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MULTIPLICITY OF POSITIVE SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION IN $\mathbb{R}_+^N$ WITH NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper, we study a class of semilinear elliptic equations in $\mathbb{R}_+^N$ with nonlinear boundary condition and sign-changing weight function. By means of the Lusternik-Schnirelman category, multiple positive solutions are obtained.

1. **Introduction.** In this paper, we consider the multiplicity results of positive solutions for the following semilinear elliptic equation:

$$\begin{cases} -\Delta u + u = g_\mu(x)\,|u|^{p-2}\,u & \text{in } \mathbb{R}_+^N, \\ \dfrac{\partial u}{\partial n} = f_\lambda(x)\,|u|^{q-2}\,u & \text{in } \partial\mathbb{R}_+^N, \end{cases} \qquad (E_{f_\lambda, g_\mu})$$

where $1 < q < 2 < p < 2^*$ $(2^* = \frac{2N}{N-2}$ if $N \geq 3,\ 2^* = \infty$ if $N = 2)$, the parameters $\lambda, \mu \geq 0$ and $\mathbb{R}_+^N = \left\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > 0\right\}$ is an upper half space in $\mathbb{R}^N$. We assume that $f_\lambda(x) = \lambda f_+(x) + f_-(x)$ and $g_\mu(x) = a(x) + \mu b(x)$ where the functions $f_\pm, a$ and b satisfy the following conditions:

$(D1)$ $f \in L^{q^*}(\partial\mathbb{R}_+^N)$ $(q^* = \frac{p}{p-q})$ with $f_\pm(x) = \pm\max\{\pm f(x), 0\} \not\equiv 0$ and there exists a positive number r_{f_-} such that

$$f_-(x) \geq -\widehat{c}\exp(-r_{f_-}|x|) \text{ for some } \widehat{c} > 0 \text{ and for all } x \in \partial\mathbb{R}_+^N;$$

$(D2)$ $a, b \in C(\overline{\mathbb{R}_+^N})$ and there are positive numbers r_a, r_b with $r_b < \min\left\{r_{f_-}, r_a, q\right\}$ such that

$$1 \geq a(x) \geq 1 - c_0\exp(-r_a|x|) \text{ for some } c_0 < 1 \text{ and for all } x \in \overline{\mathbb{R}_+^N}$$

and

$$b(x) \geq d_0\exp(-r_b|x|) \text{ for some } d_0 > 0 \text{ and for all } x \in \overline{\mathbb{R}_+^N};$$

$(D3)$ $b(x) \to 0$ and $a(x) \to 1$ as $|x| \to \infty$.

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The semilinear elliptic equations in bounded domains with nonlinear boundary condition has been the focus of a great deal of research in recent years. Garcia-Azorero, Peral and Rossi [15] have investigated the following equation:

$$\begin{cases} -\Delta u + u = |u|^{p-2} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega, \end{cases} \qquad (\widetilde{E}_\lambda)$$

where $1 < q < 2 < p < 2^*$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and the parameter $\lambda > 0$. They found that there exist positive numbers Λ_1, Λ_2 with $\Lambda_1 \leq \Lambda_2$ such that the equation $(\widetilde{E}_\lambda)$ admits at least two positive solutions for $\lambda \in (0, \Lambda_1)$ and no positive solution exists for $\lambda > \Lambda_2$. Also see Chipot, Shafrir and Fila [8], Flores and del Pino [13], Hu [16], Pierrotti and Terracini [20] and Terraccini [22] where equations similar to the equation $(\widetilde{E}_\lambda)$ are studied. Generalizations of the result of equation $(\widetilde{E}_\lambda)$ (involving sign-changing weight) were done by Brown and Wu [5] and Wu [25]. However, little has been done for this type problems in unbounded domains. We are only aware of the works Chipot, Chlebik, Fila and Shafrir [7] which studied existence of solutions for some related semilinear elliptic equations in $\mathbb{R}^N_+$ with nonlinear boundary condition (not involving sign-changing weight). Furthermore, we do not know of any results for semilinear elliptic equations in $\mathbb{R}^N_+$ with nonlinear boundary condition and sign-changing weight function. In this paper, we will study this issue.

Note that the sublinear boundary condition in equation (E_{f_λ,g_μ}) is homogeneous of the same degree $q-1$ and so the equation (E_{f_λ,g_μ}) is similar to the Ambrosetti, Brezis and Cerami problem [2] (a semilinear elliptic equation involving concave and convex nonlinearities). Thus, the existence of more than one nontrivial solution for the equation (E_{f_λ,g_μ}) is expected. Our main result in the paper is the following.

Theorem 1.1. *Suppose that the functions $f_\pm, a$ and b satisfy the conditions $(D1)-(D3)$. Let $\Lambda_0 = (2-q)^{2-q}(\frac{p-2}{\|f_+\|_{L^{q^*}}})^{p-2}(\frac{S_p}{p-q})^{\frac{p(2-q)}{2}}(\frac{C_p}{p-q})^{\frac{q(p-2)}{2}}$, where S_p and C_p the best Sobolev embedding and trace constants for the operators $H^1(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$ and $H^1(\mathbb{R}^N_+) \hookrightarrow L^p(\partial\mathbb{R}^N_+)$, respectively. Then*
(i) for each $\lambda > 0$ and $\mu > 0$ with $\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$, the equation (E_{f_λ,g_μ}) has at least two positive solutions;
(ii) there exist positive numbers λ_0, μ_0 with $\lambda_0^{p-2}(1 + \mu_0 \|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$ such that for $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$, the equation (E_{f_λ,g_μ}) has at least three positive solutions.

In the following sections, we proceed to prove Theorem 1.1. We use the variational methods to find the positive solutions of equation (E_{f_λ,g_μ}). Associated with the equation (E_{f_λ,g_μ}), we consider the energy functional J_{f_λ,g_μ} in $H^1(\mathbb{R}^N_+)$

$$J_{f_\lambda,g_\mu}(u) = \frac{1}{2}\|u\|^2_{H^1} - \frac{1}{q}\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q \, d\sigma - \frac{1}{p}\int_{\mathbb{R}^N_+} g_\mu |u|^p \, dx,$$

where $d\sigma$ is the measure on the boundary and $\|u\|_{H^1} = (\int_{\mathbb{R}^N_+} |\nabla u|^2 + u^2 dx)^{1/2}$ is the standard norm in $H^1(\mathbb{R}^N_+)$. It is well known that the solutions of equation (E_{f_λ,g_μ}) are the critical points of the energy functional J_{f_λ,g_μ} in $H^1(\mathbb{R}^N_+)$(see Rabinowitz [21]).

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we discussion some concentration behavior in the Nehari manifold. In Section 4, we prove Theorem 1.1.

2. **Notations and Preliminaries.** Throughout this section, we denote by S_p, C_p the best Sobolev embedding and trace constants for the operators $H^1(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+), H^1(\mathbb{R}^N_+) \hookrightarrow L^p(\partial\mathbb{R}^N_+)$, respectively. In particular,

$$(\int_{\mathbb{R}^N_+} |u|^p \, dx)^{\frac{1}{p}} \leq S_p^{\frac{-1}{2}} \|u\|_{H^1} \text{ for all } u \in H^1(\mathbb{R}^N_+) \backslash \{0\} \tag{1}$$

and

$$(\int_{\partial\mathbb{R}^N_+} |u|^p \, d\sigma)^{\frac{1}{p}} \leq C_p^{\frac{-1}{2}} \|u\|_{H^1} \text{ for all } u \in H^1(\mathbb{R}^N_+) \backslash \{0\} .$$

We define the Palais–Smale (simply (PS)–) sequences, (PS)–values, and (PS)–conditions in $H^1(\mathbb{R}^N_+)$ for J_{f_λ, g_μ} as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\text{PS})_\beta$–sequence in $H^1(\mathbb{R}^N_+)$ for J_{f_λ, g_μ} if $J_{f_\lambda, g_\mu}(u_n) = \beta + o(1)$ and $J'_{f_\lambda, g_\mu}(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N_+)$ as $n \to \infty$.
(ii) J_{f_λ, g_μ} satisfies the $(\text{PS})_\beta$–condition in $H^1(\mathbb{R}^N_+)$ if every $(\text{PS})_\beta$–sequence in $H^1(\mathbb{R}^N_+)$ for J_{f_λ, g_μ} contains a convergent subsequence.

As the energy functional J_{f_λ, g_μ} is not bounded below on $H^1(\mathbb{R}^N_+)$, it is useful to consider the functional on the Nehari manifold

$$\mathbf{N}_{f_\lambda, g_\mu} = \left\{ u \in H^1(\mathbb{R}^N_+) \backslash \{0\} \mid \left\langle J'_{f_\lambda, g_\mu}(u), u \right\rangle = 0 \right\} .$$

Thus, $u \in \mathbf{N}_{f_\lambda, g_\mu}$ if and only if

$$\|u\|^2_{H^1} - \int_{\partial\mathbb{R}^N_+} f_\lambda \, |u|^q \, d\sigma - \int_{\mathbb{R}^N_+} g_\mu \, |u|^p \, dx = 0.$$

Furthermore, we have the following results.

Lemma 2.2. *The energy functional J_{f_λ, g_μ} is coercive and bounded below on $\mathbf{N}_{f_\lambda, g_\mu}$.*

Proof. If $u \in \mathbf{N}_{f_\lambda, g_\mu}$, then, by the Hölder and Sobolev trace inequalities,

$$\begin{aligned} J_{f_\lambda, g_\mu}(u) &= \frac{p-2}{2p} \|u\|^2_{H^1} - \frac{p-q}{pq} \int_{\partial\mathbb{R}^N_+} (\lambda f_+ + f_-) \, |u|^q \, d\sigma \\ &\geq \frac{p-2}{2p} \|u\|^2_{H^1} - \lambda(\frac{p-q}{pq}) \int_{\partial\mathbb{R}^N_+} f_+ \, |u|^q \, d\sigma \\ &\geq \frac{p-2}{2p} \|u\|^2_{H^1} - \lambda(\frac{p-q}{pq}) \|f_+\|_{L^{q^*}} C_p^{-\frac{q}{2}} \|u\|^q_{H^1} . \end{aligned} \tag{2}$$

Thus, J_{f_λ, g_μ} is coercive and bounded below on $\mathbf{N}_{f_\lambda, g_\mu}$. □

The Nehari manifold $\mathbf{N}_{f_\lambda, g_\mu}$ is closely linked to the behavior of the function of the form $h_u : t \to J_{f_\lambda, g_\mu}(tu)$ for $t > 0$. Such maps are known as fibering maps and were introduced by Drábek and Pohozaev in [10] and are also discussed in Brown and Zhang [6] and Brown and Wu [5]. If $u \in H^1(\mathbb{R}^N_+)$, we have

$$\begin{aligned} h_u(t) &= \frac{t^2}{2}\|u\|^2_{H^1} - \frac{t^q}{q}\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q\, d\sigma - \frac{t^p}{p}\int_{\mathbb{R}^N_+} g_\mu |u|^p\, dx; \\ h'_u(t) &= t\|u\|^2_{H^1} - t^{q-1}\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q\, d\sigma - t^{p-1}\int_{\mathbb{R}^N_+} g_\mu |u|^p\, dx; \\ h''_u(t) &= \|u\|^2_{H^1} - (q-1)t^{q-2}\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q\, d\sigma - (p-1)t^{p-2}\int_{\mathbb{R}^N_+} g_\mu |u|^p\, dx. \end{aligned}$$

