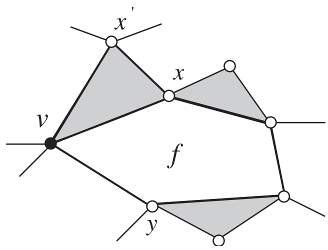

FIGURE 5. The situation when $V(f)$ has exactly one major vertex. Thick lines indicate configuration (b).

that these two major vertices together send at least 1 to f (Facts 2 and 3) and u sends at least $\frac{1}{3}$ to f, and thus $w'(f)>0$. We conclude that $w'(f)>0$ for Case 3.

We have shown so far that $w'(x)\geq 0$ for every $x\in V(G)\cup F(G)$, and we prove now that there is a face f with $w'(f)>0$. Suppose that $w'(f)=0$ for every face f. From previous arguments we know that $w'(f)>0$ for any 5-face with at least two major vertices (Case 3) or any face f with $d(f)\geq 6$. Therefore G has the following additional property:

Every face is either a 3*-face or* 5*-face, and every* 5*-face has at most one major vertex, which implies that G has no special vertex, neither source nor sink.*

Let f be a 5-face of G. If f has no major vertex, then f has the configuration in Figure 3 which is impossible as it contains sources. Therefore f has exactly one major vertex v.

Note that if $d(v)\geq 7$ then v sends f more than 1, which makes $w'(f)>0$. Therefore we may assume that $d(v)\leq 6$. If $d(v)=5$, then v sends at least $\frac{2}{3}$ to f. By Fact 1 and the fact that $w'(f)=0$, we deduce that all but one 4-vertex y in $V(f)$ are incident with two 3-faces, which implies that y is adjacent with v and we have the situation in Figure 5. Since every 5-face has exactly one major vertex, vertex x' in Figure 5 is a 4-vertex, which yields configuration (b) in G, a contradiction.

Otherwise $d(v)=6$. Let $v_0, v_1, v_2, v_3, v_4, v_5$ be the neighbors of v ordered clockwisely, and let P_i be the (v_i, v_{i+1})-path (modulo 6) along the boundary of the face containing vertices v_i, v, v_{i+1}. Since every 5-face has exactly one major vertex and no 3-faces have common edges, all vertices in each P_i are 4-vertices. Furthermore, all internal vertices of P_i's are distinct as $\delta(G)\geq 4$ and G has no 4-cycles, and therefore $P_0P_1P_2P_3P_4P_5$ is an even cycle consisting of 4-vertices only as each P_i contains an odd number of edges. This contradicts the assumption that G has no configuration (a). It follows that such G does not exist, and hence the truth of the lemma. ■

Lemma 2. *Let G be a toroidal graph without* 5*-cycles. Then* $\delta(G)\leq 4$; *and* $\delta(G)=4$ *if and only if G is* 4*-regular.*

Proof. Suppose that the lemma is false. Let G be a connected counterexample which is embedded on the torus. Thus, $\delta(G)\geq 4$, $\Delta(G)\geq 5$ and without 5-cycles. In particular, G contains neither 5-face, nor 3-face adjacent with a 4-face, nor three consecutive adjacent 3-faces. For each vertex $x \in V(G)$, let $w(x) = 2d(x) - 6$ and, for each face $x \in F(G)$, let $w(x) = d(x) - 6$. By applying Euler's formula, $\sum_{x \in V(G) \cup F(G)} w(x) = 0$. Similarly as the proof of Lemma 1, we use the discharging method to derive a contradiction. The discharging rule is the following:

(R) If v is a vertex of degree at least 4, then we, from v, transfer 1 to each incident 3-face and $\frac{1}{2}$ to each incident 4-face.

Let w' denote the resulting weight function after discharging. Suppose $f \in F(G)$. Then $d(f) \neq 5$. If $d(f) = 6$, then $w'(f) = w(f) = 0$. If $d(f) \geq 7$, then $w'(f) = w(f) \geq 1$. If $d(f) = 3$, then f receives the amount 1 from each of its boundary vertices and hence $w'(f) \geq -3 + 3 = 0$. If $d(f) = 4$, f receives $\frac{1}{2}$ from each of its boundary vertices and $w'(f) \geq -2 + 4 \cdot \frac{1}{2} = 0$.

