Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 428293, 27 pages
doi:10.1155/2010/428293

Research Article

A New Iterative Method for Finding Common Solutions of a System of Equilibrium Problems, Fixed-Point Problems, and Variational Inequalities

Jian-Wen Peng,[1] Soon-Yi Wu,[2] and Jen-Chih Yao[3]

[1] *School of Mathematics, Chongqing Normal University, Chongqing 400047, China*
[2] *Department of Mathematics, National Cheng Kung University, Tainan 701, Taiwan*
[3] *Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan*

Correspondence should be addressed to Jian-Wen Peng, jwpeng6@yahoo.com.cn

Received 18 February 2010; Revised 22 May 2010; Accepted 22 June 2010

Academic Editor: Simeon Reich

We introduce a new iterative scheme based on extragradient method and viscosity approximation method for finding a common element of the solutions set of a system of equilibrium problems, fixed point sets of an infinite family of nonexpansive mappings, and the solution set of a variational inequality for a relaxed cocoercive mapping in a Hilbert space. We prove strong convergence theorem. The results in this paper unify and generalize some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty, closed, and convex subset of H. Let $\{F_k\}_{k\in\Gamma}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Combettes and Hirstoaga [1] considered the following system of equilibrium problems:

$$\text{Find } x \in C \text{ such that } (\forall k \in \Gamma),\ (\forall y \in C),\ F_k(x, y) \geq 0. \tag{1.1}$$

If Γ is a singleton, problem (1.1) becomes the following equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solutions set of (1.2) is denoted by $\mathrm{EP}(F)$. And clearly the solutions set of problem (1.1) can be written as $\bigcap_{k\in\Gamma} \mathrm{EP}(F_k)$.

Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; see for instance, [1–4].

Recall that a mapping S of a closed and convex subset C into itself is nonexpansive if

$$\|Sx - Sy\| \le \|x - y\| \quad \forall x, y \in C. \tag{1.3}$$

We denote fixed-points set of S by $\mathrm{Fix}(S)$. A mapping $f : C \to C$ is called contraction if there exists a constant $\alpha \in (0,1)$ such that

$$\|fx - fy\| \le \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

A bounded linear operator B on H is strongly positive, if there is a constant $\overline{\gamma} > 0$ such that $\langle Bx, x\rangle \ge \overline{\gamma}\|x\|^2$ for all $x \in H$.

Combettes and Hirstoaga [1] introduced an iterative scheme for finding a common element of the solutions set of problem (1.1) in a Hilbert space and obtained a weak convergence theorem. Peng and Yao [2] introduced a new viscosity approximation scheme based on the extragradient method for finding a common element in the solutions set of the problem (1.1), fixed-points set of an infinite family of nonexpansive mappings and the solutions set of the variational inequality for a monotone and Lipschitz continuous mapping in a Hilbert space and obtained a strong convergence theorem. Colao et al. [3] introduced an implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed-points of infinite family of nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Saeidi [4] introduced some iterative algorithms for finding a common element of the solutions set of a system of equilibrium problems and of fixed-points set of a finite family and a left amenable semigroup of nonexpansive mappings in a Hilbert space and obtained some strong convergence theorems.

Several algorithms for problem (1.2) have been proposed (see [5–20]). S. Takahashi and W. Takahashi [5] introduced and studied the following iterative scheme by the viscosity approximation method for finding a common element of the solutions set of problem (1.2) and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$ define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \quad \forall n \in N. \end{gathered} \tag{1.5}$$

Shang et al. [6] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the solutions set of problem (1.2) and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$
\begin{gathered}
F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S u_n, \quad \forall n \in N.
\end{gathered} \tag{1.6}
$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.6) converge strongly to the unique solution of the variational inequality

$$
\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(S) \cap \operatorname{EP}(F), \tag{1.7}
$$

which is the optimality condition for the minimization problem

$$
\min_{x \in \operatorname{Fix}(S) \cap \operatorname{EP}(F)} \frac{1}{2}\langle Bx, x\rangle - h(x), \tag{1.8}
$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). If $C = H$, the algorithm (1.6) was also studied by Plubtieng and Punpaeng [7].

Let $A : C \to H$ be a monotone mapping. The variational inequality problem is to find a point $x \in C$ such that

$$
\langle Ax, y - x \rangle \geq 0 \tag{1.9}
$$

for all $y \in C$. The solutions set of the variational inequality problem is denoted by $\operatorname{VI}(C, A)$. Qin et al. [8] introduced the following general iterative scheme for finding a common element of the solutions set of problem (1.2), the solutions set of a variational inequality and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$
\begin{gathered}
F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S P_C (I - s_n A) u_n, \quad \forall n \in N.
\end{gathered} \tag{1.10}
$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{s_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.10) converge strongly to the unique solution of the variational inequality

$$
\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(S) \cap \operatorname{VI}(C, A) \cap \operatorname{EP}(F). \tag{1.11}
$$

Qin et al. [9] introduced the following general iterative scheme for finding a common element of the solutions set of problem (1.2) and fixed-points set of a finite family of nonexpansive mappings in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (1 - \alpha_n B) W_n P_C (I - s_n A) u_n, \quad \forall n \in N, \end{gathered} \tag{1.12}$$

where W_n is the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN}$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{s_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.12) converge strongly to the unique solution of the variational inequality

$$\left\langle (B - \gamma f)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \cap \mathrm{EP}(F). \tag{1.13}$$

A typical problem is to minimize a quadratic function over the fixed-points set of a nonexpansive mapping S on a real Hilbert space H, that is,

$$\min_{x \in \mathrm{Fix}(S)} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{1.14}$$

where b is a given point in H. In 2003, Xu [21] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial point $x_0 \in H$, chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n B) S x_n + \alpha_n u, \quad n \geq 0, \tag{1.15}$$

converges strongly to the unique solution of the minimization problem (1.15) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Marino and Xu [22] combine the iterative method (1.15) with the viscosity approximation in [23] and consider the following general iterative method: with the initial point $x_0 \in H$, chosen arbitrarily:

$$x_{n+1} = (1 - \alpha_n B) S x_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.16}$$

They proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.16) converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \mathrm{Fix}(S) \tag{1.17}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \mathrm{Fix}(S)} \frac{1}{2}\langle Bx, x\rangle - h(x), \tag{1.18}$$

where h is a potential function for γf.

Recently, Qin et al. [24] introduced the following general iterative process: with the initial point $x_1 \in C$, chosen arbitrarily:

$$\begin{aligned} y_n &= P_C(I - s_n A)x_n, \\ x_{n+1} &= \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B) W_n P_C(I - r_n A) y_n, \quad \forall n \in N, \end{aligned} \tag{1.19}$$

where W_n is the W-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nN}$. They proved that if the sequences of parameters $\{\alpha_n\}$, $\{r_n\}$ and $\{s_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$, $\{y_n\}$ generated by (1.19) converge strongly to a point x^* which is the unique solution of the variational inequality

$$\left\langle (B - \gamma f)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{VI}(C, A). \tag{1.20}$$

Inspired and motivated by above works, we introduce a new iterative scheme based on extragradient method and viscosity approximation method for finding a common element of the solutions set of a system of equilibrium problems, fixed-points set of a family of infinitely nonexpansive mappings and the solutions set of a variational inequality for a relaxed cocoercive mapping in a Hilbert space. We prove strong convergence theorem. The results in this paper unify, generalize and extend some well-known results in [6–9, 21, 22, 24].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed, and convex subset of H. Let symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively. It is well known that

$$\left\| \lambda x + (1 - \lambda) y \right\|^2 = \lambda \|x\|^2 + (1 - \lambda) \left\| y \right\|^2 - \lambda(1 - \lambda) \left\| x - y \right\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C, $P_C(x) \in C$ and

$$\left\langle x - P_C(x), P_C(x) - y \right\rangle \geq 0 \tag{2.2}$$

for all $x, y \in H$.