It is easy to see that

$$th'_u(t) = \|tu\|^2_{H^1} - \int_{\partial\mathbb{R}^N_+} f_\lambda |tu|^q\, d\sigma - \int_{\mathbb{R}^N_+} g_\mu |tu|^p\, dx$$

and so, for $u \in H^1(\mathbb{R}^N_+)\backslash\{0\}$ and $t > 0$, $h'_u(t) = 0$ if and only if $tu \in \mathbf{N}_{f_\lambda,g_\mu}$, i.e., positive critical points of h_u correspond to points on the Nehari manifold. In particular, $h'_u(1) = 0$ if and only if $u \in \mathbf{N}_{f_\lambda,g_\mu}$. Thus, it is natural to split $\mathbf{N}_{f_\lambda,g_\mu}$ into three parts corresponding to local minima, local maxima and points of inflection. Accordingly, we define

$$\begin{aligned} \mathbf{N}^+_{f_\lambda,g_\mu} &= \left\{u \in \mathbf{N}_{f_\lambda,g_\mu} \mid h''_u(1) > 0\right\}; \\ \mathbf{N}^0_{f_\lambda,g_\mu} &= \left\{u \in \mathbf{N}_{f_\lambda,g_\mu} \mid h''_u(1) = 0\right\}; \\ \mathbf{N}^-_{f_\lambda,g_\mu} &= \left\{u \in \mathbf{N}_{f_\lambda,g_\mu} \mid h''_u(1) < 0\right\}. \end{aligned}$$

We now derive some basic properties of $\mathbf{N}^+_{f_\lambda,g_\mu}$, $\mathbf{N}^0_{f_\lambda,g_\mu}$ and $\mathbf{N}^-_{f_\lambda,g_\mu}$.

Lemma 2.3. *Suppose that u_0 is a local minimizer for J_{f_λ,g_μ} on $\mathbf{N}_{f_\lambda,g_\mu}$ and that $u_0 \notin \mathbf{N}^0_{f_\lambda,g_\mu}$. Then $J'_{f_\lambda,g_\mu}(u_0) = 0$ in $H^{-1}(\mathbb{R}^N_+)$.*

Proof. The proof is essentially the same as that in Brown and Zhang [6, Theorem 2.3] (or see Binding, Drábek and Huang [4]). □

For each $u \in \mathbf{N}_{f_\lambda,g_\mu}$ we have

$$\begin{aligned} h''_u(1) &= \|u\|^2_{H^1} - (q-1)\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q\, d\sigma - (p-1)\int_{\mathbb{R}^N_+} g_\mu |u|^p\, dx \\ &= (2-p)\|u\|^2_{H^1} - (q-p)\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q\, d\sigma \qquad (3) \\ &= (2-q)\|u\|^2_{H^1} - (p-q)\int_{\mathbb{R}^N_+} g_\mu |u|^p\, dx. \qquad (4) \end{aligned}$$

Then we have the following result.

Lemma 2.4. (*i*) *For any $u \in \mathbf{N}^+_{f_\lambda,g_\mu} \cup \mathbf{N}^0_{f_\lambda,g_\mu}$, we have $\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q\, d\sigma > 0$.*
(*ii*) *for any $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$, we have $\int_{\mathbb{R}^N_+} g_\mu |u|^p\, dx > 0$.*

Proof. The results now follows immediately from (3) and (4). □

Let

$$\Lambda_0 = (2-q)^{2-q}\left(\frac{p-2}{\|f_+\|_{L^{q^*}}}\right)^{p-2}\left(\frac{S_p}{p-q}\right)^{\frac{p(2-q)}{2}}\left(\frac{C_p}{p-q}\right)^{\frac{q(p-2)}{2}}.$$

Then we have the following results.

Lemma 2.5. *For each* $\lambda > 0$ *and* $\mu \geq 0$ *with* $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$, *we have* $\mathbf{N}^0_{f_\lambda,g_\mu} = \emptyset$.

Proof. Suppose the contrary. Then there exist $\lambda > 0$ and $\mu \geq 0$ with

$$\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$$

such that $\mathbf{N}^0_{f_\lambda,g_\mu} \neq \emptyset$. Then for $u \in \mathbf{N}^0_{f_\lambda,g_\mu}$, by (3) and the Hölder and Sobolev trace inequalities, we have

$$\|u\|^2_{H^1} = \frac{p-q}{p-2}\int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q \, d\sigma \leq \lambda C_p^{\frac{-q}{2}} \frac{p-q}{p-2}\|f_+\|_{L^{q^*}} \|u\|^q_{H^1}$$

and so

$$\|u\|^2_{H^1} \leq C_p^{\frac{q}{q-2}}\left[\lambda\|f_+\|_{L^{q^*}}\frac{p-q}{p-2}\right]^{\frac{2}{2-q}}.$$

Similarly, using (4) and the Sobolev inequality we have

$$\frac{2-q}{p-q}\|u\|^2_{H^1} = \int_{\mathbb{R}^N_+}[a+\mu b]\,|u|^p \, dx \leq (1+\mu\|b\|_\infty) S_p^{\frac{-p}{2}}\|u\|^p_{H^1},$$

which implies

$$\|u\|^2_{H^1} \geq S_p^{\frac{p}{p-2}}\left[\frac{2-q}{(1+\mu\|b\|_\infty)(p-q)}\right]^{\frac{2}{p-2}} \quad \text{for all } \mu \geq 0.$$

Hence, we must have

$$\begin{aligned} & \lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} \\ \geq \quad & (2-q)^{2-q}\left(\frac{p-2}{\|f_+\|_{L^{q^*}}}\right)^{p-2}\left(\frac{S_p}{p-q}\right)^{\frac{p(2-q)}{2}}\left(\frac{C_p}{p-q}\right)^{\frac{q(p-2)}{2}} = \Lambda_0 \end{aligned}$$

which is a contradiction. This completes the proof. □

In order to get a better understanding of the Nehari manifold and fibering maps, we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_u(t) = t^{2-q}\|u\|^2_{H^1} - t^{p-q}\int_{\mathbb{R}^N_+} g_\mu |u|^p \, dx \text{ for } t > 0.$$

Clearly $tu \in \mathbf{N}_{f_\lambda,g_\mu}$ if and only if $m_u(t) = \int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q \, d\sigma$. Moreover,

$$m_u'(t) = (2-q)t^{1-q}\|u\|^2_{H^1} - (p-q)t^{p-q-1}\int_{\mathbb{R}^N_+} g_\mu |u|^p \, dx \tag{5}$$

and so it is easy to see that, if $tu \in \mathbf{N}_{f_\lambda,g_\mu}$, then $t^{q-1}m_u'(t) = h_u''(t)$. Hence $tu \in \mathbf{N}^+_{f_\lambda,g_\mu}$(or $\mathbf{N}^-_{f_\lambda,g_\mu}$) if and only if $m_u'(t) > 0$(or < 0).

Suppose $u \in H^1(\mathbb{R}^N_+)\backslash\{0\}$. Then by (5), m_u has a unique critical point at $t = t_{\max,\mu}(u)$ where

$$t_{\max,\mu}(u) = \left(\frac{(2-q)\|u\|^2_{H^1}}{(p-q)\int_{\mathbb{R}^N_+} g_\mu |u|^p \, dx}\right)^{\frac{1}{p-2}} > 0 \tag{6}$$

and clearly m_u is strictly increasing on $(0, t_{\max,\mu}(u))$ and strictly decreasing on $(t_{\max,\mu}(u), \infty)$ with $\lim_{t\to\infty} m_u(t) = -\infty$. Moreover, if $\lambda^{p-2}(1+\mu\left\|b\right\|_\infty)^{2-q} < \Lambda_0$, then

$$\begin{aligned}
m_u(t_{\max,\mu}(u)) &= \left[(\frac{2-q}{p-q})^{\frac{2-q}{p-2}} - (\frac{2-q}{p-q})^{\frac{p-q}{p-2}}\right] \frac{\left\|u\right\|_{H^1}^{\frac{2(p-q)}{p-2}}}{(\int_{\mathbb{R}^N_+} g_\mu \left|u\right|^p dx)^{\frac{2-q}{p-2}}} \\
&= \left\|u\right\|_{H^1}^q (\frac{p-2}{p-q})(\frac{2-q}{p-q})^{\frac{2-q}{p-2}} (\frac{\left\|u\right\|_{H^1}^p}{\int_{\mathbb{R}^N_+} g_\mu \left|u\right|^p dx})^{\frac{2-q}{p-2}} \\
&\geq \frac{\Lambda_0}{\lambda^{p-2}(1+\mu\left\|b\right\|_\infty)^{2-q}} \int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma \\
&> \int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma.
\end{aligned}$$

Thus, we have the following lemma.

Lemma 2.6. *For each $u \in H^1(\mathbb{R}^N_+)\setminus\{0\}$ we have the following.*
(i) If $\int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma \leq 0$, then there is a unique $t^- = t^-(u) > t_{\max,\mu}(u)$ such that $t^-u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ and m_u is increasing on $(0,t^-)$ and decreasing on (t^-,∞). Moreover,

$$J_{f_\lambda,g_\mu}(t^-u) = \sup_{t\geq 0} J_{f_\lambda,g_\mu}(tu). \tag{7}$$

(ii) If $\int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma > 0$, then there are unique $0 < t^+ = t^+(u) < t_{\max,\mu}(u) < t^-$ such that $t^+u \in \mathbf{N}^+_{f_\lambda,g_\mu}$, $t^-u \in \mathbf{N}^-_{f_\lambda,g_\mu}$, m_u is decreasing on $(0,t^+)$, increasing on (t^+,t^-) and decreasing on (t^-,∞). Moreover,

$$J_{f_\lambda,g_\mu}(t^+u) = \inf_{0\leq t\leq t_{\max,\mu}(u)} J_{f_\lambda,g_\mu}(tu); J_{f_\lambda,g_\mu}(t^-u) = \sup_{t\geq t^+} J_{f_\lambda,g_\mu}(tu). \tag{8}$$

(iii) $t^-(u)$ is a continuous function for $u \in H^1(\mathbb{R}^N_+)$;
(iv) $\mathbf{N}^-_{f_\lambda,g_\mu} = \left\{u \in H^1(\mathbb{R}^N_+) \mid \frac{1}{\left\|u\right\|_{H^1}} t^-(\frac{u}{\left\|u\right\|_{H^1}}) = 1\right\}$.