Next suppose $v \in V(G)$. Then $d(v) \geq 4$ by the assumption. Assume $d(v) = 4$. Then $|T(v)| \leq 2$ (as G has no three consecutive 3-faces). If $|T(v)| = 0$, then $w'(v) \geq 2 - 4 \cdot \frac{1}{2} = 0$. If $|T(v)| = 1$, then $|Q(v)| \leq 1$ (otherwise G contains a 3-face adjacent with a 4-face). Thus $w'(v) \geq 2 - 1 - \frac{1}{2} = \frac{1}{2}$. If $|T(v)| = 2$, then $|Q(v)| = 0$ and hence $w'(v) = 2 - 1 - 1 = 0$. Assume $d(v) = 5$. Then $|T(v)| \leq 3$. If $|T(v)| \leq 2$, then $w'(v) \geq 4 - 2 - 3 \cdot \frac{1}{2} = \frac{1}{2}$. If $|T(v)| = 3$, then $|Q(v)| = 0$ and hence $w'(v) \geq 4 - 3 = 1$. Finally, assume that $d(v) \geq 6$. Note that $|T(v)| \leq \lfloor 2d(v)/3 \rfloor \leq 2d(v)/3$ and $|Q(v)| \leq d(v) - |T(v)|$. It follows that $w'(v) \geq w(v) - |T(v)| - \frac{1}{2}|Q(v)| \geq 2d(v) - 6 - |T(v)| - \frac{1}{2}(d(v) - |T(v)|) = \frac{3}{2}d(v) - \frac{1}{2}|T(v)| - 6 \geq \frac{3}{2}d(v) - \frac{1}{2} \cdot \frac{2}{3}d(v) - 6 = \frac{7}{6}d(v) - 6 > 0$.

The preceding argument shows actually that $w'(x) > 0$ if x is either a face of degree at least 7 or a vertex of degree at least 5, and $w'(x) \geq 0$ for all other vertices and faces x. Since G is not a 4-regular graph and $\delta(G) \geq 4$, it follows that

$$\sum_{x \in V(G) \cup F(G)} w(x) = \sum_{x \in V(G) \cup F(G)} w'(x) > 0,$$

a contradiction. ■

Lemma 3. *If G is a toroidal graph without 6-cycles, then $\delta(G) \leq 4$.*

Proof. Assume to the contrary that G is a connected graph embedded on the torus without 6-cycles such that $\delta(G) \geq 5$. Let $w(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$. Then $\sum_{x \in V(G) \cup F(G)} w(x) = 0$. Apply the following discharging rule:

(R) Every vertex v transfers $\frac{1}{3}$ to each incident 3-face.

Let w' denote the new weight function after discharging. Suppose $v \in V(G)$. Then $d(v) \geq 5$. Since G contains no 6-cycles, v is not incident with four consecutive adjacent 3-faces. If $d(v) \geq 6$, then $|T(v)| \leq d(v) - 2$ and hence $w'(v) = d(v) - 4 - \frac{1}{3}|T(v)| \geq d(v) - 4 - \frac{1}{3}(d(v) - 2) = \frac{2}{3}(d(v) - 5) > 0$. Assume $d(v) = 5$, so $w(v) = 1$ and $|T(v)| \leq 3$. When $|T(v)| \leq 2$, $w'(v) \geq 1 - 2 \cdot \frac{1}{3} = \frac{1}{3}$. When $|T(v)| = 3$, $w'(v) = 1 - 3 \cdot \frac{1}{3} = 0$.

Suppose $f \in F(G)$. If $d(f) = 3$, then $w'(f) \geq -1 + 3 \cdot \frac{1}{3} = 0$. If $d(f) = 4$, then $w'(f) = w(f) = 0$. If $d(f) \geq 5$, then $w'(f) = w(f) = d(f) - 4 \geq 1$.

Therefore, it follows that $w'(x) \geq 0$ for every $x \in V(G) \cup F(G)$. Since $\sum_{x \in V(G) \cup F(G)} w'(x) = \sum_{x \in V(G) \cup F(G)} w(x) = 0$, we conclude that $w'(x) = 0$ for every $x \in V(G) \cup F(G)$. The above argument implies that G is a 5-regular graph with each vertex being incident with exactly three 3-faces and two 4-faces. We shall show such graphs always have 6-cycles, which contradicts our assumption.

