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Rainbow domination on trees

Gerard J. Chang [a,b,c,*], Jiaojiao Wu [a], Xuding Zhu [c,d]

[a] Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan
[b] Taida Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan
[c] National Center for Theoretical Sciences, Taiwan
[d] Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

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ABSTRACT

This paper studies a variation of domination in graphs called rainbow domination. For a positive integer k, a k-rainbow dominating function of a graph G is a function f from $V(G)$ to the set of all subsets of $\{1, 2, \dots, k\}$ such that for any vertex v with $f(v) = \emptyset$ we have $\cup_{u \in N_G(v)} f(u) = \{1, 2, \dots, k\}$. The 1-rainbow domination is the same as the ordinary domination. The k-rainbow domination problem is to determine the k-rainbow domination number $\gamma_{rk}(G)$ of a graph G, that is the minimum value of $\sum_{v \in V(G)} |f(v)|$ where f runs over all k-rainbow dominating functions of G. In this paper, we prove that the k-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs. We then give a linear-time algorithm for the k-rainbow domination problem on trees. For a given tree T, we also determine the smallest k such that $\gamma_{rk}(T) = |V(T)|$.

1. Introduction

Domination and its variations in graphs are natural models for the location problems in operations research. They have been extensively studied in the literature; see [4,7,8]. A *dominating set* of a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The following variation of domination was introduced by Brešar, Henning and Rall [2].

For a positive integer k, we use $[k]$ to denote the set $\{1, 2, \dots, k\}$, and $2^{[k]}$ the set of all subsets of $[k]$. A *k-rainbow dominating function* of G is a function $f : V(G) \to 2^{[k]}$ such that for every vertex v, either $f(v) \neq \emptyset$ or $f(N_G(v)) = [k]$, where $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $f(S) = \cup_{x \in S} f(x)$ for any subset S of $V(G)$. The *weight* of f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The *$k$-rainbow domination number* $\gamma_{rk}(G)$ of G is the minimum weight of a k-rainbow dominating function. A k-rainbow dominating function f of G is *optimal* if $w(f) = \gamma_{rk}(G)$. The *k-rainbow domination problem* is to determine the k-rainbow domination number of a given graph. Notice that the ordinary domination is the same as the 1-rainbow domination if we view a dominating set D as a 1-rainbow dominating function f defined by $f(v) = \{1\}$ when $v \in D$ and $f(v) = \emptyset$ otherwise.

The *Cartesian product* of two graphs G and H is the graph $G \square H$ with vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H) = \{(u, v)(u', v') : u = u' \text{ with } vv' \in E(H) \text{ or } uu' \in E(G) \text{ with } v = v'\}$. Rainbow domination of a graph G coincides with the ordinary domination of the Cartesian product of G with the complete graph, that is $\gamma_{rk}(G) = \gamma(G \square K_k)$ (see [2]).

* Corresponding author at: Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan.
E-mail addresses: gjchang@math.ntu.edu.tw (G.J. Chang), wujj0007@yahoo.com.tw (J. Wu), zhu@math.nsysu.edu.tw (X. Zhu).

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Hartnell and Rall [6] established several results on rainbow domination. In particular, it was proved in [6] that

$$\min\{|V(G)|, \gamma(G) + k - 2\} \le \gamma_{rk}(G) \le k\gamma(G)$$

for any $k \ge 2$ and any graph G. Their attempt to characterize graphs with $\gamma(G) = \gamma_{r2}(G)$ was inspired by the following famous conjecture by Vizing [9].

Vizing's Conjecture. For any graph G and H, we have $\gamma(G\Box H) \ge \gamma(G)\gamma(H)$.

One of the related problems posted by Hartnell and Rall [5] is to find classes of graphs that achieve the equality. They showed that $\gamma(G\Box H) = \gamma(G)\gamma(H)$ if G is a graph with $\gamma(G) = \gamma_{r2}(G)$ and H is a so-called *generalized comb*.