Proof. Fix $u \in H^1(\mathbb{R}^N_+)\setminus\{0\}$.
(i) Suppose $\int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma \leq 0$. Then $m_u(t) = \int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma$ has a unique solution $t^- > t_{\max,\mu}(u)$ and $m_u'(t^-) < 0$. Hence, by $t^{q-1}m_u'(u) = h_u''(t)$, h_u has a unique critical point at $t = t^-$ and $h_u''(t^-) < 0$. Thus, $t^-u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ and (7) holds.
(ii) Suppose $\int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma > 0$. Since $m_u(t_{\max,\mu}(u)) > \int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma$, the equation $m_u(t) = \int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma$ has exactly two solutions $t^+ < t_{\max,\mu}(u) < t^-$ such that $m_u'(t^+) > 0$ and $m_u'(t^-) < 0$. Hence, there are exactly two multiples of u lying in $\mathbf{N}_{f_\lambda,g_\mu}$, that is, $t^+u \in \mathbf{N}^+_{f_\lambda,g_\mu}$ and $t^-u \in \mathbf{N}^-_{f_\lambda,g_\mu}$. Thus, by $t^{q-1}m_u'(u) = h_u''(t)$, h_u has critical points at $t = t^+$ and $t = t^-$ with $h_u''(t^+) > 0$ and $h_u''(t^-) < 0$. Thus, h_u is decreasing on $(0,t^+)$, increasing on (t^+,t^-) and decreasing on (t^+,∞). Therefore, (8) must hold.
(iii) By the uniqueness of $t^-(u)$ and the extremal property of $t^-(u)$, we have $t^-(u)$ is a continuous function for $u \in H^1(\mathbb{R}^N_+)\setminus\{0\}$.
(iv) For $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$. Let $v = \frac{u}{\left\|u\right\|_{H^1}}$. By parts $(i),(ii)$, there is a unique $t^-(v) > 0$ such that $t^-(v)v \in \mathbf{N}^-_{f_\lambda,g_\mu}$ or $t^-(\frac{u}{\left\|u\right\|_{H^1}})\frac{1}{\left\|u\right\|_{H^1}}u \in \mathbf{N}^-_{f_\lambda,g_\mu}$. Since $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$, we have

$t^-(\frac{u}{\|u\|_{H^1}})\frac{1}{\|u\|_{H^1}} = 1$, and this implies

$$\mathbf{N}^-_{f_\lambda,g_\mu} \subset \left\{ u \in H^1(\mathbb{R}^N_+) \mid \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1 \right\}.$$

Conversely, let $u \in H^1(\mathbb{R}^N_+)$ such that $\frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1$. Then

$$t^-(\frac{u}{\|u\|_{H^1}})\frac{u}{\|u\|_{H^1}} \in \mathbf{N}^-_{f_\lambda,g_\mu}.$$

Thus,

$$\mathbf{N}^-_{f_\lambda,g_\mu} = \left\{ u \in H^1(\mathbb{R}^N_+) \mid \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1 \right\}.$$

This completes the proof. □

Remark 1. (i) If $\lambda = 0$, then by Lemma 2.6 (i), $\mathbf{N}^+_{f_0,g_\mu} = \emptyset$, and so $\mathbf{N}_{f_0,g_\mu} = \mathbf{N}^-_{f_0,g_\mu}$ for all $\mu \geq 0$.
(ii) If $\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < \Lambda_0$, then, by (3), for each $u \in \mathbf{N}^+_{f_\lambda,g_\mu}$ we have

$$\begin{aligned} \|u\|^2_{H^1} &< \frac{p-q}{p-2} \int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q \, d\sigma \leq \lambda \frac{p-q}{p-2} \int_{\partial\mathbb{R}^N_+} f^+ |u|^q \, d\sigma \\ &\leq \Lambda_0^{1/(p-2)} C_p^{\frac{-q}{2}} \frac{p-q}{p-2} \|f_+\|_{L^{q^*}} \|u\|^q_{H^1}, \end{aligned}$$

and so

$$\|u\|_{H^1} \leq (\Lambda_0^{1/(p-2)} C_p^{\frac{-q}{2}} \frac{p-q}{p-2} \|f_+\|_{L^{q^*}})^{1/(2-q)} \text{ for all } u \in \mathbf{N}^+_{f_\lambda,g_\mu}. \tag{9}$$

We remark that it follows from Lemma 2.5 that $\mathbf{N}_{f_\lambda,g_\mu} = \mathbf{N}^+_{f_\lambda,g_\mu} \cup \mathbf{N}^-_{f_\lambda,g_\mu}$ for all $\lambda > 0$ and $\mu \geq 0$ with $\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < \Lambda_0$. Furthermore, by Lemma 2.6 it follows that $\mathbf{N}^+_{f_\lambda,g_\mu}$ and $\mathbf{N}^-_{f_\lambda,g_\mu}$ are non-empty and by Lemma 2.2, we may define

$$\alpha^+_{f_\lambda,g_\mu} = \inf_{u \in \mathbf{N}^+_{f_\lambda,g_\mu}} J_{f_\lambda,g_\mu}(u) \text{ and } \alpha^-_{f_\lambda,g_\mu} = \inf_{u \in \mathbf{N}^-_{f_\lambda,g_\mu}} J_{f_\lambda,g_\mu}(u).$$

Then we have the following results.

Theorem 2.7. *We have the following:*
(i) $\alpha^+_{f_\lambda,g_\mu} < 0$ *for all* $\lambda > 0$ *and* $\mu \geq 0$ *with* $\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < \Lambda_0$.
(ii) *If* $\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$, *then* $\alpha^-_{f_\lambda,g_\mu} > c_0$ *for some* $c_0 > 0$.
In particular, for each $\lambda > 0$ *and* $\mu \geq 0$ *with* $\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$, *we have* $\alpha^+_{f_\lambda,g_\mu} = \inf_{u \in \mathbf{N}_{f_\lambda,g_\mu}} J_{f_\lambda,g_\mu}(u)$.

Proof. (i) Let $u \in \mathbf{N}^+_{f_\lambda,g_\mu}$. Then, by (3),

$$\|u\|^2_{H^1} < \frac{p-q}{p-2} \int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q \, d\sigma.$$

Hence, by Lemma 2.4

$$\begin{aligned} J_{f_\lambda,g_\mu}(u) &= \frac{p-2}{2p} \|u\|^2_{H^1} - \frac{p-q}{pq} \int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q \, d\sigma \\ &< -\frac{(p-q)(2-q)}{2pq} \int_{\partial\mathbb{R}^N_+} f_\lambda |u|^q \, d\sigma < 0 \end{aligned}$$

and so $\alpha^+_{f_\lambda,g_\mu} < 0$.
(ii) Let $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$. Then, by (4) and the Sobolev inequality,

$$\frac{2-q}{p-q}\|u\|^2_{H^1} < \int_{\mathbb{R}^N_+} g_\mu |u|^p\,dx \le (1+\mu\|b\|_\infty)S_p^{\frac{-p}{2}}\|u\|^p_{H^1},$$

which implies

$$\|u\|_{H^1} > \left(\frac{(2-q)S_p^{\frac{p}{2}}}{(1+\mu\|b\|_\infty)(p-q)}\right)^{\frac{1}{p-2}} \text{ for all } u \in \mathbf{N}^-_{f_\lambda,g_\mu}. \tag{10}$$

By (2) and (10), we have

$$\begin{aligned} J_{f_\lambda,g_\mu}(u) &> \left(\frac{(2-q)S_p^{\frac{p}{2}}}{(1+\mu\|b\|_\infty)(p-q)}\right)^{\frac{q}{p-2}} \\ &\quad\cdot\left(\frac{p-2}{2p}\left(\frac{(2-q)S_p^{\frac{p}{2}}}{(1+\mu\|b\|_\infty)(p-q)}\right)^{\frac{2-q}{p-2}} - \lambda\|f_+\|_{L^{q^*}} C_p^{-\frac{q}{2}}\left(\frac{p-q}{pq}\right)\right). \end{aligned}$$

Thus, if $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$, then

$$\alpha^-_{f_\lambda,g_\mu} > c_0 \text{ for some } c_0 > 0.$$

This completes the proof. □

Now, we consider the following elliptic problems:

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{E^∞}$$

and

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u = 0. \end{cases} \tag{$\widetilde{E}^\infty$}$$

Associated with the equations (E^∞) and $(\widetilde{E}^\infty)$, we consider the energy functionals J^∞ in $H^1(\mathbb{R}^N_+)$ and $\widetilde{J}^\infty$ in $H^1(\mathbb{R}^N)$

$$J^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla u|^2 + u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N_+} |u|^p\,dx$$

and

$$\widetilde{J}^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx,$$

respectively. Consider minimizing problems:

$$\inf_{u\in\mathbf{N}^\infty} J^\infty(u) = \alpha^\infty \text{ and } \inf_{u\in\widetilde{\mathbf{N}}^\infty} \widetilde{J}^\infty(u) = \widetilde{\alpha}^\infty$$

where

$$\mathbf{N}^\infty = \left\{u \in H^1(\mathbb{R}^N_+)\backslash\{0\} \mid \langle (J^\infty)'(u), u\rangle = 0\right\}$$

and

$$\widetilde{\mathbf{N}}^\infty = \left\{u \in H^1(\mathbb{R}^N)\backslash\{0\} \mid \left\langle (\widetilde{J}^\infty)'(u), u\right\rangle = 0\right\}.$$

It is known that equation (E^∞) has a least energy positive solution $w(x)$ such that $J^\infty(w) = \alpha^\infty = \widetilde{\alpha}^\infty/2$ and $w(0) = \max_{x\in\partial\mathbb{R}^N_+} w(x)$ (see [9, 12, 17]). We observe that solution $w(x)$, we can construct a solution $\widetilde{w}(x)$ of equation $(\widetilde{E}^\infty)$ by reflection with respect to $\partial\mathbb{R}^N_+$. Then we have the following proposition provides a precise description for the (PS)–sequence of J_{f_λ,g_μ}.

Proposition 1. *If $\{u_n\}$ is a $(PS)_\beta$–sequence in $H^1(\mathbb{R}^N_+)$ for J_{f_λ,g_μ} with $\beta < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$, then there exist a subsequence $\{u_n\}$ and a non-zero u_0 in $H^1(\mathbb{R}^N_+)$ such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^N_+)$ and $J_{f_\lambda,g_\mu}(u_0) = \beta$. Moreover, u_0 is a solution of equation (E_{f_λ,g_μ}).*

Proof. Similarly to the argument in [25, Proposition 4.6] (or see Adachi and Tanaka [3, Proposition 3.1]). $\square$

Then we can show that the following result.