It is easy to see that (2.2) is equivalent to

$$\left\| x - y \right\|^2 \geq \|x - P_C(x)\|^2 + \left\| y - P_C(x) \right\|^2 \tag{2.3}$$

for all $x, y \in H$. It is also known that P_C has the following firmly nonexpansive property:

$$\left\langle x - y, P_C x - P_C y \right\rangle \geq \left\| P_C x - P_C y \right\|^2 \tag{2.4}$$

for all $x, y \in H$.

Recall also that a mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \tag{2.5}$$

for all $x, y \in C$. A is said to be μ-cocoercive, if for each $x, y \in C$, we have

$$\langle Ax - Ay, x - y\rangle \geq \mu \|Ax - Ay\|^2, \tag{2.6}$$

for a constant $\mu > 0$. A is said to be relaxed (u, v)-cocoercive, if there exist two constants $u, v > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-u)\|Ax - Ay\|^2 + v\|x - y\|^2, \quad \forall x, y \in C. \tag{2.7}$$

Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.2) implies the following:

$$\begin{gathered} u \in \mathrm{VI}(C, A) \Longrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0, \\ u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0 \Longrightarrow u \in \mathrm{VI}(C, A). \end{gathered} \tag{2.8}$$

It is also known that H satisfies the Opial's condition [25], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.9}$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.10}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$ (see [26]).

For solving the problem (1.1), let us assume that the bifunction F satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y); \tag{2.11}$$

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex;

(A5) for each $x \in C, y \mapsto F(x, y)$ is lower semicontinuous.

We recall some lemmas needed later.

Lemma 2.1 (see [1, 10]). *Let C be a nonempty, closed, and convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A5). For $\beta > 0$ and $x \in H$, define the mapping $T_\beta^F : H \to C$ as follows:*

$$T_\beta^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{\beta}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{2.12}$$

for all $x \in H$. Then, the following statements hold:

(1) $T_\beta^F(x) \neq \emptyset$;

(2) T_β^F *is single-valued;*

(3) T_β^F *is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_\beta^F(x) - T_\beta^F(y) \right\|^2 \leq \left\langle T_\beta^F(x) - T_\beta^F(y), x - y \right\rangle; \tag{2.13}$$

(4) $\mathrm{Fix}(T_\beta^F) = \mathrm{EP}(F)$;

(5) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.2 (see [27]). *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n + \delta_n, \quad n \geq 1, \tag{2.14}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences of numbers which satisfy the conditions:

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^\infty \alpha_n = \infty$, *or equivalently,* $\prod_{i=1}^\infty (1 - \alpha_n) = 0$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;

(iii) $\delta_n \geq 0 (n \geq 1)$, $\sum_{n=1}^\infty \delta_n < \infty$;

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{2.15}$$

for all $x, y \in H$.

Lemma 2.4 (see [22]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\tilde{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\tilde{\gamma}$.*

Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 \leq \xi_i \leq 1$ for every $i \in N$. For any $n \in N$, define a mapping W_n of C into C as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \xi_n S_n U_{n,n+1} + (1 - \xi_n) I, \\
U_{n,n-1} &= \xi_{n-1} S_{n-1} U_{n,n} + (1 - \xi_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \xi_k S_k U_{n,k+1} + (1 - \xi_k) I, \\
U_{n,k-1} &= \xi_{k-1} S_{k-1} U_{n,k} + (1 - \xi_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \xi_2 S_2 U_{n,3} + (1 - \xi_2) I, \\
W_n = U_{n,1} &= \xi_1 S_1 U_{n,2} + (1 - \xi_1) I.
\end{aligned} \tag{2.16}$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$; see [28, 29].

Lemma 2.5 (see [28]). *Let C be a nonempty, closed, and convex subset of a Banach space E. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strict convex, then $F(W_n) = \bigcap_{i=1}^{n} \mathrm{Fix}(S_i)$.*

Lemma 2.6 (see [29]). *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. Then for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Remark 2.7. Using Lemma 2.6, one can define mappings $U_{\infty,k}$ and W of C into itself as follows:

$$U_{\infty,k} x = \lim_{n\to\infty} U_{n,k} x, \tag{2.17}$$

and $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$ for every $x \in C$. Such a mapping W is called the W-mapping generated by $S_1, S_2, \ldots$ and $\xi_1, \xi_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|. \tag{2.18}$$

If $\{x_n\}$ is a bounded sequence in C, then we have

$$\lim_{n\to\infty} \|Wx - W_n x\| = 0. \tag{2.19}$$

Lemma 2.8 (see [29]). *Let C be a nonempty, closed and convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le d < 1$ for every $i \in N$. Then $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$.*

3. Strong Convergence Theorem

In this section, we prove strong convergence theorem which solve the problem of finding a common element of the solutions set of a system of equilibrium problems, fixed-points set of a family of infinitely nonexpansive mappings, and the solutions set of a variational inequality for a relaxed cocoercive mapping in Hilbert space.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of H. Let $F_1, F_2, \ldots, F_m$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A5). Let $A : C \to H$ be relaxed (u, v)-cocoercive and μ-Lipschitz continuous and B a strongly positive linear bounded operator on H with coefficient $\tilde{\gamma} > 0$. Assume that $0 < \gamma < \tilde{\gamma}/\alpha$. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Omega = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A) \cap \bigcap_{k=1}^{m} \mathrm{EP}(F_k) \ne \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le \delta < 1$ for every $i \in N$, and let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered}
x_1 = x \in H, \\
u_n = T_{\beta_n}^{F_m} T_{\beta_n}^{F_{m-1}} \cdots T_{\beta_n}^{F_2} T_{\beta_n}^{F_1} x_n, \\
y_n = P_C(I - s_n A) u_n, \\
x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B) W_n P_C (I - r_n A) y_n
\end{gathered} \tag{3.1}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{s_n\}$ are sequences of numbers which satisfy the conditions:

(C1) *$\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;*

(C2) *$\{r_n\} \subset [a, b]$ and $\{s_n\} \subset [a, b]$ for some a, b with $0 \le a \le b \le 2(v - u\mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, and $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;*

(C3) *$\liminf_{n\to\infty} \beta_n > 0$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.*

Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to a point $q \in \Omega$ which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \le 0, \quad \forall p \in \Omega. \tag{3.2}$$

Equivalently, one has $q = P_\Omega(\gamma f + (I - B))(q)$.