Brešar, Henning and Rall [2] introduced rainbow domination to study the relation with paired-domination; also see [1]. They gave a linear-time algorithm for finding a minimum weighted 2-rainbow dominating function of a tree. On the other hand, Brešar and Šumenjak [3] proved that the 2-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs. They also established exact values of the 2-rainbow domination numbers for paths, cycles and suns, and upper and lower bounds for the generalized Petersen graphs.

The purpose of this paper is to study k-rainbow domination for a general k. In Section 2, we prove that the k-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs, and then give a linear-time algorithm for the k-rainbow domination problem on trees. For a graph G on n vertices, $\gamma_{ri}(G) \le \gamma_{ri+1}(G) \le n$ for any i and $\gamma_{rn}(G) = n$. In Section 3, we determine the minimum k such that $\gamma_{rk}(T) = |V(T)|$ for any tree T.

2. Complexity in k-rainbow domination

In this section, we prove that the k-rainbow domination problem is NP-complete even when restricted to chordal graphs or bipartite graphs. We then give a linear-time algorithm for the k-rainbow domination problem on trees.

The domination problem is known to be NP-complete not only for general graphs but also for chordal graphs and bipartite graphs and many other classes of graphs; see [4]. This is also the case for the k-rainbow domination problem.

Theorem 1. *For any positive integer k, the k-rainbow domination problem is NP-complete for general graphs.*

Proof. We shall prove the theorem by reducing the k-rainbow domination problem to the domination problem. Given a graph G on n vertices, consider the graph G' with the vertex set $V(G') = V(G) \cup \{v_2, v_3, \dots, v_k : v \in V(G)\}$ and edge set $E(G') = E(G) \cup \{vv_i : v \in V(G), 2 \le i \le k\}$. Namely, we add $n(k-1)$ leaves to G by joining $k-1$ leaves to each vertex of G (we call a degree-1 vertex of G a *leaf* of G). We claim that G has a dominating set of cardinality at most s if and only if G' has a k-rainbow dominating function of weight at most $s + n(k-1)$.

Suppose G has a dominating set D of cardinality at most s. Consider the function f from $V(G')$ to $2^{[k]}$ defined by

$$f(u) = \begin{cases} \{1\}, & \text{if } u \in D, \\ \emptyset, & \text{if } u \in V(G) - D, \\ \{i\}, & \text{if } u = v_i \text{ for some } v \in V(G) \text{ and } 2 \le i \le k. \end{cases}$$

Suppose $f(u) = \emptyset$. By the definition of f, $u \in V(G) - D$ and so u has a neighbor $v \in D$. Since $f(v) = \{1\}$ and $f(u_i) = \{i\}$ for $2 \le i \le k$, we have $f(N_{G'}(u)) = [k]$. Therefore, f is a k-rainbow dominating function of G'. Also, the weight of f is $|D| + n(k-1) \le s + n(k-1)$.

On the other hand, suppose G' has a k-rainbow dominating function f of weight at most $s + n(k-1)$. For each $v \in V(G)$, we may assume that $\sum_{2\le i\le k} |f(v_i)| \le k-1$ for otherwise if $\sum_{2\le i\le k} |f(v_i)| \ge k$ then we replace each $f(v_i)$ by $\{i\}$ and add 1 to the set $f(v)$ to obtain a k-rainbow dominating function of weight at most $s + n(k-1)$. Now, consider the set $D = \{v \in V(G) : f(v) \ne \emptyset\}$. For any vertex $v \in V(G) - D$, we have $f(v) = \emptyset$ and so $f(N_{G'}(v)) = [k]$. As $\sum_{2\le i\le k} |f(v_i)| \le k-1$, we have $f(u) \ne \emptyset$ for some $u \in N_G(v)$ which implies $u \in D$. Therefore, D is a dominating set of G. Next, we calculate the cardinality of D. Suppose there are n' vertices $v \in V(G)$ such that $f(v_i) = \emptyset$ for some i. For these n' vertices v we have $f(v) = [k]$. Therefore, the weight of f is at least $n'k + (|D| - n') + n'0 + (n - n')(k-1) = |D| + n(k-1)$ implying $|D| + n(k-1) \le s + n(k-1)$ and so $|D| \le s$. □

Notice that if G is chordal or bipartite, then so is G' in the proof above. We thus have