Theorem 2.8. *For each $\lambda > 0$ and $\mu \geq 0$ with $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$, the equation (E_{f_λ,g_μ}) has a positive solution $u^+_{\lambda,\mu} \in \mathbf{N}^+_{f_\lambda,g_\mu}$ such that $J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) = \alpha_{f_\lambda,g_\mu}$.*

Proof. By the Ekeland variational principle [11] (or see Wu [25, Proposition 3.3]), there exist $\{u^+_n\} \subset \mathbf{N}^+_{f_\lambda,g_\mu}$ such that they are $(\mathrm{PS})_{\alpha^+_{f_\lambda,g_\mu}}$–sequence for J_{f_λ,g_μ}. Then, by Theorem 2.7 and Proposition 1, there exist a subsequences $\{u^+_n\}$ and $u^+_{\lambda,\mu} \in \mathbf{N}^+_{f_\lambda,g_\mu}$ a non-zero solution of equation (E_{f_λ,g_μ}) such that $u^+_n \to u^+_{\lambda,\mu}$ strongly in $H^1(\mathbb{R}^N_+)$ and $J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) = \alpha^+_{f_\lambda,g_\mu}$. Since $J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) = J_{f_\lambda,g_\mu}(\left|u^+_{\lambda,\mu}\right|)$ and $\left|u^+_{\lambda,\mu}\right| \in \mathbf{N}^+_{f_\lambda,g_\mu}$, by Lemma 2.3 and the maximum principle, we may assume that $u^+_{\lambda,\mu}$ is a positive solutions of equation (E_{f_λ,g_μ}). $\square$

We need the following lemmas.

Lemma 2.9. *We have*

$$\inf_{u\in\mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u) = \inf_{u\in\mathbf{N}^\infty} J^\infty(u) = \alpha^\infty.$$

Furthermore, the equation (E_{f_0,g_0}) does not admit any solution u_0 such that

$$J_{f_0,g_0}(u_0) = \inf_{u\in\mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u).$$

Proof. Let $w(x)$ be a least energy solution of equation (E^∞) and let $w_l(x) = w(x+le)$, where $l \in \mathbb{R}$ and $e \in \mathbb{S} = \left\{x \in \partial\mathbb{R}^N_+ \mid |x| = 1\right\}$. Then, by Lemma 2.6, there is a unique $t^-(w_l) > (\frac{2-q}{p-q})^{1/(p-2)}$ such that $t^-(w_l)w_l \in \mathbf{N}_{f_0,g_0}$ for all $l > 0$, that is

$$\left\|t^-(w_l)w_l\right\|^2_{H^1} = \int_{\partial\mathbb{R}^N_+} f_- \left|t^-(w_l)w_l\right|^q d\sigma + \int_{\mathbb{R}^N_+} g_0 \left|t^-(w_l)w_l\right|^p dx.$$

Since

$$\begin{aligned} \int_{\partial\mathbb{R}^N_+} f_- |w_l|^q \, d\sigma \quad &\to \quad 0 \text{ as } l \to \infty, \\ \int_{\mathbb{R}^N_+} (1-g_0) |w_l|^p \, dx \quad &\to \quad 0 \text{ as } l \to \infty \end{aligned}$$

and

$$\|w_l\|^2_{H^1} = \int_{\mathbb{R}^N_+} |w_l|^p \, dx = \frac{2p}{p-2}\alpha^\infty \text{ for all } l \geq 0,$$

we have $t^-(w_l) \to 1$ as $l \to \infty$. Thus,

$$\lim_{l\to\infty} J_{f_0,g_0}(t^-(w_l)w_l) = \lim_{l\to\infty} J^\infty(t^-(w_l)w_l) = \alpha^\infty.$$

Then

$$\inf_{u\in \mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u) \leq \inf_{u\in \mathbf{N}^\infty} J^\infty(u) = \alpha^\infty.$$

Let $u \in \mathbf{N}_{f_0,g_0}$. Then, by Lemma 2.6 (i), $J_{f_0,g_0}(u) = \sup_{t\geq 0} J_{f_0,g_0}(tu)$. Moreover, there is a unique $t^\infty > 0$ such that $t^\infty u \in \mathbf{N}^\infty$. Thus,

$$J_{f_0,g_0}(u) \geq J_{f_0,g_0}(t^\infty u) \geq J^\infty(t^\infty u) \geq \alpha^\infty$$

and so $\inf_{u\in \mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u) \geq \alpha^\infty$. Therefore,

$$\inf_{u\in \mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u) = \inf_{u\in \mathbf{N}^\infty} J^\infty(u) = \alpha^\infty.$$

Next, we will show that equation (E_{f_0,g_0}) does not admit any solution u_0 such that $J_{f_0,g_0}(u_0) = \inf_{u\in \mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u)$. Suppose the contrary. Then we can assume that there exists $u_0 \in \mathbf{N}_{f_0,g_0}$ such that $J_{f_0,g_0}(u_0) = \inf_{u\in \mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u)$. Then, by Lemma 2.6 (i), $J_{f_0,g_0}(u_0) = \sup_{t\geq 0} J_{f_0,g_0}(tu_0)$. Moreover, there is a unique $t_{u_0} > 0$ such that $t_{u_0}u_0 \in \mathbf{N}^\infty$. Thus,

$$\begin{aligned}\alpha^\infty &= \inf_{u\in \mathbf{N}_{f_0,g_0}} J_{f_0,g_0}(u) = J_{f_0,g_0}(u_0) \geq J_{f_0,g_0}(t_{u_0}u_0) \\ &\geq J^\infty(t_{u_0}u_0) - \frac{t_{u_0}^q}{q}\int_{\partial\mathbb{R}^N_+} f_-\, |u_0|^q\, d\sigma \geq \alpha^\infty - \frac{t_{u_0}^q}{q}\int_{\partial\mathbb{R}^N_+} f_-\, |u_0|^q\, d\sigma.\end{aligned}$$

This implies $\int_{\partial\mathbb{R}^N_+} f_-\, |u_0|^q\, d\sigma = 0$ and so $u_0 \equiv 0$ in $\left\{x \in \partial\mathbb{R}^N_+ \mid f_-(x) \neq 0\right\}$, from the condition $(D1)$. Therefore,

$$\alpha^\infty = \inf_{u\in \mathbf{N}^\infty} J^\infty(u) = J^\infty(t_{u_0}u_0).$$

Since $|t_{u_0}u_0| \in \mathbf{N}^\infty$ and $J^\infty(|t_{u_0}u_0|) = J^\infty(t_{u_0}u_0) = \alpha^\infty$, by Willem [24, Theorem 4.3] and the maximum principle, we can assume that $t_{u_0}u_0$ is a positive solution of (E^∞). This contradicts

$$u_0 \equiv 0 \text{ in } \left\{x \in \partial\mathbb{R}^N_+ \mid f_-(x) \neq 0\right\}.$$

This completes the proof. $\square$

Lemma 2.10. *Suppose that* $\{u_n\}$ *is a minimizing sequence in* $\mathbf{N}_{f_0,g_0}$ *for* J_{f_0,g_0}. *Then*

(i) $\int_{\partial\mathbb{R}^N_+} f_-\, |u_n|^q\, d\sigma = o(1)$;

(ii) $\int_{\mathbb{R}^N_+} (1-g_0)\, |u_n|^p\, dx = o(1)$.

Furthermore, $\{u_n\}$ *is a* $(PS)_{\alpha^\infty}$*–sequence for* J^∞ *in* $H^1(\mathbb{R}^N_+)$.

Proof. For each n, there is a unique $t_n > 0$ such that $t_nu_n \in \mathbf{N}^\infty$, that is

$$t_n^2\, \|u_n\|^2_{H^1} = t_n^p \int_{\mathbb{R}^N_+} |u_n|^p\, dx.$$

Then, by Lemma 2.6 (i),

$$\begin{aligned}J_{f_0,g_0}(u_n) &\geq J_{f_0,g_0}(t_nu_n) \\ &= J^\infty(t_nu_n) + \frac{t_n^p}{p}\int_{\mathbb{R}^N_+}(1-g_0)\, |u_n|^p\, dx - \frac{t_n^q}{q}\int_{\partial\mathbb{R}^N_+} f_-\, |u_n|^q\, d\sigma \\ &\geq \alpha^\infty + \frac{t_n^p}{p}\int_{\mathbb{R}^N_+}(1-g_0)\, |u_n|^p\, dx - \frac{t_n^q}{q}\int_{\partial\mathbb{R}^N_+} f_-\, |u_n|^q\, d\sigma.\end{aligned}$$

Since $J_{f_0,g_0}(u_n) = \alpha^\infty + o(1)$ from Lemma 2.9, we have

$$\frac{t_n^q}{q}\int_{\partial\mathbb{R}^N_+} f_-\,|u_n|^q\,d\sigma = o(1)$$

and

$$\frac{t_n^p}{p}\int_{\mathbb{R}^N_+}(1-g_0)\,|u_n|^p\,dx = o(1).$$

We will show that there exists $c_0 > 0$ such that $t_n > c_0$ for all n. Suppose the contrary. Then we may assume $t_n \to 0$ as $n \to \infty$. Since $J_{f_0,g_0}(u_n) = \alpha^\infty + o(1)$, by Lemma 2.2, $\|u_n\|_{H^1}$ is uniformly bounded and so $\|t_n u_n\|_{H^1} \to 0$ or $J^\infty(t_n u_n) \to 0$ and this contradicts $J^\infty(t_n u_n) \geq \alpha^\infty > 0$. Thus,

$$\int_{\partial\mathbb{R}^N_+} f_-\,|u_n|^q\,d\sigma = o(1)$$

and

$$\int_{\mathbb{R}^N_+}(1-g_0)\,|u_n|^p\,dx = o(1),$$

this implies

$$\|u_n\|^2_{H^1} = \int_{\mathbb{R}^N_+}|u_n|^p\,dx + o(1)$$

and

$$J^\infty(u_n) = \alpha^\infty + o(1).$$

Moreover, by Wang and Wu [23, Lemma 7], we have $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$–sequence for J^∞ in $H^1(\mathbb{R}^N_+)$. $\square$

Let $P : \overline{\mathbb{R}^N_+} \to \mathbb{R}^{N-1}$ be a projection defined by $P(x', x_N) = x'$ for $(x', x_N) \in \overline{\mathbb{R}^N_+}$. Then we have the following result.

