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New approach for the nonlinear programming with transient stability constraints arising from power systems

Xiaojiao Tong · Soon-Yi Wu · Renjun Zhou

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Abstract This paper presents a new approach for solving a class of complicated nonlinear programming problems arises from optimal power flow with transient stability constraints (denoted by OTS) in power systems. By using a functional transformation technology proposed in Chen et al. (IEEE Trans. Circuits Syst. I Fundam. Theory Appl. 48:327–339, 2001), the OTS problem is transformed to a semi-infinite programming (SIP). Then based on the KKT (Karush-Kuhn-Tucker) system of the reformulated SIP problem and the finite approximation technology, an iterative method is presented, which develops Wu-Li-Qi-Zhou' (Optim. Methods Softw. 20:629–643, 2005) method. In order to save the computing cost, some typical computing technologies, such as active set strategy, quasi-Newton method for the subproblems coming from the finite approximation model, are addressed. The global convergence of

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X. Tong
College of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, China
e-mail: tongxj@csust.edu.cn

S.-Y. Wu (✉)
Institute of Applied Mathematics, National Cheng-Kung University, Tainan, Taiwan
e-mail: soonyi@mail.ncku.edu.tw

S.-Y. Wu
National Center for Theoretical Sciences, Tainan, Taiwan

R. Zhou
College of Electrical and Information, Changsha University of Science and Technology, Changsha 410076, China
e-mail: zrj0731@163.com

the proposed algorithm is established. Numerical examples from power systems are tested. The computing results show the efficiency of the new approach.

Keywords Optimal power flow (OPF) · Transient stability constraint (TSC) · Semi-infinite programming (SIP) · Quasi-Newton algorithm

1 Introduction

A practical electric power system can be described by a system of differential and algebraic equations (DAE) as follows:

$$\begin{aligned} &\dot{x} = F(x(t), y(t), \bar{z}) \\ &G(x(t), y(t), \bar{z}) = 0 \\ &H(x(t), y(t), \bar{z}) \le 0, \end{aligned} \tag{1.1}$$

where t means time, vectors $x(t) \in R^{n_x}$ and $y(t) \in R^{n_y}$ are state variables such as power outputs of generators, voltage values and angles, while $\bar{z} \in R^{n_z}$ is control variable such as transformer tap positions, phase shifter angle positions and shunt capacitor/reactors. Generally, $\bar{z}$ is independent of t. The inequality in (1.1) includes some line current limits, all variables limits and stability requirements of power systems; $F = (F_1, \ldots, F_{n_x})^T : R^{n_x+n_y+n_z} \to R^{n_x}$, $G = (G_1, \ldots, G_{n_y})^T : R^{n_x+n_y+n_z} \to R^{n_y}$ and $H = (H_1, \ldots, G_m)^T : R^{n_x+n_y+n_z} \to R^m$ are assumed continuously differentiable.

The steady-state operation of power systems is independent to the time t and satisfies

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0, \\ H(x, y, z) \le 0. \end{cases} \tag{1.2}$$

One of typical research problems in power systems is *Optimal Power Flow* (*OPF*). An OPF problem finds the optimal operation with particular objective, such as the minimizing the cost of generation, the maximal social profit, etc. We assume that the control variable $\bar{z}$ is unchanged during the operation process, then based on the system description of (1.1) and the initial steady-state operation (1.2), general OPF problems in power systems can be described as

$$\begin{aligned} \min_{(\bar{x},\bar{y},\bar{z},x(t),y(t))} \quad & f(\bar{x}, \bar{y}, \bar{z}) \\ \text{s.t.} \quad & F^0(\bar{x}, \bar{y}, \bar{z}) = 0 \\ & G^0(\bar{x}, \bar{y}, \bar{z}) = 0 \\ & H^0(\bar{x}, \bar{y}, \bar{z}) \le 0, \\ & \dot{x}(t) = F(x(t), y(t), \bar{z}), \quad \text{with } x(0) = \bar{x} \\ & G(x(t), y(t), \bar{z}) = 0, \\ & H(x(t), y(t), \bar{z}) \le 0, \quad t \in [0, T], \end{aligned} \tag{1.3}$$

where $(\bar{x}, \bar{y}, \bar{z}) \in R^{n_x} \times R^{n_y} \times R^{n_z}$ indicates the variable in the steady-state (or initial operation state), i.e., the corresponding state variable at $t = 0$; G^0 and H^0 are the descriptions of the system and the operating constraint at the steady-state case, respectively. The requirements of the system operation and stability are involved in the inequalities of (1.3).

