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Rotating spiral waves in λ–ω systems on circular domains

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ABSTRACT

λ–ω systems are reaction–diffusion systems whose reaction kinetics admit a stable limit cycle. It is known that λ–ω systems can possess various types of solutions. Among them, spiral waves are the most fascinating pattern. However, the effects of the diffusivity, the sizes of the domains, and the reaction kinetics on spiral waves are largely unknown. In this paper, we investigate how these quantities affect the properties of m-armed spiral waves in a generalized class of λ–ω system on a circular disk with no-flux boundary condition. First we derive a criterion for the existence of m-armed spiral waves. Specifically, we show that m-armed spiral waves do not exist for $d \geq \lambda_0 R^2/j_m^2$, while for $d \in (0, \lambda_0 R^2/j_m^2)$, there exists an m-armed spiral wave if the twist parameter q is small. Here d is the diffusivity for the λ–ω system, R is the radius of the circular disk, λ_0 is the value of the function $\lambda(A)$ at $A = 0$, and j_m is the first positive zero of the first derivative of the Bessel function of the first kind of order m. We also show that the critical diffusivity $d = \lambda_0 R^2/j_m^2$ is a bifurcation point. Next we use the numerical simulation to show that, for small twist parameter, the rotational frequency increases with increasing domain size, while for large twist parameter, the dependence of the rotational frequency on the domain size is not monotonic. Moreover, small circular domains may change the properties of spiral waves drastically. These numerical results are in contrast to those in excitable media. Finally, the stability of spiral waves is investigated numerically.

1. Introduction

Rotating spiral waves arise naturally in various chemical systems and biological processes. Specific examples of spiral waves can be found in the Belousov–Zhabotinskii reaction [1,2], aggregating slime-mould cells [3], and cardiac muscle tissue [4,5]. It is now widely accepted that spiral waves can arise from the interplay between the chemical process of kinetic reaction and the physical process of molecular diffusion (see [6,1]). Hence one may use reaction–diffusion systems to model these fascinating patterns. In general, the reaction kinetics can be either excitable or oscillatory [7,5].

For excitable media, there are two major approaches for the study of spiral waves [8–11]. In the first approach, which is applicable for a weakly excitable medium [12], one needs to divide the spiral wave into two parts: the far field of the spiral and the core region of the spiral. The far field of the spiral can be viewed as a modified version of periodic plane waves. To connect the far field and the core region of the spiral smoothly, the so-called kinematic theory has been developed [13]. The second approach is to use singular perturbation theory to reduce the full model to a free boundary problem for the shape and the rotational frequency of the spiral [14–17]. On the other hand, the fact that spiral waves are rotating waves suggests that spiral waves are most likely to arise in oscillatory models having rotational symmetry. Motivated by these, we will study spiral waves in a generic λ–ω system which is essentially an oscillatory system. Unlike excitable media, the additional rotational symmetry in the λ–ω system allows us to use the ordinary differential equation (ODE) approach to study spiral waves. Specifically, the λ–ω system reads

$$\begin{aligned} u_t &= d\nabla^2 u + \lambda(A)u - \omega(A)v, \\ v_t &= d\nabla^2 v + \omega(A)u + \lambda(A)v, \end{aligned} \tag{1.1}$$

where $\lambda(\cdot)$ and $\omega(\cdot)$ are the given functions of $A = \sqrt{u^2 + v^2}$. Since the λ–ω system may be viewed as the normal form of an oscillatory system near a Hopf bifurcation (see [18,19]), the study of the λ–ω system is of relevance to general oscillatory reaction–diffusion systems whose reaction kinetics admit a limit cycle via a Hopf bifurcation (see [5,20,42]). It is also reported by Cohen et al. [21] that λ–ω systems play a dominant role in the asymptotic analysis of a general class of reaction–diffusion systems modelling realistic physical phenomena. Furthermore, λ–ω systems have a variety of important applications in biology, ranging from calcium signalling in cell biology to population dynamics in ecology (see [5,22,23]).

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Now we give a precise definition of m-armed spiral waves. An m-armed spiral wave is a solution (u, v) of (1.1) in the form

$$\begin{aligned} u &= A(r)\cos\left(\hat{\Omega}t + m\theta + \int_0^r k(s)\mathrm{d}s\right), \\ v &= A(r)\sin\left(\hat{\Omega}t + m\theta + \int_0^r k(s)\mathrm{d}s\right), \end{aligned} \tag{1.1a}$$

where r and θ are polar coordinates of the plane. Here $\hat{\Omega}$ is the rotational frequency, which is a constant to be determined; m is the number of arms on the spiral under study. We require that $A(r)$ is positive in its defined interval except at $r = 0$, and $A(r)$ and $k(r)$ are regular at $r = 0$. This implies that $A(0) = 0$ and $k(0) = 0$ (see Section 3). The function $k(\cdot)$ is closely related to the shape of a spiral wave. In the case of the whole plane, the function $k(\cdot)$ satisfies that $k(r) \to k_\infty$ as $r \to +\infty$ for some constant k_∞ (see [19]). Hence, for sufficiently large r, the contour curves of constant concentration satisfy $\hat{\Omega}t + m\theta + k_\infty r = C$ for some constant C, which implies that spiral waves are approximately the well-known Archimedean spirals at large distances from the origin.

The existence of spiral wave solutions of (1.1) in the whole plane has been studied by numerous researchers. By assuming that $\lambda(A) = 1 - A$ and $\omega = 1 + \omega_1(A-1)$ with $0 < \omega_1 \ll 1$, Greenberg [24] has used a formal perturbation technique to construct single-armed spiral waves to (1.1) in the whole plane. Using the matched asymptotic expansions, Hagan [19] constructed multi-armed spiral wave solutions of (1.1) in the whole plane where $\lambda(A) = 1 - A^2$, $\omega(A) = qA^2$, and the twist parameter q can be small or large. Under the hypothesis that $\lambda(1) = 0$, $\lambda'(\cdot) < 0$, $\omega'(\cdot) < 0$ and $|\omega'| \ll 1$, Kopell and Howard [18] have used the geometrical singular perturbation theory to establish the existence of spiral waves rigorously. For recent works on spiral waves, we refer readers to [20,25,26]. We remark that by assuming a degenerate form on the function $\omega(\cdot)$, one can obtain the so-called logarithmic spiral waves which are essentially different from the ones we discuss here (see [21]).

The ubiquitous spiral waves are mainly studied in the infinite domain [1,12]. Although ignoring the influence of the geometry and the size of the medium may simplify the consideration of spiral waves, the presence of the boundary may have the crucial effect on the properties of spiral waves [27]. For examples, recent experimental and theoretical studies reveal that certain domains with sharp corners can give rise to new spirals [28], and that the domain size may affect the existence of spiral waves [29,30,27]. Furthermore, for experimental and numerical studies, one needs to perform every procedure in the finite domain, not in the infinite plane. In particular, this consideration might become important and essential when one deals with a medium with very small size, such as a single oocyte cell [31]. Therefore, unlike in earlier works, we will focus on spiral wave solutions of (1.1) on a disk, not the whole plane. Specifically, we will look for spiral wave solutions of (1.1) subject to the Neumann boundary condition

$$\begin{aligned} \mathbf{n} \cdot \nabla u &= 0 \quad \text{on } \partial\Gamma_R, \\ \mathbf{n} \cdot \nabla v &= 0 \quad \text{on } \partial\Gamma_R, \end{aligned} \tag{1.2}$$

where Γ_R is the disk with the origin as the center and the radius of the disk equal to R, and $\mathbf{n}$ is the outer unit normal to Γ_R. Note that, according to Scheel's analysis [20], the existence of rotating waves in the whole plane for a reaction–diffusion system possessing a Hopf bifurcation can be reduced to that for the λ–ω system. Hence the investigation of spiral wave solutions of system (1.1)–(1.2) might enable a good understanding of spiral wave solutions of a reaction–diffusion system admitting a Hopf bifurcation in the circular domain. The restriction of our consideration to finite circular disks is quite natural, and might be a first step towards the understanding of the way in which the characteristics of the domains influence the properties of rotating spiral waves. The consideration of spiral waves on a disk can raise several interesting questions, which include the following.

1. For what the radius, R, we can have spiral waves?
2. Does the diffusivity d prevent the existence of spiral waves?
3. How does the reaction kinetics (λ, ω) affect the existence of spiral waves?
4. What is the effect of the domain size R on the rotational frequency $\hat{\Omega}$?

In contrast, these questions may be missed in the analysis of spiral waves in the whole plane. From previous works [24,32,19,18], we see that the diffusivity (d) does not prevent the existence of spiral waves in the whole plane. The reaction kinetics (λ, ω) does enter into the analysis of spiral waves in the whole plane, but it is involved in a complicated way. Evidently, the radius R has no role in the full plane analysis of spiral waves. Therefore, one of the aims of this paper is to explore the dependence of the existence of spiral waves on the diffusivity, the domain size, and the reaction kinetics (λ, ω) in the case of circular domains. In particular, our study can give partial answers to these four questions above for system (1.1) on a circular disk.

Along this direction, we note that Paullet et al. [33] has considered problem (1.1)–(1.2) with $\lambda(A) = 1 - A^2$ and $\omega(A) = 1 + qA^2$. They rigorously showed that, for small diffusivity d and small q, there is a single-armed spiral wave solution of (1.1)–(1.2) on the unit disk, while for $d \geq 1$, there is no single-armed spiral wave solution of (1.1)–(1.2) on the unit disk. By numerical studies, they also conjectured that $d = 1/j_1^2 \in (0, 1)$ may be the critical diffusivity above which there is no single-armed spiral wave solution of (1.1)–(1.2) on the unit disk, and below which there is a single-armed spiral wave solution of (1.1)–(1.2) on the unit disk. Here j_1 is the first positive zero of the first derivative of the Bessel function of the first kind of order one. We will address these conjectures, and give a criterion for the existence of multi-armed spiral waves on the unit disk. We also remark that this paper is mainly motivated by this conjecture, and that for excitable media, we refer readers to [29,30,27] for spiral waves on the circular domain and [34] for rotating waves on the annulus.

Since the consideration of general types of nonlinearities (λ, ω) may avoid the consequence that the analysis relies on the specific form of nonlinearities, we will give a general, but physically reasonable, setting for the reaction kinetics (λ, ω). Indeed, motivated by the above-mentioned works, throughout the remainder of this paper, we will impose the following two assumptions on the functions $\lambda(\cdot)$ and $\omega(\cdot)$.

(H1) $\lambda(A)$ is defined and continuously differentiable for $A \in [0, a]$ for some $a > 1$, $\lambda(A) > 0$ for $A \in [0, 1)$, $\lambda(1) = 0$, and $\lambda'(\cdot) < 0$ on $[0, a]$. We will retain the notation $\lambda_0 := \lambda(0)$.

(H2) $\omega(A) = \omega_0 + q\varpi(A)$, where $\varpi(A)$ is defined and differentiable for $A \in [0, a]$, and positive for $A \in (0, a]$ and $\varpi(0) = 0$. Here ω_0 and q are constants.

Note that the assumption $\lambda(1) = 0$ implies that (1.1) admits a homogeneous limit cycle solution with amplitude $A = 1$ and frequency $\hat{\Omega} = \omega(1)$, and that the condition $\lambda'(\cdot) < 0$ ensures that the spatially independent solution of (1.1) asymptotically approaches this limit cycle. Here the parameter q is the so-called twist parameter. Without the twist term ($q = 0$), one can show that the arms of the corresponding spiral waves are straight (see [33]). Hence such spiral waves are rotating straight lines, and so are more like pinwheels. Therefore, we always assume that the twist parameter q is nonzero. A typical example for $(\lambda(\cdot), \omega(\cdot))$ is $(\lambda(A), \omega(A)) = (1 - A^2, 1 + qA^2)$ (see [33,19]). Before proceeding any further, we make a crucial remark. Specifically, if (u, v) is an

m-armed spiral wave solution of (1.1)–(1.2) with the amplitude function A, the function k, the parameter q, and the rotational frequency $\hat{\Omega}$, then one can check that $(\tilde{u}, \tilde{v})$ given by (1.1a) with $(A, k, q, \hat{\Omega})$ replaced by $(A, -k, -q, 2\omega_0 - \hat{\Omega})$ is also an m-armed spiral wave solution of (1.1)–(1.2). In view of this fact, *throughout the remainder of this paper we will assume that the parameter q is positive*.

Finally, our study will consist of two major parts. The first one is to give a criterion for the existence of m-armed spiral wave solutions of (1.1)–(1.2). The second one is to numerically study the effects of the diffusivity and the (effective) domain size on the amplitude and the rotational frequency of spiral waves, and the stability of spiral waves. To begin with, we will derive the criterion for the existence of m-armed spiral waves on circular domains.

2. The criterion for the existence of *m*-armed spiral waves on circular domains

To state the criterion for the existence of m-armed spiral waves, we first let $J_m(\cdot)$ be the Bessel function of the first kind of order m, and $j_{m,0}$ (resp. j_m) be the first positive zero of $J_m(\cdot)$ (resp. $J_m'(\cdot)$). Then the necessary condition for the existence of m-armed spiral wave solutions of (1.1)–(1.2) can be stated as follows.

Theorem 1. *Suppose that the functions $\lambda(\cdot)$ and $\omega(\cdot)$ satisfy the assumptions* (H1)–(H2). *If $d \in [\frac{\lambda_0}{j_m^2} \cdot R^2, +\infty)$, then there exists no m-armed spiral wave solution of* (1.1)–(1.2).

On the other hand, for $d \in (0, \lambda_0 R^2/j_m^2)$, we have the following result on the existence of m-armed spiral wave solutions of (1.1)–(1.2).

Theorem 2. *Suppose that the functions $\lambda(\cdot)$ and $\omega(\cdot)$ satisfy the assumptions* (H1)–(H2). *Let the quantities q_0 and Ω_0 be defined by*

$$q_0 = q_0(m, \lambda_0, d, R) := \min\left\{ \frac{\sqrt{d\lambda_0}/R}{\sup_{x\in[0,1]} \varpi(x)}, \frac{\lambda_0}{\sup_{x\in[0,1]} \varpi(x)} \times \max_{s\in\left[j_m, \min\left\{j_{m,0}, \sqrt{\frac{\lambda_0}{d}}\cdot R\right\}\right]} \sqrt{\frac{-J_m'(s)}{\int_0^s \xi^2 J_m(\xi)\mathrm{d}\xi}} \right\}$$

and

$$\Omega_0 = \Omega_0(m, \lambda_0, d, R) := -\min\left\{ \frac{\sqrt{d\cdot\lambda\left(\frac{1}{2}\right)}}{R}, \frac{\lambda_0}{2} \cdot \sqrt{\frac{-J_m'(\sqrt{\lambda_0/d}\cdot R)}{\int_0^{\sqrt{\lambda_0/d}\cdot R} \xi^2 J_m(\xi)\mathrm{d}\xi}} \right\}.$$

Then the following hold:

(i) *for each $d \in (0, \frac{\lambda_0}{j_m^2} \cdot R^2)$, if $q \in (0, q_0)$, then we can choose a $\hat{\Omega} \in (\omega_0, \omega_0 + q \sup_{x\in[0,1]} \varpi(x))$ for which there is an m-armed spiral wave solution of* (1.1)–(1.2)*;*

(ii) *for each $q > 0$, if $d \in (0, \frac{\lambda_0}{j_m^2} \cdot R^2)$ is sufficiently close to $\frac{\lambda_0}{j_m^2} \cdot R^2$, then we can choose an $\hat{\Omega} \in (\omega_0, \omega_0 - \Omega_0)$ for which there is an m-armed spiral wave solution of* (1.1)–(1.2). *Moreover, as $d \to (\frac{\lambda_0}{j_m^2} \cdot R^2)^-$, such an m-armed spiral wave shrinks to 0 and the corresponding rotational frequency $\hat{\Omega}$ tends to ω_0.*

As we will see in the coming proposition, the bound for the rotational frequency of the spiral wave given in the first part of Theorem 2 is necessary, and the amplitude of any spiral wave solution of (1.1)–(1.2) is less than 1.

Proposition 1. *Suppose that the functions $\lambda(\cdot)$ and $\omega(\cdot)$ satisfy the assumptions* (H1)–(H2). *Let (u, v) be an m-armed spiral wave solution of* (1.1)–(1.2) *with the rotational frequency $\hat{\Omega}$. Then we have that $\hat{\Omega} \in (\omega_0, \omega_0 + q \sup_{x\in[0,1]} \varpi(x))$ and $u^2(x, y) + v^2(x, y) \in (0, 1)$ for all $(x, y) \in \Gamma_R \setminus \{(0, 0)\}$.*

The first part of Theorem 2 asserts that, for each $d \in (0, \frac{\lambda_0}{j_m^2} \cdot R^2)$, if q is small, then system (1.1)–(1.2) admits an m-armed spiral wave solution, while Theorem 1 asserts that, for each $d \geq \frac{\lambda_0}{j_m^2} \cdot R^2$, system (1.1)–(1.2) has no m-armed spiral wave solutions. Hence $d = \frac{\lambda_0}{j_m^2} \cdot R^2$ is the critical diffusivity for the existence of m-armed spiral wave solutions of system (1.1)–(1.2). The exact characteristics of this critical diffusivity are given by the second part of Theorem 2. Specifically, the second part of Theorem 2 indicates that there is a branch of m-armed spiral waves bifurcating from the uniform steady state $(0, 0)$ when $d = \frac{\lambda_0}{j_m^2} \cdot R^2$, and that the rotational frequency of m-armed spiral waves on this branch tends to ω_0 as $d \to (\frac{\lambda_0}{j_m^2} \cdot R^2)^-$.

Now we would like to make two crucial remarks. First, although we impose an additional constraint on the twist parameter q in the first part of Theorem 2 to guarantee the existence of m-armed spiral wave solutions of system (1.1)–(1.2), this requirement is technical and may not be necessary. In fact, according to the numerical study (see Figs. 1–3 and Section 6), q is not necessarily less than q_0. This raises a question: does a large twist parameter q prevent the existence of spiral wave solutions of system (1.1)–(1.2)? Although we cannot give an affirmative answer to this, the numerical stability analysis in Section 6 indicates that the spiral waves associated with large twist parameter q are unstable. Hence we might conclude that Theorems 1 and 2 give a very elegant criterion for the existence of m-armed spiral wave solutions of the λ–ω system on a disk with no-flux boundary condition. A more complete structure of spiral waves solutions will be given in Section 6.

Second, one might use the bifurcation theory developed by Auchmuty [35] and symmetry arguments to prove the second part of Theorem 2. However, even if one can use such a argument to show that there is a branch of m-armed spiral waves bifurcating from the uniform steady state, one can only establish the existence of m-armed spiral wave solutions of system (1.1)–(1.2) for diffusivity d close to the critical diffusivity $\frac{\lambda_0}{j_m^2} \cdot R^2$, *not for all values of diffusivity $d \in (0, \frac{\lambda_0}{j_m^2} \cdot R^2)$*. Hence it cannot be employed to show the first part of Theorem 2, which is a global result. Moreover, such a bifurcation theory cannot be used to prove the non-existence of spiral waves, and so cannot be employed to show Theorem 1. Here we will use the shooting argument to prove both parts of Theorem 2. With the use of the shooting method, we not only can prove both parts of Theorem 2 in an unified way, but also give the quantitative estimates of the allowed parameters (q and d). The non-existence of spiral waves is shown via an useful identity (see Eq. (3.11)). Since our results do not rely on the specific form of the nonlinearities (λ, ω), our results give a picture for the existence of spiral waves for a generalized class of λ–ω systems.