Lemma 2.11. *There exists $d_0 > 0$ such that if $u \in \mathbf{N}_{f_0,g_0}$ with $J_{f_0,g_0}(u) \leq \alpha^\infty + d_0$, then*

$$P\Big(\int_{\mathbb{R}^N_+}\frac{x}{|x|}(|\nabla u|^2 + u^2)dx\Big) \neq 0.$$

Proof. Suppose the contrary. Then there exists sequence $\{u_n\} \subset \mathbf{N}_{f_0,g_0}$ such that $J_{f_0,g_0}(u) = \alpha^\infty + o(1)$ and

$$P\Big(\int_{\mathbb{R}^N_+}\frac{x}{|x|}(|\nabla u_n|^2 + u_n^2)dx\Big) = 0.$$

Moreover, by Lemma 2.10, we have $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$–sequence in $H^1(\mathbb{R}^N_+)$ for J^∞. It follows from Lemma 2.2 that there exist a subsequence $\{u_n\}$ and $u_0 \in H^1(\mathbb{R}^N_+)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N_+)$. By the concentration–compactness principle (see Lions [18, 19] or del Pino and Flores [9, proof of proposition 2.1]) and $\alpha^\infty = \widetilde{\alpha}^\infty/2$, there exist a sequence $\{x_n\} \subset \partial\mathbb{R}^N_+$, and a positive solution $w_0 \in H^1(\mathbb{R}^N_+)$ of equation (E^∞) such that

$$\|u_n(x) - w_0(x - x_n)\|_{H^1} \to 0 \text{ as } n \to \infty. \tag{11}$$

Now we will show that $|x_n| \to \infty$ as $n \to \infty$. Suppose the contrary. Then we may assume that $\{x_n\}$ is bounded and $x_n \to x_0$ for some $x_0 \in \partial\mathbb{R}^N_+$. Thus, by (11),

$$\begin{aligned}\int_{\partial\mathbb{R}^N_+} f_- |u_n|^q \, d\sigma &= \int_{\partial\mathbb{R}^N_+} f_-(x) |w_0(x-x_n)|^q \, d\sigma + o(1) \\ &= \int_{\partial\mathbb{R}^N_+} f_-(x+x_0) |w_0(x)|^q \, d\sigma + o(1),\end{aligned}$$

this contradicts the result of Lemma 2.10: $\int_{\partial\mathbb{R}^N_+} f_- |u_n|^q \, d\sigma = o(1)$. Hence we may assume $\frac{x_n}{|x_n|} \to e$ as $n \to \infty$, where $e \in \mathbb{S} = \left\{x \in \partial\mathbb{R}^N_+ \mid |x| = 1\right\}$. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}0 &= P\Big(\int_{\mathbb{R}^N_+} \frac{x}{|x|}(|\nabla u_n|^2 + u_n^2)dx\Big) \\ &= P\Big(\int_{\mathbb{R}^N_+} \frac{x+x_n}{|x+x_n|}(|\nabla w_0|^2 + w_0^2)dx\Big) + o(1) \\ &= \frac{2p}{p-2}\alpha^\infty P(e) + o(1),\end{aligned}$$

which is a contradiction. This completes the proof. □

3. **Concentration Behavior.** First, let $w(x)$ be a least energy positive solution of equation (E^∞) such that $J^\infty(w) = \alpha^\infty$. Then by Gidas, Ni and Nirenberg [14] and Kwong [17], for any $\varepsilon > 0$, there exist positive numbers A_ε and B_0 such that

$$A_\varepsilon \exp(-(1+\varepsilon)|x|) \le w(x) \le B_0 \exp(-|x|) \text{ for all } x \in \overline{\mathbb{R}^N_+}. \tag{12}$$

Let

$$w_l(x) = w(x + le), \text{ for } l \in \mathbb{R} \text{ and } e \in \mathbb{S},$$

where $\mathbb{S} = \left\{x \in \partial\mathbb{R}^N_+ \mid |x| = 1\right\}$. Clearly, w_l is also a least energy positive solution of equation (E^∞) for all $l \ge 0$, and $\int_{\partial\mathbb{R}^N_+} f_\lambda |w_l|^q \, d\sigma = 0$ as $l \to \infty$. Then we have the following result.

Proposition 2. *For each $\lambda > 0$ and $\mu > 0$ with $\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < \Lambda_0$, we have*

$$\alpha^-_{f_\lambda, g_\mu} < \alpha^+_{f_\lambda, g_\mu} + \alpha^\infty.$$

Proof. Let $u^+_{\lambda,\mu}$ be a positive solution of equation (E_{f_λ,g_μ}) as in Theorem 2.8. Then

$$\begin{aligned}
&J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}+tw_l)\\
\leq\ & \alpha^+_{f_\lambda,g_\mu}+J^\infty(tw)-\frac{\mu}{p}\int_{\mathbb{R}^N_+}bt^pw_l^p dx+\frac{1}{p}\int_{\mathbb{R}^N_+}(1-g_0)t^pw_l^p dx\\
&-\int_{\partial\mathbb{R}^N_+}(\lambda f_++f_-)\left\{\int_0^{tw_l}(u^+_{\lambda,\mu}+\eta)^{q-1}-(u^+_{\lambda,\mu})^{q-1}d\eta\right\}d\sigma\\
&-\frac{1}{p}\int_{\mathbb{R}^N_+}\left[(u^+_{\lambda,\mu}+tw_l)^p-(u^+_{\lambda,\mu})^p-t^pw_l^p-p(u^+_{\lambda,\mu})^{p-1}tw_l\right]dx\\
\leq\ & \alpha^+_{f_\lambda,g_\mu}+\alpha^\infty-\frac{\mu t^p}{p}\int_{\mathbb{R}^N_+}bw_l^p dx\\
&+\frac{t^p}{p}\int_{\mathbb{R}^N_+}(1-g_0)w_l^p dx+\frac{t^q}{q}\int_{\partial\mathbb{R}^N_+}|f_-|\,w_l^q d\sigma\\
&-\frac{1}{p}\int_{\mathbb{R}^N_+}\left[(u^+_{\lambda,\mu}+tw_l)^p-(u^+_{\lambda,\mu})^p-t^pw_l^p-p(u^+_{\lambda,\mu})^{p-1}tw_l\right]dx. \qquad (13)
\end{aligned}$$

We remark that

$$(u+v)^p-u^p-v^p-pu^{p-1}v\geq 0 \text{ for all } (u,v)\in[0,\infty)\times[0,\infty),$$

this implies

$$\int_{\mathbb{R}^N_+}\left[(u^+_{\lambda,\mu}+tw_l)^p-(u^+_{\lambda,\mu})^p-t^pw_l^p-p(u^+_{\lambda,\mu})^{p-1}tw_l\right]dx\geq 0. \qquad (14)$$

Thus, by (13) and (14), we have

$$\begin{aligned}
J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}+tw_l)\ \leq\ & \alpha^+_{f_\lambda,g_\mu}+\alpha^\infty-\frac{\mu t^p}{p}\int_{\mathbb{R}^N_+}bw_l^p dx\\
&+\frac{t^p}{p}\int_{\mathbb{R}^N_+}(1-g_0)w_l^p dx+\frac{t^q}{q}\int_{\partial\mathbb{R}^N_+}|f_-|\,w_l^q d\sigma. \qquad (15)
\end{aligned}$$

Since

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}+tw_l)\to J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu})=\alpha^+_{f_\lambda,g_\mu}<0 \text{ as } t\to 0$$

and

$$\begin{aligned}
&J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}+tw_l)\\
\leq\ & \left\|u^+_{\lambda,\mu}\right\|^2_{H^1}+t^2\left\|w_l\right\|^2_{H^1}+\frac{1}{q}\int_{\partial\mathbb{R}^N_+}|f_-|\left|u^+_{\lambda,\mu}+tw_l\right|^q d\sigma\\
&-\frac{t^p\min_{x\in\overline{\mathbb{R}^N_+}}a(x)}{p}\int_{\mathbb{R}^N_+}|w_l|^p dx\\
\leq\ & \left\|u^+_{\lambda,\mu}\right\|^2_{H^1}+t^2\left\|w\right\|^2_{H^1}+\frac{2^{q-1}}{q}\left\|f_-\right\|_{L^{q^*}}(\left\|u^+_{\lambda,\mu}\right\|^q_{L^p}+t^q\left\|w\right\|^q_{L^p})\\
&-\frac{t^p\min_{x\in\overline{\mathbb{R}^N_+}}a(x)}{p}\int_{\mathbb{R}^N_+}|w|^p dx\\
\to\ & -\infty \text{ as } t\to\infty,
\end{aligned}$$

we can easily find $0 < t_1 < t_2$ such that

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty \text{ for all } t \in [0,t_1] \cup [t_2,\infty). \tag{16}$$

Thus, we only need to show that there exists $l_0 > 0$ such that for $l > l_0$,

$$\sup_{t_1 \leq t \leq t_2} J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty. \tag{17}$$

From the condition $(D2)$ and (12), we have

$$\begin{aligned} \int_{\mathbb{R}^N_+} (1-g_0)t^p w_l^p dx &\leq c_0 \int_{\mathbb{R}^N_+} \exp(-r_a |x|) B_0^p \exp(-p|x+le|)dx \\ &\leq C_0 \exp(-\min\{r_a, p\} l) \end{aligned} \tag{18}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N_+} b(x) w_l^p(x) dx &= \int_{\mathbb{R}^N_+} b(x-le) w^p(x) dx \\ &\geq (\min_{x \in B^N_+(1)} w^p(x)) \int_{B^N(1)} b(x-le)dx \\ &\geq (\min_{x \in B^N_+(1)} w^p(x)) C_1 \exp(-r_b l), \end{aligned} \tag{19}$$

where $B^N_+(1) = \{x \in \mathbb{R}^N_+ \mid |x| < 1\}$. From the condition $(D1)$ and the same argument of inequality (18), we also have

$$\begin{aligned} \int_{\partial\mathbb{R}^N_+} |f_-| w_l^q d\sigma &\leq \widehat{c} B_0^q \int_{\partial\mathbb{R}^N_+} \exp(-r_{f_-}|x|) \exp(-q|x+le|)d\sigma \\ &\leq C_2 \exp(-\min\{r_{f_-}, q\} l). \end{aligned} \tag{20}$$

Since $r_b < \min\{r_{f_-}, r_a, q\} \leq \min\{r_{f_-}, r_a, p\}$ and $t_1 \leq t \leq t_2$, by (15) – (20), we can find $l_1 > 0$ such that

$$\sup_{t \geq 0} J_{f_\lambda,g_\mu}(u^+_{f_\lambda,g_\mu} + tw_l) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty \text{ for all } l \geq l_1.$$

To complete the proof of Proposition 2, it remains to show that there exists a positive number t_* such that $u^+_{f_\lambda,g_\mu} + t_* w_l \in \mathbf{N}^-_{f_\lambda,g_\mu}$. Let

$$\begin{aligned} U_1 &= \left\{ u \in H^1(\mathbb{R}^N_+) \;\middle|\; \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) > 1 \right\} \cup \{0\}; \\ U_2 &= \left\{ u \in H^1(\mathbb{R}^N_+) \;\middle|\; \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) < 1 \right\}. \end{aligned}$$