In power systems, when the system suffers some big disturbances, the structure of the system will happen change. This means that the functions defined in DAE of (1.3) (i.e., F, G, H) have different versions. For that case, we need to consider the recovery operation in the system, which indicates by the DAE with the inequality limit H in the model. This is the transient stability analysis in power systems. Combining with the OPF consideration, problem (1.3) is called OPF with transient stability constraints (for abbreviation OTS) in power systems. Since the variable $(x(t)^T, y(t)^T)^T$ is controlled (or coupled) by the initial operation point $(\bar{x}^T, \bar{y}^T, \bar{z}^T)^T$, this implies that the aim of (1.3) is to find $(\bar{x}^T, \bar{y}^T, \bar{z}^T)^T$ under the optimal and the stable operation requirement during $t \in [0, T]$. It can be seen that when the system is just considered the steady-state operation condition (1.2), OTS is reduced to normal OPF problems, which belongs to nonlinear optimization problems and is studied extensively from theory to computing methods (see [12] and references therein).

The OTS research has been attracted many researchers and obtained plentiful research results in literature, including power system engineers and nonlinear optimization researchers, see [1, 3, 9, 12, 14] and references therein. On the one hand, an OTS describes the practical operation and operating requirements; on the other hand, OTS can analyze some disturbances happened in the system (see [1, 24]). From the viewpoint of applications, the most concerned question for OTS problems is how to design effective numerical methods for solving the problem. The ordinary used numerical method for OTS is discretization approach (see [24] and references therein), which discretizes the differential equations to algebraic equations and reduces (1.3) to a nonlinear programming problem, then uses some numerical methods of nonlinear optimization problems to get the solution. The main difficult in discretization approach is the high dimension problem due to the discretization variables, which limits the practical application in power systems, especially for large numbers of disturbances. Recently, a new method was presented in [1] where the OTS problem was converted into semi-infinite programming (SIP) problems by using functional analysis. The advantage of new method is that the variable dimension of the transformed problem is equal to one of the steady-state, i.e., the dimension of $(\bar{x}^T, \bar{y}^T, \bar{z}^T)^T$.

We note that by using the functional transformed method for solving the OTS problems, the key work is the approach for solving the reformulated SIP problems. Hence, our attention in this research is to design efficient numerical methods for OTS, combing with the special structure of the reformulated SIP and the new approach in SIP problems. Motivated by the recently works for SIP problems (see [6, 7, 17, 20, 21] and reference therein), especially for the recent Newton-type methods [11, 13, 16, 23], we develop the method proposed in [23] for solving the OTS problems. We first reformulate the OTS problem to a SIP problem by using a similar way in [1]. Then the KKT system of the reformulated SIP problem is addressed. By using a complementarity function, the KKT system of the reformulated SIP problem is further converted to semismooth equations (see [15]). Based on the system of semismooth equations, we extend the method of [23] to the KKT system of the reformulated problem. Some special strategies for saving the computing cost are discussed, and the global convergence of the algorithm is analyzed. A global quasi-Newton method is presented to solve the subproblem occurred in the iterative approach. Numerical examples from power systems are tested the new method.

The rest of this paper is organized as follows. Section 2 reformulates the OTS problem to a SIP problem and considers its correlative computations. In Sect. 3, an iterative method is presented for solving the reformulated SIP problem, and the convergence is addressed. In Sect. 4, a quasi-Newton approach is discussed for subproblems occurred in the iterations. Two types of OTS problems in power systems are tested in Sect. 5. Some comments are drawn in the last section.