Our results improve the results of Paullet et al. [33] where the authors considered system (1.1)–(1.2) on the unit disk with the specific nonlinearity $(\lambda(A), \omega(A)) = (1-A^2, 1+qA^2)$. Their results can be stated as follows:

(i) for each $d \geq 1$, system (1.1)–(1.2) has no single-armed spiral wave solutions;

(ii) for each sufficiently small d, if q is small, then system (1.1)–(1.2) admits a single-armed spiral wave solution.

On the other hand, when we apply our results to the case of Paullet et al. [33], the corresponding results can be stated as follows:

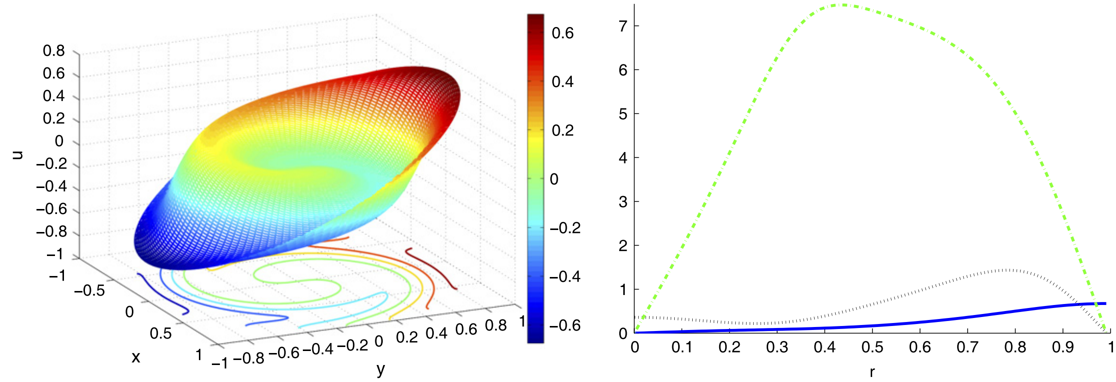

Fig. 1. The left panel shows the profile and contours of the component $u(x, y, t)$ of a single-armed spiral wave solution (u, v) of system (1.1)–(1.2) at a fixed time t for $d = 0.03$, $q = 7$, $R = 1$, $\lambda(A) = 1 - A^2$, and $\omega(A) = 1 + qA^2$. The right panel shows the corresponding profiles of the functions $A(r)$, $A'(r)$, and $k(r)$ defined by (1.1a). The blue solid line: $A(r)$. The black dotted line: $A'(r)$. The green dash–dotted line: $k(r)$.

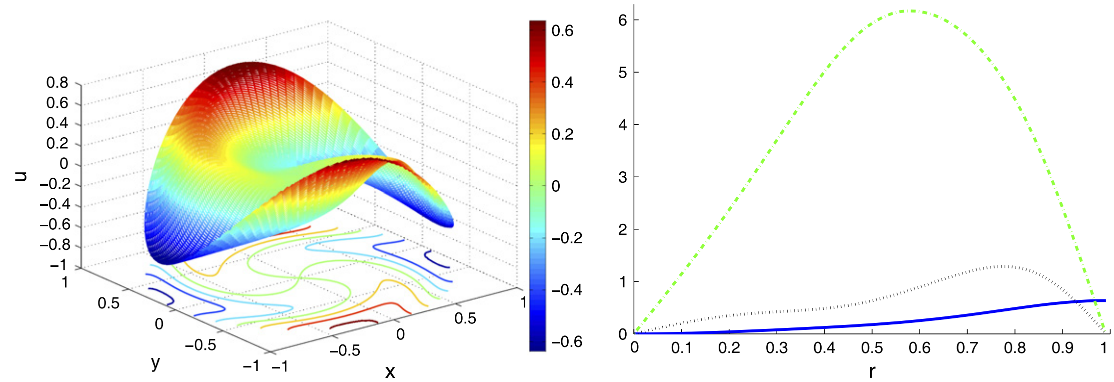

Fig. 2. The left panel shows the profile and contours of the component $u(x, y, t)$ of a two-armed spiral wave solution (u, v) of system (1.1)–(1.2) at a fixed time t for $d = 0.03$, $q = 7$, $R = 1$, $\lambda(A) = 1 - A^2$, and $\omega(A) = 1 + qA^2$. The right panel shows the corresponding profiles of the functions $A(r)$, $A'(r)$, and $k(r)$ defined by (1.1a). The blue solid line: $A(r)$. The black dotted line: $A'(r)$. The green dash–dotted line: $k(r)$.

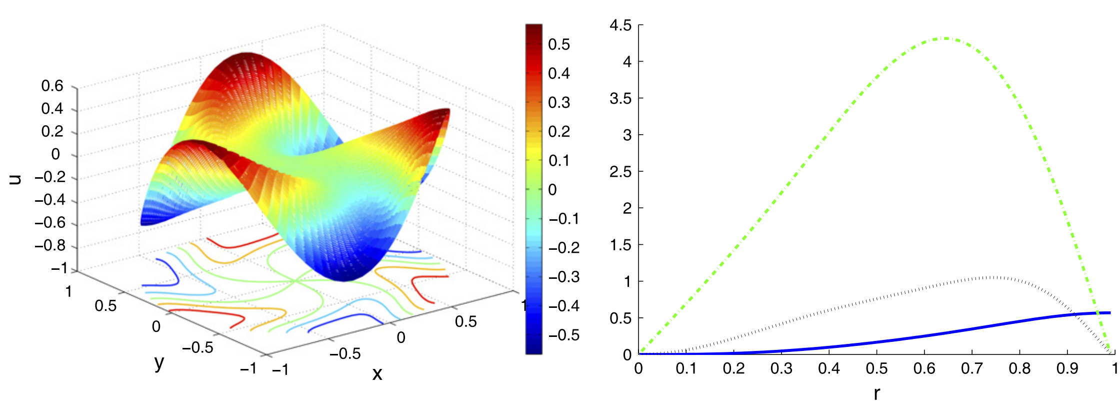

Fig. 3. The left panel shows the profile and contours of the component $u(x, y, t)$ of a three-armed spiral wave solution (u, v) of system (1.1)–(1.2) at a fixed time t for $d = 0.03$, $q = 7$, $R = 1$, $\lambda(A) = 1 - A^2$, and $\omega(A) = 1 + qA^2$. The right panel shows the corresponding profiles of the functions $A(r)$, $A'(r)$, and $k(r)$ defined by (1.1a). The blue solid line: $A(r)$. The black dotted line: $A'(r)$. The green dash–dotted line: $k(r)$.

(i) for each $d \in (\frac{1}{j_m^2}, +\infty)$, system (1.1)–(1.2) has no m-armed spiral wave solutions;
(ii) for each $d \in (0, \frac{1}{j_m^2})$, if q is small, then system (1.1)–(1.2) admits an m-armed spiral wave solution.

Hence our results improve the results of Paullet et al. [33].

We note that the quantity j_m is increasing in m. Hence, there are a number of interesting implications for Theorems 1 and 2, among them the following.

- An m-armed spiral wave solution of (1.1)–(1.2) can exist only if

$$0 < d < \frac{\lambda_0}{j_m^2} \cdot R^2.$$

In particular, for the given diffusivity d and the reaction kinetics parameter λ_0, the quantity

$$R_m := \sqrt{\frac{d}{\lambda_0}} \cdot j_m$$

is the critical radius below which the corresponding circular domain cannot support the formation of m-armed spiral waves. Similarly, we have the critical diffusivity $d_m := \lambda_0 R^2 / j_m^2$ and the critical reaction kinetics parameter $\lambda_{0,m} := d j_m^2 / R^2$.

- For any given diffusivity d which is less than the critical diffusivity $\frac{\lambda_0}{j_m^2} \cdot R^2$ (equivalently, the domain size R is above the critical domain size $\sqrt{\frac{d}{\lambda_0}} \cdot j_m$), if the twist parameter q is small, then (1.1)–(1.2) admits an m-armed spiral wave solution.

- For any given twist parameter $q > 0$, if the diffusivity d is close to the critical diffusivity $\frac{\lambda_0}{j_m^2} \cdot R^2$ (equivalently, the domain size R is close to the critical domain size $\sqrt{\frac{d}{\lambda_0}} \cdot j_m$), then (1.1)–(1.2) admits an m-armed spiral wave solution.
- An increase in the diffusivity d can eliminate spiral waves, while a large domain can aid in the existence of spiral waves.
- A large kinetics parameter (λ_0) can help the existence of spiral waves.

  This is perhaps surprising, since the reaction kinetics (λ, ω) can affect the existence of spiral waves through this simple quantity λ_0.
- The larger the parameter m is, the more space one needs to generate an m-armed spiral wave.

Hence these implications give partial answers to the questions proposed in the introduction. We remark that the spiral waves in the above statements are in the sense of (1.1a). We would like to point out that the above implications for spiral wave solutions of the λ–ω system on a circular disk should not be taken further to imply that these implications hold for general reaction–diffusion systems on a circular disk. Other reaction–diffusion systems, in particular, excitable media, may have very different properties (see also [27,36,37]).

The plan of the remaining parts of this paper is as follows. In Section 3, we give the setting of the proof for Theorem 1, Theorem 2, and Proposition 1. In fact, we will give the equivalent theorems (Theorem 3, Theorem 4, and Proposition 2) for Theorem 1, Theorem 2, and Proposition 1. Then in Section 4 we use an identity inspired by the Sturm–Liouville theory to prove Theorem 3. Proposition 2 and Theorem 4 are shown by the two-parameter shooting method in Section 5. A numerical study of spiral waves is given in Section 6, where the effects of the diffusivity and the (effective) domain size on the amplitude and the rotational frequency of spiral waves, and the stability of spiral waves are discussed. Finally, the conclusion and a short discussion are given in Section 7, and the behavior of m-armed spiral waves for d close to $\frac{\lambda_0}{j_m^2} \cdot R^2$ is discussed in the Appendix.

3. Formulation of the *m*-armed spiral waves problem

3.1. The ODE setting

The rotation symmetry of (1.1)–(1.2) allows us to reduce the question of the existence of spiral waves to a boundary value problem for ordinary differential equations (see [21,24,32,19,18,33]). Indeed, we look for m-armed spiral wave solutions of (1.1)–(1.2) in the form

$$\begin{aligned} u &= A(r)\cos\left((\omega_0 - \Omega)t + m\theta + \int_0^r k(s)\mathrm{d}s\right), \\ v &= A(r)\sin\left((\omega_0 - \Omega)t + m\theta + \int_0^r k(s)\mathrm{d}s\right), \end{aligned} \tag{1.1b}$$

where $\hat{\Omega} := \omega_0 - \Omega$ is the rotational frequency, ω_0 is given in the definition of the function $\omega(\cdot)$, and Ω is a constant which is not known a priori. As before, we require that $A(r)$ is positive in its defined interval except at $r = 0$, and $A(r)$ and $k(r)$ are regular at $r = 0$. It is clear that (1.1a) is equivalent to (1.1b). However, it is easier to work in the frame of (1.1b).

Substitution of (1.1b) into (1.1) and (1.2) yields the following boundary value problem:

$$\begin{aligned} &dA'' + \frac{d}{r}A' + A\left[\lambda(A) - \frac{dm^2}{r^2} - dk^2\right] = 0, \\ &dk' + \left(\frac{d}{r} + 2d\frac{A'}{A}\right)k + (\Omega + q\varpi(A)) = 0, \\ &A(0) = k(0) = 0, \qquad A'(R) = k(R) = 0, \end{aligned} \tag{3.1}$$

where the prime denotes d/dr. Since $A(r)$ and $k(r)$ are regular at $r = 0$, we have that $A(0) = k(0) = 0$ holds. Now introduce the new variables, parameters, and functions by

$$\begin{aligned} &\tilde{r} = \frac{r}{R}, \qquad \tilde{d} = \frac{d}{R^2}, \qquad \tilde{q} = q, \qquad \tilde{\Omega} = \Omega, \\ &\tilde{A}(\tilde{r}) = A(r), \qquad \tilde{k}(\tilde{r}) = Rk(r), \qquad \tilde{\lambda}(\tilde{A}) = \lambda(A), \\ &\tilde{\varpi}(\tilde{A}) = \varpi(A). \end{aligned} \tag{3.2}$$

Substituting these rescaled variables, parameters, and functions into problem (3.1) and dropping the tildes, we get the following equivalent boundary value problem:

$$dA'' + \frac{d}{r}A' + A\left[\lambda(A) - \frac{dm^2}{r^2} - dk^2\right] = 0, \tag{3.3a}$$

$$dk' + \left(\frac{d}{r} + 2d\frac{A'}{A}\right)k + (\Omega + q\varpi(A)) = 0, \tag{3.3b}$$

and

$$A(0) = k(0) = 0, \tag{3.4a}$$

$$A'(1) = k(1) = 0. \tag{3.4b}$$

Therefore, it suffices to consider problem (3.1) for $R = 1$. In view of the above discussion, we can conclude that for each positive twist parameter q, the problem of constructing m-armed spiral wave solutions of (1.1) and (1.2) is equivalent to selecting the constant Ω such that the solution (A, k) of the boundary value problem (3.3a)–(3.3b) and (3.4a)–(3.4b) satisfies that $A(\cdot) > 0$ on $(0, 1]$. Furthermore, in order to establish Theorem 1 (resp. Theorem 2 and Proposition 1), it suffices to show Theorem 3 (resp. Theorem 4 and Proposition 2), which we state below (see Figs. 1–3).

Theorem 3. *Suppose that the functions $\lambda(\cdot)$ and $\omega(\cdot)$ satisfy the assumptions* (H1)–(H2). *If $d \in [\frac{\lambda_0}{j_m^2}, +\infty)$, then, for any $\Omega \in \mathbb{R}$, there exists no solution (A, k) of problem* (3.3a)–(3.3b) *and* (3.4a)–(3.4b) *which satisfies $A > 0$ on $(0, 1]$.*

Theorem 4. *Suppose that the functions $\lambda(\cdot)$ and $\omega(\cdot)$ satisfy the assumptions* (H1)–(H2). *Let q_0 and Ω_0 be defined as in Theorem 2. Then the following hold:*

(i) *for each $d \in (0, \frac{\lambda_0}{j_m^2})$, if $q \in (0, q_0)$, then there exists an $\Omega \in (-q\sup_{x\in[0,1]}\varpi(x), 0)$ such that problem* (3.3a)–(3.3b) *and* (3.4a)–(3.4b) *admits a solution (A, k). Moreover, this solution satisfies $A' > 0$ on $(0, 1)$;*

(ii) *for each $q > 0$, if $d \in (0, \frac{\lambda_0}{j_m^2})$ is sufficiently close to $\frac{\lambda_0}{j_m^2}$, then there exists an $\Omega \in (\Omega_0, 0)$ such that problem* (3.3a)–(3.3b) *and* (3.4a)–(3.4b) *admits a solution (A, k). Moreover, this solution (A, k) satisfies that $A' > 0$ on $(0, 1)$ and $|A(\cdot)| \le |\Omega_0|$ on $[0, 1]$, and that as $d \to (\frac{\lambda_0}{j_m^2})^-$, $\sup_{r\in[0,1]} A(r) \to 0$ and $\Omega \to 0$.*

Proposition 2. *Suppose that the functions $\lambda(\cdot)$ and $\omega(\cdot)$ satisfy the assumptions* (H1)–(H2). *If the solution (A, k) of problem* (3.3a)–(3.3b) *and* (3.4a)–(3.4b) *with $\Omega \in \mathbb{R}$ satisfies that $A(\cdot) > 0$ on $(0, 1]$, then we have $A(r) \in (0, 1)$ for $r \in (0, 1]$ and $\Omega \in (-q\sup_{x\in[0,1]}\varpi(x), 0)$.*

3.2. Associated initial value problems and useful identities

We observe that any solution (A, k) of (3.3a)–(3.3b) and (3.4a) which is regular at $r = 0$ must satisfy $A^{(i)}(0) = 0$ for $i = 1, \ldots, m-1$, and so for such a solution (A, k), we have

$$A(r) \sim \alpha r^m \quad \text{as } r \to 0^+, \qquad k(0) = 0 \tag{3.5}$$

for some $\alpha > 0$. Hence, for each positive q, the problem of constructing m-armed spiral wave solutions of (1.1) and (1.2) can be reduced to selecting the constants α and Ω such that the solution (A, k) of the initial value problem (3.3a)–(3.3b) and (3.5) satisfies that the boundary condition (3.4b) holds and $A(\cdot) > 0$ on $(0, 1]$. Motivated by this, we need to study the initial value problem (3.3a)–(3.3b) and (3.5), which we denote by $(P_{\alpha,\Omega,q})$. We also let $(A(r; \alpha, \Omega, q), k(r; \alpha, \Omega, q))$ be the solution of $(P_{\alpha,\Omega,q})$ defined on the maximal existence interval $[0, R_{\alpha,\Omega,q})$. *If there is no ambiguity, we will omit the dependence of* $(A(r; \alpha, \Omega, q), k(r; \alpha, \Omega, q))$ *on the parameters* α, Ω *and* q. Now, multiplying (3.3b) by rA^2 and integrating from 0 to r, we obtain the equality (see [21,19,33])

$$rA^2(r; \alpha, \Omega, q) \cdot k(r; \alpha, \Omega, q) = -\int_0^r \frac{sA^2(s; \alpha, \Omega, q)}{d} \cdot (\Omega + q\varpi(A(s; \alpha, \Omega, q)))\, ds. \tag{3.6}$$

From this, we can conclude that for the solution $(A(r; \alpha, \Omega, q), k(r; \alpha, \Omega, q))$ of the problem $(P_{\alpha,\Omega,q})$, $k(\cdot; \alpha, \Omega, q)$ is bounded as long as $A(\cdot; \alpha, \Omega, q)$ is bounded.

Before we go further, we need another version of the problem $(P_{\alpha,\Omega,q})$. We recall that J_m is the Bessel function of the first kind of order m. Hence J_m satisfies the Bessel equation of order m:

$$\frac{d^2J}{ds^2} + \frac{1}{s} \cdot \frac{dJ}{ds} + \left(1 - \frac{m^2}{s^2}\right) J = 0. \tag{3.7}$$

Introduce the new independent variable s and the dependent variables $(\tilde{A}, \tilde{k})$ by

$$s := \sqrt{\frac{\lambda_0}{d}}\, r \quad \text{and} \quad \tilde{A}(s) := A(r), \qquad \tilde{k}(s) := \sqrt{\frac{d}{\lambda_0}}\, k(r). \tag{3.8}$$

Then the initial value problem $(P_{\alpha,\Omega,q})$ is transformed into the initial value problem $(\tilde{P}_{\alpha,\Omega,q})$:

$$\frac{d^2\tilde{A}}{ds^2} + \frac{1}{s} \cdot \frac{d\tilde{A}}{ds} + \left(1 - \frac{m^2}{s^2}\right)\tilde{A} + \left(\frac{\lambda(\tilde{A}) - \lambda_0}{\lambda_0} - \tilde{k}^2\right)\tilde{A} = 0, \tag{3.9a}$$

$$\tilde{k}' + \left(\frac{1}{s} + \frac{2\tilde{A}'}{\tilde{A}}\right)\tilde{k} + \frac{\Omega + q\varpi(\tilde{A})}{\lambda_0} = 0, \tag{3.9b}$$

and

$$\tilde{A}(s) \sim \alpha \sqrt{\left(\frac{d}{\lambda_0}\right)^m}\, s^m \quad \text{as } s \to 0^+, \qquad \tilde{k}(0) = 0. \tag{3.10}$$

Now, multiplying (3.7) by $s\tilde{A}$ and (3.9a) by sJ_m, and then subtracting, we get the identity:

$$\frac{d}{ds}\left(s\tilde{A}'J_m - s\tilde{A}J_m'\right) = s\tilde{A}J_m \cdot \left(\frac{\lambda_0 - \lambda(\tilde{A})}{\lambda_0} + \tilde{k}^2\right). \tag{3.11}$$

4. The necessary condition: Proof of Theorem 3

In this section, we will prove Theorem 3. To do this, it suffices to show the following lemma.