Proof. Since $\alpha_n \to 0$ from condition (C1), we may assume, with no loss of generality, that $\alpha_n \leq \|B\|^{-1}$ for all n. Lemma 2.4 implies $\|I - \alpha_n B\| \leq 1 - \alpha_n \widetilde{\gamma}$. Next, we will assume that $\|I - B\| \leq 1 - \widetilde{\gamma}$. Now, we show that the mappings $I - s_n A$ and $I - r_n A$ are nonexpansive. Indeed, from the relaxed (u, v)-cocoercivity and μ-Lipschitz continuity of A and condition (C2), we have

$$\begin{aligned}
\left\|(I - s_n A)x - (I - s_n A)y\right\|^2 &= \left\|(x - y) - s_n(Ax - Ay)\right\|^2 \\
&= \|x - y\|^2 - 2s_n\langle x - y, Ax - Ay\rangle + s_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2s_n\left[-u\|Ax - Ay\|^2 + v\|x - y\|^2\right] + s_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + 2s_n\mu^2 u\|x - y\|^2 - 2s_n v\|x - y\|^2 + \mu^2 s_n^2\|x - y\|^2 \\
&= \left(1 + 2s_n\mu^2 u - 2s_n v + \mu^2 s_n^2\right)\|x - y\|^2 \\
&\leq \|x - y\|^2,
\end{aligned} \tag{3.3}$$

which implies the mapping $I - s_n A$ is nonexpansive, so is $I - r_n A$.

For $k \in \{0, 1, 2, \ldots, m\}$, and for any positive integer number n, we define the operator $\Theta_{\beta_n}^k : H \to C$ as follows:

$$\begin{aligned}
\Theta_{\beta_n}^0 x &= x, \\
\Theta_{\beta_n}^k x &= T_{\beta_n}^{F_k} T_{\beta_n}^{F_{k-1}} \cdots T_{\beta_n}^{F_2} T_{\beta_n}^{F_1} x, \quad k = 1, 2, \ldots, m.
\end{aligned} \tag{3.4}$$

Next, we show that the sequence $\{x_n\}$ is bounded. Let $p \in \Omega$. Then from Lemma 2.1(3), we know that for $k \in \{1, 2, \ldots, m\}$, $T_{\beta_n}^{F_k}$ is nonexpansive and $p = T_{\beta_n}^{F_k} p$, and

$$\|u_n - p\| = \left\|\Theta_{\beta_n}^m x_n - p\right\| = \left\|\Theta_{\beta_n}^m x_n - \Theta_{\beta_n}^m p\right\| \leq \|x_n - p\| \tag{3.5}$$

for all $n = 1, 2, \ldots$. By $p = P_C(I - s_n A)p$ and (3.5), we have

$$\begin{aligned}
\|y_n - p\| &= \|P_C(I - s_n A)u_n - P_C(I - s_n A)p\| \\
&\leq \|(I - s_n A)u_n - (I - s_n A)p\| \leq \|u_n - p\| \leq \|x_n - p\|.
\end{aligned} \tag{3.6}$$

Since $x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - r_n A)y_n$ and $p = W_n p$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\gamma f(W_n x_n) - Bp) + (I - \alpha_n B)(W_n P_C(I - r_n A)y_n - p)\| \\
&\leq \alpha_n\|\gamma f(W_n x_n) - Bp\| + (1 - \alpha_n\widetilde{\gamma})\|P_C(I - r_n A)y_n - p\| \\
&\leq \alpha_n\gamma\|f(W_n x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\widetilde{\gamma})\|y_n - p\| \\
&\leq [1 - \alpha_n(\widetilde{\gamma} - \alpha\gamma)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|.
\end{aligned} \tag{3.7}$$

By inductions, we have

$$\|x_n - p\| \le \max\left\{ \|x_0 - p\|, \frac{\|\gamma f(p) - Bp\|}{\widetilde{\gamma} - \alpha\gamma} \right\}, \tag{3.8}$$

which proves that the sequence $\{x_n\}$ is bounded. It follows from (3.5) and (3.6) that $\{y_n\}$ and $\{u_n\}$ are also bounded.

Since $\Theta^k_{\beta_n} x_n = T^{F_k}_{\beta_n} \Theta^{k-1}_{\beta_n} x_n$ and $\Theta^k_{\beta_{n+1}} x_{n+1} = T^{F_k}_{\beta_{n+1}} \Theta^{k-1}_{\beta_{n+1}} x_{n+1}$ for each $k = 1, 2, \ldots, m$, by Lemma 2.1, we have

$$F_k\left(\Theta^k_{\beta_n} x_n, y\right) + \frac{1}{\beta_n}\left\langle y - \Theta^k_{\beta_n} x_n, \Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n \right\rangle \ge 0 \quad \forall y \in C, \tag{3.9}$$

$$F_k\left(\Theta^k_{\beta_{n+1}} x_{n+1}, y\right) + \frac{1}{\beta_{n+1}}\left\langle y - \Theta^k_{\beta_{n+1}} x_{n+1}, \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1} \right\rangle \ge 0 \quad \forall y \in C, \tag{3.10}$$

Setting $y = \Theta^k_{\beta_{n+1}} x_{n+1}$ in (3.9) and $y = \Theta^k_{\beta_n} x_n$ in (3.10), we have

$$\begin{aligned} &F_k\left(\Theta^k_{\beta_n} x_n, \Theta^k_{\beta_{n+1}} x_{n+1}\right) + \frac{1}{\beta_n}\langle \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n, \Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n \rangle \ge 0, \\ &F_k\left(\Theta^k_{\beta_{n+1}} x_{n+1}, \Theta^k_{\beta_n} x_n\right) + \frac{1}{\beta_{n+1}}\left\langle \Theta^k_{\beta_n} x_n - \Theta^k_{\beta_{n+1}} x_{n+1}, \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1} \right\rangle \ge 0. \end{aligned} \tag{3.11}$$

Adding the two inequalities and from the monotonicity of F, we get

$$\left\langle \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n, \frac{\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n}{\beta_n} - \frac{\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1}}{\beta_{n+1}} \right\rangle \ge 0 \tag{3.12}$$

and hence

$$\begin{aligned} &\left\|\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n\right\|^2 \\ &\quad \le \left\langle \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n, \left(\Theta^{k-1}_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_n} x_n\right) + \left(1 - \frac{\beta_n}{\beta_{n+1}}\right)\left(\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1}\right) \right\rangle. \end{aligned} \tag{3.13}$$

Without loss of generality, let us assume that there exists a real number d such that $\beta_n > d > 0$ for all $n = 1, 2, \ldots$. Hence, for each $k = 1, 2, \ldots, m$ we have

$$\begin{aligned} \left\|\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n\right\| &\le \left\|\Theta^{k-1}_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_n} x_n\right\| + \frac{1}{\beta_{n+1}}\left|\beta_{n+1} - \beta_n\right| \left\|\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1}\right\| \\ &\le \left\|\Theta^{k-1}_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_n} x_n\right\| + \frac{1}{d}\left|\beta_{n+1} - \beta_n\right| M_0, \end{aligned} \tag{3.14}$$

where M_0 is an approximate constant such that

$$M_0 \geq \max\left\{ \sup_{n\geq 1}\left\{ \left\| \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1} \right\| \right\},\ k = 1, 2, \ldots, m \right\}. \tag{3.15}$$