Some notation are used in this paper. For a smooth (continuously differentiable) function $\Phi : R^n \to R^m$, we denote the Jacobian of Φ at $x \in R^n$ by $\Phi'(x)$, which is an $m \times n$ matrix. We denote the transposed Jacobian of Φ by $\nabla\Phi(x)$. Moreover, for a smooth function $F(x, y) : R^{n_x+n_y} \to R^n$, F_x and F_y mean the partial derivative of F with respect to x and y, respectively. For a nonsmooth function $G(x)$, $\partial G(x)$ means the generalized Jacobian of G at x in the sense of Clarke [2]. The Euclidean norm is denoted by $\|\cdot\|$.

2 Reformulation of OTS problems

In this section, we transform OTS to a SIP problem via a similar technology proposed in [1], some related problems such as the derivative of functions, the stability constraints, are also discussed.

2.1 Reformulated SIP problem

We first make the following assumption.

Assumption 2.1 For any points $(x(t)^T, y(t)^T)^T$ satisfying $G(x(t), y(t), \bar{z}) = 0$, $H(x(t), y(t), \bar{z}) \le 0$, it holds that $\det(G_y(x(t), y(t), \bar{z})) \ne 0$ for $t \in [0, T]$, where $G_y = \partial G/\partial y$.

Assumption 2.1 is a typical one in DAE problems, which can realize the transformation of the DAE problem to a differential equation. In the stability analysis of power systems, except for the singularity bifurcation study, this assumption is a common one and the system satisfies it for the most cases [1, 8]. Under above assumption, from $G(x(t), y(t), \bar{z}) = 0$ and the implicit function theorem, it exists an unique C^1 mapping q for each $t \in [0, T]$

$$y(t) = q(x(t), \bar{z}), \quad t \in [0, T] \tag{2.1}$$

in the neighborhoods of $(x(t)^T, \bar{z}^T)^T$ satisfying $G(x(t), q(x(t), \bar{z}), \bar{z}) = 0$. Then the constraints dependent on the time t in (1.3) are changed equivalently to the follows:

$$\dot{x} = F(x(t), q(x(t), \bar{z}), \bar{z}), \tag{2.2}$$

$$H(x(t), q(x(t), \bar{z}), \bar{z}) \le 0 \tag{2.3}$$

with $x(0)=\bar{x}$ at $t=0$. Furthermore, the differential equations can be handled by an integral approach. Let $\phi_t(\bar{x},\bar{z})$ denote the trajectory of $x(t)$. Then, (2.2) implies

$$x(t)\equiv\phi_t(\bar{x},\bar{z})=\bar{x}+\int_0^t F(\phi_t(\bar{x},\bar{z}),q(\phi_t(\bar{x},\bar{z}),\bar{z}),\bar{z})dt. \tag{2.4}$$

Finally, the OTS problem (1.3) is transformed to the following type of nonlinear programming problem:

$$\begin{aligned}
\min_{(\bar{x},\bar{y},\bar{z})}\quad & f(\bar{x},\bar{y},\bar{z})\equiv\bar{f}(\bar{x},\bar{y})\\
\text{s.t.}\quad & F^0(\bar{x},\bar{y},\bar{z})=0\\
& G^0(\bar{x},\bar{y},\bar{z})=0\\
& H^0(\bar{x},\bar{y},\bar{z})\le 0,\\
& H(\phi_t(\bar{x},\bar{z}),q(\phi_t(\bar{x},\bar{z}),\bar{z}),\bar{z})\le 0,\qquad t\in[0,\bar{t}].
\end{aligned} \tag{2.5}$$

The remarkable advantage of this equivalent model (2.5) is that it has the same variables $(\bar{x}^T,\bar{y}^T,\bar{z}^T)^T$ as ordinary OPF problems with steady-state stable constraints. Since the second inequality constraint in (2.5) includes time $t\in[0,T]$, this is a typical *semi-infinite programming* (*SIP*) problem. Furthermore, this methodology can handle multi-disturbances where H in (2.5) is denoted by $H^k(\cdot)$ $(k=1,\dots,l)$ with l time disturbances in power systems.