Then $\mathbf{N}^-_{f_\lambda,g_\mu}$ separates $H^1(\mathbb{R}^N_+)$ into two connected components U_1 and U_2 and $H^1(\mathbb{R}^N_+) \backslash \mathbf{N}^-_{f_\lambda,g_\mu} = U_1 \cup U_2$. For each $u \in \mathbf{N}^+_{f_\lambda,g_\mu}$, we have

$$1 < t_{\max,\mu}(u) < t^-(u).$$

Since $t^-(u) = \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}})$, then $\mathbf{N}^+_{f_\lambda,g_\mu} \subset U_1$. In particular, $u^+_{\lambda,\mu} \in U_1$. We claim that there exists $t_0 > 0$ such that $u^+_{f_\lambda,g_\mu} + t_0 w_l \in U_2$. First, we find a constant $c > 0$ such that $0 < t^-(\frac{u^+_{\lambda,\mu} + tw_l}{\|u^+_{\lambda,\mu} + tw_l\|_{H^1}}) < c$ for each $t \geq 0$. Suppose the contrary. Then there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $t^-(\frac{u^+_{\lambda,\mu} + t_n w_l}{\|u^+_{\lambda,\mu} + t_n w_l\|_{H^1}}) \to \infty$

as $n \to \infty$. Let $v_n = \frac{u^+_{\lambda,\mu} + t_n w_l}{\left\| u^+_{\lambda,\mu} + t_n w_l \right\|_{H^1}}$. Since $t^-(v_n)v_n \in \mathbf{N}^-_{f_\lambda, g_\mu}$ and by the Lebesgue dominated convergence theorem,

$$\begin{aligned}
\int_{\mathbb{R}^N_+} g_\mu v_n^p dx &= \frac{1}{\left\| u^+_{\lambda,\mu} + t_n w_l \right\|^p_{H^1}} \int_{\mathbb{R}^N_+} g_\mu (u^+_{\lambda,\mu} + t_n w_l)^p dx \\
&= \frac{1}{\left\| \frac{u^+_{\lambda,\mu}}{t_n} + w_l \right\|^p_{H^1}} \int_{\mathbb{R}^N_+} g_\mu (\frac{u^+_{\lambda,\mu}}{t_n} + w_l)^p dx \\
&\to \frac{\int_{\mathbb{R}^N_+} g_\mu w_l^p dx}{\|w_l\|^p_{H^1}} \text{ as } n \to \infty,
\end{aligned}$$

we have

$$\begin{aligned}
& J_{f_\lambda, g_\mu}(t^-(v_n)v_n) \\
= \ & \frac{1}{2} \left[t^-(v_n) \right]^2 - \frac{[t^-(v_n)]^q}{q} \int_{\partial \mathbb{R}^N_+} f_\lambda v_n^q d\sigma - \frac{[t^-(v_n)]^p}{p} \int_{\mathbb{R}^N_+} g_\mu v_n^p dx \\
\to \ & -\infty \text{ as } n \to \infty,
\end{aligned}$$

this contradicts the fact that J_{f_λ, g_μ} is bounded below on $\mathbf{N}_{f_\lambda, g_\mu}$. Let

$$t_0 = (\frac{p-2}{2p\alpha^\infty} \left| c^2 - \left\| u^+_{\lambda,\mu} \right\|^2_{H^1} \right|)^{\frac{1}{2}} + 1.$$

Then

$$\begin{aligned}
\left\| u^+_{\lambda,\mu} + t_0 w_l \right\|^2_{H^1} &= \left\| u^+_{\lambda,\mu} \right\|^2_{H^1} + t_0^2 \|w_l\|^2_{H^1} + o(1) \\
&> \left\| u^+_{\lambda,\mu} \right\|^2_{H^1} + \left| c^2 - \left\| u^+_{\lambda,\mu} \right\|^2_{H^1} \right| + o(1) \\
&> c^2 + o(1) > \left[t^-(\frac{u^+_{\lambda,\mu} + t_0 w_l}{\left\| u^+_{\lambda,\mu} + t_0 w_l \right\|_{H^1}}) \right]^2 + o(1) \text{ as } l \to \infty.
\end{aligned}$$

Thus, there exists $l_2 \geq l_1$ such that for $l \geq l_2$,

$$\frac{1}{\left\| u^+_{\lambda,\mu} + t_0 w_l \right\|_{H^1}} t^-(\frac{u^+_{\lambda,\mu} + t_0 w_l}{\left\| u^+_{\lambda,\mu} + t_0 w_l \right\|_{H^1}}) < 1$$

or $u^+_{\lambda,\mu} + t_0 w_l \in U_2$. Define a path $\gamma_l(s) = v_\lambda + s t_0 w_l$ for $s \in [0,1]$. Then

$$\gamma_l(0) = u^+_{\lambda,\mu} \in U_1, \gamma_l(1) = u^+_{\lambda,\mu} + t_0 w_l \in U_2.$$

Since $\frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}})$ is a continuous function for non-zero u and $\gamma_l([0,1])$ is connected, there exists $s_l \in (0,1)$ such that $u^+_{\lambda,\mu} + s_l t_0 w_l \in \mathbf{N}^-_{f_\lambda, g_\mu}$. This completes the proof. □

Then we can show that the following result.

Theorem 3.1. *For each $\lambda > 0$ and $\mu > 0$ with*

$$\lambda^{p-2}(1 + \mu \|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2} \Lambda_0,$$

the equation (E_{f_λ,g_μ}) *has a positive solution* $u^-_{\lambda,\mu} \in \mathbf{N}^-_{f_\lambda,g_\mu}$ *such that* $J_{f_\lambda,g_\mu}(u^-_{\lambda,\mu}) = \alpha^-_{f_\lambda,g_\mu}$.

Proof. Similarly to the argument in the proof of Theorem 2.8. $\square$

By (4), (6) and Lemma 2.6, for each $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ there is a unique $t_0^-(u) > 0$ such that $t_0^-(u)u \in \mathbf{N}_{f_0,g_0}$ and

$$t_0^-(u) > t_{\max,0}(u) = \left(\frac{(2-q)\left\|u\right\|^2_{H^1}}{(p-q)\int_{\mathbb{R}^N_+} g_0 \left|u\right|^p dx}\right)^{\frac{1}{p-2}} > 0.$$

Let

$$\theta_\mu = \left[A_\mu\left(1 + \frac{\|f_-\|_{L^{q^*}}}{C_p^{\frac{q}{2}}}\left(\frac{A_\mu}{S_p^{\frac{p}{2}}}\right)^{\frac{2-q}{p-2}}\right)\right]^{\frac{p}{p-2}},$$

where $A_\mu = \frac{(1+\mu\|b/a\|_\infty)(p-q)}{2-q}$. Then we have the following results.

Lemma 3.2. *For each* $\lambda > 0$ *and* $\mu > 0$ *with* $\lambda^{p-2}(1+\mu\left\|b\right\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$ *we have the following.*
(i) $\left[t_0^-(u)\right]^p < \theta_\mu$ *for all* $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ *with* $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$.
(ii) $\int_{\mathbb{R}^N_+} g_0 \left|u\right|^p dx \geq \frac{qp}{\theta_\mu(p-q)}\alpha^\infty$ *for all* $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ *with* $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$.

Proof. (i) For $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$, we have

$$\left\|u\right\|^2_{H^1} - \int_{\partial\mathbb{R}^N_+} f_\lambda \left|u\right|^q d\sigma - \int_{\mathbb{R}^N_+} g_\mu \left|u\right|^p dx = 0.$$

We distinguish two cases.
Csae (I) : $t_0^-(u) < 1$. Since $\theta_\mu > 1$ for all $\mu > 0$, we have

$$\left[t_0^-(u)\right]^p < 1 < \theta_\mu.$$

Case (II) : $t_0^-(u) \geq 1$. Since

$$\begin{aligned}\left[t_0^-(u)\right]^p \int_{\mathbb{R}^N_+} g_0 \left|u\right|^p dx &= \left[t_0^-(u)\right]^2 \left\|u\right\|^2_{H^1} - \left[t^-(u)\right]^q \int_{\partial\mathbb{R}^N_+} f_- \left|u\right|^q d\sigma \\ &\leq \left[t_0^-(u)\right]^2 (\left\|u\right\|^2_{H^1} + \int_{\partial\mathbb{R}^N_+} \left|f_-\right| \left|u\right|^q d\sigma),\end{aligned}$$

we have

$$\left[t_0^-(u)\right]^{p-2} \leq \frac{\left\|u\right\|^2_{H^1} + \int_{\partial\mathbb{R}^N_+} \left|f_-\right| \left|u\right|^q d\sigma}{\int_{\mathbb{R}^N_+} g_0 \left|u\right|^p dx}. \tag{21}$$

Moreover, by (4) and the Sobolev inequality,

$$\begin{aligned}\left\|u\right\|^2_{H^1} &< \frac{p-q}{2-q}\int_{\mathbb{R}^N_+} g_\mu \left|u\right|^p dx \leq \frac{p-q}{2-q}(1+\mu\left\|b/a\right\|_\infty)\int_{\mathbb{R}^N_+} g_0 \left|u\right|^p dx &(22)\\ &\leq (1+\mu\left\|b/a\right\|_\infty) S_p^{-\frac{p}{2}} \frac{p-q}{2-q}\left\|u\right\|^p_{H^1} &(23)\end{aligned}$$

and so

$$\left\|u\right\|_{H^1} \geq \left(\frac{(2-q)S_p^{\frac{p}{2}}}{(1+\mu\left\|b/a\right\|_\infty)(p-q)}\right)^{\frac{1}{p-2}}. \tag{24}$$

Thus, by (21) – (24) and the Sobolev inequality,

$$
\begin{aligned}
& \left[t_0^-(u)\right]^{p-2} \\
\leq \quad & (1+\mu\|b/a\|_\infty)(\frac{p-q}{2-q})(1+\frac{\int_{\partial\mathbb{R}^N_+} f_- |u|^q \, d\sigma}{\|u\|^2_{H^1}}) \\
\leq \quad & \frac{(1+\mu\|b/a\|_\infty)(p-q)}{2-q}(1+\frac{\|f_-\|_{L^{q^*}}}{C_p^{\frac{q}{2}}\|u\|^{2-q}_{H^1}}) \\
\leq \quad & \frac{(1+\mu\|b/a\|_\infty)(p-q)}{2-q}(1+\frac{\|f_-\|_{L^{q^*}}}{C_p^{\frac{q}{2}}}(\frac{(1+\mu\|b/a\|_\infty)(p-q)}{(2-q)S_p^{\frac{p}{2}}})^{\frac{2-q}{p-2}})
\end{aligned}
$$

or $\left[t_0^-(u)\right]^p \leq \theta_\mu$.

(ii) By Lemma 2.9 and $t_0^-(u)u \in \mathbf{N}_{f_0,g_0}$,

$$
\begin{aligned}
\alpha^\infty \quad & \leq \quad J_{f_0,g_0}(t_0^-(u)u) \\
& = \quad (\frac{1}{2}-\frac{1}{q})\left[t_0^-(u)\right]^2\|u\|^2_{H^1} + (\frac{1}{q}-\frac{1}{p})\left[t_0^-(u)\right]^p \int_{\mathbb{R}^N_+} g_0|u|^p\,dx \\
& < \quad (\frac{1}{q}-\frac{1}{p})\left[t_0^-(u)\right]^p \int_{\mathbb{R}^N_+} g_0|u|^p\,dx,
\end{aligned}
$$

and this implies

$$\int_{\mathbb{R}^N_+} g_0|u|^p\,dx \geq \frac{1}{\left[t_0^-(u)\right]^p}(\frac{pq}{p-q})\alpha^\infty.$$

By part (i), we can conclude that

$$\int_{\mathbb{R}^N_+} g_0|u|^p\,dx \geq \frac{pq}{\theta_\mu(p-q)}\alpha^\infty$$

for all $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$. This completes the proof. □

By the proof of Proposition 2, there exist positive numbers t_* and l_2 such that $u^+_{\lambda,\mu} + t_* w_l \in \mathbf{N}^-_{f_\lambda,g_\mu}$ and

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + t_* w_l) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty \text{ for all } l \geq l_2.$$

Then we have the following result.