Lemma 4.1. *Suppose that* (A, k) *is a solution of problem* (3.3a)–(3.3b) *and* (3.4a)–(3.4b) *with the property that* $A(\cdot) > 0$ *on* $(0, 1]$. *Then we have*

$$0 < d < \frac{\lambda_0}{j_m^2}.$$

Proof. Let $(\tilde{A}, \tilde{k})$ be the solution of Eqs. (3.9a) and (3.9b) corresponding to (A, k). Note that $\tilde{A}$ satisfies that $\tilde{A} > 0$ on $(0, \sqrt{\frac{\lambda_0}{d}}]$, and

$$\tilde{A}(0) = \tilde{A}'\left(\sqrt{\frac{\lambda_0}{d}}\right) = 0. \tag{4.1}$$

Integrating (3.11) from 0 to $\sqrt{\frac{\lambda_0}{d}}$ and using (4.1), we get

$$-\sqrt{\frac{\lambda_0}{d}}\,\tilde{A}\left(\sqrt{\frac{\lambda_0}{d}}\right)J_m'\left(\sqrt{\frac{\lambda_0}{d}}\right) = \int_0^{\sqrt{\frac{\lambda_0}{d}}} s\tilde{A}(s)J_m(s) \cdot \left(\frac{\lambda_0 - \lambda(\tilde{A}(s))}{\lambda_0} + \tilde{k}^2(s)\right) ds. \tag{4.2}$$

For contradiction, we assume that $d \geq \frac{\lambda_0}{j_m^2}$. Hence $j_m \geq \sqrt{\frac{\lambda_0}{d}}$. This, together with the facts that $\lambda(\cdot)$ is decreasing and that $\tilde{A} > 0$ on $(0, \sqrt{\frac{\lambda_0}{d}}]$, implies that the right-hand side of (4.2) is positive. Therefore we have $J_m'(\sqrt{\frac{\lambda_0}{d}}) < 0$, which implies that $\sqrt{\frac{\lambda_0}{d}} > j_m$. This is a contradiction. Hence the assertion of this lemma is established. □

5. The sufficient condition: Proof of Theorem 4

In this section, we will prove Theorem 4. In Sections 5.1–5.4, we will show the first part of Theorem 4, while the second part of Theorem 4 is proven in Section 5.5.

5.1. The proof for the existence of m-armed spiral waves when $q \in (0, q_0)$

We shall use the two-parameter shooting scheme [33] to prove the first part of Theorem 4. Specifically, we will vary the free parameters α and Ω. We briefly describe the plan of the proof, which consists of three steps. First, for ease of notation, we define the quantity R_1 as follows.

Definition 1. Let (A, k) be the solution of the problem $(P_{\alpha,\Omega,q})$. We set $R_1 = R_1(\alpha, \Omega, q)$ as the first positive zero of A' if it exists.

We also need the following two sets:

$$A_{1,q} := \{(\alpha, \Omega) \in (0, \infty) \times [-q \sup_{x\in[0,1]} \varpi(x), 0] \mid R_{\alpha,\Omega,q} \in (0, 1] \text{ and } A'(\cdot; \alpha, \Omega, q) > 0 \text{ on } (0, R_{\alpha,\Omega,q})\} \bigcup \{(\alpha, \Omega) \in (0, \infty) \times [-q \sup_{x\in[0,1]} \varpi(x), 0] \mid A'(\cdot; \alpha, \Omega, q) > 0 \text{ on } (0, 1]\},$$

$$A_{2,q} := \{(\alpha, \Omega) \in (0, \infty) \times [-q \sup_{x\in[0,1]} \varpi(x), 0] \mid R_1(\alpha, \Omega, q) \in (0, 1)\}.$$

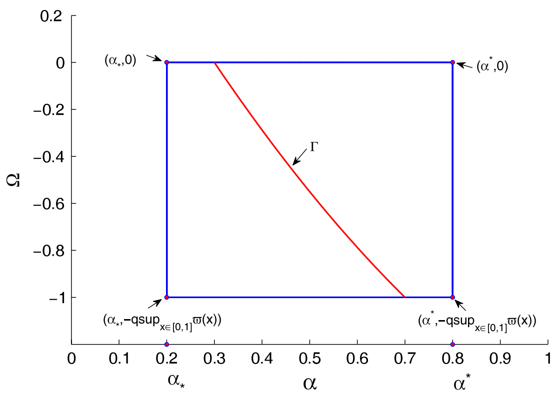

Fig. 4. The shooting region $\mathcal{A}$.

The restriction on Ω is motivated by Proposition 2. Also recall the quantity q_0 defined in Theorem 2. Now fix a $q \in (0, q_0)$. Then the plan of the proof consists of the following three steps.

Step 1. We find a large positive number α^*, which is independent of $q > 0$, such that $\{\alpha^*\} \times [-q \sup_{x\in[0,1]} \varpi(x), 0] \subset A_{1,q}$ (see Lemma 5.4 in Section 5.2).

Step 2. We find a small positive number $\alpha_* < \alpha^*$, which may depend on $q \in (0, q_0)$, such that $\{\alpha_*\} \times [-q \sup_{x\in[0,1]} \varpi(x), 0] \subset A_{2,q}$ (see Lemma 5.6 in Section 5.3). We remark that we use the arguments based on a comparison with the Bessel equation and a detailed estimate on $k(\cdot)$ to reach *Step* 1–*Step* 2, and so these arguments are different from those in [33] where the authors use the rotating straight lines ("pinwheels") as building bricks.

Step 3. This step is a slight modification of that in Paullet et al. [33, pp. 1393–1395]. From *Step* 1 and *Step* 2, we use a topological theorem of McLeod and Serrin [38] to conclude that, for any fixed $q \in (0, q_0]$, there exists a continuum $\Gamma \subseteq \{\alpha \in [\alpha_*, \alpha^*], \Omega \in [-q \sup_{x\in[0,1]} \varpi(x), 0]\}$ which connects the lines $\{\alpha > 0, \Omega = 0\}$ and $\{\alpha > 0, \Omega = -q \sup_{x\in[0,1]} \varpi(x)\}$, and satisfies that $A'(\cdot; \alpha, \Omega, q) > 0$ on $(0, 1)$ and $A'(1; \alpha, \Omega, q) = 0$ for all $(\alpha, \Omega) \in \Gamma$ (see Fig. 4). Then we show that there exists $(\alpha_1, \Omega_1) \in \Gamma$ (resp. $(\alpha_2, \Omega_2) \in \Gamma$) for which $k(1; \alpha_1, \Omega_1, q) < 0$ (resp. $k(1; \alpha_2, \Omega_2, q) > 0$). This implies that there exists a $(\bar{\alpha}, \bar{\Omega}) \in \Gamma$ such that $k(1; \bar{\alpha}, \bar{\Omega}, q) = 0$. Therefore, $(A(\cdot; \bar{\alpha}, \bar{\Omega}, q), k(\cdot; \bar{\alpha}, \bar{\Omega}, q))$ is the desired solution of problem (3.3a)–(3.3b) and (3.4a)–(3.4b), thereby completing the proof of the first part of Theorem 4 (see Section 5.4).

In the following, we collect some lemmas which concern the properties of the sets $A_{1,q}$ and $A_{2,q}$. First, we show that $R_1(\alpha, \Omega, q)$ has a positive lower bound.

Lemma 5.1. *Suppose that* (A, k) *is the solution of the initial value problem* $(P_{\alpha,\Omega,q})$*, and that* R_1 *is the first positive zero of* A'*. Then we have* $R_1 > \sqrt{\frac{d}{\lambda_0}} j_m$ *and* $A(r) \in (0, 1)$ *for* $r \in (0, R_1]$.

Proof. The assertion for the estimate of R_1 can be proved by a similar argument as in Lemma 4.1, and so we omit it.

For the second assertion, we note that $A'(\cdot) > 0$ on $(0, R_1)$ and $A'(R_1) = 0$. Hence we have $A''(R_1) \le 0$, which, together with (3.3a), implies that

$$d \cdot \frac{m^2}{R_1^2} + dk^2(R_1) - \lambda(A(R_1)) \le 0.$$

Therefore, we have $\lambda(A(R_1)) > 0$. Together with the fact that $\lambda(\cdot)$ is decreasing and $\lambda(1) = 0$, it follows that $A(R_1) \in (0, 1)$. Finally, the second assertion follows from the fact that $A(R_1) \in (0, 1)$ and the definition of R_1. This completes the proof of this lemma. □

Following the proof of the second assertion in Lemma 5.1, we have the following lemma which, together with the continuous dependence on the parameters α and Ω, shows that the set $A_{1,q}$ is open.

Lemma 5.2. *Suppose that* (A, k) *is the solution of the initial value problem* $(P_{\alpha,\Omega,q})$*, and that for some* $r_1 \in (0, R_{\alpha,\Omega,q})$*,* $A(r) \in [0, 1)$ *for all* $r \in [0, r_1)$ *and* $A(r_1) = 1$*. Then we have that* $A'(r) > 0$ *for all* $r \in [r_1, R_{\alpha,\Omega,q})$.

Finally, we give the proof for Proposition 2. To do this, it suffices to show the following lemma.

Lemma 5.3. *If the solution* $(A(r), k(r))$ *of the problem* $(P_{\alpha,\Omega,q})$ *satisfies that* (3.4b) *holds and* $A(\cdot) > 0$ *on* $(0, 1]$*, then we have* $A(r) \in (0, 1)$ *for* $r \in (0, 1]$ *and* $\Omega \in (-q \sup_{x\in[0,1]} \varpi(x), 0)$.

Proof. We first prove the first assertion. For contradiction, we assume that there is a first $r_1 \in (0, 1]$ such that $A(r) \in [0, 1)$ for all $r \in [0, r_1)$ and $A(r_1) = 1$. Then, by Lemma 5.2, we have $A'(r) > 0$ for all $r \in [r_1, 1]$. This is a contradiction to the fact that $A'(1) = 0$, thereby completing the proof of the first assertion.

For the second assertion, we use the method of [33, Lemma 3.1]. Indeed, by evaluating (3.6) at $r = 1$ and using (3.4b), we obtain $\int_0^1 sA^2(s) \cdot (\Omega + q\varpi(A(s)))\,ds = 0$, which implies that

$$\frac{-\Omega}{q} = \frac{\int_0^1 sA^2(s)\varpi(A(s))ds}{\int_0^1 sA^2(s)ds}.$$

Recall that $\varpi(x) > 0$ for $x \in (0, 1]$ and that $A(r) \in (0, 1)$ for $r \in (0, 1]$. Then it follows from the above inequality that $\Omega \in (-q \sup_{x\in[0,1]} \varpi(x), 0)$. This proves the second assertion of this lemma, thereby completing the proof of this lemma. □

5.2. The nonemptiness of the set $A_{1,q}$

The following lemma concerns the nonemptiness of the set $A_{1,q}$ for any $q > 0$.

Lemma 5.4. *Let* $d \in (0, \frac{\lambda_0}{j_m^2})$*,* $q > 0$*,* $\Omega \le 0$*, and* $(A(\cdot; \alpha, \Omega, q), k(\cdot; \alpha, \Omega, q))$ *be the solution of the initial value problem* $(P_{\alpha,\Omega,q})$*. Then there is a sufficiently large* $\alpha^* = \alpha^*(d, \lambda_0)$*, which is independent of* (Ω, q)*, such that, for each* $\alpha \ge \alpha^*$*,* $A(\cdot; \alpha, \Omega, q) > 0$ *and* $A'(\cdot; \alpha, \Omega, q) > 0$ *on* $(0, \hat{r}]$*, and* $A(\hat{r}; \alpha, \Omega, q) = 1$ *for some* $\hat{r} = \hat{r}(\alpha) \in (0, \sqrt{\frac{d}{\lambda_0}} j_m)$.

Proof. Let $(\tilde{A}(\cdot; \alpha, \Omega, q), \tilde{k}(\cdot; \alpha, \Omega, q))$ be the solution of the initial value problem $(\tilde{P}_{\alpha,\Omega,q})$ corresponding to $(A(\cdot; \alpha, \Omega, q), k(\cdot; \alpha, \Omega, q))$. Then with the use of (3.8) and Lemma 5.1, we see that $\tilde{A}'(\cdot; \alpha, \Omega, q) > 0$ on $(0, \min\{\tilde{R}_{\alpha,\Omega,q}, j_m\})$, where $[0, \tilde{R}_{\alpha,\Omega,q})$ is the maximal existence interval of $(\tilde{A}(\cdot; \alpha, \Omega, q), \tilde{k}(\cdot; \alpha, \Omega, q))$.

From (3.10), $\tilde{A}(\cdot; \alpha, \Omega, q)$ satisfies that $\tilde{A}(s; \alpha, \Omega, q) \sim \alpha \sqrt{(\frac{d}{\lambda_0})^m} s^m$ as $s \to 0^+$, and note that $J_m(s) \sim s^m/(2^m \cdot m!)$ as $s \to 0^+$. Hence we have $\lim_{s\to 0^+} \tilde{A}(s; \alpha, \Omega, q)/J_m(s) = \alpha \cdot (2^m \cdot m!)\sqrt{(\frac{d}{\lambda_0})^m}$. Since $\lambda(\cdot)$ is decreasing, it follows from (3.11) that $s\tilde{A}'(s; \alpha, \Omega, q)J_m(s) - s\tilde{A}(s; \alpha, \Omega, q)J_m'(s) > 0$ for all $s \in (0, \tilde{R}_{\alpha,\Omega,q})$. Hence $(\tilde{A}(\cdot; \alpha, \Omega, q)/J_m(\cdot))' > 0$ on $(0, \min\{\tilde{R}_{\alpha,\Omega,q}, j_m\})$. Taken together, we obtain that $\tilde{A}(s; \alpha, \Omega, q) > \alpha \cdot (2^m \cdot m!)\sqrt{(\frac{d}{\lambda_0})^m} J_m(s)$ for all $s \in (0, \min\{\tilde{R}_{\alpha,\Omega,q}, j_m\})$. From this and Lemma 5.1, we can conclude that there exists a sufficiently large $\alpha^* = \alpha^*(d, \lambda_0)$ such that, for each $\alpha \ge \alpha^*$, $\tilde{A}'(\cdot; \alpha, \Omega, q) > 0$ on $(0, \hat{s}]$ and $\tilde{A}(\hat{s}; \alpha, \Omega, q) = 1$ for some $\hat{s} = \hat{s}(\alpha) \in (0, j_m)$. Finally, by transforming back to the origin variable $(A(\cdot; \alpha, \Omega, q), k(\cdot; \alpha, \Omega, q))$, the assertion of this lemma follows. □

5.3. The nonemptiness of the set $A_{2,q}$

Before showing that $A_{2,q}$ is nonempty, we need some estimates on $(A(\cdot;\alpha,\Omega,q), k(\cdot;\alpha,\Omega,q))$ for small positive α.

Lemma 5.5. *Let $q \in (0, \sqrt{d\lambda(\varepsilon_0)}/\sup_{x\in[0,1]}\varpi(x)]$ with $\varepsilon_0 \in (0, 1/2)$. Then, for each $\alpha \in (0,\varepsilon_0)$, we have*

$$0 < A(r;\alpha,\Omega,q) \le \alpha r^m \quad \text{and}$$
$$|k(r;\alpha,\Omega,q)| \le \left(q \sup_{x\in[0,1]}\varpi(x)/d\right) r \tag{5.1}$$

for all $r \in (0, \min\{R_1(\alpha,\Omega,q), 1\}]$ and $\Omega \in [-q\sup_{x\in[0,1]}\varpi(x), 0]$.

Proof. First, for ease of use, we set $\hat{R} = \hat{R}(\alpha,\Omega,q) := \min\{R_1(\alpha,\Omega,q), 1\}$, and write $A(\cdot;\alpha,\Omega,q)$ (rep. $k(\cdot;\alpha,\Omega,q)$) as A (resp. k). Now, with the help of (3.3a), one can check that the following equality holds.

$$A'(r) = \delta_m\alpha + \frac{m^2-1}{2}\int_0^r\left(\frac{1}{r^2}+\frac{1}{s^2}\right)A(s)\mathrm{d}s$$
$$+\frac{1}{2d}\int_0^r\left(1+\frac{s^2}{r^2}\right)A(s)\left(dk^2(s)-\lambda(A(s))\right)\mathrm{d}s, \tag{5.2}$$

where $\delta_m = 1$ if $m = 1$, and $\delta_m = 0$ if $m > 1$.

We consider the case when $m = 1$. Since $\lambda(0) > 0$ and $k(0) = 0$, from (5.2), it follows that $0 < A'(r) < \alpha$ for small r. Hence we have $0 < A(r) < \alpha r$ for small r. Set $\hat{r} := \sup\{r \in (0,\hat{R}] \mid 0 < A(r') < \alpha r' \text{ for } r' \in (0,r]\}$. Then we have $\hat{r} \le \hat{R} \le 1$. We claim that $\hat{r} = \hat{R}$. Suppose not. Then we have $\hat{r} < \hat{R}$, and, from the definition of $\hat{r}$, we have $0 < A(r) < \alpha r < \varepsilon_0$ for $r \in (0,\hat{r})$ and $A(\hat{r}) = \alpha\hat{r} < \varepsilon_0$. Now we give an estimate on $k(\cdot)$. To do this, we use Eq. (3.6) to estimate $k(\cdot)$ on $(0,\hat{r}]$ as follows:

$$\begin{aligned}|k(r)| &\le \frac{1}{d}\int_0^r \frac{sA^2(s)}{rA^2(r)}\cdot|-\Omega - q\varpi(A(s))|\mathrm{d}s\\
&\le \frac{1}{d}\int_0^r \max\{|\Omega|, q\varpi(A(s))\}\mathrm{d}s\\
&\quad (A(\cdot) \text{ is increasing on } [0,\hat{r}] \text{ and } \Omega \le 0)\\
&\le \frac{q\sup_{x\in[0,1]}\varpi(x)}{d}\cdot r \quad (\text{since } A(\cdot)\in[0,1) \text{ on } [0,\hat{r}]\\
&\quad \text{and } |\Omega| \le q\sup_{x\in[0,1]}\varpi(x)).\end{aligned} \tag{5.3}$$

With these estimates on (A, k) and the choice of q, we can conclude that $dk^2(r) - \lambda(A(r)) < 0$ for $r \in (0,\hat{r}]$. This, together with (5.2), implies that $A'(r) < \alpha$ for $r \in (0,\hat{r}]$, which yields that $A(\hat{r}) < \alpha\hat{r}$. This is a contradiction, hence completing the proof of this claim. With the use of this claim, we can proceed as in (5.3) to conclude that $|k(r)| \le (q\sup_{x\in[0,1]}\varpi(x)/d)r$ for $r \in [0,\hat{R}]$.