2.2 Jacobian computation

In this subsection, we consider the Jacobian computation of function H in (2.5), which will be used in the proposed algorithm. We have the following conclusion.

Proposition 2.1 *Suppose that Assumption 2.1 holds. The Jacobian of function H has the following computing formulas*:

$$\frac{\partial H}{\partial\bar{x}}=\left[H_x-H_y(G_y)^{-1}G_x\right]\frac{\partial\phi_t}{\partial\bar{x}}, \tag{2.6}$$

$$\frac{\partial H}{\partial\bar{z}}=\left[H_x-H_y(G_y)^{-1}G_x\right]\frac{\partial\phi_t}{\partial\bar{z}}-H_y(G_y)^{-1}G_{\bar{z}}+H_{\bar{z}}, \tag{2.7}$$

where H_x indicates $\frac{\partial H}{\partial x}$, i.e., the derivative of H with respect to x. Other notations have the same meaning; *$\frac{\partial\phi_t}{\partial\bar{x}}$ and $\frac{\partial\phi_t}{\partial\bar{z}}$ satisfy*

$$\begin{cases}\frac{d}{dt}(\frac{\partial\phi_t}{\partial\bar{x}})=[F_x-F_y(G_y)^{-1}G_x]\frac{\partial\phi_t}{\partial\bar{x}}\\ (\frac{\partial\phi_t}{\partial\bar{x}})|_{t=0}=I\end{cases} \tag{2.8}$$

$$\begin{cases}\frac{d}{dt}(\frac{\partial\phi_t}{\partial\bar{z}})=[F_x-F_y(G_y)^{-1}G_x]\frac{\partial\phi_t}{\partial\bar{z}}-F_y(G_y)^{-1}G_{\bar{z}}+F_{\bar{z}}\\ (\frac{\partial\phi_t}{\partial\bar{z}})|_{t=0}=0\end{cases} \tag{2.9}$$

with $I\in R^{n_x\times n_x}$ to be the identical matrix.

Proof Since $y(t)=q(x(t),\bar{z})$ satisfies $G(x(t),y(t),\bar{z})=0$, it holds

$$\begin{cases} G_x + G_y \frac{\partial q}{\partial x} = 0, \\ G_{\bar{z}} + G_y \frac{\partial q}{\partial \bar{z}} = 0. \end{cases} \tag{2.10}$$

Then from the related variables of H and the derivative computation, we derive that

$$\frac{\partial H}{\partial \bar{x}} = H_x \frac{\partial \phi_t}{\partial \bar{x}} + H_y \frac{\partial q}{\partial x}\frac{\partial \phi_t}{\partial \bar{x}} = \left[H_x - H_y (G_y)^{-1} G_x\right] \frac{\partial \phi_t}{\partial \bar{x}}, \tag{2.11}$$

where the second equality is due to the result in (2.10).

Similar to the derivation above, we also have

$$\begin{aligned} \frac{\partial H}{\partial \bar{z}} &= H_x \frac{\partial \phi_t}{\partial \bar{z}} + H_y \left(\frac{\partial q}{\partial x}\frac{\partial \phi_t}{\partial \bar{z}} + \frac{\partial q}{\partial \bar{z}}\right) + H_{\bar{z}} \\ &= \left[H_x - H_y (G_y)^{-1} G_x\right] \frac{\partial \phi_t}{\partial \bar{z}} - H_y (G_y)^{-1} G_{\bar{z}} + H_{\bar{z}}, \end{aligned} \tag{2.12}$$

where the second equality comes from (2.10). This proves the result (2.6)–(2.7).