Corollary 2. *For any positive integer k, the k-rainbow domination problem is NP-complete for chordal graphs and for bipartite graphs.*

In the rest of this section, we establish a linear-time algorithm for the k-rainbow domination problem on trees. For technical reasons, we in fact dealing with a more general problem. A *k-rainbow assignment* is a mapping L that assigns each vertex v a label $L(v) = (a_v, b_v)$ with $a_v, b_v \in \{0\} \cup [k]$. A *$k$-$L$-rainbow dominating function* is a function $f : V(G) \to 2^{[k]}$ such that for every vertex v in G we have

(L1) $|f(v)| \ge a_v$, and
(L2) $|f(N_G(v))| \ge b_v$ whenever $f(v) = \emptyset$.

The *k-L-rainbow domination number* $\gamma_{rkL}(G)$ of G is the minimum weight of a k-L-rainbow dominating function. A k-L-rainbow dominating function f of G is *optimal* if $w(f) = \gamma_{rkL}(G)$. Notice that k-rainbow domination is the same as k-L-rainbow domination if $L(v) = (0, k)$ for each $v \in V(G)$.

Theorem 3. *Suppose x is a leaf adjacent to y in a graph G with a k-rainbow assignment L. Let $G' = G - x$ and L' be the restriction of L on $V(G')$, except that when $a_x > 0$ we let $b'_y = \max\{0, b_y - a_x\}$. Then the following hold.*

(1) *If $a_x > 0$, then $\gamma_{rkL}(G) = \gamma_{rkL'}(G') + a_x$.*
(2) *If $a_x = 0$ and $a_y \geq b_x$, then $\gamma_{rkL}(G) = \gamma_{rkL'}(G')$.*

Proof. (1) Suppose f is an optimal k-L-rainbow dominating function of G. By condition (L1), $|f(x)| \geq a_x > 0$. We may assume that $|f(x)| = a_x$. Otherwise, if $|f(x)| > a_x$, then let $\hat{f}$ be the same as f except that $\hat{f}(x)$ is a subset of $f(x)$ of cardinality a_x and $\hat{f}(y) = f(y) \cup \{1\}$. It is easy to verify that $\hat{f}$ is a k-L-rainbow dominating function and $w(\hat{f}) \leq w(f)$. So, we may replace f by $\hat{f}$.

Now, let f' be the restriction of f on $V(G')$. We shall check that f' is a k-L'-rainbow dominating function of G'. First, condition (L1) for f implies condition (L1) for f', since $f'(v) = f(v)$ and $a'_v = a_v$ for all vertices v in G'. Secondly, condition (L2) holds for f' on vertex y, since $|f'(N_{G'}(y))| \geq \max\{0, |f(N_G(y))| - a_x\} \geq \max\{0, b_y - a_x\} = b'_y$. Also, condition (L2) for f implies condition (L2) for f' on vertices v in $G' - y$, since $f'(N_{G'}(v)) = f(N_G(v))$ and $b'_v = b_v$. Therefore, $\gamma_{rkL}(G) - a_x \geq \gamma_{rkL'}(G')$.

On the other hand, suppose f' is an optimal k-L'-rainbow dominating function of G'. Extend f' to f by letting $f(x)$ be an a_x-subset of $[k]$ that contains as many elements in $[k] - \cup_{v \in N_{G'}(y)} f(v)$ as possible. We shall check that f is a k-L-rainbow dominating function of G. First, condition (L1) for f' implies condition (L1) for f, since $|f(x)| = a_x$ and for all vertices v in G' we have $|f(v)| = |f'(v)| \geq a'_v = a_v$. Secondly, condition (L2) holds for f on vertex y, since $|f(N_G(y))| = |f(N_{G'}(y))| + a_x \geq b'_y + a_x \geq b_y$. Also, condition (L2) for f' implies condition (L2) for f on vertices v in $G' - y$, since $f'(N_{G'}(v)) = f(N_G(v))$ and $b'_v = b_v$. Therefore, $\gamma_{rkL'}(G') + a_x \geq \gamma_{rkL}(G)$.