Let $f = [x_1x_2x_3x_4]$ be a 4-face. For $i = 1,2,3,4$, let f_i denote the adjacent face of f with x_ix_{i+1} as a boundary edge, where the indices are taken modulo 4. If f is adjacent with two consecutive 3-faces, say f_1, f_2, then it is easy to see that the vertices in $V(f) \cup V(f_1) \cup V(f_2)$ give rise to a 6-cycle. If f is adjacent with two consecutive 4-faces, say f_3, f_4, then the vertex x_3 is incident with at least three 4-faces, contradicting the previous paragraph. Without loss of generality, we may assume that $f_1 = [x_1y_1x_2]$ and $f_3 = [x_3y_2x_4]$ are 3-faces, and $f_2 = [x_2u_1u_2x_3]$ and f_4 are 4-faces. If $y_1 \neq y_2$, then $x_1y_1x_2x_3y_2x_4x_1$ is a 6-cycle. If $y_1 = y_2$, then either $u_1 \neq x_4$ or $u_2 \neq x_1$. When $u_1 \neq x_4$ and $u_2 \neq x_1$, $x_1x_2u_1u_2x_3x_4x_1$ is a 6-cycle. When $u_1 = x_4$ and $u_2 \neq x_1$, $x_1y_1x_2x_3u_2x_4x_1$ is a 6-cycle. When $u_1 \neq x_4$ and $u_2 = x_1$, $x_1u_1x_2x_3y_2x_4x_1$ is a 6-cycle. ■

Lemma 4. *Let G be a toroidal graph without 7-cycles. If $\Delta(G) \geq 6$ and $K_6 \not\subseteq G$, then $\delta(G) \leq 4$.*

Proof. By contradiction. Assume that G is a connected counterexample, which is embedded on the torus. Again, let $w(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$, so $\sum_{x \in V(G) \cup F(G)} w(x) = 0$. Before introducing the discharging rules, we need a new concept. Let f be a face adjacent with a 3-face $f' = [xyz]$ such that $xy \in b(f) \cap b(f')$. Then the vertex z is called a *leaf* of the face f. The discharging rules are as follows:

(R1) Every vertex v transfers $\frac{1}{3}$ to each incident 3-face.
(R2) Every face f of degree at least 6 transfers $\frac{1}{3}$ to each of its leaves.

Let w' be the new weight function. Let $f \in F(G)$. If $d(f) = 3$, then f receives $\frac{1}{3}$ from each of its boundary vertices by **(R1)**, and thus $w'(f) \geq -1 + 3 \cdot \frac{1}{3} = 0$. If $4 \leq d(f) \leq 5$, then $w'(f) = w(f) \geq 0$. If $d(f) \geq 6$, then, since f has at most $d(f)$ leaves, $w'(f) \geq d(f) - 4 - \frac{1}{3}d(f) = \frac{2}{3}(d(f) - 6) \geq 0$.

Next let $v \in V(G)$. If $d(v) \geq 6$, then f is incident with at most $d(v) - 1$ 3-faces for otherwise the induced subgraph $G[N(v) \cup \{v\}]$ contains a 7-cycle. Therefore, $w'(v) \geq d(v) - 4 - \frac{1}{3}(d(v) - 1) = \frac{1}{3}(2d(v) - 11) > 0$. Finally assume that $d(v) = 5$. Then $w(v) = 1$. If $|T(v)| \leq 3$, then $w'(v) \geq 1 - 3 \cdot \frac{1}{3} = 0$ by **(R1)**. Suppose $|T(v)| \geq 4$. Let $x_1, x_2, \ldots, x_5$ denote the neighbors of v in clockwise order and let $f_1 = [vx_1x_2]$, $f_2 = [vx_2x_3]$, $f_3 = [vx_3x_4]$, $f_4 = [vx_4x_5]$, and f_5 denote the faces of G incident with the vertex v. For $i = 1,2,3,4$, let g_i denote the adjacent face of f_i with $x_ix_{i+1} \in b(f_i) \cap b(g_i)$. If $|T(v)| = 5$, we further denote by g_5 the adjacent face of f_5 with $x_5x_1 \in b(f_5) \cup b(g_5)$.