Equations (2.6)–(2.7) shows that the Jacobian of H with respect to $(\bar{x},\bar{z})$ can be obtained via the computation of $\frac{\partial \phi_t}{\partial \bar{x}}$ and $\frac{\partial \phi_t}{\partial \bar{z}}$. To this end, we make the further derivations.

We compute the derivative of (2.4) with respective to $\bar{x}$ and obtain that

$$\begin{aligned} \frac{\partial \phi_t}{\partial \bar{x}} &= I + \int_0^t \left[F_x \frac{\partial \phi_t}{\partial \bar{x}} + F_y \frac{\partial q}{\partial x}\frac{\partial \phi_t}{\partial \bar{x}}\right] dt \\ &= I + \int_0^t \left[F_x - F_y (G_y)^{-1} G_x\right] \frac{\partial \phi_t^k}{\partial \bar{x}} dt, \end{aligned} \tag{2.13}$$

where the second equality comes from (2.10). Then (2.8) follows by computing the derivative of (2.13) with respective to t.

By using the same way to the system (2.4) with respective to $\bar{x}$, it is not difficult to get

$$\frac{\partial \phi_t}{\partial \bar{z}} = \int_0^t \left[(F_x - F_y G_y^{-1} G_x)\frac{\partial \phi_t}{\partial \bar{z}} - F_y G_y^{-1} G_{\bar{z}} + F_{\bar{z}}\right] dt.$$

From above expression, we can derive the expression (2.9) by calculating the derivative of $\frac{\partial \phi_t}{\partial \bar{z}}$ with respect to t. We prove the conclusion. □

Proposition 2.1 presents the Jacobian computing formula of the constraint function H by solving the first ordinary differential equations.

3 Iterative algorithm

This section will develop the iteration method of [23] to the reformulated SIP problem (2.5). The optimal condition is studied, and a new algorithm is constructed. The convergence of the algorithm is also investigated.

3.1 Optimal conditions of the reformulated SIP problem

Denote $n = n_x + n_y + n_z$, $w = (\bar{x}^T, \bar{y}^T, \bar{z}^T)^T$ and $h = (h_1, \dots, h_m)^T$ with

$$h_i(w,t) \equiv H_i(\phi_t(\bar{x},\bar{z}), q(\phi_t(\bar{x},\bar{z}),\bar{z}),\bar{z}), \quad (i = 1,\dots,m),\ t \in [0,T]. \tag{3.1}$$

Let $w^* \equiv ((\bar{x}^*)^T, (\bar{y}^*)^T, (\bar{z}^*)^T)^T$ be a solution of the reformulated SIP problem (2.5). Under some constrained qualification conditions, for each $i = 1,\dots,m$, there exists a nonnegative integer $p_i \le n$, multipliers $\lambda^* \in R^{n_x}$, $\mu^* \in R^{n_y}$, $\gamma^* \in R^{m_0}$, $(u_i^j)^*$ and attainers $(t_i^j)^*$ for $i = 1,\dots,m$; $j = 1,\dots,p_i$ satisfying the following so-called KKT system of (2.5) (see [6, 23])

$$\begin{cases} \nabla f(w^*) + \sum_{i=1}^{n_x} \lambda_i^* \nabla F_i^0(w^*) + \sum_{i=1}^{n_y} \mu_i^* \nabla G_i^0(w^*) + \sum_{i=1}^{m_0} \gamma_i^* \nabla H_i^0(w^*) \\ \quad + \sum_{i=1}^{m} (\sum_{j=1}^{p_i} (u_i^j)^* \nabla h_i(w^*, (t_i^j)^*)) = 0, \\ F^0(w^*) = 0, \\ G^0(w^*) = 0, \\ \gamma_i^* \ge 0, \qquad H_i^0(w^*) \le 0, \qquad \gamma_i^* H_i^0(w^*) = 0, \quad i = 1,\dots,m_0, \\ (u_i^j)^* > 0, \qquad h_i(w^*, (t_i^j)^*) = 0, \quad i = 1,\dots,m,\ j = 1,\dots,p_i, \\ h_i(w^*, t) \le 0, \quad i = 1,\dots,m,\ t \in [0,T]. \end{cases} \tag{3.2}$$

It is well-known that under some second order sufficient conditions at w^*, the point w^* satisfying (3.2) is a solution of the reformulated SIP problem (2.5) (see Theorem 3.3 in [19] and Theorem 4.1 in [18]).