Now we turn to the case when $m > 1$. Fix an arbitrary $\beta \in (\alpha,\varepsilon_0)$. From (3.5), we have that $A(r) \sim \alpha r^m$ as $r \to 0^+$. Hence $0 < A(r) < \beta r^m$ for small r. Set $\tilde{r} := \sup\{r \in (0,\hat{R}] \mid 0 < A(r') < \beta(r')^m \text{ for } r' \in (0,r]\}$. We claim that $\tilde{r} = \hat{R}$. Suppose not. Then we have $0 < \tilde{r} < \hat{R}$, and from the definition of $\tilde{r}$, we have $0 < A(r) < \beta r^m < \varepsilon_0$ for $r \in (0,\tilde{r})$ and $A(\tilde{r}) = \beta\tilde{r}^m < \varepsilon_0$. Proceeding as in the case when $m = 1$, we can show that $|k(r)| \le (q\sup_{x\in[0,1]}\varpi(x)/d)r$ for $r \in [0,\tilde{r}]$, and hence we have $dk^2(r) - \lambda(A(r)) < 0$ for $r \in (0,\tilde{r}]$. This, together with (5.2), implies that the following estimate holds:

$$\begin{aligned}A'(r) &< \frac{m^2-1}{2}\int_0^r\left(\frac{1}{r^2}+\frac{1}{s^2}\right)A(s)\mathrm{d}s\\
&< \frac{m^2-1}{2}\int_0^r\left(\frac{1}{r^2}+\frac{1}{s^2}\right)\beta s^m\mathrm{d}s = m\beta r^{m-1}\end{aligned}$$

for $r \in [0,\tilde{r}]$. Integrating the above inequality from 0 to $\tilde{r}$, we obtain $A(\tilde{r}) < \beta\tilde{r}^m$. This is a contradiction. Hence we have $\tilde{r} = \hat{R}$, and so $0 < A(r) < \beta r^m$ for all $r \in (0,\hat{R})$ and any $\beta \in (\alpha,\varepsilon_0)$. Letting $\beta \to \alpha^+$ and using the definition of $\hat{R}$, we obtain $0 < A(r) \le \alpha r^m$ for all $r \in (0,\hat{R}]$. Finally, the estimate for $k(\cdot)$ follows from the same lines as in the case when $m = 1$. This completes the proof of this lemma. □

Now we are in a position to show that $A_{2,q}$ is nonempty. Intuitively, $R_1(\alpha,\Omega,q)$ can exist for small α. However, we need the fact that $R_1(\alpha,\Omega,q)$ exists and $R_1(\alpha,\Omega,q) \in (0,1)$ *for small* α, which will be established in the following lemma. We also note that Paullet et al. [33] have used a restrictive method to show the following lemma for small d, $m = 1$, and $\lambda(A) = 1 - A^2$ and $\varpi(A) \equiv 0$. Here, with the use of different arguments, we can prove this lemma for $d \in (0, \frac{\lambda_0}{j_m^2})$ (not only for small d), and for general $m \in \mathbb{N}$ and reaction kinetics $(\lambda(\cdot),\omega(\cdot))$.

Lemma 5.6. *Let $d \in (0, \frac{\lambda_0}{j_m^2})$ and $q \in (0, q_0)$, where q_0 is defined in Theorem 2. Then there exists an $\alpha_* = \alpha_*(d,q) > 0$ such that $R_1(\alpha,\Omega,q) \in (0,1)$ holds for each $\alpha \in (0,\alpha_*]$ and $\Omega \in [-q\sup_{x\in[0,1]}\varpi(x), 0]$.*

Proof. By the definition of q_0 and the fact that $\lambda(\cdot)$ is decreasing, we can choose a small number $\varepsilon_0 \in (0,1/2)$ such that $q \in (0, \sqrt{d\lambda(\varepsilon_0)}/\sup_{x\in[0,1]}\varpi(x))$. Now, for contradiction, we assume that there are two sequences $\{\alpha_n\} \subset (0,\varepsilon_0)$ and $\{\Omega_n\} \subset [-q\sup_{x\in[0,1]}\varpi(x), 0]$ with $\alpha_n \to 0$ as $n \to +\infty$ such that the solution $(A(r;\alpha_n,\Omega_n), k(r;\alpha_n,\Omega_n))$ of $(P_{\alpha_n,\Omega_n,q})$ satisfies that $A'(r;\alpha_n,\Omega_n) > 0$ for all $r \in (0, \min\{1, R_{\alpha_n,\Omega_n,q}\})$. Here, for simplicity, we omit the dependence of $(A(r;\alpha_n,\Omega_n), k(r;\alpha_n,\Omega_n))$ on the parameter q. From Lemma 5.5, we may assume that $A'(r;\alpha_n,\Omega_n) > 0$ for each $r \in (0,1)$ and $n \in \mathbb{N}$, and that the estimates on $(A(\cdot;\alpha_n,\Omega_n), k(\cdot;\alpha_n,\Omega_n))$ given by (5.1) hold for each $r \in (0,1]$ and $n \in \mathbb{N}$.

Now, for each $n \in \mathbb{N}$, let $(\tilde{A}(s;\alpha_n,\Omega_n), \tilde{k}(s;\alpha_n,\Omega_n))$ be the solution of the problem $(\tilde{P}_{\alpha_n,\Omega_n,q})$ corresponding to $(A(r;\alpha_n,\Omega_n), k(r;\alpha_n,\Omega_n))$. Recall that

$$s := \sqrt{\frac{\lambda_0}{d}}r \quad \text{and} \quad \tilde{A}(s;\alpha_n,\Omega_n) := A(r;\alpha_n,\Omega_n),$$
$$\tilde{k}(s;\alpha_n,\Omega_n) := \sqrt{\frac{d}{\lambda_0}}k(r;\alpha_n,\Omega_n).$$

Hence we have $\tilde{A}'(\cdot;\alpha_n,\Omega_n) > 0$ on $(0, \sqrt{\frac{\lambda_0}{d}})$. Also note that $\sqrt{\frac{\lambda_0}{d}} > j_m$ and $\tilde{A}(\cdot;\alpha_n,\Omega_n)$ satisfies that $\tilde{A}(s;\alpha_n,\Omega_n) \sim \alpha_n\sqrt{(\frac{d}{\lambda_0})^m}s^m$ as $s \to 0^+$. Integrating (3.11) from 0 to s, we have that

$$\begin{aligned}&s\tilde{A}'(s;\alpha_n,\Omega_n)J_m(s) - s\tilde{A}(s;\alpha_n,\Omega_n)J_m'(s)\\
&\quad = \int_0^s \xi\tilde{A}(\xi;\alpha_n,\Omega_n)J_m(\xi)\\
&\qquad \times\left(\frac{\lambda_0 - \lambda(\tilde{A}(\xi;\alpha_n,\Omega_n))}{\lambda_0} + \tilde{k}^2(\xi;\alpha_n,\Omega_n)\right)\mathrm{d}\xi\end{aligned}$$

holds for $s \in [0, \sqrt{\frac{\lambda_0}{d}}]$ and $n \in \mathbb{N}$. Dividing the above equation by $s\tilde{A}(s;\alpha_n,\Omega_n)J_m(s)$, we obtain

$$\begin{aligned}\frac{\tilde{A}'(s;\alpha_n,\Omega_n)}{\tilde{A}(s;\alpha_n,\Omega_n)} &= \int_0^s \frac{\xi}{s}\cdot\frac{\tilde{A}(\xi;\alpha_n,\Omega_n)}{\tilde{A}(s;\alpha_n,\Omega_n)}\\
&\quad \times\frac{J_m(\xi)}{J_m(s)}\cdot\left(\frac{\lambda_0 - \lambda(\tilde{A}(\xi;\alpha_n,\Omega_n))}{\lambda_0}\right)\mathrm{d}\xi\end{aligned}$$

$$+\left(\frac{J_m'(s)}{J_m(s)}+\int_0^s\frac{\xi}{s}\cdot\frac{\tilde{A}(\xi;\alpha_n,\Omega_n)}{\tilde{A}(s;\alpha_n,\Omega_n)}\times\frac{J_m(\xi)}{J_m(s)}\cdot\tilde{k}^2(\xi;\alpha_n,\Omega_n)\mathrm{d}\xi\right)$$
$$:=I(s)+II(s) \tag{5.4}$$

for each $s\in(0,\min\{\sqrt{\frac{\lambda_0}{d}},j_{m,0}\})$, where $j_{m,0}$ denotes the first positive zero of $J_m(\cdot)$. In the following, we will estimate the right-hand side of (5.4). To do this, we let $\hat{j}_m$ be the maximal point of the function $\sqrt{\frac{-J_m'(s)}{\int_0^s\xi^2J_m(\xi)\mathrm{d}\xi}}$ on $[j_m,\min\{j_{m,0},\sqrt{\frac{\lambda_0}{d}}\}]$. Since $J_m'(j_m)=0$ (resp. $J_m'(j_{m,0})<0$ and $J_m''(j_{m,0})>0$), we have $\hat{j}_m\neq j_m$ (resp. $\hat{j}_m\neq j_{m,0}$). Hence we can conclude that $\hat{j}_m\in(j_m,j_{m,0})$. Now we choose a sufficiently large number M_m such that

$$0<J_m(s)/J_m(\hat{j}_m)\le M_m$$

for $s\in(0,\hat{j}_m]$. With the use of these facts, we first estimate $I(\hat{j}_m)$ as follows:

$$\begin{aligned}I(\hat{j}_m)&=\int_0^{\hat{j}_m}M_m\cdot\frac{\lambda_0-\lambda(\tilde{A}(\xi;\alpha_n,\Omega_n))}{\lambda_0}\mathrm{d}\xi\\&\quad\left(\text{since }\tilde{A}(\cdot;\alpha_n,\Omega_n)\text{ is increasing on }\left(0,\sqrt{\frac{\lambda_0}{d}}\right)\right)\\&\le\int_0^{\hat{j}_m}\frac{M_m\sup_{x\in[0,1]}|\lambda'(x)|}{\lambda_0}\cdot\tilde{A}(\xi;\alpha_n,\Omega_n)\mathrm{d}\xi\\&\quad(\text{the mean-value theorem, }\lambda_0=\lambda(0),\text{ and }\lambda(\cdot)\text{ is decreasing})\\&\le\int_0^{\hat{j}_m}\frac{M_m\sup_{x\in[0,1]}|\lambda'(x)|}{\lambda_0}\cdot\alpha_n\sqrt{\left(\frac{d}{\lambda_0}\right)^m}\xi^m\mathrm{d}\xi\\&\quad(\text{by Lemma 5.5})\\&\le\alpha_nM_m\hat{j}_m^{m+1}\sqrt{\left(\frac{d}{\lambda_0}\right)^m}\cdot\frac{\sup_{x\in[0,1]}|\lambda'(x)|}{(m+1)\lambda_0}.\end{aligned}$$

Next we estimate $II(\hat{j}_m)$ as follows:

$$\begin{aligned}II(\hat{j}_m)&=\frac{1}{J_m(\hat{j}_m)}\cdot\left(J_m'(\hat{j}_m)+\int_0^{\hat{j}_m}\frac{\xi}{s}\cdot\frac{\tilde{A}(\xi;\alpha_n,\Omega_n)}{\tilde{A}(s;\alpha_n,\Omega_n)}\times J_m(\xi)\cdot\tilde{k}^2(\xi;\alpha_n,\Omega_n)\mathrm{d}\xi\right)\\&<\frac{1}{J_m(\hat{j}_m)}\cdot\left(J_m'(\hat{j}_m)+\int_0^{\hat{j}_m}J_m(\xi)\cdot\tilde{k}^2(\xi;\alpha_n,\Omega_n)\mathrm{d}\xi\right)\\&\quad\left(\text{since }\tilde{A}(\cdot;\alpha_n,\Omega_n)\text{ is increasing on }\left(0,\sqrt{\frac{\lambda_0}{d}}\right)\right)\\&\le\frac{1}{J_m(\hat{j}_m)}\cdot\left(J_m'(\hat{j}_m)+\frac{q^2(\sup_{x\in[0,1]}\varpi(x))^2}{\lambda_0^2}\int_0^{\hat{j}_m}\xi^2J_m(\xi)\mathrm{d}\xi\right)\\&\quad(\text{by Lemma 5.5})\\&<0.\quad(\text{by the choice of }q).\end{aligned}$$

In view of the estimates on $I(\hat{j}_m)$ and $II(\hat{j}_m)$, we can conclude from (5.4) that

$$\frac{\tilde{A}'(\hat{j}_m;\alpha_n,\Omega_n)}{\tilde{A}(\hat{j}_m;\alpha_n,\Omega_n)}\le\alpha_nM_m\hat{j}_m^{m+1}\sqrt{\left(\frac{d}{\lambda_0}\right)^m}\cdot\frac{\sup_{x\in[0,1]}|\lambda'(x)|}{(m+1)\lambda_0}+II(\hat{j}_m)$$

holds for each $n\in\mathbb{N}$. Note that $II(\hat{j}_m)$ is bounded above by a negative number which is independent of $n\in\mathbb{N}$. Together with the above inequality, this estimate implies that $\tilde{A}'(\hat{j}_m;\alpha_n,\Omega_n)/\tilde{A}(\hat{j}_m;\alpha_n,\Omega_n)$ must be negative for all $n>n_0$ and for some large n_0. This is a contradiction to the assumption that $\tilde{A}'(\cdot;\alpha_n,\Omega_n)>0$ on $(0,\sqrt{\frac{\lambda_0}{d}})$. The proof is thus completed. □

5.4. Proof of the first part of Theorem 4

With the preparation in the previous subsections, we can slightly modify the arguments of Paullet et al. [33, pp. 1393–1395] to complete the proof of the first part of Theorem 4, and we include these arguments here for the sake of completeness and later use in Section 5.5.1. To begin with, let q_0 be defined as in Theorem 2. Throughout the remainder of this section, we fix $d\in(0,\lambda_0/j_m^2)$ and $q\in(0,q_0)$. We assume that the pair of parameters (α,Ω) lies in the set

$$\mathcal{A}:=\left\{(\alpha,\Omega)\mid\alpha\in[\alpha_*,\alpha^*],\ \Omega\in[-q\sup_{x\in[0,1]}\varpi(x),0]\right\},$$

and for simplicity we set

$$\tilde{A}_{1,q}:=A_{1,q}\bigcap\mathcal{A}\quad\text{and}\quad\tilde{A}_{2,q}:=A_{2,q}\bigcap\mathcal{A}.$$

We repeat again that the restriction on Ω is motivated by Proposition 2. Our goal is to look for $(\bar{\alpha},\bar{\Omega})$ for which the solution $(A(r;\bar{\alpha},\bar{\Omega},q),k(r;\bar{\alpha},\bar{\Omega},q))$ of the problem $(P_{\alpha,\Omega,q})$ satisfies (3.4b). Now we summarize the associated properties on the sets $\tilde{A}_{1,q}$ and $\tilde{A}_{2,q}$ as follows.

- $\tilde{A}_{1,q}\supset\{\alpha^*\}\times[-q\sup_{x\in[0,1]}\varpi(x),0]$ by Lemma 5.4.
- $\tilde{A}_{1,q}$ is relatively open in $\mathcal{A}$ by Lemma 5.2 and the continuous dependence on the parameters α and Ω.
- $\tilde{A}_{2,q}\supset\{\alpha_*\}\times[-q\sup_{x\in[0,1]}\varpi(x),0]$ by Lemma 5.6.
- $\tilde{A}_{2,q}$ is relatively open in $\mathcal{A}$ by the continuous dependence on the parameters α and Ω.
- $\tilde{A}_{1,q}\bigcap\tilde{A}_{2,q}=\emptyset$ by the definitions of the sets $A_{1,q}$ and $A_{2,q}$.

These properties, together with the theorem due to McLeod and Serrin [38], immediately imply that we can find a continuum $\Gamma\subset\mathcal{A}\setminus(\tilde{A}_{1,q}\cup\tilde{A}_{2,q})$ which connects the segments $\{\alpha\in[\alpha_*,\alpha^*],\ \Omega=0\}$ and $\{\alpha\in[\alpha_*,\alpha^*],\ \Omega=-q\cdot\sup_{x\in[0,1]}\varpi(x)\}$. From the definition of Γ, we can conclude that, for each $(\alpha,\Omega)\in\Gamma$, $A(\cdot;\alpha,\Omega,q)$ and $k(\cdot;\alpha,\Omega,q)$ are defined on $[0,1]$, and

$$A'(\cdot;\alpha,\Omega,q)>0\quad\text{on }(0,1),\qquad\text{and}\quad A'(1;\alpha,\Omega,q)=0$$
$$\text{for all }(\alpha,\Omega)\in\Gamma.$$

Furthermore, together with Lemma 5.1, $A(\cdot;\alpha,\Omega,q)\in(0,1)$ on $(0,1]$.

Note that $k(1;\alpha,\Omega,q)$ depends continuously on $(\alpha,\Omega)\in\Gamma$. Suppose that we can show that $k(1;\alpha,\Omega,q)<0$ (resp. $k(1;\alpha,\Omega,q)>0$) for some $(\alpha_1,\Omega_1)\in\Gamma$ (resp. $(\alpha_2,\Omega_2)\in\Gamma$). Then as (α,Ω) varies along Γ from $\Omega=0$ to $\Omega=-q\cdot\sup_{x\in[0,1]}\varpi(x)$, there must exist a $(\bar{\alpha},\bar{\Omega})\in\Gamma$ such that $k(1;\bar{\alpha},\bar{\Omega},q)=0$. This in turn implies that $(A(\cdot;\bar{\alpha},\bar{\Omega},q),k(\cdot;\bar{\alpha},\bar{\Omega},q))$ is the desired solution of problem (3.3a)–(3.3b) and (3.4a)–(3.4b), thereby completing the proof of the first part of Theorem 4.

Therefore, it remains to show the existence of (α_i,Ω_i) with the desired properties for $i=1,2$. We first study $k(1;\alpha,\Omega,q)$ for $(\alpha,\Omega)\in\Gamma$ with Ω close to 0.

Lemma 5.7. *There exists an $\Omega_1\in(-q\sup_{x\in[0,1]}\varpi(x),0)$ such that, if $\Omega\in[\Omega_1,0)$ and $(\alpha,\Omega)\in\Gamma$, then we have $k(1;\alpha,\Omega,q)<0$.*

Proof. Let $\hat{\alpha} \in [\alpha_*, \alpha^*]$ be such that $(\hat{\alpha}, 0) \in \Gamma$. With the use of (3.6) and $(\alpha, \Omega) = (\hat{\alpha}, 0)$, we have

$$rA^2(r; \hat{\alpha}, 0, q) \cdot k(r; \hat{\alpha}, 0, q) = -\int_0^r \frac{sA^2(s; \hat{\alpha}, 0, q)}{\mathrm{d}} \cdot q\varpi(A(s; \hat{\alpha}, 0, q))\mathrm{d}s$$

for $r \in [0, 1]$. Since $(\hat{\alpha}, 0) \in \Gamma$, we have $A(r; \hat{\alpha}, 0, q) \in (0, 1)$ for $r \in (0, 1]$. This, together with the above equality and the fact that $\varpi(\cdot)$ is positive on $(0, 1]$, shows that $k(1; \hat{\alpha}, 0, q) < 0$. Hence the assertion of this lemma follows from this and an application of the continuous dependence on (α, Ω). $\square$

We next study $k(1; \alpha, \Omega, q)$ for $(\alpha, \Omega) \in \Gamma$ with Ω close to $-q\sup_{x\in[0,1]} \varpi(x)$.