Claim 1. *$d(g_i) \neq 5$ for all $i \in \{1,2,3,4\}$.*

Assume to the contrary that some g_i is a 5-face. We shall show that G contains a 7-cycle, contrary to our assumption. Note that the boundary of a 5-face is a simple cycle. By symmetry, it suffices to consider two cases as follows.

(i) $g_1=[x_1y_1y_2y_3x_2]$. Since $\delta(G)\geq 5$, it is easy to derive that $v\notin\{y_1,y_2,y_3\}$ and $|\{y_1,y_2,y_3\}\cap\{x_3,x_4,x_5\}|\leq 1$. In particular, since $d(x_2)\geq 5$, we have $x_3\neq y_3$. If $x_3\notin\{y_1,y_2\}$, then G contains a 7-cycle $vx_1y_1y_2y_3x_2x_3v$. If $x_3=y_2$, then G contains a 7-cycle $vx_1x_2y_3x_3x_4x_5v$. If $x_3=y_1$, then G contains a 7-cycle $vx_1x_2y_3y_2x_3x_4v$. A contradiction is always established.

(ii) $g_2=[x_2y_1y_2y_3x_3]$. Similarly, we first note that $v\neq y_1,y_2,y_3$. Furthermore, at most one of x_1,x_4 belongs to $\{y_1,y_2,y_3\}$. Consequently, a 7-cycle is contained in the induced subgraph $G[(V(g_2)\cup N(v))\setminus\{x_5\}]$. This proves Claim 1.

Claim 2. $d(g_i)\neq 4$ *for* $i=2,3$.

Assume the Claim is false, and without loss of generality, assume that $g_2=[x_2y_1y_2x_3]$ is a 4-face. Note that $v\neq y_1$, y_2 and at most one of y_1, y_2 is identical to some vertex in $\{x_1,x_4,x_5\}$. If $y_1=x_4$, then $vx_1x_2x_3y_2x_4x_5v$ is a 7-cycle. If $y_1=x_5$ or $y_2=x_5$, then $vx_1x_2y_1y_2x_3x_4v$ is a 7-cycle. If $y_2=x_1$, then $vx_1y_1x_2x_3x_4x_5v$ is a 7-cycle. This proves Claim 2.

The remaining part of the proof is split into two cases.

Case 1. $|T(v)|=5$.

By Claims 1 and 2, $d(g_i)=3$ or $d(g_i)\geq 6$ for all $i=1,2,\ldots,5$. First consider the case that there exist four g_i's of degree 3, say $g_i=[x_iz_ix_{i+1}]$ for $i=1,2,3,4$. If there is some $z_i\notin N(v)$, then $vx_iz_ix_{i+1}x_{i+2}\ldots x_{i-1}x_iv$ is a 7-cycle. Thus assume that $\{z_1,z_2,z_3,z_4\}\subseteq N(v)$. It is easy to observe that $z_1=x_4$, $z_2=x_5$, $z_3=x_1$, and $z_4=x_2$. It follows that the edges x_1x_4, x_4x_2, x_2x_5, x_5x_3, $x_3x_1\in E(G)$ and $G[\{v,x_1,x_2,x_3,x_4,x_5\}]$ induces a K_6, contrary to our assumption. Thus we may assume that there exist $k\neq j$ such that $d(g_k)\geq 6$ and $d(g_j)\geq 6$. Since both g_k and g_j have the same leaf v, $w'(v)\geq 1+2\cdot\frac{1}{3}-5\cdot\frac{1}{3}=0$ by **(R1)** and **(R2)**.

Case 2. $|T(v)|=4$.