3.2 Iterative approach

Our objective is to find the solution of (3.2). Based on the KKT system of SIP, some Newton-type methods are presented (see [11, 13, 16, 23]). In our recent work [22], we presented a smoothing quasi-Newton algorithm for the reformulated SIP problem. However, the feasibility of infinite constraints did not take into account in [22]. Moreover, it exists the difficulty to decide p_i in (3.2). Whereas [23] handled the infinite constraints better for a simple SIP problem, and p_i did not give for the algorithm. Motivated by the work of [23], this subsection will present a new algorithm for solving the reformulated SIP problem. We main state the algorithm, the detailed description for the algorithm is referred to [23].

Let Ω_k be a finite subset of $\Omega = [0,T]$. We make an approximate nonlinear programming problem of (2.5) as follows:

$$\begin{aligned} \min_{(\bar{x},\bar{y},\bar{z})} \quad & f(\bar{x},\bar{y},\bar{z}),\bar{z}) \equiv \bar{f}(\bar{x},\bar{y}) \\ \text{s.t.} \quad & F^0(\bar{x},\bar{y},\bar{z}) = 0 \\ & G^0(\bar{x},\bar{y},\bar{z}) = 0 \\ & H^0(\bar{x},\bar{y},\bar{z}) \le 0, \\ & h(\bar{x},\bar{y},\bar{z},t_i) \le 0, \quad t_i \in \Omega_k, \end{aligned} \tag{3.3}$$

where h is defined in (3.1). Then we have an approximate system of (3.2) according to problem (3.3) as

$$\begin{cases} \nabla f(w) + \sum_{i=1}^{n_x} \lambda_i \nabla F_i^0(w) + \sum_{i=1}^{n_y} \mu_i \nabla G_i^0(w) + \sum_{i=1}^{m_0} \gamma_i \nabla H_i^0(w) \\ \quad + \sum_{i=1}^{m} (\sum_{t_i^j \in \Omega_k} u_i^j \nabla h_i(w, t_i^j)) = 0, \\ F^0(w) = 0, \\ G^0(w) = 0, \\ \gamma_i \geq 0, \qquad H_i^0(w) \leq 0, \qquad \gamma_i H_i^0(w) = 0, \quad i = 1, \dots, m_0, \\ u_i^j \geq 0, \qquad h_i(w, t_i^j) \leq 0, \qquad u_i^j h_i(w, t_i^j) = 0, \quad i = 1, \dots, m, \ t_i^j \in \Omega_k. \end{cases} \tag{3.4}$$

We call (3.4) the subproblem for solving (2.5). Denote the solution of (3.4) as

$$v^k \equiv \big((w^k)^T, (\lambda^k)^T, (\mu^k)^T, (\gamma^k)^T, (u^k)^T\big)^T \in R^{n+n_x+n_y+m_0+mm_k}$$

with $m_k = |\Omega_k|$. It is easy to see that: (i) since the system (3.4) is the KKT system of nonlinear programming problems with finite constraints, there exist many effective methods for solving it; (ii) if w^k is feasible, i.e., $h(w^k, t) \leq 0, \ \forall t \in \Omega = [0, T]$, then v^k is a KKT point of (2.5).

Based on the observation, our aim is to design some iterative methods for the finite KKT system (3.4) and obtain the solution of the KKT system (3.2) with infinite constraints. Combining with our observation and objective, two computation related problems will be considered:

- The number of Ω_k is large for a better approximation of Ω, especially for the large k, which will result the computing difficult for solving the subproblem (3.4) due to the large inequality constraints;
- To check the feasibility of w^k is equivalent to test whether the following holds for each $i \in \{1, \dots, m\}$

$$\max_{t \in \Omega} h_i(w^k, t) \leq 0. \tag{3.5}$$

  This will be done by finding a global solution of $\max_{t \in \Omega} h_i(w^k, t)$ and is not an easy work.