Lemma 5.8. *There exists an $\Omega_2 \in (-q\sup_{x\in[0,1]} \varpi(x), \Omega_1)$ such that, if $\Omega \in (-q\sup_{x\in[0,1]} \varpi(x), \Omega_2]$ and $(\alpha, \Omega) \in \Gamma$, then we have $k(1; \alpha, \Omega, q) > 0$.*

Proof. For simplicity, we set $\tilde{\Omega} = -q\sup_{x\in[0,1]} \varpi(x)$. Let $\tilde{\alpha} \in [\alpha_*, \alpha^*]$ be such that $(\tilde{\alpha}, \tilde{\Omega}) \in \Gamma$. Since $(\tilde{\alpha}, \tilde{\Omega}) \in \Gamma$, we have $A(r; \tilde{\alpha}, \tilde{\Omega}, q) \in (0, 1)$ for $r \in (0, 1]$. Now, with the use of (3.6) and $(\alpha, \Omega) = (\tilde{\alpha}, \tilde{\Omega})$, we have

$$rA^2(r; \tilde{\alpha}, \tilde{\Omega}, q) \cdot k(r; \tilde{\alpha}, \tilde{\Omega}, q) = q\int_0^r \frac{sA^2(s; \tilde{\alpha}, \tilde{\Omega}, q)}{\mathrm{d}} \cdot \left(\sup_{x\in[0,1]} \varpi(x) - \varpi(A(s; \tilde{\alpha}, \tilde{\Omega}, q))\right)\mathrm{d}s$$

for $r \in (0, 1]$. This, with the fact that $A(r; \tilde{\alpha}, \tilde{\Omega}, q) \in (0, 1)$ for $r \in (0, 1]$, yields $k(1; \tilde{\alpha}, \tilde{\Omega}, q) > 0$. Finally, an application of the continuous dependence on (α, Ω) shows that, if $(\alpha, \Omega) \in \Gamma$ and (α, Ω) close to $(\tilde{\alpha}, \tilde{\Omega})$, then we have $k(1; \alpha, \Omega, q) > 0$. This completes the proof of this lemma. $\square$

5.5. *The proof for the existence of m-armed spiral waves when d is close to $\frac{\lambda_0}{j_m^2} \cdot R^2$*

In this subsection, we will establish the second part of Theorem 4.

5.5.1. *Outline of the proof of the second part of Theorem 4*

The proof of the second part of Theorem 4 follows the same plan as in the first part. But it needs some modifications. For simplicity of presentation, we will only sketch the proof and show the necessary lemmas. To begin with, we recall the quantity

$$\Omega_0 = \Omega_0(m, \lambda_0, d) := -\min\left\{\sqrt{d\cdot\lambda\left(\frac{1}{2}\right)}, \frac{\lambda_0}{2} \cdot \sqrt{\frac{-J_m'(\sqrt{\lambda_0/d})}{\int_0^{\sqrt{\lambda_0/d}} \xi^2 J_m(\xi)\mathrm{d}\xi}}\right\}.$$

For each fixed $q > 0$, we also need the following two sets:

$$B_{1,q} := \{(\alpha, \Omega) \in (0, \infty) \times [\Omega_0, 0] \mid R_{\alpha,\Omega,q} \in (0, 1] \text{ and } A'(\cdot; \alpha, \Omega, q) > 0 \text{ on } (0, R_{\alpha,\Omega,q})\} \bigcup \{(\alpha, \Omega) \in (0, \infty) \times [\Omega_0, 0] \mid A'(\cdot; \alpha, \Omega, q) > 0 \text{ on } (0, 1]\},$$

$$B_{2,q} := \{(\alpha, \Omega) \in (0, \infty) \times [\Omega_0, 0] \mid R_1(\alpha, \Omega, q) \in (0, 1)\}.$$

The difference between the sets $A_{i,q}$ and $B_{i,q}$ is that we restrict the parameter Ω to the smaller interval $[\Omega_0, 0]$, not the interval $[-q\sup_{x\in[0,1]} \varpi(x), 0]$. Now fix a $\gamma \in (0, 1)$ and recall that $j_{m,0}$ denotes the first positive zero of $J_m(\cdot)$. Since $\varpi(A)$ is differentiable at $A = 0$ and $\Omega_0 \to 0$ as $d \to (\frac{\lambda_0}{j_m^2})^-$, we can choose a sufficiently small $\delta_0 > 0$ such that, for each $d \in (\frac{\lambda_0}{j_m^2} - \delta_0, \frac{\lambda_0}{j_m^2})$ and $\alpha \in (0, |\Omega_0|^{1+\gamma}]$, we have

$$|\Omega_0|^{1+\gamma} < \frac{1}{2} \quad \text{and} \quad q\varpi(\alpha) \le |\Omega_0|. \tag{5.5}$$

Step 1. Set $\alpha^* := |\Omega_0|^{1+\gamma}$. We find a small positive number $\delta_1 \in (0, \delta_0)$ such that, for each $d \in (\frac{\lambda_0}{j_m^2} - \delta_1, \frac{\lambda_0}{j_m^2})$, we have $\{\alpha^*\} \times [\Omega_0, 0] \subset B_{1,q}$ (see Lemma 5.10 in Section 5.5.2).

Step 2. For each $d \in (\frac{\lambda_0}{j_m^2} - \delta_1, \frac{\lambda_0}{j_m^2})$, we find a small positive number $\alpha_* < \alpha^*$ such that $\{\alpha_*\} \times [\Omega_0, 0] \subset B_{2,q}$ (see Lemma 5.11 in Section 5.5.2).

Step 3. This step is a slight modification of that in Paullet et al. [33, pp. 1393–1395]. From *Step* 1 and *Step* 2, we can conclude that, for each fixed $d \in (\frac{\lambda_0}{j_m^2} - \delta_1, \frac{\lambda_0}{j_m^2})$, there exists a continuum $\Gamma \subseteq \{\alpha \in [\alpha_*, \alpha^*], \Omega \in [\Omega_0, 0]\}$ which connects the lines $\{\alpha > 0, \Omega = 0\}$ and $\{\alpha > 0, \Omega = \Omega_0\}$, and satisfies that $A'(\cdot; \alpha, \Omega, q) > 0$ on $(0, 1)$ and $A'(1; \alpha, \Omega, q) = 0$ for all $(\alpha, \Omega) \in \Gamma$. Then, as before, we can show that there exists $(\alpha_1, \Omega_1) \in \Gamma$ (resp. $(\alpha_2, \Omega_2) \in \Gamma$) for which $k(1; \alpha_1, \Omega_1, q) < 0$ (resp. $k(1; \alpha_2, \Omega_2, q) > 0$). This in turn implies that there exists a $(\bar{\alpha}, \bar{\Omega}) \in \Gamma$ such that $k(1; \bar{\alpha}, \bar{\Omega}, q) = 0$. Therefore, $(A(\cdot; \bar{\alpha}, \bar{\Omega}, q), k(\cdot; \bar{\alpha}, \bar{\Omega}, q))$ is the desired solution of problem (3.3a)–(3.3b) and (3.4a)–(3.4b). Now we give some details for this step. First, the existence of (α_1, Ω_1) follows the same argument as in Lemma 5.7. Next, to show the existence of (α_2, Ω_2), we note that, since $\alpha \le \alpha^*(= |\Omega_0|^{1+\gamma})$ for all $(\alpha, \Omega) \in \Gamma$ and $|\Omega_0| < 1/2$, we have $q\varpi(A(\cdot; \alpha, \Omega, q)) \in [0, |\Omega_0|]$ on $[0, 1]$ (see Lemma 5.9). Using this fact and the argument of Lemma 5.8, we can establish the existence of (α_2, Ω_2). Finally, from the fact that $A(\cdot; \alpha, \Omega, q) \in [0, |\Omega_0|]$ on $[0, 1]$ for each $(\alpha, \Omega) \in \Gamma$ and $\Omega_0 \to 0$ as $d \to (\frac{\lambda_0}{j_m^2})^-$, it follows that $\sup_{r\in[0,1]} A(r; \bar{\alpha}, \bar{\Omega}, q) \to 0$ as $d \to (\frac{\lambda_0}{j_m^2})^-$. This implies that, as $d \to (\frac{\lambda_0}{j_m^2})^-$, the corresponding m-armed spiral waves and the rotational frequency shrink to 0 and ω_0, respectively. This completes the proof of the second part of Theorem 4. In the next subsection, we will show the necessary lemmas which are mentioned in the above plan.

5.5.2. *Auxiliary lemmas*

Throughout this subsection, we will retain the notations δ_0, γ, α^*, and Ω_0 defined in the previous section, Section 5.5.1. First, by following a similar arguments as in Lemma 5.5 and using Eq. (5.5), we have the following lemma.

Lemma 5.9. *Let $q > 0$ and $d \in (\frac{\lambda_0}{j_m^2} - \delta_0, \frac{\lambda_0}{j_m^2})$. Then, for each $\alpha \in (0, \alpha^*]$, we have*

$$0 < A(r; \alpha, \Omega, q) \le \alpha r^m (\le |\Omega_0|),$$
$$0 < q\varpi(A(r; \alpha, \Omega, q)) \le |\Omega_0|,$$
$$|k(r; \alpha, \Omega, q)| \le \frac{|\Omega_0|}{d} \cdot r$$

for all $r \in (0, \min\{R_1(\alpha, \Omega, q), 1\}]$ and $\Omega \in [\Omega_0, 0]$.

The following lemma concerns the nonemptiness of the set $A_{1,q}$.

Lemma 5.10. *Let $q > 0$ and $d \in (\frac{\lambda_0}{j_m^2} - \delta_0, \frac{\lambda_0}{j_m^2})$. Then there is a sufficiently small $\delta_1 \in (0, \delta_0)$ such that, for each $d \in (\frac{\lambda_0}{j_m^2} - \delta_1, \frac{\lambda_0}{j_m^2})$, we have that $A(\cdot; \alpha^*, \Omega, q) > 0$ and $A'(\cdot; \alpha^*, \Omega, q) > 0$ on $(0, 1]$ for all $\Omega \in [\Omega_0, 0]$.*

Proof. We will work in the frame of $(\tilde{P}_{\alpha^*,\Omega,q})$. Indeed, let $(\tilde{A}(s;\alpha^*,\Omega),\tilde{k}(s;\alpha^*,\Omega))$ be the solution of the problem $(\tilde{P}_{\alpha^*,\Omega,q})$ corresponding to $(A(r;\alpha^*,\Omega,q),k(r;\alpha^*,\Omega,q))$. Here, for simplicity, we omit the dependence of $(\tilde{A}(\cdot;\alpha^*,\Omega),\tilde{k}(\cdot;\alpha^*,\Omega))$ on the parameter q. Let $[0,\tilde{R}_{\alpha^*,\Omega})$ be the maximal existence interval of $(\tilde{A}(\cdot;\alpha^*,\Omega),\tilde{k}(\cdot;\alpha^*,\Omega))$. Note that

$$s:=\sqrt{\frac{\lambda_0}{d}}r \quad\text{and}\quad \tilde{A}(s;\alpha^*,\Omega):=A(r;\alpha^*,\Omega,q),$$
$$\tilde{k}(s;\alpha^*,\Omega):=\sqrt{\frac{d}{\lambda_0}}k(r;\alpha^*,\Omega,q).$$

Hence we have $\tilde{A}'(\cdot;\alpha^*,\Omega)>0$ on $(0,\min\{\tilde{R}_1(\alpha^*,\Omega),\sqrt{\frac{\lambda_0}{d}}\})$, where $\tilde{R}_1(\alpha^*,\Omega)$ is the first positive zero of $\tilde{A}(\cdot;\alpha^*,\Omega)$ if it exists. Also note that $\sqrt{\frac{\lambda_0}{d}}>j_m$, and $\tilde{R}_1(\alpha^*,\Omega)>j_m$ by Lemma 5.1. Note that $\tilde{A}(\cdot;\alpha^*,\Omega)$ cannot blow up before $s=\sqrt{\frac{\lambda_0}{d}}$ by virtue of Lemma 5.9. Integrating (3.11) from 0 to s, we have that

$$\begin{aligned}&s\tilde{A}'(s;\alpha^*,\Omega)J_m(s)-s\tilde{A}(s;\alpha^*,\Omega)J_m'(s)\\&\quad=\int_0^s\xi\tilde{A}(\xi;\alpha^*,\Omega)J_m(\xi)\\&\qquad\times\left(\frac{\lambda_0-\lambda(\tilde{A}(\xi;\alpha^*,\Omega))}{\lambda_0}+\tilde{k}^2(\xi;\alpha^*,\Omega)\right)\mathrm{d}\xi\end{aligned}\tag{5.6}$$

holds for $s\in[0,\tilde{R}_{\alpha^*,\Omega})$.

From the proof of Lemma 5.4, we see that $\tilde{A}(s;\alpha^*,\Omega)>\alpha^*\cdot(2^m\cdot m!)\sqrt{(\frac{d}{\lambda_0})^m}J_m(s)$ for all $s\in[0,\tilde{R}_{\alpha^*,\Omega})$ and $\Omega\in[\Omega_0,0]$. From Lemma 5.9, we have that $\tilde{A}(s;\alpha^*,\Omega)\le\alpha^*\sqrt{(\frac{d}{\lambda_0})^m s^m}\le|\Omega_0|$ for $s\in[0,\min\{\tilde{R}_1(\alpha^*,\Omega),\sqrt{\frac{\lambda_0}{d}}\}]$. Using these facts and Eq. (5.6), we can estimate $\tilde{A}'(j_m;\alpha^*,\Omega)$ as follows:

$$\begin{aligned}&j_m\cdot J_m(j_m)\tilde{A}'(j_m;\alpha^*,\Omega)\\&\quad\ge\int_0^{j_m}\xi\tilde{A}(\xi;\alpha^*,\Omega)J_m(\xi)\cdot\left(\frac{\lambda_0-\lambda(\tilde{A}(\xi;\alpha^*,\Omega))}{\lambda_0}\right)\mathrm{d}\xi\\&\quad\ge\int_0^{j_m}\frac{\xi J_m(\xi)\inf_{x\in[0,|\Omega_0|]}|\lambda'(x)|}{\lambda_0}\cdot\tilde{A}^2(\xi;\alpha^*,\Omega)\mathrm{d}\xi\\&\qquad\text{(the mean-value theorem, }\lambda_0=\lambda(0)\text{, and }\lambda(\cdot)\text{ is decreasing)}\\&\quad>(\alpha^*)^2\cdot\left(\frac{d}{\lambda_0}\right)^m\cdot\inf_{x\in[0,|\Omega_0|]}|\lambda'(x)|\\&\qquad\times\left(\frac{2^{2m}(m!)^2}{\lambda_0}\cdot\int_0^{j_m}\xi J_m^3(\xi)\mathrm{d}\xi\right)\\&\quad:=I(\alpha^*).\end{aligned}\tag{5.7}$$

Note that the quantity $I(\alpha^*)$ is independent of $\Omega\in[\Omega_0,0]$. By the definition of Ω_0 and α^*, there exists a sufficiently small $\delta_1\in(0,\delta_0)$ such that, for each $d\in(\frac{\lambda_0}{j_m^2}-\delta_1,\frac{\lambda_0}{j_m^2})$, we have $\sqrt{\frac{\lambda_0}{d}}<j_{m,0}$ and

$$I(\alpha^*)>\alpha^*\sqrt{\frac{\lambda_0}{d}}\max_{\xi\in[j_m,j_{m,0}]}|J_m''(\xi)|\cdot\left|\sqrt{\frac{\lambda_0}{d}}-j_m\right|.\tag{5.8}$$

Now we claim that $\tilde{A}'(\cdot;\alpha^*,\Omega)>0$ on $(0,\sqrt{\frac{\lambda_0}{d}}]$ for each $d\in(\frac{\lambda_0}{j_m^2}-\delta_1,\frac{\lambda_0}{j_m^2})$ and $\Omega\in[\Omega_0,0]$. For contradiction, we can assume that $\tilde{R}_1(\alpha^*,\Omega)$ exists and lies in the interval $(0,\sqrt{\frac{\lambda_0}{d}})$ for some $d\in(\frac{\lambda_0}{j_m^2}-\delta_1,\frac{\lambda_0}{j_m^2})$ and $\Omega\in[\Omega_0,0]$. Then, from Lemmas 5.1 and 5.9, we have that $\tilde{R}_1(\alpha^*,\Omega)\in(j_m,\sqrt{\frac{\lambda_0}{d}})$ and $\tilde{A}(\tilde{R}_1(\alpha^*,\Omega);\alpha^*,\Omega)\le\alpha^*\sqrt{(\frac{d}{\lambda_0})^m s^m}$ for $s\in[0,\tilde{R}_1(\alpha^*,\Omega)]$. Together with the mean-value theorem, the following holds:

$$\begin{aligned}&|\tilde{R}_1(\alpha^*,\Omega)\tilde{A}(\tilde{R}_1(\alpha^*,\Omega);\alpha^*,\Omega)J_m'(\tilde{R}_1(\alpha^*,\Omega))|\\&\quad\le\sqrt{\frac{\lambda_0}{d}}\cdot\left(\alpha^*\sqrt{\left(\frac{d}{\lambda_0}\right)^m}\cdot\tilde{R}_1^m(\alpha^*,\Omega)\right)|J_m'(\tilde{R}_1(\alpha^*,\Omega))|\\&\quad\le\alpha^*\sqrt{\frac{\lambda_0}{d}}\max_{\xi\in[j_m,j_{m,0}]}|J_m''(\xi)|\cdot\left|\sqrt{\frac{\lambda_0}{d}}-j_m\right|,\end{aligned}$$

which, together with (5.8), yields

$$\begin{aligned}&-\tilde{R}_1(\alpha^*,\Omega)\tilde{A}(\tilde{R}_1(\alpha^*,\Omega);\alpha^*,\Omega)J_m'(\tilde{R}_1(\alpha^*,\Omega))\\&\quad-j_m\cdot J_m(j_m)\tilde{A}'(j_m;\alpha^*,\Omega)<0.\end{aligned}$$

On the other hand, since $A'(\tilde{R}_1(\alpha^*,\Omega);\alpha^*,\Omega)=J_m'(j_m)=0$, it follows from Eq. (5.6) that

$$\begin{aligned}&-\tilde{R}_1(\alpha^*,\Omega)\tilde{A}(\tilde{R}_1(\alpha^*,\Omega);\alpha^*,\Omega)J_m'(\tilde{R}_1(\alpha^*,\Omega))\\&\quad-j_m\cdot J_m(j_m)\tilde{A}'(j_m;\alpha^*,\Omega)=\int_{j_m}^{\tilde{R}_1(\alpha^*,\Omega)}\xi\tilde{A}(\xi;\alpha^*,\Omega)J_m(\xi)\\&\quad\times\left(\frac{\lambda_0-\lambda(\tilde{A}(\xi;\alpha^*,\Omega))}{\lambda_0}+\tilde{k}^2(\xi;\alpha^*,\Omega)\right)\mathrm{d}\xi>0.\end{aligned}$$

This is a contradiction, and hence the assertion of the claim is established. This completes the proof. □

Finally, the following lemma concerns the nonemptiness of the set $A_{2,q}$.

Lemma 5.11. *Let $q>0$ and $d\in(\frac{\lambda_0}{j_{m,0}^2},\frac{\lambda_0}{j_m^2})$. Then there exists an $\alpha_*=\alpha_*(d,q)>0$ such that $R_1(\alpha,\Omega,q)\in(0,1)$ holds for each $\alpha\in(0,\alpha_*]$ and $\Omega\in[\Omega_0,0]$.*

Proof. Note that the assumption that $d\in(\frac{\lambda_0}{j_{m,0}^2},\frac{\lambda_0}{j_m^2})$ implies that $\sqrt{\frac{\lambda_0}{d}}\in(j_m,j_{m,0})$. Then, by setting $\hat{j}_m:=\sqrt{\frac{\lambda_0}{d}}$ and using Lemma 5.9, we can follow the argument of Lemma 5.6 to complete the proof. Hence we omit the details of this proof. □

6. Numerical study of spiral waves

In this section, we will use the reaction kinetics $(\lambda(A),\omega(A))=(1-A^2,1+qA^2)$ to numerically study the spiral wave solutions of (1.1)–(1.2). Note that $\lambda_0=\omega_0=1$ for this example, and that this choice of reaction kinetics (λ,ω) is the same as that in [33,19].