It follows from (3.14) that

$$\|u_{n+1} - u_n\| = \left\| \Theta^m_{\beta_{n+1}} x_{n+1} - \Theta^m_{\beta_n} x_n \right\| \leq \|x_{n+1} - x_n\| + \frac{m}{d}\left|\beta_{n+1} - \beta_n\right| M_0. \tag{3.16}$$

Put $\rho_n = P_C(I - r_n A)y_n$. We have

$$\begin{aligned} \left\|y_n - y_{n+1}\right\| &= \|P_C(I - s_n A)u_n - P_C(I - s_{n+1}A)u_{n+1}\| \\ &\leq \|(I - s_n A)u_n - (I - s_{n+1}A)u_{n+1}\| \\ &= \|(u_n - s_n A u_n) - (u_{n+1} - s_n A u_{n+1}) + (s_{n+1} - s_n)A u_{n+1}\| \\ &\leq \|u_n - u_{n+1}\| + |s_{n+1} - s_n| M_1, \end{aligned} \tag{3.17}$$

where M_1 is an approximate constant such that $M_1 \geq \max\{\sup_{n\geq 1}\{\|Au_n\|\}, M_0\}$.

Substituting (3.16) into (3.17), we have

$$\left\|y_n - y_{n+1}\right\| \leq \|x_{n+1} - x_n\| + \left[\frac{m}{d}\left|\beta_{n+1} - \beta_n\right| + |s_{n+1} - s_n|\right] M_1. \tag{3.18}$$

It follows from (3.18) that

$$\begin{aligned} \left\|\rho_n - \rho_{n+1}\right\| &= \left\|P_C(I - r_n A)y_n - P_C(I - r_{n+1}A)y_{n+1}\right\| \\ &\leq \left\|(I - r_n A)y_n - (I - r_{n+1}A)y_{n+1}\right\| \\ &= \left\|\left(y_n - r_n A y_n\right) - \left(y_{n+1} - r_n A y_{n+1}\right) + (r_{n+1} - r_n)A y_{n+1}\right\| \\ &\leq \left\|y_n - y_{n+1}\right\| + |r_{n+1} - r_n| M_2 \\ &\leq \|x_n - x_{n+1}\| + \left[\frac{m}{d}\left|\beta_{n+1} - \beta_n\right| + |s_{n+1} - s_n| + |r_{n+1} - r_n|\right] M_2, \end{aligned} \tag{3.19}$$

where M_2 is an approximate constant such that $M_2 \geq \max\{M_1, \sup_{n\geq 1}\{\|Ay_{n+1}\|\}\}$.

Observe that

$$\begin{aligned} x_{n+1} &= \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n \rho_n, \\ x_{n+2} &= \alpha_{n+1}\gamma f(W_{n+1}x_{n+1}) + (I - \alpha_{n+1}B)W_{n+1}\rho_{n+1}, \end{aligned} \tag{3.20}$$

we have

$$\begin{aligned} x_{n+2} - x_{n+1} = {} & \alpha_{n+1}\gamma\left[f(W_{n+1}x_{n+1}) - f(W_n x_n)\right] + (I - \alpha_{n+1}B)\left(W_{n+1}\rho_{n+1} - W_n\rho_n\right) \\ & + (\alpha_{n+1} - \alpha_n)\left[\gamma f(W_n x_n) - BW_n\rho_n\right]. \end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| \leq{}& \alpha_{n+1}\gamma\alpha\|W_{n+1}x_{n+1}-W_nx_n\| + \left(1-\alpha_{n+1}\widetilde{\gamma}\right)\left\|W_{n+1}\rho_{n+1}-W_n\rho_n\right\| \\
&+ |\alpha_{n+1}-\alpha_n|\left\|\gamma f(W_nx_n)-BW_n\rho_n\right\| \\
\leq{}& \alpha_{n+1}\gamma\alpha(\|x_{n+1}-x_n\| + \|W_{n+1}x_n-W_nx_n\|) \\
&+ \left(1-\alpha_{n+1}\widetilde{\gamma}\right)\left(\left\|\rho_{n+1}-\rho_n\right\| + \left\|W_{n+1}\rho_n-W_n\rho_n\right\|\right) \\
&+ |\alpha_{n+1}-\alpha_n|\left\|\gamma f(W_nx_n)-BW_n\rho_n\right\|.
\end{aligned} \tag{3.22}$$

Next we estimate $\|W_{n+1}x_n - W_nx_n\|$ and $\|W_{n+1}\rho_n - W_n\rho_n\|$. It follows from the definition of W_n and nonexpansiveness of S_i that

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &= \|U_{n+1,1}x_n - U_{n,1}x_n\| \\
&= \|\xi_1S_1U_{n+1,2}x_n + (1-\xi_1)x_n - \{\xi_1S_1U_{n,2}x_n + (1-\xi_1)x_n\}\| \\
&= \xi_1\|S_1U_{n+1,2}x_n - S_1U_{n,2}x_n\| \\
&\leq \xi_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\
&= \xi_1\|\xi_2S_2U_{n+1,3}x_n + (1-\xi_2)x_n - \{\xi_2S_2U_{n,3}x_n + (1-\xi_2)x_n\}\| \\
&= \xi_1\xi_2\|S_2U_{n+1,3}x_n - S_2U_{n,3}x_n\| \\
&\leq \xi_1\xi_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\
&\vdots \\
&\leq \Pi_{i=1}^{n}\xi_i\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&= \Pi_{i=1}^{n}\xi_i\|\xi_{n+1}S_{n+1}x_n + (1-\xi_{n+1})x_n - x_n\| \\
&= \Pi_{i=1}^{n+1}\xi_i\|S_{n+1}x_n - x_n\| \\
&\leq \Pi_{i=1}^{n+1}\xi_iM_3,
\end{aligned} \tag{3.23}$$

where M_3 is an approximate constant such that

$$M_3 \geq \max\left\{M_2, \sup_{n\geq 1}\{\|S_{n+1}x_n - x_n\|\}, \sup_{n\geq 1}\left\{\left\|S_{n+1}\rho_n - \rho_n\right\|\right\}\right\}. \tag{3.24}$$

Similarly, we have

$$\left\|W_{n+1}\rho_n - W_n\rho_n\right\| \leq \Pi_{i=1}^{n+1}\xi_iM_3. \tag{3.25}$$

Substituting (3.19), (3.23), and (3.25) into (3.22) yields that

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\quad\le \alpha_{n+1}\gamma\alpha\Big(\|x_{n+1}-x_n\|+\Pi_{i=1}^{n+1}\xi_i M_3\Big) \\
&\qquad+\left(1-\alpha_{n+1}\widetilde{\gamma}\right)\Big(\|x_{n+1}-x_n\|+\Big[\frac{m}{d}\left|\beta_{n+1}-\beta_n\right|+|s_{n+1}-s_n|+|r_{n+1}-r_n|\Big]M_2+\Pi_{i=1}^{n+1}\xi_i M_3\Big) \\
&\qquad+|\alpha_{n+1}-\alpha_n|\left\|\gamma f(W_nx_n)-BW_n\rho_n\right\| \\
&\quad\le \left[1-\alpha_{n+1}(\widetilde{\gamma}-\gamma\alpha)\right]\|x_{n+1}-x_n\|+M_4\Big(\frac{m}{d}\left|\beta_{n+1}-\beta_n\right|+|s_{n+1}-s_n|+|r_{n+1}-r_n|+|\alpha_{n+1}-\alpha_n|\Big), \\
&\qquad+\Pi_{i=1}^{n+1}\xi_i
\end{aligned}\tag{3.26}$$

where M_4 is an approximate constant such that

$$M_4\ge\max\left\{M_3,\sup_{n\ge1}\left\{\left\|\gamma f(W_nx_n)-BW_n\rho_n\right\|\right\}\right\}.\tag{3.27}$$