(2) Suppose f is an optimal k-L-rainbow dominating function of G. We may assume that $f(x) = \emptyset$. Otherwise, if $f(x) \neq \emptyset$, then let $\hat{f}$ be the same as f except that $\hat{f}(x) = \emptyset$ and $\hat{f}(y) = f(y) \cup \{1\}$. It is easy to verify that $\hat{f}$ is a k-L-rainbow dominating function and $w(\hat{f}) \leq w(f)$. So, we may replace f by $\hat{f}$.

Now, let f' be the restriction of f on $V(G')$. We shall check that f' is a k-L'-rainbow dominating function of G'. First, condition (L1) for f implies condition (L1) for f', since $f'(v) = f(v)$ and $a'_v = a_v$ for all vertices v in G'. Secondly, condition (L2) for f implies condition (L2) for f', since $f'(N_{G'}(v)) = f(N_G(v))$ and $b'_v = b_v$ for all vertices v in G'. Therefore, $\gamma_{rkL}(G) \geq \gamma_{rkL'}(G')$.

On the other hand, suppose f' is an optimal k-L'-rainbow dominating function of G'. Extend f' to f by letting $f(x) = \emptyset$. We shall check that f is a k-L-rainbow dominating function of G. First, condition (L1) for f' implies condition (L1) for f, since $|f(x)| = a_x$ and for all vertices v in G' we have $|f(v)| = |f'(v)| \geq a'_v = a_v$. Secondly, condition (L2) holds for f on vertex x, since $|f(N_G(x))| = |f(y)| \geq a'_y = a_y \geq b_x$. Also, condition (L2) for f' implies condition (L2) for f on vertices v in G', since $f'(N_{G'}(v)) = f(N_G(v))$ and $b'_v = b_v$. Therefore, $\gamma_{rkL'}(G') \geq \gamma_{rkL}(G)$. □

Theorem 4. *Suppose $N_G(x) = \{z, x_1, x_2, \ldots, x_s\}$ such that $x_1, x_2, \ldots, x_s$ are leaves in a graph G with a k-rainbow assignment L. Assume $a_{x_i} = 0$ for $1 \leq i \leq s$ and $b_{x_1} \geq b_{x_2} \geq \cdots \geq b_{x_s} > a_x$. Let $b^* = \min\{b_{x_i} + i - 1 : 1 \leq i \leq s + 1\} = b_{x_{i^*}} + i^* - 1$, where $b_{x_{s+1}} = a_x$ and i^* is chosen as small as possible. If $G' = G - \{x_1, x_2, \ldots, x_s\}$ and L' is the restriction of L on $V(G')$ with modifications that $a'_x = b_{x_{i^*}}$ and $b'_x = \max\{0, b_x - i^* + 1\}$, then $\gamma_{rkL}(G) = \gamma_{rkL'}(G') + i^* - 1$.*

Remarks. For the case when the component of G containing x is a star, the vertex z is null. There in fact one does not have the vertex x_{s+1}. The assignment of $b_{x_{s+1}} = a_x$ is for the purpose of convenience. In the case of $i^* = s + 1$, this just means that a'_x is the same as a_x.

Proof. Suppose f is an optimal k-L-rainbow dominating function of G such that $|f(x)|$ is as large as possible. We may assume that $|f(x_i)| \leq 1$ for $1 \leq i \leq s$. Otherwise, if $|f(x_i)| \geq 2$, then let $\hat{f}$ be the same as f except that $\hat{f}(x_i) = \{1\}$ and $\hat{f}(x) = f(x) \cup \{1\}$. It is easy to verify that $\hat{f}$ is a k-L-rainbow dominating function and $w(\hat{f}) \leq w(f)$. So, we may replace f by $\hat{f}$.

We may also assume that $|f(x_{i+1})| \leq |f(x_i)|$ for $1 \leq i \leq s - 1$. Otherwise, if $f(x_i) = \emptyset$ and $f(x_{i+1}) \neq \emptyset$, then let $\hat{f}$ be the same as f except that $\hat{f}(x_i) = f(x_{i+1})$ and $\hat{f}(x_{i+1}) = \emptyset$. It is easy to verify that $\hat{f}$ is a k-L-rainbow dominating function and $w(\hat{f}) = w(f)$. So, we may replace f by $\hat{f}$.