6.1. The choice of reaction kinetics

We first discuss this choice of reaction kinetics $(\lambda(A),\omega(A))$. Consider the new variables, parameters, and functions

$$\tilde{t}=\lambda_0 t,\qquad\tilde{r}=\frac{r}{R},\qquad\tilde{\theta}=\theta,\qquad\tilde{d}=\frac{d}{\lambda_0R^2},\qquad\tilde{\hat{\Omega}}=\frac{\hat{\Omega}}{\lambda_0},$$
$$\tilde{u}(\tilde{r},\tilde{\theta},\tilde{t})=u(r,\theta,t),\qquad\tilde{v}(\tilde{r},\tilde{\theta},\tilde{t})=v(r,\theta,t),$$
$$\tilde{A}(\tilde{r})=A(r),\qquad\tilde{k}(\tilde{r})=R\cdot k(r),$$
$$\tilde{\lambda}(\tilde{A})=\frac{\lambda(A)}{\lambda_0},\qquad\tilde{\varpi}(\tilde{A})=\frac{\varpi(A)}{\lambda_0},\qquad\tilde{q}=\frac{q}{\lambda_0},\qquad\tilde{\omega_0}=\frac{\omega_0}{\lambda_0}.$$

Substituting these rescaled variables, parameters, and functions into problem (1.1)–(1.2) (resp. problem (3.3a)–(3.3b) and (3.4a)–(3.4b)) and dropping the tildes, we get the same governing equations as in (1.1)–(1.2) (resp. (3.3a)–(3.3b) and (3.4a)–(3.4b)) with R replaced by $R = 1$. Hence for the choice of the reaction kinetics $(\lambda(A), \omega(A)) = (1 - A^2, 1 + qA^2)$, we can view the twist parameter q as the ratio of q to λ_0. Similar explanations can be made for the diffusivity d and the rotational frequency $\hat{\Omega}$. Finally, with the use of Eq. (1.1b) and the relation $\hat{\Omega} = \omega_0 - \Omega$, we can conclude from (3.3a) and (3.3b) that ω_0 has only the "shift" effect on the rotational frequency $\hat{\Omega}$, and will not affect the qualitative dependence of spiral waves on the other parameters.

6.2. Spiral wave solutions diagram

In this subsection, we will focus on spiral waves on the unit disk.

6.2.1. Numerical tools and continuation of spiral waves

If we are given a solution (A, k, Ω) of problem (3.3)–(3.4), then we can track a branch of spiral wave solutions and graphically depict the dependence of various properties of spiral wave solutions on the parameters such as d or q by using the continuation software AUTO which is implemented in XPPAUT by Ermentrout [39]. As a 'starting" m-armed spiral wave solution, we employ the two-parameter shooting scheme discussed in the previous section to look for a solution of problem (3.3)–(3.4) for the given diffusivity d and the twist parameter q.

6.2.2. Numerical results

Fig. 5 shows spiral wave solutions diagram including the amplitude $(A(1) = \sup_{r\in[0,1]} A(r))$ and the rotational frequency $(\hat{\Omega})$ from continuation on the diffusivity d for three fixed values of the twist parameter q ($q = 1, 4, 7$). It indicates that, for $m = 1, 2, 3$, a branch of m-armed spiral waves on the unit disk emanates from $(u, v) = (0, 0)$ when $d = d_m := 1/j_m^2$, which has been predicted by the second part of Theorem 2. Moreover, the second part of Theorem 2 also shows that as $d \to (\frac{1}{j_m^2})^-$, the corresponding m-armed spiral waves on the unit disk will shrink to 0 and their rotational frequency tends to $\omega_0 = 1$ in the case of $(\lambda(A), \omega(A)) = (1 - A^2, 1 + qA^2)$, which is consistent with the numerical results presented in Fig. 5. From Fig. 5, we see that, as q increases, the amplitude $(A(1))$ decreases and the rotational frequency $(\hat{\Omega})$ increases.

Fig. 5 also indicates that, for large twist parameter q (e.g., $q = 4$ or 7), as the diffusivity d decreases from d_m to some small critical diffusivity, it seems that m-armed spiral waves on the unit disk cease to exist. On the other hand, no matter how small the diffusivity d is, we can use the first part of Theorem 2 to conclude that there always exists a small twist parameter q for which there is an m-armed spiral wave.

Now we give explanation on why branches of spiral waves terminate at some small critical diffusivity in the continuation procedures. To begin with, these small critical diffusivity are labelled as MX points by XPPAUT. There are two possible reasons for this. The first possible reason is that compared with the magnitude of the diffusivity d (equivalently, the domain size is too large), a large twist parameter q might prevent the existence of m-armed spiral waves. The second possible reason is due to the restriction of numerical computation. For this, we first observe that from Lemma 5.2 one can see that, if the component $A(\cdot)$ of the solution (A, k) to problem (3.3)–(3.4a) exceeds 1 at some point $r = r_1$, then $A'(\cdot) > 0$ for all $r > r_1$ such that (A, k) exists. This implies that (A, k) cannot be a solution of problem (3.3)–(3.4), and so spiral wave solutions cannot exist for this set of parameters q and d. Now Fig. 5 and the left panels of Figs. 6 and 7 indicate that for small twist parameter q ($q = 0.3$ and 1), as the diffusivity d decreases from d_m to some critical diffusivity $\hat{d}_{q,m}$, the amplitude $(A(1) = \sup_{r\in[0,1]} A(r))$ of the corresponding spiral wave increases to some number which is very close to 1. Hence if we continue to decrease the diffusivity d, the component A of the solution (A, k) to problem (3.3)–(3.4a) via numerical computation may exceed 1 at some point $r = r_1 \in (0, 1)$, and so (A, k) cannot be a solution of problem (3.3)–(3.4) by the previous remark.

Finally, for large twist parameter q ($q \geq 4$) and small diffusivity d, Fig. 5 indicates that, as the diffusivity d decreases from d_m to some critical diffusivity $\hat{d}_{q,m}$, the amplitude $(A(1) = \sup_{r\in[0,1]} A(r))$ of the corresponding spiral wave will tend to some small number. Specifically, the right panels of Figs. 6 and 7 indicate that the solution (A, k) of problem (3.3)–(3.4) has the property that the component $A(\cdot)$ is close to 0 and has a boundary layer near the vicinity of $r = 1$, while the component $k(\cdot)$ is very large (compared with $A(\cdot)$). Hence we may conclude that for large twist parameter q, termination of branches of spiral waves at some small critical diffusivity is due to the restriction of numerical computation.

In the next subsection, we will use the perturbation method to construct m-armed spiral waves for $d \ll 1$ and $d/q \ll 1$, which formally verifies that termination of branches of spiral waves at some small diffusivity in the continuation procedure is due to the restriction of numerical computation.

6.3. Perturbation analysis

We now construct spiral waves by undertaking an asymptotic analysis for both small q and large q. The method employed in this subsection is motivated by those in [24,32,19].

6.3.1. Small twist parameter q and small diffusivity d

We first discuss the case where $0 < q \ll 1$, $0 < d \ll 1$, and $d/q \ll 1$. To do this, we introduce the new variables, parameters, and functions by

$$s = \frac{r}{\sqrt{d}}, \qquad \tilde{A}(s) = A(r), \qquad \tilde{k}(s) = \sqrt{d}k(r), \qquad \tilde{\Omega} = \frac{\Omega}{q}. \tag{6.1}$$

Substituting these rescaled variables, parameters, and functions into problem (3.3)–(3.4), we get the following equivalent boundary value problem:

$$\tilde{A}'' + \frac{1}{s}\tilde{A}' + \tilde{A}\left[1 - \tilde{A}^2 - \frac{m^2}{s^2} - \tilde{k}^2\right] = 0, \tag{6.2a}$$

$$\tilde{k}' + \left(\frac{1}{s} + 2\frac{\tilde{A}'}{\tilde{A}}\right)\tilde{k} + q(\tilde{\Omega} + \tilde{A}^2) = 0, \tag{6.2b}$$

and

$$\tilde{A}(0) = \tilde{k}(0) = 0, \tag{6.3a}$$

$$\tilde{A}'(1/\sqrt{d}) = \tilde{k}(1/\sqrt{d}) = 0, \tag{6.3b}$$

where the prime denotes d/ds. We remark that problem (6.2)–(6.3) is identical to that in [19], but with the condition (6.3b) replaced by $\tilde{A}(\infty) = 1$ and $\tilde{k}(s)$ bounded as $s \to +\infty$.

Fig. 5 indicates that, for small twist parameter q, $A(1) \approx 1$ as $d \to 0^+$. Also, Figs. 6 and 7 suggest that the magnitude of $k(\cdot)$ is an increasing function of q (see also Fig. 4 in [33]). Therefore we substitute the expansions

$$\tilde{A}(s) = \tilde{A}_0(s) + q\tilde{A}_1(s) + q^2\tilde{A}_2(s) + \cdots,$$
$$\tilde{k}(s) = q\tilde{k}_0(s) + q^2\tilde{k}_1(s) + q^3\tilde{k}_2(s) + \cdots$$

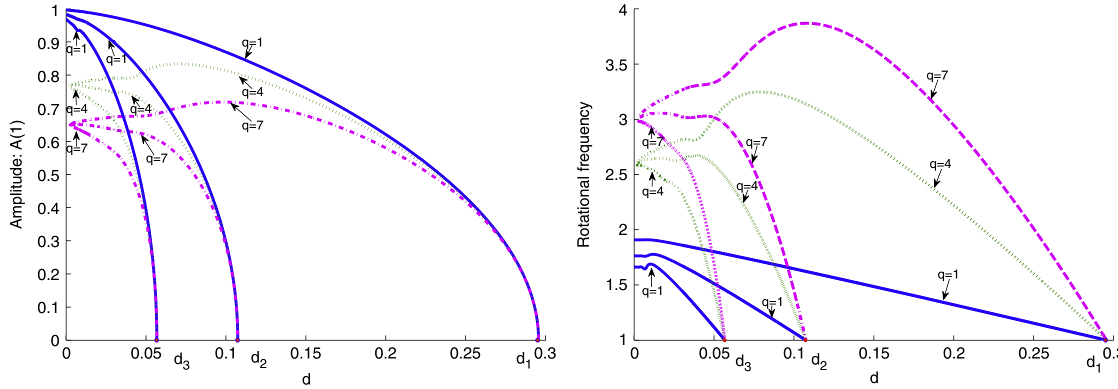

Fig. 5. The amplitude $A(1)(=\sup_{r\in[0,1]}A(r))$ of m-armed spiral waves on the unit disk versus the diffusivity d (left panel) and the rotational frequency $\hat{\Omega}$ of m-armed spiral waves on the unit disk versus the diffusivity d (right panel) for three fixed twist parameter values q $(q = 1, 4, 7)$. For $m = 1, 2, 3$, a branch of m-armed spiral waves bifurcates from the uniform steady state $(u, v) = (0, 0)$ when $d = d_m := 1/j_m^2$. The blue solid line: $q = 1$. The green dotted line: $q = 4$. The pink dash–dotted line: $q = 7$.

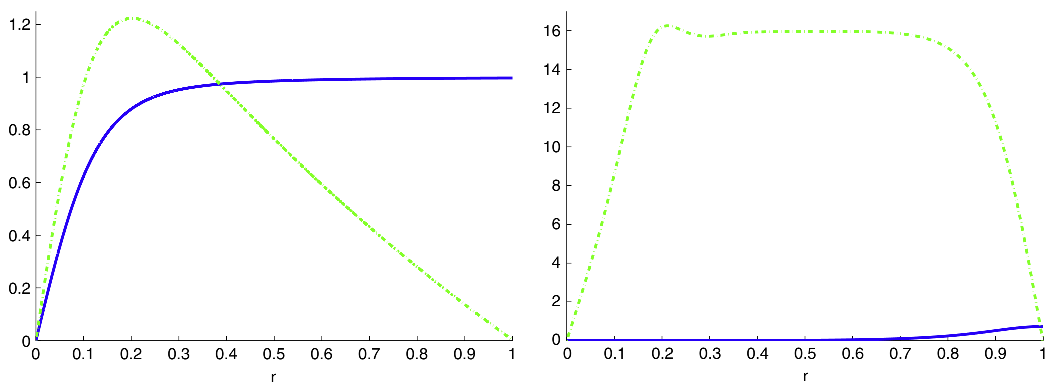

Fig. 6. The profiles of the functions $A(r)$ and $k(r)$ which are defined by (1.1a) and correspond to a single-armed spiral wave (u, v) of system (1.1)–(1.2) on the unit disk with small diffusivity $d = 0.006$ and $(\lambda(A), \omega(A)) = (1 - A^2, 1 + qA^2)$. The blue solid line: $A(r)$. The green dash–dotted line: $k(r)$. The twist parameter $q = 0.3$ for the left panel and $q = 4$ for the right panel.

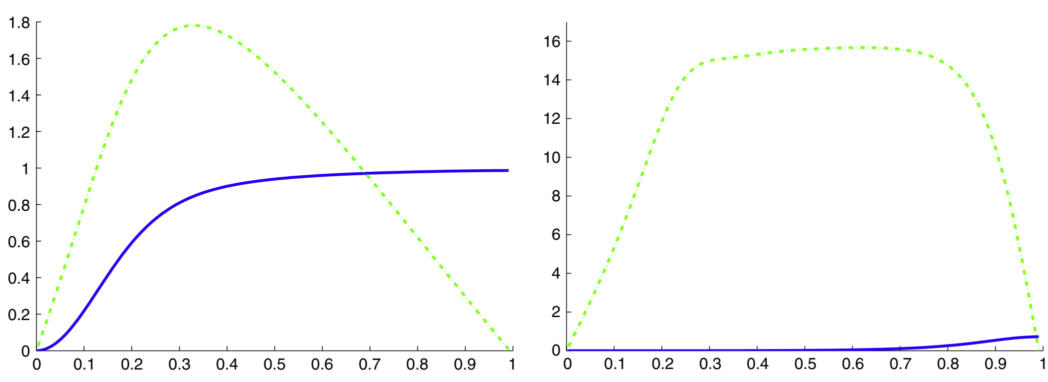

Fig. 7. The profiles of the functions $A(r)$ and $k(r)$ which are defined by (1.1a) and correspond to a two-armed spiral wave (u, v) of system (1.1)–(1.2) on the unit disk with small diffusivity $d = 0.006$ and $(\lambda(A), \omega(A)) = (1 - A^2, 1 + qA^2)$. The blue solid line: $A(r)$. The green dash–dotted line: $k(r)$. The twist parameter $q = 0.3$ for the left panel and $q = 4$ for the right panel.

into (6.2)–(6.3) and equate to zero the coefficient of q, which leads to the consideration of the following problem:

$$\tilde{A}_0'' + \frac{1}{s}\tilde{A}_0' + \tilde{A}_0\left[1 - \tilde{A}_0^2 - \frac{m^2}{s^2}\right] = 0, \tag{6.4a}$$

$$\tilde{k}_0' + \left(\frac{1}{s} + 2\frac{\tilde{A}_0'}{\tilde{A}_0}\right)\tilde{k}_0 + (\tilde{\Omega} + \tilde{A}_0^2) = 0, \tag{6.4b}$$

and

$$\tilde{A}_0(s) \sim \alpha s^m \quad \text{as } s \to 0^+, \qquad \tilde{A}_0'(1/\sqrt{d}) = 0, \tag{6.5a}$$

$$\tilde{k}_0(0) = 0, \qquad \tilde{k}_0(1/\sqrt{d}) = 0. \tag{6.5b}$$

Here α is a positive number. It have been shown [24] that Eq. (6.4a) with $\tilde{A}_0(0) = 0$ admits a unique solution $\mathbb{A}_m(s)$ such that

$$\mathbb{A}_m'(s) > 0 \quad \text{for all } s > 0, \qquad \mathbb{A}_m(s) \to 1 \quad \text{as } s \to +\infty.$$

Hence, for $0 < d \ll 1$, $\mathbb{A}_m(\cdot)$ is an approximate solution of problem (6.4a) and (6.5a).

Now we turn to problem (6.4b) and (6.5b) with $\tilde{A}_0 = \mathbb{A}_m$. With the use of a simple integration, the solution $\mathbb{K}_m(\cdot; \tilde{\Omega})$ of Eq. (6.4b)

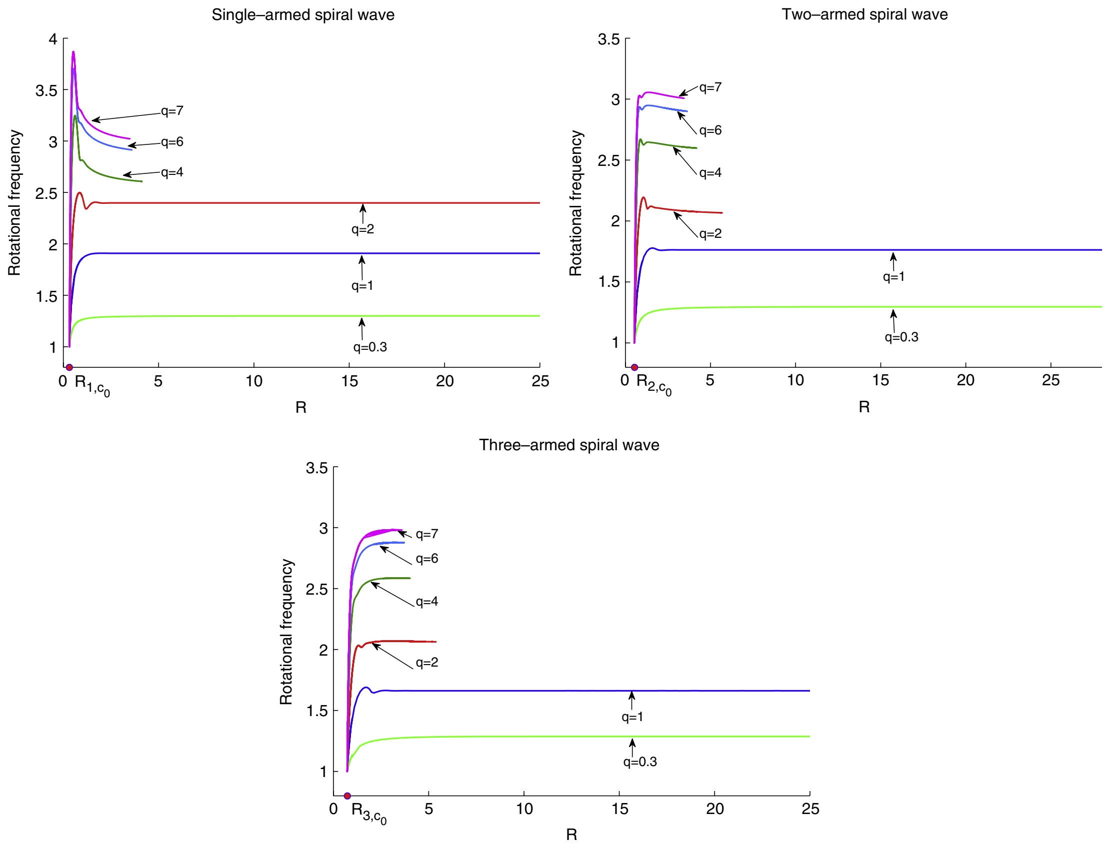

Fig. 8. Dependence of the rotational frequency $\hat{\Omega}$ of m-armed spiral waves on the radius R for various twist parameter values q (from the top to the bottom curve, the q values are 7, 6, 4, 2, 1, and 0.3) for fixed $d = 0.03$, $\lambda(A) = 1 - A^2$, and $\omega(A) = 1 + qA^2$. The left panel is for the case of single-armed spiral wave, the right panel is for the case of two-armed spiral wave, and the bottom panel is for the case of three-armed spiral wave.

with $\tilde{A}_0 = \mathbb{A}_m$ and $\mathbb{K}_m(0; \tilde{\Omega}) = 0$ is given by

$$\mathbb{K}_m(s; \tilde{\Omega}) = \frac{1}{s\mathbb{A}_m^2(s)} \int_0^s t\mathbb{A}_m^2(t)[-\tilde{\Omega} - \mathbb{A}_m^2(t)]dt.$$

Using Greenberg's result [24], we have

$$\mathbb{A}_m(s) = 1 - \frac{m^2}{2s^2} + \cdots, \quad \text{for } s \gg 1,$$

and so $\mathbb{K}_m(\cdot; -1) > 0$ on $(0, +\infty)$ and $\mathbb{K}_m(s; -1) \to 0$ as $s \to +\infty$. Furthermore, from the expansion of $\mathbb{A}_m(\cdot)$ and the monotonicity of Eq. (6.4b) on the parameter $\tilde{\Omega}$, we see that, for each fixed $d \ll 1$, there exists a unique negative number $\tilde{\Omega}(d) \in (-1, 0)$ such that $\mathbb{K}_m(\cdot; \tilde{\Omega}(d)) > 0$ on $(0, 1/\sqrt{d})$ and $\mathbb{K}_m(1/\sqrt{d}; \tilde{\Omega}(d)) = 0$. This implies that $\mathbb{K}_m(\cdot; \tilde{\Omega}(d))$ solves problem (6.4b) and (6.5b). Moreover, $\tilde{\Omega}(d)$ is an increasing function of d and $\tilde{\Omega}(d) \to -1$ as $d \to 0^+$.