It follows from conditions (C1)–(C3) and $\Pi_{i=1}^{n+1}\xi_i\le\delta^{n+1}$ and Lemma 2.2 that

$$\|x_{n+1}-x_n\|\longrightarrow 0.\tag{3.28}$$

Observe that

$$x_{n+1}-W_n\rho_n=\alpha_n\left(\gamma f(W_nx_n)-BW_n\rho_n\right),\tag{3.29}$$

it follows from (C1) that

$$\lim_{n\to\infty}\left\|W_n\rho_n-x_{n+1}\right\|=0.\tag{3.30}$$

For $p\in\Omega$, we have

$$\begin{aligned}
\left\|y_n-p\right\|^2&=\left\|P_C(I-s_nA)u_n-P_C(I-s_nA)p\right\|^2 \\
&\le\left\|(u_n-p)-s_n(Au_n-Ap)\right\|^2 \\
&=\left\|u_n-p\right\|^2-2s_n\langle u_n-p,Au_n-Ap\rangle+s_n^2\left\|Au_n-Ap\right\|^2 \\
&\le\left\|x_n-p\right\|^2-2s_n\Big[-u\left\|Au_n-Ap\right\|^2+v\left\|u_n-p\right\|^2\Big]+s_n^2\left\|Au_n-Ap\right\|^2 \\
&\le\left\|x_n-p\right\|^2+\left(2s_nu+s_n^2-\frac{2s_nv}{\mu^2}\right)\left\|Au_n-Ap\right\|^2.
\end{aligned}\tag{3.31}$$

Similarly, we have

$$\|\rho_n - p\|^2 \le \|x_n - p\|^2 + \left(2r_n u + r_n^2 - \frac{2r_n v}{\mu^2}\right)\|Ay_n - Ap\|^2. \tag{3.32}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma f(W_n x_n) - Bp) + (I - \alpha_n B)(W_n \rho_n - p)\|^2 \\
&\le \left(\alpha_n\|\gamma f(W_n x_n) - Bp\| + (1 - \alpha_n\tilde{\gamma})\|\rho_n - p\|\right)^2 \\
&\le \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + \|\rho_n - p\|^2 + 2\alpha_n\|\rho_n - p\|\|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.33}$$

Substituting (3.32) into (3.33), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 \le{}& \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 + \left(2r_n u + r_n^2 - \frac{2r_n v}{\mu^2}\right)\|Ay_n - Ap\|^2 \\
&+ 2\alpha_n\|\rho_n - p\|\|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.34}$$

It follows from condition (C2) that

$$\begin{aligned}
&\left(\frac{2av}{\mu^2} - 2bu - b^2\right)\|Ay_n - Ap\|^2 \\
&\quad\le \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 \\
&\qquad - \|x_{n+1} - p\|^2 + 2\alpha_n\|\rho_n - p\|\|\gamma f(W_n x_n) - Bp\| \\
&\quad\le \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| \\
&\qquad + 2\alpha_n\|\rho_n - p\|\|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.35}$$

As $\|x_{n+1} - x_n\| \to 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0 \tag{3.36}$$

It is easy to see that $\|\rho_n - p\| \le \|y_n - p\|$. Using (3.33) again, we have

$$\|x_{n+1} - p\|^2 \le \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + \|y_n - p\|^2 + 2\alpha_n\|\rho_n - p\|\|\gamma f(W_n x_n) - Bp\|. \tag{3.37}$$

Substituting (3.31) into (3.37), we can obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 \le{}& \alpha_n\|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 + \left(2s_n u + s_n^2 - \frac{2s_n v}{\mu^2}\right)\|Au_n - Ap\|^2 \\
&+ 2\alpha_n\|\rho_n - p\|\|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.38}$$

It follows from (C2) that

$$\begin{aligned}
&\left(\frac{2av}{\mu^2}-2bv-b^2\right)\|Au_n-Ap\|^2\\
&\quad\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\qquad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\
&\quad\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|-\|x_{n+1}-p\|)\|x_{n+1}-x_n\|\\
&\qquad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.
\end{aligned}\tag{3.39}$$

As $\|x_{n+1}-x_n\| \to 0$ and $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\|Au_n-Ap\|=0.\tag{3.40}$$

Observe that

$$\begin{aligned}
\|\rho_n-p\|^2&=\|P_C(I-r_nA)y_n-P_C(I-r_nA)p\|^2\\
&\le\langle(I-r_nA)y_n-(I-r_nA)p,\rho_n-p\rangle\\
&=\frac{1}{2}\Big\{\|(I-r_nA)y_n-(I-r_nA)p\|^2+\|\rho_n-p\|^2\\
&\qquad-\|(I-r_nA)y_n-(I-r_nA)p-(\rho_n-p)\|^2\Big\}\\
&\le\frac{1}{2}\Big\{\|y_n-p\|^2+\|\rho_n-p\|^2-\|(y_n-\rho_n)-r_n(Ay_n-Ap)\|^2\Big\}\\
&\le\frac{1}{2}\Big\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|y_n-\rho_n\|^2-r_n^2\|Ay_n-Ap\|^2\\
&\qquad+2r_n\langle y_n-\rho_n,Ay_n-Ap\rangle\Big\},
\end{aligned}\tag{3.41}$$

which yields that

$$\|\rho_n-p\|^2\le\|x_n-p\|^2-\|y_n-\rho_n\|^2+2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|.\tag{3.42}$$

Substituting (3.42) into (3.33) we have

$$\begin{aligned}
\|x_{n+1}-p\|^2&\le\alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|y_n-\rho_n\|^2\\
&\quad+2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|,
\end{aligned}\tag{3.43}$$

which implies that

$$\begin{aligned}\|y_n-\rho_n\|^2 &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|\\ &\quad +2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.\end{aligned} \tag{3.44}$$

It follows from (C1), $\|x_{n+1}-x_n\| \to 0$, and $\|Ay_n-Ap\| \to 0$ that $\lim_{n\to\infty}\|y_n-\rho_n\|=0$.