In this case, $d(f_5)\geq 4$. If either $d(g_2)\geq 6$ or $d(g_3)\geq 6$, then v receives at least $\frac{1}{3}$ from g_2 and g_3 by **(R2)** and hence $w'(v)\geq 1+\frac{1}{3}-4\cdot\frac{1}{3}=0$. Otherwise, by Claims 1 and 2, we may assume that $d(g_2)=d(g_3)=3$. Similarly to the proof of Case 1, we derive that $g_2=[x_2x_3x_5]$ and $g_3=[x_1x_3x_4]$. It follows that $x_1x_3,x_1x_4,x_2x_5,x_3x_5\in E(G)$. This implies that the 3-cycles vx_2x_5v, vx_3x_5v, vx_1x_3v, and vx_1x_4v are all noncontractible. By Claim 1, $d(g_1)\neq 5$. Also $d(g_1)\neq 3$ by the embedding property of G on the torus. Assume that $g_1=[x_1u_1u_2x_2]$ is a 4-face. Thus $v\neq u_1,u_2$ and at most one of u_1,u_2 is identical to some of x_3,x_4,x_5. If $u_1,u_2\notin\{x_3,x_4,x_5\}$, G has a 7-cycle $vx_1u_1u_2x_2x_3x_4v$. If $u_1=x_4$, G has a 7-cycle $vx_1x_2u_2x_4x_5x_3v$. If $u_1=x_5$, G has a 7-cycle $vx_1x_2u_2x_5x_4x_3v$. If $u_2=x_4$, G has a 7-cycle $vx_1u_1x_4x_5x_3x_2v$. If $u_2=x_5$, G has a 7-cycle $vx_1u_1x_5x_4x_3x_2v$. So we have $d(g_1)\geq 6$, i.e., v is a leaf of g_1. By **(R2)**, $w'(v)\geq 1+\frac{1}{3}-4\cdot\frac{1}{3}=0$.

We have proved that $w'(x)\geq 0$ for all $x\in V(G)\cup F(G)$. Hence $w'(x)=0$ for all $x\in V(G)\cup F(G)$. This implies that $\Delta(G)\leq 5$, contrary to our assumption. This completes the proof of the lemma. ■

Obviously, if a graph G embedded on the torus satisfies $d(x)=4$ for all $x \in V(G) \cup F(G)$, then G can be expressed as a $p \times q$ grid, where p, $q \geq 3$.

Lemma 5. *If G is a toroidal graph without 3-cycles, then $\delta(G) \leq 4$; and moreover $\delta(G)=4$ if and only if G is a grid.*

This lemma is easily proved by taking the weight assignment $w(x)=d(x)-4$ for all $x \in V(G) \cup F(G)$.

It is easy to see that every subgraph of a graph G without k-cycles is a graph without k-cycles. Note that each of the subgraphs (a) and (b) in Lemma 1 contains a vertex of degree 4 in the original graph G. These facts, altogether with Lemmas 1 to 5, imply the following result.

Theorem 6. *Let G be a toroidal graph without k-cycles. Then the following statements hold.*

(a) *For $k=3$, G is 3-degenerate unless it contains an induced grid.*
(b) *For $k=5$, G is 3-degenerate unless it contains an induced 4-regular subgraph.*
(c) *For $k \in \{4,6\}$, G is 4-degenerate.*
(d) *For $k=7$, G is 5-degenerate.*
(e) *For $k \geq 8$, G is 6-degenerate.*

For a fixed $n \in \{4,5,6,7\}$, K_n is a toroidal graph with $\delta(K_n)=n-1$ and without $(n+1)$-cycles. Observe that the line graph of a dodecahedron is a 4-regular toroidal graph without 4-cycles. Indeed, this graph also is a planar graph. In addition, there exist infinitely many toroidal graphs without 3-cycles and having the minimum degree 3. These examples illustrate that Theorem 6 is best possible in certain sense.

3. CHOOSABILITY

In this section, we give tight upper bounds of $\chi_l(G)$ when the toroidal graph G contains no k-cycles for some $3 \leq k \leq 7$. First we note the following two results due to Erdős et al. [6] and Böhme et al. [4], respectively.

Lemma 7. *If a connected graph G is neither a complete graph nor an odd cycle, then $\chi_l(G) \leq \Delta(G)$.*

Theorem 8. *For any toroidal graph G, $\chi_l(G) \leq 7$ with $\chi_l(G)=7$ if and only if $K_7 \subseteq G$.*

Our main result in the paper is the following tight upper bounds for $\chi_l(G)$ when the toroidal graph G contains no k-cycles for some $3 \leq k \leq 7$. For the tightness of the bound, we note that the Mycielski graph of a 5-cycle is a (nonplanar) toroidal graph without 3-cycles and having list chromatic number 4. Aksionov and Mel'nikov [1] constructed many planar graphs without 4-cycles that have list chromatic number 4. For $n=4,5,6$, K_n is a toroidal graph with $\chi_l(K_n)=n$ but without $(n+1)$-cycles. Moreover, it should be noted that excluding cycles of length eight does not forbid the complete graph K_7 and thus the values of k between 3 and 7 are of the most interest.

