Decomposition of sparse graphs, with application to game coloring number

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\textbf{A B S T R A C T}

Let $k$ be a nonnegative integer, and let $m_k = \frac{4k+1+2k^2}{k^2+6k+6}$. We prove that every simple graph with maximum average degree less than $m_k$ decomposes into a forest and a subgraph with maximum degree at most $k$ (furthermore, when $k \leq 3$ both subgraphs can be required to be forests). It follows that every simple graph with maximum average degree less than $m_k$ has game coloring number at most $4 + k$.

\section{1. Introduction}

The game coloring number of a graph $G$ is defined using a two-person game to produce an ordering of the vertices of $G$. In the ordering game on $G$, Alice and Bob take turns choosing vertices from the set of unchosen vertices of $G$. This places the vertices in a linear order $L$, with $x < y$ if $x$ is chosen before $y$. The back degree of a vertex $x$ with respect to $L$, written $b_L(x)$, is the number of neighbors of $x$ that precede $x$ in $L$. The back degree of $L$, written $b(L)$, is $\max_{x \in V(G)} b_L(x)$. Alice’s goal is to minimize $b(L)$, and Bob’s goal is to maximize it.

The game coloring number $\text{col}_g(G)$ of $G$ is defined to be $1 + k$, where $k$ is the least integer such that Alice can guarantee $b(L) \leq k$. Equivalently, $k$ is the greatest integer such that Bob can guarantee $b(L) \geq k$. The game coloring number was first formally defined in [9] as a tool for proving upper bounds on the game chromatic number [2]. It is the game version of the coloring number, which is defined to be $1 + \min_L b(L)$ and received its somewhat unfortunate name because it is an upper bound on the chromatic number. A more accurate and less confusable term might be “(game) coloring bound”, but we will use the traditional term and notation. The definition of back degree makes multi-edges and loops irrelevant in the game, so we use the model of “graph” that forbids these.

Recently, Zhu [10] proved that $\text{col}_g(G) \leq 17$ when $G$ is planar. Borodin et al. [3], He et al. [6], and Kleitman [7] improved this for planar graphs with large girth by proving structural properties of planar graphs with large girth. A decomposition of a graph $G$ is a set of edge-disjoint subgraphs whose union is $G$.

\textbf{Theorem 1.} Let $G$ be a planar graph with girth at least $g$.

1. [3] If $g \geq 9$, then $G$ decomposes into a forest and a matching.
2. [7] If \( g \geq 6 \), then \( G \) decomposes into a forest and a graph with maximum degree 2.
3. [6] If \( g \geq 5 \), then \( G \) decomposes into a forest and a graph with maximum degree 4.

Nash-Williams [8] proved that every planar graph decomposes into three forests. Balogh et al. [1] conjectured that one of the three forests can be required to have maximum degree at most 4, which is sharp infinitely often. They proved several results in this direction, and Gonçalves [5] proved the full conjecture. In addition, he showed that planar graphs with girth at least 6 (at least 7) decompose into two forests with one having maximum degree at most 4 (at most 2).

Two lemmas show the importance, for game coloring number, of decomposing a graph into a forest and a graph with small maximum degree.

**Lemma 1** (Zhu [9]). If a graph \( G \) decomposes into subgraphs \( G_1 \) and \( G_2 \), then \( \text{col}_g(G) \leq \text{col}_g(G_1) + \Delta(G_2) \).

**Lemma 2** (Faigle et al. [4]). If \( T \) is a forest, then \( \text{col}_g(T) \leq 4 \).

Combining these two lemmas with **Theorem 1** yields:

**Corollary 1** ([3,6,7]). If \( G \) is a planar graph with girth at least 5, then \( \text{col}_g(G) \leq 8 \). The upper bound decreases to 6 for girth at least 6 and to 5 for girth at least 9.

In this note, we bound the game coloring number of sparse graphs using this decomposition approach. We measure sparseness by avoidance of dense subgraphs. The maximum average degree of a graph \( G \), written \( \text{Mad}(G) \), is the largest average degree among the subgraphs of \( G \). That is,

\[
\text{Mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \subseteq G \right\}.
\]

We can now state our main result.

**Theorem 2.** Every graph \( G \) satisfying \( \text{Mad}(G) < \frac{4(k+1)(k+3)}{k^2 + 6k + 6} \) decomposes into a forest and a subgraph with maximum degree at most \( k \). When \( k \leq 3 \), both subgraphs can be required to be forests.