In order to overcome the computing difficulties, two computation technologies are adopted in our iterative method. One is the active set method; other is the approximate solved method for test the feasibility of w^k. The algorithm is stated as follows.

Algorithm 3.1 (The iterative Algorithm)

Step 0. Given a prescribed number $\delta_0 > 0$ and $\epsilon \in (0, 1)$. Given the finite subset $\Omega_0 = \{t_1^0, \dots, t_{p_0}^0\} \subset \Omega$ such that for all $i \in \{1, \dots, m\}$, $\{w \in R^n : h_i(w, t) \leq 0, \forall t \in \Omega_0\} \neq \emptyset$. Let $k := 0$.

Step 1. Solve the KKT system (3.4) to get $v^{k,0} \equiv ((w^{k,0})^T, (\lambda^{k,0})^T, (\mu^{k,0})^T, (\gamma^{k,0})^T, (u^{k,0})^T)^T$. Stop if $h_i(w^{k,0}, t) \leq 0$ for all $t \in \Omega = [0, T]$ and $i = 1, \ldots, m$. Let $l := 1$, go to Step 3.

Step 2. Solve the KKT system (3.4) to get $v^{k,l} \equiv ((w^{k,l})^T, (\lambda^{k,l})^T, (\mu^{k,l})^T, (\gamma^{k,l})^T, (u^{k,l})^T)^T$. Stop if $h_i(w^{k,l}, t) \leq 0$ for all $t \in \Omega = [0, T]$ and $i = 1, \ldots, m$. Set $v^k = v^{k,l}$.

Step 3. Find a $t^k_{p_k+l} \in \Omega$ such that for some $i \in \{1, \ldots, m\}$, it has $h_i(w^{k,l}, t^k_{p_k+l}) > \delta_k$. Then let $\Omega_k := \Omega_k \cup \{t^k_{p_k+l}\}$ and set $l := l + 1$, go to Step 2. If there does not exist such $t^k_{p_k+l} \in \Omega$ satisfying $h_i(w^{k,l}, t^k_{p_k+l}) > \delta_k$ for all $i = 1, \ldots, m$, then let $v^k = v^{k,l}$ and go to Step 4.

Step 4. Set $k := k + 1$. Let $E_{k-1} = \{t \in \Omega_k | h_i(w^{k-1,l}, t) = 0, i \in \{1, \ldots, m\}\} = \{t^k_1, \ldots, t^k_{p_k}\}$ and $\delta_k = \epsilon\delta_{k-1}$, $l := 1$, and go to Step 5.

Step 5. Find a $t^k_{p_k+l} \in \Omega$ such that $h_i(w^{k,l}, t^k_{p_k+l}) > \delta_k$ for some $i \in \{1, \ldots, m\}$ and let $\Omega_k = E_{k-1} \cup \{t^k_{p_k+l}\}$. Set $l := l + 1$, go to Step 2.

Step 6. If there do not exist i and $t^k_{p_k+l} \in \Omega$ such that $h_i(w^{k,l}, t^k_{p_k+l}) > \delta_k$, then let $\delta_k := \epsilon\delta_k$, go to Step 5.

We give some explanations for Algorithm 3.1.

(i) The finite set Ω_k is an approximate set of the infinite set Ω; The constant ϵ is the descend scale of δ_k;

(ii) E_k defined in Algorithm 3.1 indicates active set at k-th iteration, which begins to work on the case $h_i(w^k, t) \leq \delta_k$ for all $t \in \Omega$ and $i \in \{1, \ldots, m\}$. This technology is to decrease the number of approximate set Ω_k;

(iii) The number δ_k is a key scale in the algorithm. On the one hand, it is used to set new approximate set Ω_k if there exists t_k and i such that $h_i(w^k, t_k) \geq \delta_k$ (see Step 2); on the other hand, it is an index to decide whether the active set E_k works or not;

(iv) The test of feasibility, i.e., test $\max_{t\in\Omega} h_i(w^k, t) \leq 0$ for all $i = 1, \ldots, m$, is replaced by *finding* t_k *such that* $h_i(w^k, t_k) \geq \delta_k$ for some i.