Theorem 9. *Let $k \geq 3$ be a fixed integer and G a toroidal graph without k-cycles. Then the following statements hold.*

(1) *If $3 \leq k \leq 5$, then $\chi_l(G) \leq 4$.*
(2) *If $k=6$, then $\chi_l(G) \leq 5$.*
(3) *If $k=7$, then $\chi_l(G) \leq 6$ with $\chi_l(G)=6$ if and only if $K_6 \subseteq G$.*

Proof. First assume that $k=6$. By Theorem 6(c), G is 4-degenerate and thus is 5-choosable. This proves (2).

Assume next that $k=7$. By Theorem 6(d), G is 5-degenerate and thus $\chi_l(G) \leq 6$. If G contains a subgraph K_6, then it is obvious that $\chi_l(G)=\chi(K_6)=6$. Conversely, we suppose that $K_6 \not\subseteq G$. To prove that $\chi_l(G) \leq 5$, we use induction on the vertex number $|V(G)|$. The result holds clearly if $|V(G)| \leq 5$. Let G be a toroidal graph without 7-cycles that satisfies $K_6 \not\subseteq G$ and $|V(G)| \geq 6$. If $\Delta(G) \leq 5$, it is easy to derive $\chi_l(G) \leq 5$ by Lemma 7. If $\Delta(G) \geq 6$, then $\delta(G) \leq 4$ by Lemma 4. According to the induction hypothesis, $G-v$ is 5-choosable, where v is a vertex of minimum degree in $V(G)$. Since $d(v) \leq 4$, every 5-list-coloring of $G-v$ can be extended to a 5-list-coloring of G. This proves (3).

Assume now that $k \in \{3,5\}$. If $|V(G)| \leq 4$, it is evident that $\chi_l(G) \leq 4$. Let G be a toroidal graph without 3-cycles or 5-cycles and with $|V(G)| \geq 5$. We note that $G \neq K_5$. If G is 4-regular, then Lemma 7 asserts that G is 4-choosable. Otherwise, G contains a vertex v of degree at most 3 by Lemma 2 or Lemma 5. By the induction hypothesis, $G-v$ is 4-choosable. Obviously, any 4-list coloring of $G-v$ can be easily extended to a 4-list coloring of G.

Finally assume that $k=4$. The result holds trivially when $|V(G)| \leq 4$. Suppose that G is a toroidal graph without 4-cycles such that $|V(G)| \geq 5$. Let L denote an assignment for G such that $|L(v)|=4$ for all $v \in V(G)$. If $\delta(G) \leq 3$, let u be a vertex of minimum degree in G. By the induction hypothesis, $G-u$ is L-colorable. Obviously, we can extend any L-coloring of $G-u$ to an L-coloring of G. If $\delta(G) \geq 4$, G contains the configurations (a) or (b) by Lemma 1. Thus the proof is divided into the following two cases.

Case 1. G contains an even cycle $C=u_1u_2\ldots u_nu_1$ such that $d(u_i)=4$ for all $i=1,2,\ldots,n$.

Let $H=G-V(C)$ and $B=G[V(C)]$. By the induction hypothesis, H has an L-coloring ϕ. For every vertex $u_i \in V(C)$, define a list $L'(u_i)=L(u_i) \setminus \{\phi(z)|z \in V(H)$ and $zu_i \in E(G)\}$. Since $|L(u_i)|=d_G(u_i)=4$, it follows that $|L'(u_i)| \geq d_G(u_i)-d_H(u_i)=d_B(u_i) \geq 2$ for all $u_i \in V(C)$. Since $n \geq 4$ is even and B contains no 4-cycles, B is neither a complete graph nor an odd cycle. By Lemma 7, B is L'-colorable and furthermore G is L-colorable.

Case 2. G contains a 6-cycle $C=x_1x_2\ldots x_6x_1$ with a chord x_1x_3 such that $d(x_1) \leq 5$ and $d(x_i)=4$ for $i=2,3,\ldots,6$.