Lemma 3.3. *There exist positive numbers λ_0 and μ_0 with*

$$\lambda_0^{p-2}(1+\mu_0\|b\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$$

such that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, we have

$$P(\int_{\mathbb{R}^N_+} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx) \neq 0$$

for all $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$.

Proof. For $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$, by Lemma 2.6 (i), there exists $t_0^-(u) > 0$ such that $t_0^-(u)u \in \mathbf{N}_{f_0,g_0}$. Moreover,

$$\begin{aligned} J_{f_\lambda,g_\mu}(u) &= \sup_{t\geq 0} J_{f_\lambda,g_\mu}(tu) \geq J_{f_\lambda,g_\mu}(t_0^-(u)u) \\ &= J_{f_0,g_0}(t_0^-(u)u) - \frac{\lambda\left[t_0^-(u)\right]^q}{q}\int_{\partial\mathbb{R}^N_+} f_+ |u|^q\, d\sigma \\ &\quad -\frac{\mu\left[t_0^-(u)\right]^p}{p}\int_{\mathbb{R}^N_+} b\,|u|^p\, dx. \end{aligned}$$

Thus, by Lemma 3.2 and the Hölder and Sobolev inequalities,

$$\begin{aligned} J_{f_0,g_0}(t_0^-(u)u) &\leq J_{f_\lambda,g_\mu}(u) + \frac{\lambda\left[t_0^-(u)\right]^q}{q}\int_{\partial\mathbb{R}^N_+} f_+ |u|^q\, d\sigma \\ &\quad +\frac{\mu\left[t_0^-(u)\right]^p}{p}\int_{\mathbb{R}^N_+} b\,|u|^p\, dx \\ &< \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty + \frac{\lambda\theta_\mu^{q/p}}{q}\left\|f_+\right\|_{L^{q^*}} C_p^{-\frac{q}{2}}\left\|u\right\|^q_{H^1} \\ &\quad +\frac{\mu\theta_\mu\left\|b\right\|_\infty}{p} S_p^{-\frac{p}{2}}\left\|u\right\|^p_{H^1}. \end{aligned}$$

Since $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty < \alpha^\infty$, by (2) in Lemma 2.2, for each $\lambda > 0$ and $\mu > 0$ with $\lambda^{p-2}(1+\mu\left\|b\right\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$, there exists a positive number $\widetilde{c}$ independent of λ, μ such that $\left\|u\right\|_{H^1} \leq \widetilde{c}$ for all $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$. Therefore,

$$J_{f_0,g_0}(t_0^-(u)u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty + \frac{\lambda\theta_\mu^{q/p}}{q}\left\|f_+\right\|_{L^{q^*}} C_p^{-\frac{q}{2}}\widetilde{c}^q + \frac{\mu\theta_\mu\left\|b\right\|_\infty}{p} S_p^{-\frac{p}{2}}\widetilde{c}^p.$$

Let $d_0 > 0$ be as in Lemma 2.11. Then there exist positive numbers λ_0 and μ_0 with $\lambda_0^{p-2}(1+\mu_0\left\|b\right\|_\infty)^{2-q} < (\frac{q}{2})^{p-2}\Lambda_0$ such that for $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$,

$$J_{f_0,g_0}(t^-(u)u) < \alpha^\infty + d_0. \tag{25}$$

Since $t_0^-(u)u \in \mathbf{N}_{f_0,g_0}$ and $t_0^-(u) > 0$, by Lemma 2.11 and (25)

$$P(\int_{\mathbb{R}^N_+}\frac{x}{|x|}(\left|\nabla(t_0^-(u)u)\right|^2 + (t_0^-(u)u)^2)dx) \neq 0,$$

and this implies

$$P(\int_{\mathbb{R}^N_+}\frac{x}{|x|}(\left|\nabla u\right|^2 + u^2)dx) \neq 0$$

for all $u \in \mathbf{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty$. □

4. **Proof of Theorem 1.1.** In the following, we use an idea of Adachi and Tanaka [3]. For $c \in \mathbb{R}^+$, we denote

$$\left[J_{f_\lambda,g_\mu} \leq c\right] = \left\{u \in \mathbf{N}^-_{f_\lambda,g_\mu} \mid u \geq 0, J_{f_\lambda,g_\mu}(u) \leq c\right\}.$$

We then try to show for a sufficiently small $\sigma > 0$

$$\operatorname{cat}(\left[J_{f_\lambda,g_\mu} \leq \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty - \sigma\right]) \geq 2. \tag{26}$$

To prove (26), we need some preliminaries. Recall the definition of Lusternik-Schnirelman category.

Definition 4.1. (i) For a topological space X, we say a non-empty, closed subset $Y \subset X$ is contractible to a point in X if and only if there exists a continuous mapping

$$\xi : [0,1] \times Y \to X$$

such that for some $x_0 \in X$

$$\xi(0,x) = x \text{ for all } x \in Y,$$

and

$$\xi(1,x) = x_0 \text{ for all } x \in Y.$$

(ii) We define

$$\begin{aligned} \mathrm{cat}(X) \quad = \quad & \min \{k \in \mathbb{N} \mid \text{there exist closed subsets } Y_1, ..., Y_k \subset X \text{ such that} \\ & Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \bigcup_{j=1}^{k} Y_j = X\}. \end{aligned}$$

When there do not exist finitely many closed subsets $Y_1, ..., Y_k \subset X$ such that Y_j is contractible to a point in X for all j and $\bigcup_{j=1}^{k} Y_j = X$, we say $\mathrm{cat}(X) = \infty$.

We need the following two lemmas.

Lemma 4.2. *Suppose that X is a Hilbert manifold and $F \in C^1(X, \mathbb{R})$. Assume that there are $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$,*
(i) $F(x)$ satisfies the Palais–Smale condition for energy level $c \le c_0$;
(ii) $\mathrm{cat}(\{x \in X \mid F(x) \le c_0\}) \ge k$.
Then $F(x)$ has at least k critical points in $\{x \in X; F(x) \le c_0\}$.

Proof. See Ambrosetti [1, Theorem 2.3]. □

Let $\mathbb{S}^{m-1} = \{x \in \mathbb{R}^m \mid |x| = 1\}$ be a unit sphere in $\mathbb{R}^m$ for $m \in \mathbb{N}$. Then we have the following results.

Lemma 4.3. *Let X be a topological space. Suppose that there are two continuous maps*

$$\Phi : \mathbb{S}^{m-1} \to X, \ \ \Psi : X \to \mathbb{S}^{m-1}$$

such that $\Psi \circ \Phi$ is homotopic to the identity map of $\mathbb{S}^{m-1}$, that is, there exists a continuous map $\zeta : [0,1] \times \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ such that

$$\begin{aligned} \zeta(0,x) &= (\Psi \circ \Phi)(x) \text{ for each } x \in \mathbb{S}^{m-1}, \\ \zeta(1,x) &= x \text{ for each } x \in \mathbb{S}^{m-1}. \end{aligned}$$

Then

$$\mathrm{cat}(X) \ge 2.$$

Proof. See Adachi and Tanaka [3, Lemma 2.5]. □

Since $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{0\}$, for $l > l_2$, we may define a map

$$\Phi_{f_\lambda, g_\mu} : \mathbb{S}^{(N-1)-1} \to H^1(\mathbb{R}^N_+)$$

by

$$\Phi_{f_\lambda, g_\mu}(\widetilde{e})(x) = u^+_{\lambda,\mu}(x) + s_l t_0 w(x + l(\widetilde{e}, 0)) \text{ for } \widetilde{e} \in \mathbb{S}^{(N-1)-1},$$

where $u^+_{\lambda,\mu} + s_l t_0 w_l$ is as in the proof of Proposition 2. Note that $\mathbb{S}^{(N-1)-1} \times \{0\} = \mathbb{S}$. Then we have the following result.

Lemma 4.4. *There exists a sequence $\{\sigma_l\} \subset \mathbb{R}^+$ with $\sigma_l \to 0$ as $l \to \infty$ such that*

$$\Phi_{f_\lambda,g_\mu}(\mathbb{S}^{(N-1)-1}) \subset \left[J_{f_\lambda,g_\mu} \le \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty - \sigma_l\right].$$

Proof. By Proposition 2, for each $l > l_2$ we have $u^+_{\lambda,\mu} + s_l t_0 w_l \in \mathbf{N}^-_{f_\lambda,g_\mu}$ and

$$\sup_{l\ge 0} J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + s_l t_0 w_l) < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty \text{ uniformly in } e \in \mathbb{S}.$$

Since $\mathbb{S} = \mathbb{S}^{(N-1)-1} \times \{0\}$ and $\Phi_{f_\lambda,g_\mu}(\mathbb{S}^{(N-1)-1})$ is compact,

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + s_l t_0 w_l) \le \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty - \sigma_l,$$

so that the conclusion holds. $\square$

From Lemma 3.3, we define

$$\Psi_{f_\lambda,g_\mu} : \left[J_{f_\lambda,g_\mu} < \alpha^+_{f_\lambda,g_\mu} + \alpha^\infty\right] \to \mathbb{S}^{(N-1)-1}$$

by

$$\Psi_{f_\lambda,g_\mu}(u) = \frac{P(\int_{\mathbb{R}^N_+} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx)}{\left|P(\int_{\mathbb{R}^N_+} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx)\right|}.$$

Then we have the following results.

Lemma 4.5. *Let λ_0, μ_0 be as in Lemma 3.3. Then for each $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$ there exists $l_* \ge l_2$ such that for $l > l_*$, the map*

$$\Psi_{f_\lambda,g_\mu} \circ \Phi_{f_\lambda,g_\mu} : \mathbb{S}^{(N-1)-1} \to \mathbb{S}^{(N-1)-1}$$

is homotopic to the identity.