**Theorem 2** combines with **Lemmas 1 and 2** to yield:

**Corollary 2.** If a graph \( G \) satisfies \( \text{Mad}(G) < \frac{4(k+1)(k+3)}{k^2 + 6k + 6} \), then \( \text{col}_g(G) \leq 4 + k \).

Let \( m_k = \frac{4(k+1)(k+3)}{k^2 + 6k + 6} \). The value \( m_k \) is the largest bound our approach can prove. However, we do not know whether the result is sharp. Let \( f(k) \) be the infimum of \( \text{Mad}(H) \) over graphs \( H \) that do not decompose into a forest and a graph with maximum degree at most \( k \). For an upper bound, the complete bipartite graph \( K_{2,2k+2} \) has average degree \( \frac{4k+4}{k+2} \) but has no such decomposition. The graph obtained from any \((2k+2)\)-regular multigraph by subdividing each edge is another such example, with the same average degree. Thus

\[
4 - \frac{8k + 12}{k^2 + 6k + 6} \leq f(k) \leq 4 - \frac{4}{k+2}.
\]

Although **Theorem 2** holds for both planar and nonplanar graphs, it does not imply **Theorem 1** by using the usual inequality \( \text{Mad}(G) \leq 2g/(g - 2) \) that holds for every planar graph \( G \) having girth at least \( g \). Those results would follow from \( f(k) = \frac{4k+4}{k+2} \) and to imply the result for girth 5 no smaller \( f(k) \) suffices. For girth at least 7, the weaker threshold \( f(2) \geq 14/5 \) would imply the result of Gonçalves [5].

Answering the following question would solve the problem completely.

**Question 1.** For every \( k \), what are the graphs with smallest maximum average degree that do not decompose into a forest and a subgraph with maximum degree at most \( k \)?

The proof of **Theorem 2** uses reducible configurations (discussed in Section 2) and a discharging procedure (discussed in Section 3). The key structures in the proof are “banks” and “cores” that allow the transfer of charge over unlimited distances.

### 2. Reducible configurations and special subgraphs

Let \( d(x) \) denote the degree of a vertex \( x \) in a graph \( G \). A \( k \)-vertex is a vertex of degree \( k \). A \( \geq k \)-vertex or \( \leq k \)-vertex is a vertex of degree at least \( k \) or at most \( k \), respectively. An \((a,b)\)-alternating cycle is an even cycle that alternates between \( a \)-vertices and \( b \)-vertices.

We prove **Theorem 2** by considering a counterexample such that \(|V(G)| + |E(G)| \) is smallest. Since \( \text{Mad}(H) \leq \text{Mad}(G) \) when \( H \) is a subgraph of \( G \), every proper subgraph of \( G \) decomposes into a forest and a graph with maximum degree at most \( k \), but \( G \) has no such decomposition. We use this to exclude various configurations from \( G \). Since \( m_0 = 2 \), and \( \text{Mad}(G) < 2 \) implies that \( G \) is a forest, we may assume that \( k \geq 1 \).
Lemma 3. A minimal counterexample $G$ to Theorem 2 has (a) no 1-vertex, (b) no edge $uv$ with $d(v) \leq k + 1$ and $d(u) = 2$ (or $d(u) = 3$ when $k > 3$), and (c) no $(k + 2, 2)$-alternating cycle.

Proof. When $G$ contains any such configuration, we decompose an appropriate subgraph of $G$ into a forest $F$ and a subgraph $D$ with maximum degree at most $k$ (when $k \leq 3$, also $D$ is a forest) and use it to obtain such a decomposition of $G$, contradicting that $G$ is a counterexample.

(a) When $d(u) = 1$, the decomposition of $G - u$ extends by adding the extra edge to $F$.

(b) If $k > 3$, then $D$ need not be a forest; we prove the stronger statement that $G$ has no adjacent $\geq (k + 1)$-vertices. If $u$ and $v$ are such, then consider the decomposition of $G - uv$. If $u$ or $v$ has $k$ neighbors in $D$, then add $uv$ to $F$; otherwise, add $uv$ to $D$.

If $k \leq 3$, then $D$ in the decomposition of $G - uv$ is a forest, and we consider only $d(u) = 2$. If $v$ has $k$ neighbors in $D$, then add $uv$ to $F$; otherwise, add $uv$ to whichever of $D$ and $F$ does not contain the edge incident to $u$ in $G - uv$.