Now, let f' be the restriction of f on $V(G')$. We shall check that f' is a k-L'-rainbow dominating function of G'. First, condition (L1) for f implies condition (L1) for f' for each vertex $v \neq x$ in G'. Assume that i is the largest index such that $|f(x_j)| = 1$ for all $j < i$. Then, $|f(x)| = f(N_G(x_i)) \geq b_{x_i}$. Hence $|f(N_G[x] \setminus z)| \geq b_{x_i} + i - 1 \geq b_{x_{i^*}} + i^* - 1$. If $|f(x)| < b_{x_{i^*}}$, then $i^* \leq s$. In this case, let $\hat{f}$ be the same as f, except that we add some labels to $f(x)$ so that $|\hat{f}(x)| = b_{x_{i^*}}$, and that $\hat{f}(x_j) = \emptyset$ for all $j \geq i^*$. It is easy to verify that $\hat{f}$ is a k-L-rainbow dominating function of G, with $w(\hat{f}) \leq w(f)$ (because $|\hat{f}(N_G[x] \setminus z)| = b_{x_{i^*}} + i^* - 1$) and $|\hat{f}(x)| > |f(x)|$, in contradiction to the choice of f. Hence we have $|f'(x)| = |f(x)| \geq b_{x_{i^*}} = a'_x$. These prove condition (L1) for f'. Condition (L2) for f implies condition (L2) for f' on vertices $v \neq x$. To see condition (L2) for f' on vertex

x, we assume $f(x) = \emptyset$. This is possible only when $b_{x_{i^*}} = 0$ and so $i^* = s+1$. Then, $|f'(N_{G'}(x))| \geq |f(N_G(x))| - s \geq b_x - i^* + 1$ and so condition (L2) holds for f' on vertex x. Therefore, $\gamma_{\mathrm{r}kL}(G) - i^* + 1 \geq \gamma_{\mathrm{r}kL'}(G')$.

On the other hand, suppose f' is an optimal k-L'-rainbow dominating function of G'. Assume that $[k] - f'(N_{G'}(x)) = \{p_1, p_2, \ldots, p_t\}$. Extend f' to f by letting $f(x_j) = \{p_{\min\{j,t\}}\}$ for all $j < i^*$ and $f(x_j) = \emptyset$ for all $j \geq i^*$. We shall check that f is a k-L-rainbow dominating function of G. First, condition (L1) for f' implies condition (L1) for f, since $|f(x_i)| \geq 0 = a_{x_i}$ for all i and for all vertices v in G' we have $|f(v)| = |f'(v)| \geq a'_v \geq a_v$. Secondly, condition (L2) holds for f on vertex x_i for $i \geq i^*$, since $|f(N_G(x_i))| \geq |f(x)| \geq b_{x_{i^*}} \geq b_{x_i}$. Condition (L2) holds for f on vertex x_i for $i < i^*$, since $f(x_i) \neq \emptyset$. Condition (L2) holds for f on vertex x, since either $|f(N_G(x))| = k$ or $|f(N_G(x))| \geq |f'(N_{G'}(x))| + i^* - 1 \geq b'_x + i^* - 1 \geq b_x$. Also, condition (L2) for f' implies condition (L2) for f on vertices v in $G' - x$, since $f'(N_{G'}(v)) = f(N_G(v))$ and $b'_v = b_v$. Therefore, $\gamma_{\mathrm{r}kL'}(G') + i^* - 1 \geq \gamma_{\mathrm{r}kL}(G)$. □

On the basis of the theorem above, we have the following linear-time algorithm for the k-L-rainbow domination problem on trees.

Algorithm.

Input. A tree $T = (V, E)$ in which each vertex v is labeled by $L(v) = (a_v, b_v)$.
Output. The minimum k-L-rainbow dominating number r of T.
Method.