For $p\in\Omega$, we have

$$\begin{aligned}&\|y_n-p\|^2\\ &=\|P_C(I-s_nA)u_n-P_C(I-s_nA)p\|^2\\ &\le\langle P_C(I-s_nA)u_n-P_C(I-s_nA)p,(I-s_nA)u_n-(I-s_nA)p\rangle\\ &=\langle y_n-p,(I-s_nA)u_n-(I-s_nA)p\rangle\\ &=\frac{1}{2}\left(\|y_n-p\|^2+\|(I-s_nA)u_n-(I-s_nA)p\|^2-\|(y_n-p)-[u_n-p-s_n(Au_n-Ap)]\|^2\right)\\ &\le\frac{1}{2}\left(\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-p-[u_n-p-s_n(Au_n-Ap)]\|^2\right)\\ &=\frac{1}{2}\left(\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\langle y_n-u_n,Au_n-Ap\rangle-s_n^2\|Au_n-Ap\|^2\right).\end{aligned} \tag{3.45}$$

This implies that

$$\begin{aligned}\|y_n-p\|^2&\le\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\langle y_n-u_n,Au_n-Ap\rangle-s_n^2\|Au_n-Ap\|^2\\ &\le\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\|y_n-u_n\|\|Au_n-Ap\|.\end{aligned} \tag{3.46}$$

By (3.46), (3.37), and (3.5), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\|y_n-u_n\|\|Au_n-Ap\|\\ &\quad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\ &\le\alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|y_n-u_n\|^2\\ &\quad+2s_n\|y_n-u_n\|\|Au_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.\end{aligned} \tag{3.47}$$

It follows that

$$\begin{aligned}\|y_n-u_n\|^2 &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2s_n\|y_n-u_n\|\|Au_n-Ap\|\\ &\quad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_n-x_{n+1}\|)\\ &\quad+2s_n\|y_n-u_n\|\|Au_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.\end{aligned} \tag{3.48}$$

It follows from (C1), $\|Au_n-Ap\|\to 0$, and $\|x_{n+1}-x_n\|\to 0$ that $\|y_n-u_n\|\to 0$. It follows from $\|\rho_n-u_n\|\le\|\rho_n-y_n\|+\|y_n-u_n\|$ that $\lim_{n\to\infty}\|u_n-\rho_n\|=0$.

We now show that

$$\lim_{n\to\infty}\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|=0,\quad k=1,2,\dots,m. \tag{3.49}$$

Indeed, let $p\in\Omega$, it follows from the firmly nonexpansiveness of $T^{F_k}_{\beta_n}$, we have for each $k\in\{1,2,\dots,m\}$,

$$\begin{aligned}\left\|\Theta^k_{\beta_n}x_n-p\right\|^2&=\left\|T^{F_k}_{\beta_n}\Theta^{k-1}_{\beta_n}x_n-T^{F_k}_{\beta_n}p\right\|^2\le\langle\Theta^k_{\beta_n}x_n-p,\Theta^{k-1}_{\beta_n}x_n-p\rangle\\ &=\frac{1}{2}\left(\left\|\Theta^k_{\beta_n}x_n-p\right\|^2+\left\|\Theta^{k-1}_{\beta_n}x_n-p\right\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2\right).\end{aligned} \tag{3.50}$$

Thus, we get

$$\left\|\Theta^k_{\beta_n}x_n-p\right\|^2\le\left\|\Theta^{k-1}_{\beta_n}x_n-p\right\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2,\quad k=1,2,\dots,m. \tag{3.51}$$

This implies that for each $k\in\{1,2,\dots,m\}$,

$$\begin{aligned}\left\|\Theta^k_{\beta_n}x_n-p\right\|^2&\le\left\|\Theta^0_{\beta_n}x_n-p\right\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2\\ &\quad-\left\|\Theta^{k-1}_{\beta_n}x_n-\Theta^{k-2}_{\beta_n}x_n\right\|^2-\cdots-\left\|\Theta^2_{\beta_n}x_n-\Theta^1_{\beta_n}x_n\right\|^2-\left\|\Theta^1_{\beta_n}x_n-\Theta^0_{\beta_n}x_n\right\|^2.\end{aligned} \tag{3.52}$$

It follows from $u_n=\Theta^m_{\beta_n}x_n$ that for each $k=1,2,\dots,m$

$$\|u_n-p\|^2\le\|x_n-p\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2. \tag{3.53}$$

By (3.37), (3.6), and (3.53), we have that for each $k = 1, 2, \ldots, m$

$$\begin{aligned}\left\|x_{n+1} - p\right\|^2 &\le \alpha_n \left\|\gamma f(W_n x_n) - Bp\right\|^2 + \left\|u_n - p\right\|^2 + 2\alpha_n \left\|\rho_n - p\right\| \left\|\gamma f(W_n x_n) - Bp\right\| \\ &\le \alpha_n \left\|\gamma f(W_n x_n) - Bp\right\|^2 + \left\|x_n - p\right\|^2 - \left\|\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n\right\|^2 \\ &\quad + 2\alpha_n \left\|\rho_n - p\right\| \left\|\gamma f(W_n x_n) - Bp\right\|.\end{aligned} \tag{3.54}$$

Thus, we have that for each $k = 1, 2, \ldots, m$

$$\begin{aligned}\left\|\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n\right\|^2 &\le \alpha_n \left\|\gamma f(W_n x_n) - Bp\right\|^2 + \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\ &\quad + 2\alpha_n \left\|\rho_n - p\right\| \left\|\gamma f(W_n x_n) - Bp\right\| \\ &\le \alpha_n \left\|\gamma f(W_n x_n) - Bp\right\|^2 + \left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right) \|x_n - x_{n+1}\| \\ &\quad + 2\alpha_n \left\|\rho_n - p\right\| \left\|\gamma f(W_n x_n) - Bp\right\|.\end{aligned} \tag{3.55}$$

It follows from (C1) and $\|x_{n+1} - x_n\| \to 0$ that for each $k = 1, 2, \ldots, m$

$$\left\|\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n\right\| \longrightarrow 0. \tag{3.56}$$

Since

$$\begin{aligned}\left\|W_n \rho_n - \rho_n\right\| &\le \left\|W_n \rho_n - x_{n+1}\right\| + \|x_{n+1} - x_n\| + \left\|x_n - \Theta^1_{\beta_n} x_n\right\| + \left\|\Theta^1_{\beta_n} x_n - \Theta^2_{\beta_n} x_n\right\| \\ &\quad + \cdots + \left\|\Theta^{m-1}_{\beta_n} x_n - \Theta^m_{\beta_n} x_n\right\| + \left\|u_n - y_n\right\| + \left\|y_n - \rho_n\right\|.\end{aligned} \tag{3.57}$$

It follows from (3.56) that

$$\lim_{n \to \infty} \left\|W_n \rho_n - \rho_n\right\| = 0. \tag{3.58}$$

Observe that

$$\left\|W \rho_n - \rho_n\right\| \le \left\|W \rho_n - W_n \rho_n\right\| + \left\|W_n \rho_n - \rho_n\right\|. \tag{3.59}$$

It follows from Remark 2.7 that

$$\lim_{n \to \infty} \left\|W \rho_n - \rho_n\right\| = 0. \tag{3.60}$$

We show that $P_\Omega(\gamma f + (I - B))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}
&\left\| P_\Omega(\gamma f + (I - B))(x) - P_\Omega(\gamma f + (I - B))(y) \right\| \\
&\quad \le \left\| (\gamma f + (I - B))(x) - (\gamma f + (I - B))(y) \right\| \\
&\quad \le \gamma \left\| f(x) - f(y) \right\| + \|I - B\| \left\| x - y \right\| \\
&\quad \le \gamma\alpha \left\| x - y \right\| + (1 - \tilde{\gamma}) \left\| x - y \right\| \\
&\quad = (\gamma\alpha + 1 - \tilde{\gamma}) \left\| x - y \right\|.
\end{aligned} \tag{3.61}$$

The Banach's Contraction Mapping Principle guarantees that $P_\Omega(\gamma f + (I - B))$ has a unique fixed point, say $q \in H$. That is, $q = P_\Omega(\gamma f + (I - B))(q)$.