Proof. Let $\Sigma = \left\{u \in H^1(\mathbb{R}^N_+) \setminus \{0\} \mid P(\int_{\mathbb{R}^N_+} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx) \ne 0\right\}$. We define

$$\overline{\Psi}_{f_\lambda,g_\mu} : \Sigma \to \mathbb{S}^{(N-1)-1}$$

by

$$\overline{\Psi}_{f_\lambda,g_\mu}(u) = \frac{P(\int_{\mathbb{R}^N_+} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx)}{\left|P(\int_{\mathbb{R}^N_+} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx)\right|}$$

as an extension of Ψ_{f_λ,g_μ}. Since $w_l \in \Sigma$ for all $(\widetilde{e}, 0) \in \mathbb{S} = \mathbb{S}^{(N-1)-1} \times \{0\}$ and for l sufficiently large, we let $\gamma : [s_1, s_2] \to \mathbb{S}^{(N-1)-1}$ be a regular geodesic between $\overline{\Psi}_{f_\lambda,g_\mu}(w_l)$ and $\overline{\Psi}_{f_\lambda,g_\mu}(\Phi_{f_\lambda,g_\mu}(\widetilde{e}))$ such that $\gamma(s_1) = \overline{\Psi}_{f_\lambda,g_\mu}(w_l), \gamma(s_2) = \overline{\Psi}_{f_\lambda,g_\mu}(\Phi_{f_\lambda,g_\mu}(\widetilde{e}))$. By an argument similar to that in Lemma 2.11, there exists a positive number $l_* \ge l_2$ such that for $l > l_*$,

$$w(x + \frac{l(\widetilde{e}, 0)}{2(1-\theta)}) \in \Sigma \text{ for all } \widetilde{e} \in \mathbb{S}^{(N-1)-1} \text{ and } \theta \in [1/2, 1).$$

We define

$$\zeta_l(\theta, \widetilde{e}) : [0, 1] \times \mathbb{S}^{(N-1)-1} \to \mathbb{S}^{(N-1)-1}$$

by

$$\zeta_l(\theta, \widetilde{e}) = \begin{cases} \gamma(2\theta(s_1 - s_2) + s_2) & \text{for } \theta \in [0, 1/2); \\ \overline{\Psi}_{f_\lambda,g_\mu}(w(x + \frac{l(\widetilde{e},0)}{2(1-\theta)})) & \text{for } \theta \in [1/2, 1); \\ \widetilde{e} & \text{for } \theta = 1. \end{cases}$$

Then $\zeta_l(0,\widetilde{e}) = \overline{\Psi}_{f_\lambda,g_\mu}(\Phi_{f_\lambda,g_\mu}(\widetilde{e})) = \Psi_{f_\lambda,g_\mu}(\Phi_{f_\lambda,g_\mu}(\widetilde{e}))$ and $\zeta_l(1,\widetilde{e}) = \widetilde{e}$. By the standard regularity, we have $u^+_{\lambda,\mu} \in C(\overline{\mathbb{R}^N_+})$. First, we claim that $\lim_{\theta\to 1^-}\zeta_l(\theta,\widetilde{e}) = \widetilde{e}$ and $\lim_{\theta\to \frac{1}{2}^-}\zeta_l(\theta,\widetilde{e}) = \overline{\Psi}_{f_\lambda,g_\mu}(w(x+l(\widetilde{e},0)))$.

(a) $\lim_{\theta\to 1^-}\zeta_l(\theta,\widetilde{e}) = \widetilde{e}$: since

$$
\begin{aligned}
& P(\int_{\mathbb{R}^N_+}\frac{x}{|x|}(\left|\nabla\left[w(x+\frac{l(\widetilde{e},0)}{2(1-\theta)})\right]\right|^2+\left[w(x+\frac{l(\widetilde{e},0)}{2(1-\theta)})\right]^2)dx) \\
= \; & P(\int_{\mathbb{R}^N_+}\frac{x-\frac{l}{2(1-\theta)}(\widetilde{e},0)}{\left|x-\frac{l}{2(1-\theta)}(\widetilde{e},0)\right|}(|\nabla\left[w(x)\right]|^2+\left[w(x)\right]^2)dx) \\
= \; & (\frac{2p}{p-2})\alpha^\infty\widetilde{e}+o(1) \text{ as } \theta\to 1^-,
\end{aligned}
$$

then $\lim_{\theta\to 1^-}\zeta_l(\theta,\widetilde{e}) = \widetilde{e}$.

(b) $\lim_{\theta\to \frac{1}{2}^-}\zeta_l(\theta,\widetilde{e}) = \overline{\Psi}_{f_\lambda,g_\mu}(w(x+l(\widetilde{e},0)))$: since $\overline{\Psi}_{f_\lambda,g_\mu}\in C(\Sigma,\mathbb{S}^{(N-1)-1})$, we obtain $\lim_{\theta\to \frac{1}{2}^-}\zeta_l(\theta,\widetilde{e}) = \overline{\Psi}_{f_\lambda,g_\mu}(w(x+l(\widetilde{e},0)))$.

Thus, $\zeta_l(\theta,\widetilde{e})\in C([0,1]\times\mathbb{S}^{(N-1)-1},\mathbb{S}^{(N-1)-1})$ and

$$
\begin{aligned}
\zeta_l(0,\widetilde{e}) &= \Psi_{f_\lambda,g_\mu}(\Phi_{f_\lambda,g_\mu}(\widetilde{e})) \text{ for all } \widetilde{e}\in\mathbb{S}^{(N-1)-1}, \\
\zeta_l(1,\widetilde{e}) &= \widetilde{e} \text{ for all } \widetilde{e}\in\mathbb{S}^{(N-1)-1},
\end{aligned}
$$

provided $l > l_*$. This completes the proof. □

Lemma 4.6. *For each* $\lambda\in(0,\lambda_0),\mu\in(0,\mu_0)$ *and* $l>l_*$, *the functional* J_{f_λ,g_μ} *has at least two critical points in* $\left[J_{f_\lambda,g_\mu} < \alpha^+_{f_\lambda,g_\mu}+\alpha^\infty\right]$. *In particular, the equation* (E_{f_λ,g_μ}) *has two positive solutions* u_1^- *and* u_2^- *such that* $u_i^-\in\mathbf{N}^-_{f_\lambda,g_\mu}$ *for* $i=1,2$.

Proof. Applying Lemmas 4.3, 4.5, we have for $\lambda\in(0,\lambda_0),\mu\in(0,\mu_0)$ and $l>l_*$,

$$\operatorname{cat}(\left[J_{f_\lambda,g_\mu}\le\alpha^+_{f_\lambda,g_\mu}+\alpha^\infty-\sigma_l\right])\ge 2.$$

By Proposition 1 and Lemma 4.2, $J_{f_\lambda}(u)$ has at least two critical points in

$$\left[J_{f_\lambda,g_\mu} < \alpha^+_{f_\lambda,g_\mu}+\alpha^\infty\right],$$

which implies that the equation (E_{f_λ,g_μ}) has two nontrivial nonnegative solutions u_1^- and u_2^- such that $u_i^-\in\mathbf{N}^-_{f_\lambda,g_\mu}$ for $i=1,2$. Moreover, by the maximum principle, we have $u_i^->0$ in $\mathbb{R}^N_+$. □

We can now complete the proof of Theorem 1.1: (i) by Theorems 2.8 and 3.1. (ii) for $\lambda\in(0,\lambda_0)$ and $\mu\in(0,\mu_0)$, from Theorem 2.8 and Lemma 4.6, equation (E_{f_λ,g_μ}) has three positive solutions $u^+_{\lambda,\mu},u_1^-,u_2^-$ such that $u^+_{\lambda,\mu}\in\mathbf{N}^+_{f_\lambda,g_\mu}$and $u_i^-\in\mathbf{N}^-_{f_\lambda,g_\mu}$ for $i=1,2$. This completes the proof of Theorem 1.1.

REFERENCES

[1] A. Ambrosetti, "Critical Points and Nonlinear Variational Problems," Bulletin Soc. Math. France, Mémoire, N.49, 1992.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal., **122** (1994), 519–543.

[3] S. Adachi and K. Tanaka, *Four positive solutions for the semilinear elliptic equation: $-\triangle u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$*, Calc. Var. Partial Diff. Eqns., **11** (2000), 63–95.
[4] P. A. Binding, P. Drabek and Y. X. Huang, *On Neumann boundary value problems for some quasilinear elliptic equations*, Electr. J. Diff. Eqns., **5** (1997), 1–11.
[5] K. J. Brown and T. F. Wu, *A fibering map approach to a potential operator equation and its applications*, Diff. Int. Eqns, **22** (2009), 1094–1114.
[6] K. J. Brown and Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Diff. Eqns, **193** (2003), 481–499.
[7] M. Chipot, M. Chlebik, M. Fila and I. Shafrir, *Existence of positive solutions of a semilinear elliptic equation in $\mathbb{R}^N_+$ with a nonlinear boundary condition*, J. Math. Anal. Appl., **223** (1998), 429–471.
[8] M. Chipot, I. Shafrir and M. Fila, *On the solutions to some elliptic equations with nonlinear Neumann boundary conditions*, Adv. Diff. Eqns, **1** (1996), 91–110.
[9] M. del Pino and C. Flores, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains*, Comm. Partial Diff. Eqns., **26** (2001), 2189–2210.
[10] P. Drábek and S. I. Pohozaev, *Positive solutions for the p–Laplacian: application of the fibering method*, Proc. Roy. Soc. Edinburgh, **127** (1997), 703–726.
[11] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., **17** (1974), 324–353.
[12] M. del Pino and P. L. Felmer, *Spike-layered Solutions of singularly perturbed elliptic problems in a degenerate setting*, Indiana Math. J., **45** (1999), 883–898.
[13] C. Flores and M. del Pino, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains*, Comm. Partial Diff. Eqns., **26** (2001), 2189–2210.
[14] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., **68** (1979), 209–243.
[15] J. Garcia-Azorero, I. Peral and J. D. Rossi, *A convex-concave problem with a nonlinear boundary condition*, J. Diff. Eqns., **198** (2004), 91–128.
[16] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Diff. Integral Eqns., **7** (1994), 301–313.
[17] M. K. Kwong, *Uniqueness of positive solution of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$*, Arch. Rat. Math. Anal., **105** (1989), 243–266.
[18] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The local compact case I*, Ann. Inst. H. Poincaré Anal. Non Lineairé, **1** (1984), 102–145.
[19] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The local compact case II*, Ann. Inst. H. Poincaré Anal. Non Lineairé, **1** (1984), 223–283.
[20] D. Pierotti and S. Terracini, *On a Neumann problem with critical exponent and critical nonlinearity on the boundary*, Comm. Partial Diff. Eqns., **20** (1995), 1155–1187.
[21] P. H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," Regional Conference Series in Mathematics, American Mathematical Society, 1986.
[22] S. Terracini, *Symmetry properties of positive solutions to some elliptic equations with nonlinear boundary conditions*, Diff. Int. Eqns., **8** (1995), 1911–1922.
[23] H. C. Wang and T. F. Wu, *Symmetry breaking in a bounded symmetry domain*, NoDEA-Nonlinear Differential Equations Appl., **11** (2004), 361–377.
[24] M. Willem, "Minimax Theorems," Birkhäuser, Boston 1996.
[25] T. F. Wu, *A semilinear elliptic problem involving nonlinear boundary condition and sign-changing potential*, Electr. J. Diff. Eqns., **131** (2006), 1–15.

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