$r \leftarrow 0$;
get a Breadth First Search ordering $v_1, v_2, \ldots, v_n$ for the tree T rooted at v_1;
for $j = 1$ **to** n **do** $s_j \leftarrow 0$; {number of children v_i with $a_{v_i} = 0$ and $b_{v_i} > a_{v_j}$}
for $j = n$ **to** 1 **step by** -1 **do**
    $s \leftarrow s_j$;
    $x \leftarrow v_j$;
    if $s > 0$ **then** {apply Theorem 4}
        let $x_1, x_2, \ldots, x_s, b^*, i^*$ be as described in Theorem 4;
        $r \leftarrow r + i^* - 1$;
        $a_x \leftarrow b_{x_{i^*}}$;
        $b_x \leftarrow \max\{0, b_x - i^* + 1\}$;
    if $j \neq 1$ **then** {apply Theorem 3}
        let $y = x_{j'}$ be the parent of x;
        if $a_x > 0$ **then** { $b_y \leftarrow \max\{0, b_y - a_x\}$; $r \leftarrow r + a_x$ };
        else if $b_x > a_y$ **then** $s_{j'} \leftarrow s_{j'} + 1$;
end do;
if $a_{v_1} > 0$ **then** $r \leftarrow r + a_{v_1}$;
else if $b_{v_1} > 0$ **then** $r \leftarrow r + 1$;

3. The smallest integer k with $\gamma_{\mathrm{r}k}(G) = |V(G)|$

For any graph G, $\gamma_{\mathrm{r}k}(G)$ is a non-decreasing function of k, as $\gamma_{\mathrm{r}k}(G) = \gamma(G \square K_k) \leq \gamma(G \square K_{k+1}) = \gamma_{\mathrm{r}k+1}(G)$. However, $\gamma_{\mathrm{r}k}(G)$ is bounded from above by $|V(G)|$.

Proposition 5. *If G is a graph on n vertices, then $\min\{k, n\} \leq \gamma_{\mathrm{r}k}(G) \leq n$. In particular, $\gamma_{\mathrm{r}n}(G) = n$*

Proof. Suppose f is an optimal k-rainbow dominating function of G. If there is some vertex v such that $f(v) = \emptyset$, then $f(N_G(v)) = [k]$ and so $k \leq \gamma_{\mathrm{r}k}(G)$. If $f(v)$ is nonempty for all vertices v, then $n \leq \gamma_{\mathrm{r}k}(G)$. These give the first inequality.

Next, consider the function g defined by $g(v) = \{1\}$ for all vertices v. Then, g is a k-rainbow dominating function of weight n. Thus, $\gamma_{\mathrm{r}k}(G) \leq n$. □

Define $m(G)$ to be the minimum k such that $\gamma_{\mathrm{r}k}(G) = |V(G)|$. In this section, we give a simple bound of $m(T)$ for any tree T. Combined with the algorithm in the previous section, this also provides a linear-time algorithm that determines $m(T)$ for any tree T.

In a graph G, for any vertex x and any nonempty subset S of $N_G(x)$ define $d^*(x, S)$ to be $|S| + \min\{d(y) : y \in S\}$; and $d^*(G)$ to be the maximum of $d^*(x, S)$ where x runs over all vertices and S runs over all nonempty subset of $N_G(x)$.

We may determine $d^*(G)$ in linear time as follows. For each vertex x with $N_G(x) = \{x_1, x_2, \ldots, x_s\}$, we may use a bucket sort to assume that $d(x_1) \geq d(x_2) \geq \cdots \geq d(x_s)$. Then, $\max_{1 \leq i \leq s}(i + d(x_i))$ is equal to the maximum value of $d^*(x, S)$ for S runs over all nonempty subset S of $N_G(x)$. Then, $d^*(G)$ can be obtained by taking maximum of above mentioned value for all vertices x.