Next, we show that

$$\limsup_{n \to \infty} \langle \gamma f(q) - Bq, x_n - q \rangle \le 0. \tag{3.62}$$

To show that, we choose a subsequence $\{x_{n_i}\}$ of x_n such that

$$\limsup_{n \to \infty} \langle \gamma f(q) - Bq, x_n - q \rangle = \lim_{i \to \infty} \langle \gamma f(q) - Bq, x_{n_i} - q \rangle. \tag{3.63}$$

As $\{x_{n_i}\}$ is bounded, we know that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to p. We may assume, without loss of generality, that $x_{n_i} \rightharpoonup p$. From $\|\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n\| \to 0$ for each $k = 1, 2, \ldots, m$, we obtain that $\Theta^k_{\beta_{n_i}} x_{n_i} \rightharpoonup p$ for $k = 1, 2, \ldots, m$. From $\|u_n - \rho_n\| \to 0$, we also obtain that $\rho_{n_i} \rightharpoonup p$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $p \in C$.

Now we show that $p \in \Omega$. Indeed, let us first show that $p \in \mathrm{VI}(C, A)$. Put

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1 & \text{if } w_1 \in C, \\ \emptyset & \text{if } w_1 \notin C. \end{cases} \tag{3.64}$$

Since A is relaxed (u, v)-cocoercive, we have

$$\langle Ax - Ay, x - y \rangle \ge (-u) \left\| Ax - Ay \right\|^2 + v \left\| x - y \right\|^2 \ge \left(v - u\mu^2 \right) \left\| x - y \right\|^2 \ge 0, \tag{3.65}$$

which yields that A is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_C w_1$ and $\rho_n \in C$, we have

$$\langle w_1 - \rho_n, w_2 - Aw_1 \rangle \ge 0. \tag{3.66}$$

On the other hand, from $\rho_n = P_C(I - r_n A) y_n$, we have

$$\langle w_1 - \rho_n, \rho_n - (I - r_n A) y_n \rangle \ge 0 \tag{3.67}$$

and hence

$$\left\langle w_1 - \rho_n, \frac{\rho_n - y_n}{r_n} + Ay_n \right\rangle \geq 0. \tag{3.68}$$

It follows that

$$\begin{aligned}
\langle w_1 - \rho_{n_i}, w_2 \rangle &\geq \langle w_1 - \rho_{n_i}, Aw_1 \rangle \\
&\geq \langle w_1 - \rho_{n_i}, Aw_1 \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} + Ay_{n_i} \right\rangle \\
&\geq \left\langle w_1 - \rho_{n_i}, Aw_1 - \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} - Ay_{n_i} \right\rangle \\
&= \langle w_1 - \rho_{n_i}, Aw_1 - A\rho_{n_i} \rangle + \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Ay_{n_i} \rangle \\
&\quad - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} \right\rangle \\
&\geq \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Ay_{n_i} \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} \right\rangle,
\end{aligned} \tag{3.69}$$

which implies that $\langle w_1 - p, w_2 \rangle \geq 0$. We have $p \in T^{-1}0$ and hence $p \in \mathrm{VI}(C, A)$.

We next show that $p \in \bigcap_{k=1}^{m} \mathrm{EP}(F_k)$. Indeed, by Lemma 2.1, we have that for each $k = 1, 2, \ldots, m$,

$$F_k\left(\Theta_{\beta_n}^k x_n, y\right) + \frac{1}{\beta_n} \left\langle y - \Theta_{\beta_n}^k x_n, \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\rangle \geq 0, \quad \forall y \in C \tag{3.70}$$

.

It follows from (A2) that

$$\frac{1}{\beta_n} \left\langle y - \Theta_{\beta_n}^k x_n, \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\rangle \geq F_k\left(y, \Theta_{\beta_n}^k x_n\right), \quad \forall y \in C. \tag{3.71}$$

Hence,

$$\left\langle y - \Theta_{\beta_{n_i}}^k x_{n_i}, \frac{\Theta_{\beta_{n_i}}^k x_{n_i} - \Theta_{\beta_{n_i}}^{k-1} x_{n_i}}{\beta_{n_i}} \right\rangle \geq F_k\left(y, \Theta_{\beta_{n_i}}^k x_{n_i}\right), \quad \forall y \in C. \tag{3.72}$$

It follows from (A4), (A5), $(\Theta_{\beta_{n_i}}^k x_{n_i} - \Theta_{\beta_{n_i}}^{k-1} x_{n_i})/\beta_{n_i} \to 0$, and $\Theta_{\beta_{n_i}}^k x_{n_i} \rightharpoonup p$ that for each $k = 1, 2, \ldots, m$,

$$F_k(y, p) \leq 0, \quad \forall y \in C. \tag{3.73}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Since $y \in C$ and $p \in C$, we obtain $y_t \in C$ and hence $F_k(y_t, p) \leq 0$. So by (A4), we have

$$0 = F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, p) \leq tF_k(y_t, y). \tag{3.74}$$

Dividing by t, we get that for each $k = 1, 2, \ldots, m$,

$$F_k(y_t, y) \geq 0. \tag{3.75}$$

Letting $t \to 0$, it follows from (A3) that for each $k = 1, 2, \ldots, m$,

$$F_k(p, y) \geq 0 \tag{3.76}$$

for all $y \in C$ and hence $p \in \mathrm{EP}(F_k)$ for $k = 1, 2, \ldots, m$. That is, $p \in \bigcap_{k=1}^{m} \mathrm{EP}(F_k)$.

We now show that $p \in \mathrm{Fix}(W)$. Assume that $p \notin \mathrm{Fix}(W)$. Since $\rho_{n_i} \rightharpoonup p$ and $p \neq Wp$, from (3.60) and the Opial condition we have

$$\begin{aligned}\liminf_{i\to\infty}\|\rho_{n_i} - p\| &< \liminf_{i\to\infty}\|\rho_{n_i} - Wp\| \\ &\leq \liminf_{i\to\infty}\{\|\rho_{n_i} - W\rho_{n_i}\| + \|W\rho_{n_i} - Wp\|\} \\ &\leq \liminf_{i\to\infty}\|\rho_{n_i} - p\|,\end{aligned} \tag{3.77}$$

which is a contradiction. So, we get $p \in \mathrm{Fix}(W) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$. This implies that $p \in \Omega$.