Since G contains no 4-cycles, C has no other chords except x_1x_3. Let $H = G - V(C)$ and $B = G[V(C)]$. By the induction hypothesis, H has an L-coloring ϕ. For each vertex $x_i \in V(C)$, we define the similar list $L'(x_i)$ as in the proof of Case 1. It is easy to show that $|L'(x_3)| \geq 3$ and $|L'(x_i)| \geq 2$ for all $i \neq 3$. If $|L'(x_3)| = 4$, we color x_1, x_2, x_6, x_5, x_4 and x_3 successively. So assume that $|L'(x_3)| = 3$.

If there exists a color $\alpha \in L'(x_4) \setminus L'(x_3)$, we assign α to x_4, then color x_5, x_6, x_1, x_2 and x_3 successively. If $L'(x_2) \setminus L'(x_3) \neq \emptyset$, we have a similar proof. If $|L'(x_k)| \geq 3$ for some $k \neq 3$, an L'-coloring of B is constructed easily. Thus assume that $|L'(x_i)| = 2$ for all $i \in \{1,2,4,5,6\}$. If $L'(x_4) \neq L'(x_5)$, we color x_5 with a color from $L'(x_5) \setminus L'(x_4)$, then color x_6, x_1, x_2, x_3 and x_4 successively. If either $L'(x_5) \neq L'(x_6)$ or $L'(x_6) \neq L'(x_1)$, we can form a similar coloring. Hence, in view of the above argument, we now assume that $L'(x_3) = \{1,2,3\}$ and $L'(x_1) = L'(x_4) = L'(x_5) = L'(x_6) = \{1,2\}$. Notice that at least one of the colors 1 and 2 belongs to $L'(x_2)$, say $1 \in L'(x_2)$. We color x_2, x_4, x_6 with 1, x_1, x_5 with 2, and x_3 with 3. We have succeeded in obtaining an L'-coloring of B in each possible case. Therefore G is L-colorable. ■

4. OPEN PROBLEMS

In this section, we give some open problems about the choosability of toroidal graphs. As mentioned in Section 1, for any $k \in \{3,4,5,6\}$, every planar graph without k-cycles is 4-choosable [16, 17, 26, 27]. In this paper, we show that for any $k \in \{3,4,5\}$, toroidal graphs without k-cycles are also 4-choosable. Note that toroidal graphs without 6-cycles (e.g., K_5) may not be 4-choosable. However, K_5 seems to be the only obstacle for such graphs to be 4-choosable, and we propose the following conjecture, whose truth implies the above result for planar graphs.

Conjecture 1. *Let G be a toroidal graph without 6-cycles. If G is K_5-free, then $\chi_l(G) \leq 4$.*

It is known [6, 24, 25] that for every $2 \leq k \leq 4$, there exist planar graphs (and hence toroidal graphs) which are k-colorable but not k-choosable. On the other hand, every 6-colorable toroidal graph is also 6-choosable, which leads to the following question:

Problem 2. Is there a toroidal graph G with $\chi(G) \leq 5$ and $\chi_l(G) = 6$?

The *girth* of a graph G is the length of a shortest cycle in G. Thomassen [21] proved that every planar graph of girth at least 5 is 3-choosable, and Voigt [25] constructed a planar graph of girth 4 that is not 3-choosable. Alon and Tarsi [2] proved that every bipartite planar graph is 3-choosable, and Kronk and White [14] showed that every toroidal graph of girth at least 6 is 3-colorable (the girth requirement was later reduced to 5 by Thomassen [20]). Here we propose the following conjecture in connection with girth.

Conjecture 3. *Every toroidal graph of girth at least 5 is 3-choosable.*

Note that the above conjecture holds if we change the girth requirement to 6. A graph G embedded on the torus is a *regular tessellation* if every vertex has degree 3 and every face has degree 6.

Theorem 10. *Every toroidal graph G of girth at least* 6 *is* 3*-choosable.*

Proof. We first show the following claim:

(*) *If G is not a regular tessellation, then* $\delta(G) \leq 2$.

Let $w(x) = d(x) - 4$ for each vertex or face x, and carry out the following discharging rule on G: *Every face f sends $\frac{1}{3}$ to each of its boundary vertices.*

It is easy to verify that the new weight function w' and each vertex or face x satisfy $w'(x) \geq 0$ with $w'(x) = 0$ only if G is a regular tessellation, which implies that $\delta(G) \leq 2$ when it is not.