To summarize, for $0 < q \ll 1$, $0 < d \ll 1$, and $d/q \ll 1$, to leading order the solution (A, k) of problem (3.3)–(3.4) is given by

$$A(r) = \mathbb{A}_m(r/\sqrt{d}) + \cdots,$$
$$k(r) = \frac{q}{\sqrt{d}} \cdot \mathbb{K}_m(r/\sqrt{d}; \tilde{\Omega}(d)) + \cdots,$$
$$\Omega = q\tilde{\Omega}(d).$$

6.3.2. *Large twist parameter q and small diffusivity d*

We now consider the other extreme case, where $q \gg 1$, $0 < d \ll 1$, and $d/q \ll 1$. To do this, we set $\bar{\Omega} = \Omega/q$. Inspired by Hagan [19], we substitute the expansions

$$A(r) = \sqrt{-\bar{\Omega}} \cdot \left(\hat{A}_0(r) + \frac{1}{q}\hat{A}_1(r) + \frac{1}{q^2}\hat{A}_2(r) + \cdots\right),$$
$$k(r) = \sqrt{1 + \bar{\Omega}} \cdot \left(\hat{k}_0(r) + \frac{1}{q}\hat{A}_1(r) + \frac{1}{q^2}\hat{A}_2(r) + \cdots\right),$$
$$\frac{\Omega}{\sqrt{1 + \bar{\Omega}}} = \hat{\Omega}_0 + \frac{1}{q}\hat{\Omega}_1 + \frac{1}{q^2}\hat{\Omega}_2 + \cdots,$$

into problem (3.3)–(3.4), and equate to zero the coefficient of $1/q$, we obtain

$$d\hat{A}_0'' + \frac{d}{r}\hat{A}_0' + \hat{A}_0\left[1 - \frac{dm^2}{r^2} - d\hat{k}_0^2\right] = 0, \tag{6.6a}$$

$$d\hat{k}_0' + \left(\frac{d}{r} + 2d\frac{\hat{A}_0'}{\hat{A}_0}\right)\hat{k}_0 + \hat{\Omega}_0(1 - \hat{A}_0^2) = 0, \tag{6.6b}$$

and

$$\hat{A}_0(r) \sim \alpha r^m \quad \text{as } r \to 0^+, \qquad \hat{k}_0(0) = 0, \tag{6.7a}$$

$$\hat{A}_0'(1) = 0, \qquad \hat{k}_0(1) = 0. \tag{6.7b}$$

Here α is a positive number. Unlike the case for small q, problem (6.6)–(6.7) cannot be solved analytically, and hence we need to resort to numerical computation. With the help of XPPAUT, we have numerically computed that, as $d \to 0^+$,

$$\hat{\Omega}_0 \approx \begin{cases} -1.342 & \text{for } m = 1, \\ -3.007 & \text{for } m = 2, \\ -2.9147 & \text{for } m = 3. \end{cases}$$

We remark that, as $d \to 0^+$, the limiting problem of problem (6.6)–(6.7) is similar to that in [19], but with the condition (6.7b) replaced by $\hat{A}_0(\infty) = 1$ and $\hat{k}_0(\infty) = 1$.

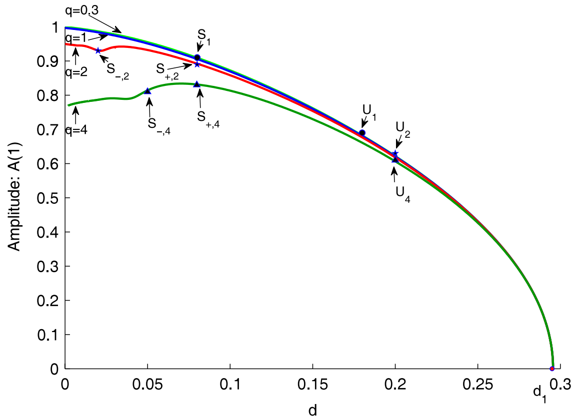

Fig. 9. The amplitude $A(1)(= \sup_{r\in[0,1]} A(r))$ of single-armed spiral waves on the unit disk versus the diffusivity d for four fixed twist parameter values q (from the top to the bottom curve, the q values are 0.3, 1, 2, and 4). The light blue solid line: $q = 0.3$. The blue solid line: $q = 1$. The red solid line: $q = 2$. The green line: $q = 4$. The single-armed spiral waves which correspond to the points between $S_{-,*}$ and $S_{+,*}$ with $* = 2, 4$, or lying on the left-hand side of the point S_1 are stable, while those corresponding to the points lying on the right-hand side of the point U_* with $* = 1, 2, 4$, are unstable. See Section 6.5 for more details. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

6.4. The dependence of the rotational frequency $\hat{\Omega}$ on the domain size R

In this subsection we would like to study the behavior of m-armed spiral waves on large circular domains. To begin with, we recall the scaling relation (3.2) and the relationship between problem (3.1) and problem (3.3)–(3.4). Hence the construction of m-armed spiral wave solutions of system (1.1)–(1.2) with diffusivity d and radius R is reduced to problem (3.3)–(3.4) with diffusivity d/R^2.

Several plots of the numerically computed rotational frequency $\hat{\Omega}$ versus the radius R of the circular domain are shown in Fig. 8. They indicate several interesting phenomena. First, for small q ($q = 0.3$), it shows that an increase of the radius R results in the increase of the rotational frequency $\hat{\Omega}$. Furthermore, when the radius R is above a critical value R_{m,c_1}, the rotational frequency seems to be independent of the radius R. This seems to suggest that, if R is bigger than R_{m,c_1}, then the domain size will not affect the properties of spiral waves. Next, there is another critical radius $R_{m,c_0} = \sqrt{d \cdot j_m^2}$ which is given by Theorem 1 (with $\lambda_0 = 1$). For the case $d = 0.03$, we have $R_{m,c_0} \approx 0.3189$ for $m = 1$, $R_{m,c_0} \approx 0.5290$ for $m = 2$, $R_{m,c_0} \approx 0.7277$ for $m = 3$. When the radius R is below R_{m,c_0}, the corresponding medium no longer supports the formation of m-armed spiral waves. On the other hand, if the radius R is between these two critical radii R_{m,c_0} and R_{m,c_1}, the corresponding rotational frequency changes substantially. By contrast, excitable media exhibit an increase of the rotational frequency with decreasing domain size (see [30,27]). However, there also exist critical radii in excitable media whose properties are similar to that of R_{m,c_1} and R_{m,c_0} (see [30,27]).

For large q ($q \geq 2$), there are more critical radii involved. In fact, the curves related to the single-armed and two-armed spiral waves in Fig. 8 indicate that there is a small interval of R on which the dependence of the rotational frequency $\hat{\Omega}$ on the radius R is no longer monotonic. Finally, Fig. 8 also indicates that a large twist parameter q ($q \geq 2$) might prevent the existence of m-armed spiral waves, if the domain size is too large. This is a reflection of the qualitative behavior seen in Fig. 5.

6.5. Stability of spiral waves

In this subsection, we numerically investigate the stability of spiral wave solutions of (1.1)–(1.2) which are computed via XPPAUT in Section 6.2. We employ the finite element method which is implemented by the PDE Toolbox package of MATLAB. For all computations, the domain under study is the unit disk which is divided into 541 triangles of equal size, and time step $\Delta t = 0.005$. For a possible stable spiral wave solution $(\bar{u}, \bar{v})$, we use $(1.1\bar{u}, 1.1\bar{v})$ as the initial condition.

Before we go any further, we would like to make one remark. Recall that spiral waves bifurcate from the unstable uniform steady state $(0, 0)$ when the diffusivity d equals the critical diffusivity $1/j_m^2$. Hence we might expect that spiral waves are unstable for diffusivity d close to the critical diffusivity $1/j_m^2$, which is confirmed numerically. Also, from Section 6.2.2 (in particular, see the right panels of Figs. 6 and 7), we see that, for large twist parameter q ($q \geq 4$) and small diffusivity d, the amplitude function $A(\cdot)$ associated with spiral waves is close to 0, and hence the corresponding spiral wave (u, v) is close to 0, which suggests that such a spiral wave is unstable. This is also confirmed numerically.

6.5.1. Single-armed spiral waves

Numerical evidence suggests the following.

(i) For twist parameters $q = 0.3$ and $q = 1$, let d_s (resp. d_u) be the abscissa of the point S_1 (resp. U_1) in Fig. 9. Then the following hold.
- If $d \in (0, d_s)$, then the corresponding single-armed spiral wave solution of (1.1)–(1.2) is stable (Fig. 10).
- If $d \in (d_u, 1/j_1^2)$, then the corresponding single-armed spiral wave solution of (1.1)–(1.2) is unstable.
- The interval (d_s, d_u) is the fuzzy region for which the stability of the corresponding single-armed spiral wave solution of (1.1)–(1.2) is uncertain.

(ii) For twist parameters $q = 2$ and $q = 4$, let $* = 2$ or 4, and $d_{s,\pm,*}$ (resp. $d_{u,*}$) be the abscissa of the point $S_{\pm,*}$ (resp. U_*) in Fig. 9. Then the following hold.
- If $d \in (0, d_{s,-,*})$, then the corresponding single-armed spiral wave solution of (1.1)–(1.2) is unstable.
- If $d \in (d_{s,-,*}, d_{s,+,*})$, then the corresponding single-armed spiral wave solution of (1.1)–(1.2) is stable.
- If $d \in (d_{u,*}, 1/j_1^2)$, then the corresponding single-armed spiral wave solution of (1.1)–(1.2) is unstable.
- The interval $(d_{s,+,*}, d_{u,*})$ is the fuzzy region for which the stability of the corresponding single-armed spiral wave solution of (1.1)–(1.2) is uncertain.

(iii) For twist parameters $q = 6$ and $q = 7$, all of the computed single-armed spiral wave solutions of (1.1)–(1.2) are unstable.

Furthermore, those unstable waves loss stability and evolve into chaotic or periodic patterns. Recall that single-armed spiral waves on the whole plane are stable only for small twist parameter q (see [19]). Hence these numerical evidences suggest some differences between single-armed spiral waves on the whole plane and those on the finite disks. These numerical evidences also lead to the following conjecture:

(i) For small twist parameter q ($q = 0.3, 1$), there exists a $d_c \in (0, 1/j_1^2)$ such that, if $d \in (0, d_c)$ (resp. $d \in (d_c, 1/j_1^2)$), then the corresponding single-armed spiral wave solution of (1.1)–(1.2) is stable (resp. unstable). Note that this case has been predicted by the formal bifurcation discussion in [33].

(ii) For twist parameter q of middle size ($q = 2$ and 4), there exist two numbers $0 < d_{s,-} < d_{s,+} < 1/j_1^2$ such that, if $d \in (d_{s,-}, d_{s,+})$ (resp. $d \in (0, d_{s,-}) \cup (d_{s,+}, 1/j_1^2)$), then the corresponding single-armed spiral wave solution of (1.1)–(1.2) is stable (resp. unstable).

(iii) For large twist parameter q ($q = 7$), all of the single-armed spiral wave solutions of (1.1)–(1.2) are unstable.

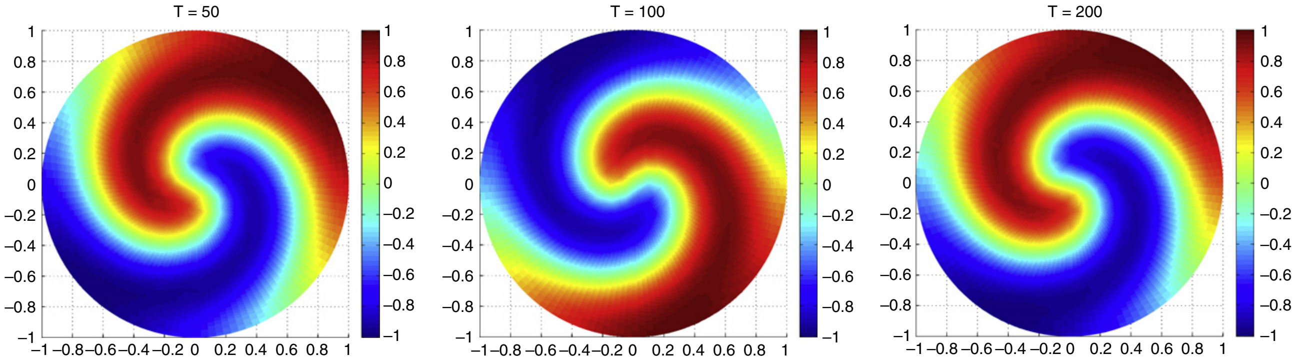

Fig. 10. Snapshots at the indicated times $T = 50$, 100 and 200 of the component $u(x, y, t)$ for the solution (u, v) of system (1.1)–(1.2) on the unit disk with $d = 0.006$, $q = 1$, $\lambda(A) = 1 - A^2$, and $\omega(A) = 1 + qA^2$. The initial data is the single-armed spiral wave which is computed in Section 6.2.

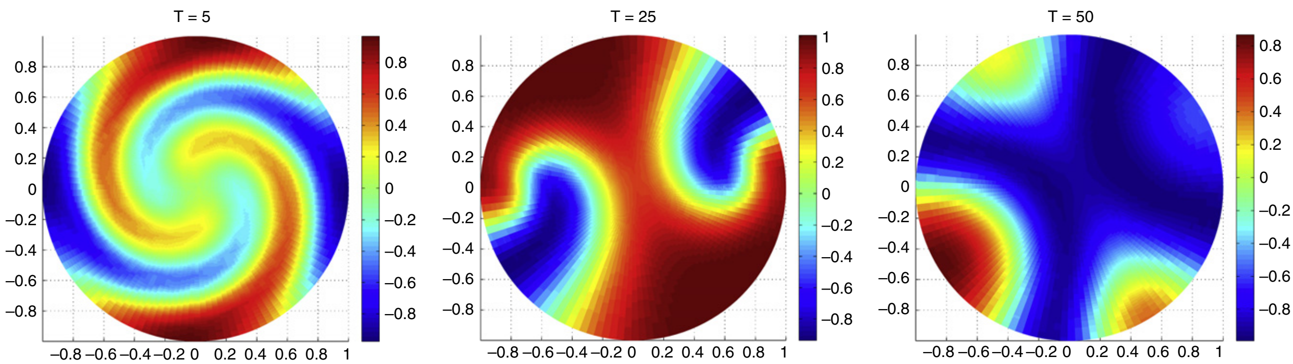

Fig. 11. Snapshots at the indicated times $T = 5$, 25 and 50 of the component $u(x, y, t)$ for the solution (u, v) of system (1.1)–(1.2) on the unit disk with $d = 0.006$, $q = 1$, $\lambda(A) = 1 - A^2$, and $\omega(A) = 1 + qA^2$. The initial data is the two-armed spiral wave which is computed in Section 6.2. These figures suggest that loss of stability of a spiral wave may result in chaotic patterns.

6.5.2. Multi-armed spiral waves

Numerical evidences show that all of the computed multi-armed spiral waves are unstable. Furthermore, those unstable waves loss stability and evolve into chaotic or periodic patterns (Fig. 11).

7. Conclusion and discussion

Although spiral waves are commonly observed in various chemical systems, the slime mould *Dioctyostelium discoideum*, and cardiac fibrillation, their existence in model equations has been proved only infrequently. Therefore, it is difficult to obtain a criterion for the existence of the general system. On the other hand, it is widely believed that λ–ω systems arise naturally near a Hopf bifurcation. Furthermore, λ–ω systems are tractable analytically.

In this paper, we give a elegant criterion for the existence of m-armed spiral wave solutions of λ–ω systems on a circular disk with no-flux boundary condition. To begin with, we note that most of previous studies only consider the specific nonlinearities (λ, ω) (see [24,32,33]). To avoid our analytical results relying on the specific form of the nonlinearities (λ, ω), we give a general setting for the reaction kinetics (λ, ω) (see assumptions H1–H2), but keep the supercritical characteristics which are the common features for the previous studies. With the help of the useful identity (3.11), we show that m-armed spiral wave solutions of (1.1)–(1.2) can exist only if

$$0 < d < \frac{\lambda_0}{j_m^2} \cdot R^2.$$

On the other hand, for any given diffusivity $d \in (0, \frac{\lambda_0}{j_m^2} \cdot R^2)$ (equivalently, the domain size R is above the critical domain size $\sqrt{\frac{d}{\lambda_0}} \cdot j_m$), if the twist parameter q is small, then (1.1)–(1.2) admits an m-armed spiral wave solution, while for any given twist parameter $q > 0$, if the diffusivity d is close to the critical diffusivity $\frac{\lambda_0}{j_m^2} \cdot R^2$ (equivalently, the domain size R is close to the critical domain size $\sqrt{\frac{d}{\lambda_0}} \cdot j_m$), then we can find m-armed spiral wave solutions of (1.1)–(1.2). We remark that for the existence of m-armed spiral waves, numerical studies indicate that q (resp. d) may not be necessarily small (resp. close to $\frac{\lambda_0}{j_m^2} \cdot R^2$). However, numerical studies suggest that spiral waves associated with large twist parameters are always unstable. Therefore, we obtain a relation between the diffusivity d, the radius R of a circular domain, and the reaction kinetic quantity λ_0, which makes it possible to forecast whether the interplay between these quantities leads to the existence of m-armed spiral wave solutions of (1.1)–(1.2). Note that we do not intend to claim that the property of spiral wave solutions of (1.1)–(1.2) depends only on these quantities d, R and λ_0. It may be the case that the qualitative properties (e.g., the shape) of a spiral wave depend on the parameters d, R, and the reaction kinetics (λ, ω) in a more complicated way. Nevertheless, the above criterion holds for this generic λ–ω system which keeps the supercritical characteristics.