(c) Let $C$ be a $(k + 2, 2)$-alternating cycle in $G$. In the decomposition of $G - E(C)$ into $F$ and $D$, we enforce that each $(k + 2)$-vertex on $C$ has an incident edge in $F$, by moving an incident edge from $D$ to $F$ if not. Now adding one perfect matching in $C$ to $D$ and the other to $F$ extends the decomposition to $G$ without creating cycles in either subgraph. \qed

We use discharging to show that every graph satisfying (a), (b), and (c) of Lemma 3 has average degree at least $m_k$, and hence there is no counterexample to Theorem 2. To apply the discharging method, we first give each vertex a “charge” equal to its degree. We then redistribute the charge (without changing the total charge) to obtain charge at least $m_k$ on each vertex. To facilitate the discharging argument, we also move some charge to special subgraphs. They start with charge 0 and will end with nonnegative charge, so the initial average degree is at least $m_k$.

Given such a graph $G$, let $X$ be the set of all $(k + 2)$-vertices in $G$ that are adjacent to at least $k + 1$ vertices of degree 2, and let $Y$ be the set of all 2-vertices adjacent to at least one vertex of $X$. Define the bank of $G$ to be the maximal bipartite subgraph of $G$ with partite sets $X$ and $Y$. When $k \geq 4$, we modify this slightly by restricting $X$ to use only the $(k + 2)$-vertices whose neighbors all have degree 2.

A cycle in the bank would be a $(k + 2, 2)$-alternating cycle in $G$, which is forbidden. Hence the bank is a forest. We call each component of the bank a core. By construction, each vertex of $X$ has at least $k + 1$ neighbors in the bank $(k + 2$ when $k \geq 4$); hence each leaf in the bank belongs to $Y$.

3. The discharging argument

The initial charge at each vertex of $G$ is its degree, and also each core has initial charge 0. We use three discharging rules (plus a special rule when $k \geq 4$) to redistribute charges. In most discharging arguments, movement of charge is local. Assigning charge to cores permits charge to move long distances.

For the computations, recall that $m_k = \frac{4k^2 + 4k + 3}{k^2 + 6k + 6}$. Each discharging rule $R_i$ moves a constant amount $r_i$ of charge. These constants $r_1, r_2, r_3, r_4$ are defined in terms of $m_k$ by

$$r_1 = \frac{m_k - 2}{2}, \quad r_2 = 1 - r_1 - \frac{m_k}{k + 3}, \quad r_3 = m_k - (k + 2)(1 - r_1), \quad r_4 = \frac{m_k - 3}{3}.$$  

The discharging rules are as follows, with $R_4$ used only when $k \geq 4$. We add $R_4$ because $m_k \geq 3$ if and only if $k \geq 4$, so when $k \geq 4$ the 3-vertices need to gain charge. A vertex belonging to no core is adjacent to a core $C$ if it is adjacent in $G$ to a leaf of $C$.

R1 Every $\geq (k + 2)$-vertex gives $r_1$ to each neighbor that is a 2-vertex.

R2 If $C$ is a core, $v$ is a $\geq (k + 2)$-vertex belonging to no core, and $v$ is adjacent to $l$ leaves of $C$, then $v$ gives $l r_2$ to $C$.

R3 Every core gives $r_1$ to each of its $(k + 2)$-vertices whose neighbors all have degree 2.

R4 (For $k \geq 4$ only.) Every $\geq (k + 2)$-vertex gives $r_4$ to each neighboring 3-vertex.

The proof of Theorem 2 is now completed by proving the following lemma.

Lemma 4. If a graph $G$ satisfies (a), (b), and (c) of Lemma 3, then $\text{Mad}(G) \geq m_k$.

Proof. As described above, we give initial charge $d(v)$ to each vertex $v$ and initial charge 0 to each core $C$. After applying the discharging rules, let $\omega(v)$ and $\omega(C)$ denote the final charges. We prove that $\omega(v) \geq m_k$ for each vertex $v$ and $\omega(C) \geq 0$ for each core $C$.

By (b), the neighbors of 2-vertices are $\geq (k + 2)$-vertices. Using R1, the final charge of each 2-vertex is $2 + 2r_1$, which equals $m_k$, as desired.

If $3 \leq d(v) \leq k + 1$, then $v$ does not give or receive charge, unless $d(v) = 3 < k$. Thus $\omega(v) = d(v) > m_k$ except in that case. If $d(v) = 3 < k$, then by (b) its neighbors are all $\geq (k + 2)$-vertices. Via $R_4$ it receives $3r_4$, and hence $\omega(v) = 3 + 3r_4 = m_k$.