Lemma 7 and statement $(*)$ together give us the theorem. ■

Finally, we consider toroidal $p \times q$ grids G, where p, $q \geq 3$. It is easy to prove by induction that G is 3-colorable for all p, $q \geq 3$. Furthermore, using the technique of Alon and Tarsi [2], we can prove that G is 3-choosable if either $p = q = 3$ or both p and q are even. However, the following question remains open.

Problem 4. Is every toroidal grid 3-choosable?

REFERENCES

[1] V. A. Aksionov and L. S. Mel'nikov, Some counterexamples associated with the three-color problem, J Combin Theory, Ser B 28 (1980), 1–9.

[2] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992), 125–134.

[3] D. Archdeacon, J. Hutchinson, A. Nakamoto, S. Negam, and K. Ota, Chromatic numbers of quadrangulations on closed surfaces, J Graph Theory 37 (2001), 100–114.

[4] T. Böhme, B. Mohar, and M. Stiebitz, Dirac's map-color theorem for choosability, J Graph Theory 32 (1999), 327–339.

[5] G. A. Dirac, Map color theorems, Canad J Math 4 (1952), 480–490.

[6] P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, Congr Numer 26 (1980), 125–157.

[7] B. Farzad, Planar graphs without 7-cycles are 4-choosable, SIAM J Discrete Math 23 (2009), 1179–1199.

[8] G. Fijavž, M. Juvan, B. Mohar, and R. Škrekovski, Planar graphs without cycles of specific lengths, European J Combin 23 (2002), 377–388.

[9] H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Wiss Z Martin Luther-Univ Halle Wittenberg, Math-Nat Reihe 8 (1959), 109–120.

[10] P. J. Heawood, Map-color theorem, Quart J Pure Appl Math 24 (1890), 332–338.

[11] J. P. Hutchinson, Three-coloring graphs embedded on surfaces with all faces even-sided, J Combin Theory, Ser B 65 (1995), 139–155.
[12] T. R. Jensen and B. Toft, Graph Coloring Problems, Wiley, New York, 1995.
[13] D. Král and R. Thomas, Coloring even-faced graphs in the torus and the Klein bottle, Combinatorica 28 (2008), 325–341.
[14] H. V. Kronk and A. T. White, A 4-color theorem for toroidal graphs, Proc Amer Math Soc 34 (1972), 83–86.
[15] A. V. Kostochka, M. Stiebitz, and B. Wirth, The colour theorems of Brooks and Gallai extended, Discrete Math 101 (1992), 189–201.
[16] B. C. P. Lam, B. Xu, and J. Liu, The 4-choosability of plane graphs without 4-cycles, J Combin Theory, Ser B 76 (1999), 117–126.
[17] B. C. P. Lam, W. C. Shiu, and B. Xu, On structure of some plane graphs with applications to choosability, J Combin Theory, Ser B 82 (2001), 285–296.
[18] C. Thomassen, Every planar graph is 5-choosable, J Combin Theory, Ser B 62 (1994), 180–181.
[19] C. Thomassen, Five-coloring graphs on the torus, J Combin Theory, Ser B 62 (1994), 11–33.
[20] C. Thomassen, Grötzsch 3-color theorem and its counterparts for the torus and the projective plane, J Combin Theory, Ser B 62 (1994), 268–279.
[21] C. Thomassen, 3-List coloring planar graphs of girth 5, J Combin Theory, Ser B 64 (1995), 101–107.
[22] C. Thomassen, Color-critical graphs on a fixed surface, J Combin Theory, Ser B 70 (1997), 67–100.
[23] V. G. Vizing, Coloring the vertices of a graph in prescribed colors, Diskret Analiz 29 (1976), 3–10 (in Russian).
[24] M. Voigt, List colorings of planar graphs, Discrete Math 120 (1993), 215–219.
[25] M. Voigt, A not 3-choosable planar graph without 3-cycles, Discrete Math 146 (1995), 325–328.
[26] W. F. Wang and K.-W. Lih, Choosability and edge-choosability of planar graphs without 5-cycles, Appl Math Lett 15 (2002), 561–565.
[27] W. F. Wang and K.-W. Lih, The 4-choosability of planar graphs without 6-cycles, Australas J Combin 24 (2001), 157–164.
[28] W. F. Wang and K.-W. Lih, Choosability and edge choosability of planar graphs without intersecting triangles, SIAM J Discrete Math 15 (2002), 538–545.