Numerical studies (see the right panel in Figs. 5 and 8) reveal that, for small q and large domains, the rotational frequency of spiral waves seems to be independent of the domain sizes. This suggests that the properties of spiral waves on large circular domains may be similar to those of spiral waves in the infinite plane. On the other hand, for small circular domains, the domain size has very strong influence on the rotational frequency of spiral waves. From Fig. 8, this is very apparent for large q and/or multi-armed spiral waves. Therefore, it is interesting to investigate how the middle domain sizes affect the properties of spiral waves, which we will leave as our future study.

Although the stability of the constructed spiral waves is not analyzed rigourously, we have used the finite element method to numerically investigate the stability of these obtained spiral waves which are computed in Section 6.2. The numerical results can be summarized concisely as follows.

(i) Single-armed spiral wave case:
- for small twist parameter q ($q = 0.3$ and 1), the single-armed spiral wave is stable for diffusivity d close to 0, and it is unstable for diffusivity d close to the bifurcation diffusivity ($d = 1/j_1^2$);
- for twist parameter q of middle magnitude ($q = 2$ and 4), the single-armed spiral wave is stable for diffusivity d in a small interval which is away from the bifurcation diffusivity ($d = 1/j_1^2$), and it is unstable for d close to the bifurcation diffusivity ($d = 1/j_1^2$) and for d close to 0;
- for large twist parameter q ($q = 5$, 6, and 7), the single-armed spiral wave is unstable.

(ii) Multiple-armed spiral waves are unstable.

According to Hagan's analysis (see [19]), single-armed spiral waves in the infinite plane are stable only for small q, while multi-armed spiral waves in the infinite plane are unstable for all q. Hence for small q and middle size q, the stability of single-armed spiral waves in finite disks is different from that in the whole plane. We also refer the readers to [33] for formal bifurcation discussion on the stability of single-armed spiral waves.

Finally, we would like to discuss the relevance of our results for general reaction–diffusion systems. To begin with, it is widely accepted that λ–ω systems arise naturally only as center-manifold reductions of Hopf bifurcations in reaction–diffusion systems, and hence might not actually correspond to some specific type of realistic models (see [21,19,20]). However, λ–ω systems are actually special cases of a general class of physical models. To see this, we set $d = 1$ and let $\lambda_0 = \lambda(0)$ be the controlled parameter in system (1.1) for ease of comparison. We can do this without loss of generality since system (1.1) enjoys a nice scaling property (see Section 3.1). Also recall that $\omega(\cdot) = \omega_0 + q\varpi(\cdot)$. Now we follow [40] to set $z = u + \mathrm{i}v$. Then system (1.1) can be transformed into a single complex equation:

$$\begin{aligned} z_t &= \nabla^2 z + (\lambda(|z|) + \mathrm{i}\omega(|z|))\, z \\ &= \nabla^2 z + (\lambda_0 + \mathrm{i}\omega_0) z + (\lambda(|z|) - \lambda_0 + \mathrm{i}q\varpi(|z|))\, z. \end{aligned} \tag{7.1}$$

For the typical nonlinearity $(\lambda(|z|), \omega(|z|)) = (\lambda_0 - |z|^2, \omega_0 + q|z|^2)$ where $\varpi(|z|) = |z|^2$, system (7.1) becomes

$$z_t = \nabla^2 z + (\lambda_0 + \mathrm{i}\omega_0) z - (1 - \mathrm{i}q)|z|^2 z,$$

which is a special case of the well-known cubic complex Ginzburg–Landau equation (CGLE)

$$z_t = (1 + \mathrm{i}b)\nabla^2 z + (\lambda_0 + \mathrm{i}\omega_0) z - (1 + \mathrm{i}c)|z|^2 z, \tag{7.2}$$

where the real parameters b and c are related to linear and nonlinear dispersion, respectively (see [41]). Hence our results can be applied to system (7.2) on the unit disk with $b = 0$ and no-flux boundary condition. Roughly speaking, m-armed spiral wave solutions of system (7.2) on the unit disk with $b = 0$ and no-flux boundary condition exist only for $\lambda_0 > j_m^2$, while system (7.2) on the unit disk with $b = 0$ and no-flux boundary condition admits physically observable single-armed spiral waves for either small c and large λ_0, or middle size c and suitably chosen λ_0.

It is also believed [41] that the CGLE is a minimal realistic model, which means that it cannot be further reduced. To include new phenomena, it is necessary to generalize the CGLE, or extra factors need to be incorporated into the CGLE. For example, the following perturbed cubic CGLE

$$z_t = (1 + \mathrm{i}b)\nabla^2 z + (\lambda_0 + \mathrm{i}\omega_0) z - (1 + \mathrm{i}c)|z|^2 z + \gamma|z|^4 z, \tag{7.3}$$

has been employed to discuss the destruction of Nozaki–Bekki holes (see [41]). We can then see that system (7.1) with the nonlinearity $(\lambda(|z|), \omega(|z|)) = (\lambda_0 - |z|^2 + \gamma|z|^4, \omega_0 + q|z|^2)$ is a special case of system (7.3). This specific example and possible generalizations of the CGLE suggest that our results can be applied to a potential generalized class of CGLE which are physically acceptable models.

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Appendix. The behavior of m-armed spiral waves when d is close to $\frac{\lambda_0}{j_m^2} \cdot R^2$

In this appendix, we will show that as the diffusivity d approaches the critical diffusivity $\frac{\lambda_0}{j_m^2} \cdot R^2$, the amplitude function $A(\cdot)$ and the related quantity Ω of m-armed spiral waves shrink to 0. Recall that the rotational frequency $\hat{\Omega}$ satisfies $\hat{\Omega} = \omega_0 - \Omega$ (see Eq. (1.1b)). By doing so, we obtain the asymptotical behavior of m-armed spiral waves when d is close to $\frac{\lambda_0}{j_m^2} \cdot R^2$, even if the uniqueness of m-armed spiral waves does not hold.

To do this, with the discussion in Section 3, we assume that $R = 1$ and consider the solution (A, k) of problem (3.3)–(3.4). To specify the dependence of solutions of problem (3.3)–(3.4) on d and Ω, we will write such solutions as $(A(\cdot; d, \Omega), k(\cdot; d, \Omega))$ in the following proposition.

Proposition 3. *Suppose that the functions $\lambda(\cdot)$ and $\omega(\cdot)$ satisfy the assumptions* (H1)–(H2). *Assume that $(A(\cdot; d, \Omega), k(\cdot; d, \Omega))$ is a solution of problem (3.3)–(3.4) which satisfies that $A(\cdot; d, \Omega) > 0$ on* $(0, 1]$. *Then it holds that*

$$\sup_{r\in[0,1]} A(r; d, \Omega) \to 0 \quad \textit{and} \quad \Omega \to 0 \quad \textit{as}\ d \to \left(\frac{\lambda_0}{j_m^2}\right)^-.$$

Proof. We first establish the first assertion. To begin with, for each $n \in \mathbb{N}$, let $(\tilde{A}(\cdot; d, \Omega), \tilde{k}(\cdot; d, \Omega))$ be the solution of Eqs. (3.9a)–(3.9b) corresponding to $(A(\cdot; d, \Omega), k(\cdot; d, \Omega))$. Recall that

$$s := \sqrt{\frac{\lambda_0}{d}}\, r \quad \text{and} \quad \tilde{A}(s; d, \Omega) := A(r; d, \Omega),$$

$$\tilde{k}(s; d, \Omega) := \sqrt{\frac{d}{\lambda_0}}\, k(r; d, \Omega).$$

Hence we have that $(\tilde{A}(s; \alpha, \Omega), \tilde{k}(s; \alpha, \Omega))$ is defined on $[0, \sqrt{\frac{\lambda_0}{d}}]$ and $\tilde{A}(\cdot; d, \Omega) \in (0, 1)$ on $(0, \sqrt{\frac{\lambda_0}{d}}]$ by Proposition 2. Also note that $\sqrt{\frac{\lambda_0}{d}} > j_m$. Hence, from Lemma 5.1, we have that $\tilde{A}'(\cdot; d, \Omega) > 0$ on $(0, j_m]$. Integrating (3.11) from 0 to s, we have that

$$\begin{aligned} & s\tilde{A}'(s; d, \Omega) J_m(s) - s\tilde{A}(s; d, \Omega) J_m'(s) \\ &\quad = \int_0^s \xi \tilde{A}(\xi; d, \Omega) J_m(\xi) \\ &\qquad \times \left(\frac{\lambda_0 - \lambda(\tilde{A}(\xi; d, \Omega))}{\lambda_0} + \tilde{k}^2(\xi; d, \Omega) \right) \mathrm{d}\xi \\ &\quad := F[s; \tilde{A}(\cdot; d, \Omega), \tilde{k}(\cdot; d, \Omega)] \end{aligned} \tag{A.1}$$

holds for $s \in [0, \sqrt{\frac{\lambda_0}{d}}]$. Since the integrand in the above integral is positive, we have that the function $F[s; \tilde{A}(\cdot; d, \Omega), \tilde{k}(\cdot; d, \Omega)]$ is increasing on $[0, \sqrt{\frac{\lambda_0}{d}}]$. Evaluating (A.1) at $s = \sqrt{\frac{\lambda_0}{d}}$, we obtain

$$F\left[\sqrt{\frac{\lambda_0}{d}}; \tilde{A}(\cdot; d, \Omega), \tilde{k}(\cdot; d, \Omega)\right] = -\sqrt{\frac{\lambda_0}{d}}\tilde{A}\left(\sqrt{\frac{\lambda_0}{d}}; d, \Omega\right)J_m'\left(\sqrt{\frac{\lambda_0}{d}}\right).$$

Fix a sufficiently small $\delta_0 \in (0, \min\{1/2, (j_{m,0} - j_m)/2\})$ such that $\sqrt{\frac{\lambda_0}{d}} \le 2j_m$ for $\sqrt{\frac{\lambda_0}{d}} \in (j_m - \delta_0, j_m + \delta_0)$. Let $C_m := 2j_m \cdot \max_{s\in[j_m-1,j_m+1]} |J_m''(s)|$. Then, for each $\sqrt{\frac{\lambda_0}{d}} \in (j_m - \delta_0, j_m + \delta_0)$, we can conclude from the above equality that

$$\begin{aligned}
0 &\le F[s; \tilde{A}(\cdot; d, \Omega), \tilde{k}(\cdot; d, \Omega)] \\
&\le F\left[\sqrt{\frac{\lambda_0}{d}}; \tilde{A}(\cdot; d, \Omega), \tilde{k}(\cdot; d, \Omega)\right] \quad \text{(since } F \text{ is increasing)} \\
&\le -2j_m\left(J_m'\left(\sqrt{\frac{\lambda_0}{d}}\right) - J_m'(j_m)\right) \\
&\qquad \left(\text{since } \tilde{A}(\cdot; d, \Omega) \in [0, 1) \text{ on } \left[0, \sqrt{\frac{\lambda_0}{d}}\right]\right) \\
&\le C_m\left(\sqrt{\frac{\lambda_0}{d}} - j_m\right) \quad \text{(by the mean-value theorem)}
\end{aligned} \tag{A.2}$$

holds for $s \in [0, \sqrt{\frac{\lambda_0}{d}}]$.

We claim that $\tilde{A}(j_m; d, \Omega) \to 0$ as $\sqrt{\frac{\lambda_0}{d}} \to j_m^+$. For contradiction, we assume that there is a $\epsilon_0 > 0$ and a sequence of solutions $\{(\tilde{A}(\cdot; d_n, \Omega_n), \tilde{k}(\cdot; d_n, \Omega_n))\}$ to problem (3.3a)–(3.3b) and (3.4a)–(3.4b) such that $\sqrt{\frac{\lambda_0}{d_n}} \to j_m^+$ as $n \to +\infty$, and $\tilde{A}(j_m; d_n, \Omega_n) \ge \epsilon_0$ for all $n \in \mathbb{N}$. Without loss of generality, we can also assume that $\sqrt{\frac{\lambda_0}{d_n}} \in (j_m - \delta_0, j_m + \delta_0)$ for each $n \in \mathbb{N}$. From the fact that F is increasing and $\tilde{A}(\cdot; d_n, \Omega_n) \in [0, 1)$ on $[0, \sqrt{\frac{\lambda_0}{d}}]$, we can use Eq. (A.1) to estimate $\tilde{A}(s; d_n, \Omega_n)$ for all $s \in [0, j_m]$ and $n \in \mathbb{N}$ as follows:

$$\begin{aligned}
&|s\tilde{A}'(s; d_n, \Omega_n)J_m(s)| \\
&\quad \le j_m \cdot \max_{\xi\in[0,j_m]} J_m'(\xi) + F[j_m; \tilde{A}(\cdot; d_n, \Omega_n), \tilde{k}(\cdot; d_n, \Omega_n)].
\end{aligned}$$

Together with (A.2) and the definition of δ_0, we have

$$\begin{aligned}
|\tilde{A}'(s; d_n, \Omega_n)| &\le \frac{2}{J_m(j_m/2)} \cdot \max_{\xi\in[0,j_m]} J_m'(\xi) + \frac{2C_m}{j_m \cdot J_m(j_m/2)} \cdot \left|\sqrt{\frac{\lambda_0}{d_n}} - j_m\right| \\
&\le \frac{2}{J_m(j_m/2)} \cdot \max_{\xi\in[0,j_m]} J_m'(\xi) + \frac{2C_m}{j_m \cdot J_m(j_m/2)} \cdot \delta_0
\end{aligned}$$

for all $s \in [j_m/2, j_m]$ and $n \in \mathbb{N}$. By the Arzelà–Ascoli theorem, we can find a subsequence of $\{(\tilde{A}(\cdot; d_n, \Omega_n), \tilde{k}(\cdot; d_n, \Omega_n))\}$, which we still denote by $\{(\tilde{A}(\cdot; d_n, \Omega_n), \tilde{k}(\cdot; d_n, \Omega_n))\}$, such that the sequence $\tilde{A}(\xi; d_n, \Omega_n) \to \tilde{A}(\xi)$ uniformly on $[j_m/2, j_m]$ as $n \to +\infty$ for some continuous function $\tilde{A}(\cdot)$ which is defined on $[j_m/2, j_m]$ and satisfies that $\tilde{A}(\cdot) \in [0, 1]$ on $[j_m/2, j_m]$ and $\tilde{A}(j_m) \ge \epsilon_0$. By using (A.1) and (A.2) with $(d, \Omega) = (d_n, \Omega_n)$ and taking the limit, we have

$$\int_{j_m/2}^{j_m} \xi\tilde{A}(\xi)J_m(\xi) \cdot \left(\frac{\lambda_0 - \lambda(\tilde{A}(\xi))}{\lambda_0}\right) \mathrm{d}\xi = 0,$$

which, together with the fact that $\lambda(\cdot)$ is decreasing, implies that $\tilde{A}(s) = 0$ for $s \in [j_m/2, j_m]$. This is a contradiction, thus completing the proof of the claim. With this claim and the fact that $\tilde{A}(\cdot; d, \Omega)$ is increasing on $[0, j_m]$, it follows that $\tilde{A}(\cdot; d, \Omega) \to 0$ uniformly on $[0, j_m]$ as $\sqrt{\frac{\lambda_0}{d}} \to j_m^+$.

Now, with the help of (3.9a) and a straightforward computation, one can check that the following equality holds:

$$\begin{aligned}
\tilde{A}'(s; d, \Omega) &= \tilde{A}'(j_m; d, \Omega) + \frac{m^2 - 1}{2} \\
&\quad \times \int_{j_m}^{s} \left(\frac{1}{s^2} + \frac{1}{\xi^2}\right)\tilde{A}(\xi; d, \Omega)\mathrm{d}\xi \\
&\quad + \frac{1}{2}\int_{j_m}^{s} \left(1 + \frac{\xi^2}{s^2}\right)\tilde{A}(\xi; d, \Omega) \\
&\quad \times \left(\tilde{k}^2(\xi; d, \Omega) - \frac{\lambda(\tilde{A}(\xi; d, \Omega))}{\lambda_0}\right)\mathrm{d}\xi \\
&= \tilde{A}'(j_m; d, \Omega) + \frac{m^2 - 1}{2}\int_{j_m}^{s} \left(\frac{1}{s^2} + \frac{1}{\xi^2}\right)\tilde{A}(\xi; d, \Omega)\mathrm{d}\xi \\
&\quad - \frac{1}{2}\int_{j_m}^{s} \left(1 + \frac{\xi^2}{s^2}\right)\tilde{A}(\xi; d, \Omega)\mathrm{d}\xi \\
&\quad + \frac{1}{2}\int_{j_m}^{s} \frac{\left(1 + \frac{\xi^2}{s^2}\right)}{\xi J_m(\xi)} \cdot \xi\tilde{A}(\xi; d, \Omega)J_m(\xi) \\
&\quad \times \left(\frac{\lambda_0 - \lambda(\tilde{A}(\xi; d, \Omega))}{\lambda_0} + \tilde{k}^2(\xi; d, \Omega)\right)\mathrm{d}\xi.
\end{aligned} \tag{A.3}$$

By using (A.1) with $s = j_m$ and (A.2), we have

$$0 < \tilde{A}'(j_m; d, \Omega) \le \frac{C_m}{j_m \cdot J_m(j_m)} \cdot \left(\sqrt{\frac{\lambda_0}{d}} - j_m\right). \tag{A.4}$$

With the use of (A.3)–(A.4) and the fact that $\tilde{A}(\cdot; d, \Omega) \in (0, 1)$ on $(0, \sqrt{\frac{\lambda_0}{d}}]$, the following estimate on $\tilde{A}'(s; d, \Omega)$ holds for $s \in [j_m, \sqrt{\frac{\lambda_0}{d}}]$:

$$\begin{aligned}
|\tilde{A}'(s; d, \Omega)| &\le \left(\frac{C_m}{j_m \cdot J_m(j_m)} + m^2\right) \cdot \left(\sqrt{\frac{\lambda_0}{d}} - j_m\right) \\
&\quad + \frac{F[s; \tilde{A}(\cdot; d, \Omega), \tilde{k}(\cdot; d, \Omega)]}{\min_{\xi\in[j_m,j_m+1]} \xi J_m(\xi)} \\
&\le \left(\frac{C_m}{j_m \cdot J_m(j_m)} + m^2 + \frac{C_m}{\min_{\xi\in[j_m,j_m+1]} \xi J_m(\xi)}\right) \\
&\quad \times \left(\sqrt{\frac{\lambda_0}{d}} - j_m\right) \quad \text{(by (A.2))}.
\end{aligned} \tag{A.5}$$

Together with the mean-value theorem and the fact that $\tilde{A}(\cdot; d, \Omega) \to 0$ uniformly on $[0, j_m]$ as $\sqrt{\frac{\lambda_0}{d}} \to j_m^+$, this yields that $\tilde{A}(\cdot; d, \Omega) \to 0$ uniformly on $[0, \sqrt{\frac{\lambda_0}{d}}]$ as $\sqrt{\frac{\lambda_0}{d}} \to j_m^+$.

Now we turn to the second assertion. From the proof of Lemma 5.3, we have

$$0 \leq \frac{-\Omega}{q} = \frac{\int_0^1 sA^2(s)\varpi(A(s))\mathrm{d}s}{\int_0^1 sA^2(s)\mathrm{d}s} \leq \sup_{x\in[0,\ \sup_{r\in[0.1]} A(r;d,\Omega)]} \varpi(x).$$

This, together with the first assertion and the continuity of $\varpi(x)$ at $x = 0$, shows the second assertion. This completes the proof of this lemma. □

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