Since $q = P_\Omega(\gamma f + (I - B))(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle &= \lim_{i\to\infty}\langle \gamma f(q) - Bq, x_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Bq, p - q\rangle \leq 0.\end{aligned} \tag{3.78}$$

That is, (3.62) holds. Next, we consider

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(W_n x_n) - Bq) + (I - \alpha_n B)(W_n\rho_n - q)\|^2 \\ &\leq (1 - \alpha_n\widetilde{\gamma})^2\|W_n\rho_n - q\|^2 + 2\alpha_n\langle \gamma f(W_n x_n) - Bq, x_{n+1} - q\rangle \\ &\leq (1 - \alpha_n\widetilde{\gamma})^2\|x_n - q\|^2 + 2\alpha_n\gamma\langle f(W_n x_n) - f(q), x_{n+1} - q\rangle \\ &\quad + 2\alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\ &\leq (1 - \alpha_n\widetilde{\gamma})^2\|x_n - q\|^2 + \alpha_n\gamma\alpha\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\ &\quad + 2\alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle\end{aligned} \tag{3.79}$$

So, we can obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \frac{(1-\alpha_n\widetilde{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&= \frac{1-2\alpha_n\widetilde{\gamma}+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{\alpha_n^2\widetilde{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 \\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\leq \left[1-\frac{2\alpha_n(\widetilde{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\right]\|x_n-q\|^2 \\
&\quad+\frac{2\alpha_n(\widetilde{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\left[\frac{1}{\widetilde{\gamma}-\alpha\gamma}\langle \gamma f(q)-Bq, x_{n+1}-q\rangle+\frac{\alpha_n\widetilde{\gamma}^2}{2(\widetilde{\gamma}-\alpha\gamma)}M\right],
\end{aligned} \tag{3.80}$$

where M is an approximate constant such that $M \geq \sup_{n\geq 1}\{\|x_n-q\|^2\}$.

Put $l_n = 2\alpha_n(\widetilde{\gamma}-\alpha\gamma)/(1-\alpha_n\gamma\alpha)$ and $t_n = (1/(\widetilde{\gamma}-\alpha\gamma))\langle \gamma f(q)-Bq, x_{n+1}-q\rangle+(\alpha_n\widetilde{\gamma}^2/2(\widetilde{\gamma}-\alpha\gamma))M$. That is,

$$\|x_{n+1}-q\|^2 \leq (1-l_n)\|x_n-q\|+l_nt_n. \tag{3.81}$$

From condition (C1) and Lemma 2.2, we concluded that $x_n \longrightarrow q \in \Omega$. It is easy to see that $u_n \longrightarrow q$ and $y_n \longrightarrow q$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of H. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfies conditions (A1)–(A5). Let $A : C \longrightarrow H$ be relaxed (u,v)-cocoercive and μ-Lipschitz continuous and B a strongly positive linear bounded operator on H with coefficient $\widetilde{\gamma}>0$. Assume that $0<\gamma<\widetilde{\gamma}/\alpha$. Let $S_1, S_2, \dots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Gamma=\bigcap_{i=1}^{\infty}\operatorname{Fix}(S_i)\cap VI(C,A)\cap EP(F)\neq\emptyset$, let $\xi_1,\xi_2,\dots$ be real numbers such that $0<\xi_i\leq\delta<1$ for every $i\in N$ and W_n be the W-mapping of C into itself generated by $S_n, S_{n-1},\dots,S_1$ and $\xi_n,\xi_{n-1},\dots,\xi_1$. Let $f : C\longrightarrow C$ be a contraction with coefficient α $(0<\alpha<1)$ and $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{gathered}
x_1 = x \in H, \\
F(u_n,y)+\frac{1}{\beta_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y\in C, \\
y_n = P_C(I-s_nA)u_n, \\
x_{n+1} = \alpha_n\gamma f(W_nx_n)+(I-\alpha_nB)W_nP_C(I-r_nA)y_n
\end{gathered} \tag{3.82}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and $\{s_n\}$ are sequences of numbers satisfying the conditions:

(C1) $\{\alpha_n\} \subset [0,1]$ *with* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a,b]$ *and* $\{s_n\} \subset [a,b]$ *for some* $a,\ b$ *with* $0 \le a \le b \le 2(v - u\mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C3) $\liminf_{n\to\infty}\beta_n > 0$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \Gamma$, which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \le 0, \quad \forall p \in \Gamma. \tag{3.83}$$

Proof. Let $m = 1$, by Theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of H. Let $A : C \to H$ be relaxed (u,v)-cocoercive and μ-Lipschitz continuous and let B be a strongly positive linear bounded operator on H with coefficient $\tilde{\gamma} > 0$. Assume that $0 < \gamma < \tilde{\gamma}/\alpha$. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Delta = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap VI(C,A) \neq \emptyset$, let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le \delta < 1$ for every $i \in N$, and let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered} x_1 = x \in C, \\ y_n = P_C(I - s_n A)x_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B) W_n P_C (I - r_n A) y_n \end{gathered} \tag{3.84}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{s_n\}$ are sequences of numbers satisfying the conditions:

(C1) $\{\alpha_n\} \subset [0,1]$ *with* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a,b]$ *and* $\{s_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v - u\mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \Delta$, which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \le 0, \quad \forall p \in \Delta. \tag{3.85}$$

Proof. Let $F(x,y) = 0$ for $x, y \in C$, by Corollary 3.2 we obtain the desired result. □

Corollary 3.4. *Let C be a nonempty, closed and convex subset of H. Let $F_1, F_2, \ldots, F_m$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfies conditions (A1)–(A5). Let $A : C \to H$ be relaxed (u,v)-cocoercive and μ-Lipschitz continuous and B a strongly positive linear bounded operator on H with coefficient $\tilde{\gamma} > 0$*

such that $\Xi = \bigcap_{k=1}^{m} EP(F_k) \cap VI(C,A) \neq \emptyset$. *Let* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$ *and* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{aligned} x_1 &= x \in H, \\ u_n &= T_{\beta_n}^{F_m} T_{\beta_n}^{F_{m-1}} \cdots T_{\beta_n}^{F_2} T_{\beta_n}^{F_1} x_n, \\ y_n &= P_C(I - s_n A) u_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n B) P_C(I - r_n A) y_n \end{aligned} \tag{3.86}$$

for every $n = 1, 2, \ldots$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ *and* $\{s_n\}$ *are sequences of numbers satisfying the conditions:*

(C1) $\{\alpha_n\} \subset [0,1]$ *with* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a,b]$ *and* $\{s_n\} \subset [a,b]$ *for some* a, b *with* $0 \leq a \leq b \leq 2(\upsilon - u\mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C3) $\liminf_{n\to\infty} \beta_n > 0$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in \Omega$, *which solves the following variational inequality:*

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \Xi. \tag{3.87}$$

Remark 3.5. (i) If $s_n = 0$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 2.1 in [9]. If $s_n = 0$ and $S_i = I$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 2.1 in [8] with $S = I$. If $s_n = 0$, $r_n = 0$ and $S_i = I$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 3.1 in [6] with $S = I$ and Theorem 3.3 in [7] with $S = I$ and $C = H$.

(ii) Corollary 3.3 extends, generalizes and improves the main results in [21, 22, 24].

(iii) It is easy to see that Theorem 3.1 is different from the main results in [1–4].

Acknowledgments

The authors are grateful to the referee and Proffessor S. Reich for the detailed comments and helpful suggestions which improved the original manuscript greatly. This research was supported by the National Center of Theoretical Sciences (South) of Taiwan, the National Natural Science Foundation of China (Grants 10771228 and 10831009), the Natural Science Foundation of Chongqing (Grant no. CSTC, 2009BB8240), the Research Project of Chongqing Normal University (Grant 08XLZ05), and the project of the Grant NSC 98-2115-M-110-001.

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