Now suppose that $d(v) \geq k + 2$. Vertex $v$ may lose charge to each neighbor, and $v$ may lose additional charge when $v$ is not in a core and its neighbors are. Since always $r_1 > r_4$, we may assume that each neighbor getting charge from $v$ is a
2-vertex. Hence the maximum charge lost from \( v \), via \( \{R1, R2, R4\} \), is \( d(v)(r1 + r2) \). Hence \( \omega(v) \geq d(v)(1 - r1 - r2) = \frac{d(v)m_k}{k+3} \).

If \( d(v) \geq k + 3 \), then \( \omega(v) \geq m_k \).

The case \( d(v) = k+2 \) is more delicate. If \( v \) is not in a core, then the definition of the bank limits the number of 2-neighbors of \( v \) (to \( k \) if \( k \leq 3 \), to \( k + 1 \) if \( k \geq 4 \)). If \( k \leq 3 \), then \( \omega(v) \geq 2 + k(1 - r1 - r2) = m_k + 2 - \frac{3m_k}{k+3} \). The formula for \( m_k \) yields

\[
2 - \frac{3m_k}{k+3} = \frac{2k^2}{2k^2 + 6k + 6} > 0, \quad \text{and hence } \omega(v) > m_k. \]

If \( k \geq 4 \), then \( v \) may have one 3-neighbor in addition to the maximum number of 2-neighbors. Hence \( \omega(v) \geq 1 + (k + 1)(1 - r1 - r2) - r4 = m_k + 2 - \frac{2m_k}{k+3} - \frac{m_k}{3} \).

The formula for \( m_k \) converts the last expression to \( m_k + \frac{2k(k-2)}{3(k^2 + 6k + 6)} \), and hence \( \omega(v) > m_k \).

Suppose now that \( d(v) = k+2 \) and \( v \) is in a core. If every neighbor of \( v \) has degree 2, then \( v \) loses \( r1 \) exactly \( k + 2 \) times, but it loses nothing by \( R2 \) and gains \( r3 \) by \( R3 \). Hence \( \omega(v) = (k + 2)(1 - r1) + r3 = m_k \). If \( v \) has a neighbor with degree more than 2, then \( k \leq 3 \). Now \( v \) loses \( r1 \) exactly \( k + 1 \) times by \( R1 \) and is unaffected by \( \{R2, R3, R4\} \). Hence \( \omega(v) = (k + 2) - (k + 1)r1 \).

Using \( r1 = m_k / 2 - 1 \) and the formula for \( m_k \), we compute

\[
(k + 2) - (k + 1)r1 - m_k = \frac{k^2}{k^2 + 6k + 6} > 0, \quad \text{and hence } \omega(v) > m_k. \]

Finally, we check that \( \omega(C) \geq 0 \) when \( C \) is a core. We have observed (using \( c \)) that \( C \) is a tree whose leaves are 2-vertices in \( G \). The non-leaves in \( X \) have degree \( k + 1 \) or \( k + 2 \) in \( C \), while the non-leaves in \( Y \) have degree 2 in \( C \). Let there be \( n_1 \) non-leaves of the first type, \( n_2 \) of the second, and \( n' \) of the third, and let \( n_0 \) be the number of leaves. Since \( C \) is a tree, its vertex degrees must sum to \( 2(n_0 + n_1 + n_2 + n') - 2 \), so we obtain \( n_0 = (k - 1)n_1 + kn_2 + 2 \). By (b), the neighbor outside \( C \) of a leaf of \( C \) is a \( n_2 \). Also, those vertices are not in cores, so \( C \) receives \( m_k \) via \( R2 \). Via \( R3 \), \( C \) distributes \( n_2 \), so

\[
kr2 - r3 = (2k + 2)(1 - r1) - \frac{2k + 3}{k + 3} m_k = 4(k + 1) - \frac{k^2 + 6k + 6}{k + 3} m_k = 0. \]

We have shown that all vertices and cores have sufficient final charge. \( \square \)

Note that \( r1 \) is defined in terms of \( m_k \) so that 2-vertices have final charge \( m_k \), and then \( r2 \) is defined in terms of \( m_k \) and \( r1 \) to give \( (k + 3) \)-vertices final charge \( m_k \). Next \( r3 \) is defined in terms of these so that \( (k + 2) \) vertices in cores whose neighbors all have degree 2 have final charge \( m_k \), and \( r4 \) is defined so that 3-vertices have final charge \( m_k \) when \( k \geq 4 \). Given all this, and the fact that \( n_1 \) may equal 0 in a core, \( m_k \) has been chosen as the largest value allowing us to guarantee nonnegative final charge for cores. In this sense the theorem cannot be improved using the present argument.

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References