Proposition 6. *For any graph G, we have $d^*(G) \geq \Delta(G) + 1$.*

Proof. Choose a vertex y of degree $\Delta(G)$ and a vertex $x \in N_G(y)$. Then, $d^*(x, \{y\}) = \Delta(G) + 1$ and so $d^*(G) \geq \Delta(G) + 1$. □

Theorem 7. *If T is a tree, then $d^*(T) \leq m(T) \leq d^*(T) + 1$.*

Proof. To see $d^*(T) \leq m(T)$, let $k = d^*(T) - 1$ and choose a vertex x and a subset S of $N_T(x)$ such that $d^*(T) = |S| + \min\{d(y) : y \in S\}$. Define a function f from $V(T)$ to $2^{[k]}$ by letting $f(x) = \{1, 2, \ldots, |S|\}$ and $f(y) = \emptyset$ for all $y \in S$. For any y in S, since $|S| + d(y) \geq d^*(T) = k + 1$ we can assign all neighbors of y other than x a subset of size 1 so that the union of all $f(z)$ over all neighbors of y, including x, is equal to $[k]$. We assign all other vertices a set of cardinality 1. This gives a k-rainbow dominating function whose weight is $|V(T)| - 1$. Thus, $m(T) \geq k + 1 = d^*(T)$.

To see $m(T) \leq d^*(T) + 1$, let $k = d^*(T) + 1$ and suppose f is an optimal k-rainbow dominating function of T. We shall prove that $w(f) \leq |V(T)|$.

Root T at a vertex r. We may assume that f is chosen so that there are as few vertices v with $f(v)$ empty as possible. We shall prove that in fact there is no such v.

Claim 1. *If v is a vertex with $f(v)$ empty but $f(u)$ nonempty for all proper descendants u of v, then $|f(u)| = 1$ for all proper descendants u of v.*

Proof of Claim 1. If there is a descendant u of v having $f(u) = \{a, b, \ldots\}$, then replacing $f(u)$ by $f(u) - \{a\}$ and $f(v)$ by $\{a\}$ will give another optimal k-rainbow dominating function having fewer vertices v with empty $f(v)$. □

Now, choose a vertex y with $d_T(r, y)$ largest and $f(y) = \emptyset$. Then, $f(u) \neq \emptyset$, and so $|f(u)| = 1$ by Claim 1, for all proper descendants u of y. Since $f(y) = \emptyset$, we have $f(N(y)) = [k]$ and so $\sum_{u \in N(y)} |f(u)| \geq k = d^*(T) + 1 \geq \Delta(T) + 2 \geq d(y) + 2$. As $|f(u)| = 1$ for all except possibly one neighbor of y, it must be the case that y has a parent x in the rooted tree.

Let S be the set of all children z of x such that $f(z) = \emptyset$. Then, $y \in S$. Also, for any $z \in S$, we have that $f(u) \neq \emptyset$, and so $|f(u)| = 1$ by Claim 1, for all proper descendants u of z. Choose a vertex $z' \in S$ such that $d(z') = \min_{z \in S} d(z)$. By $f(z') = \emptyset$, we have $k \leq \sum_{u \in N(z')} |f(u)| = d(z') - 1 + |f(x)|$ where $d(z') = d^*(x, S) - |S| \leq d^*(T) - |S| = k - 1 - |S|$. Thus, $|f(x)| \geq |S| + 2$. Now, modify f to a function f' such that $f'(z) = f'(x) = \{1\}$ for all $z \in S$, and if x has a parent x' then $f'(x') = f(x') \cup \{1\}$. Then, the resulting function f' is a k-rainbow dominating function of T whose weight is less than or equal to $w(f)$, but has fewer vertices v with $f(v)$ empty, a contradiction. So, in fact $f(v) \neq \emptyset$ for all vertices as desired. □

Remark that there are examples attaining both bounds in the theorem above. For instance, $d^*(K_{1,n-1}) = m(K_{1,n-1}) = n$ for any star $K_{1,n-1}$. For integer $p \geq 2$, consider the tree T_p rooted at r such that r has two children v_1 and v_2, and each v_i has $p - 1$ children each of which has p leaf-children. It can be shown that $d^*(T_p) = 2p$ and $m(T_p) = 2p + 1$.

We may determine the exact value of $m(T)$ by applying the algorithm in the last section to evaluate $\gamma_{rk}(T)$ with $k = d^*(T)$. If the value is $|V(T)|$, then $m(T) = d^*(T)$; otherwise $m(T) = d^*(T) + 1$.

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