In order to prove the global convergence of Algorithm 3.1, we need to introduce some measures for discrete sets. Let $\Omega_k = \{\bar{t}^k_0, \bar{t}^k_1, \ldots, \bar{t}^k_{p^*_k}\}$ and $\Omega^* = \{t^*_1, t^*_2, \ldots, t^*_p\}$. Here Ω^* is the index satisfying the KKT system (3.2). Define the discrete measure σ^* and σ_k with finite support as

$$\sigma^*(t) = \begin{cases} u^*_i, & \text{if } t = t^*_i,\ i = 1, 2, \ldots, p \\ 0, & \text{if } t \notin \Omega^*, \end{cases}$$

$$\sigma_k(t) = \begin{cases} u^k_i, & \text{if } t = \bar{t}^k_i,\ i = 0, 1, \ldots, p^*_k \\ 0, & \text{if } t \notin \Omega_k. \end{cases}$$

Then the KKT systems (3.2) and (3.4) can be written as

$$\begin{cases} \nabla f(w^*) + \sum_{i=1}^{n_x} \lambda_i^* \nabla F_i^0(w^*) + \sum_{i=1}^{n_y} \mu_i^* \nabla G_i^0(w^*) + \sum_{i=1}^{m_0} \gamma_i^* \nabla H_i^0(w^*) \\ \quad + \sum_{i=1}^{m} \int \nabla h_i(w^*, t) d\sigma^* = 0, \\ F^0(w^*) = 0, \\ G^0(w^*) = 0, \\ \gamma_i^* \geq 0, \qquad H_i^0(w^*) \leq 0, \qquad \gamma_i^* H_i^0(w^*) = 0, \quad i = 1, \dots, m_0, \\ \int h_i(w^*, t) d\sigma^* = 0, \quad i = 1, 2, \dots, m, \\ h(w^*, t) \leq 0, \quad t \in \Omega = [0, T]. \end{cases} \tag{3.6}$$

$$\begin{cases} \nabla f(w) + \sum_{i=1}^{n_x} \lambda_i \nabla F_i^0(w) + \sum_{i=1}^{n_y} \mu_i \nabla G_i^0(w) + \sum_{i=1}^{m_0} \gamma_i \nabla H_i^0(w) \\ \quad + \sum_{i=1}^{m} \int \nabla h_i(w, t) d\sigma_k = 0, \\ F^0(w) = 0, \\ G^0(w) = 0, \\ \gamma_i \geq 0, \qquad H_i^0(w) \leq 0, \qquad \gamma_i H_i^0(w) = 0, \quad i = 1, \dots, m_0, \\ \int h_i(w, t) d\sigma_k = 0, \quad i = 1, \dots, m, \\ h(w, t) \leq 0, \quad t \in \Omega_k. \end{cases} \tag{3.7}$$

The global convergence of Algorithm 3.1 is stated as follows.

Theorem 3.1 *Let* $\{v^k\} = \{((w^k)^T, (\lambda^k)^T, (\mu^k)^T, (\gamma^k)^T, (u^k)^T)\}$. *Suppose that the infinite sequence* $\{v^k\}$ *generated by Algorithm* 3.1 *is bounded. Then there exists a subsequence of* $\{v^k\}$ *converging to the KKT point of* (2.5).

Proof (i) First we prove that at each iteration k with given $\delta_k > 0$, the cycling between Step 2 and Step 3 is finite under the condition of the theorem. Let the interior cycling index be l. Suppose that the interior iteration between Step 2 and Step 3 is infinite.

Let $w^{k,n_l} \to w^{k,*}$ as $l \to \infty$. We claim that for all $i = 1, \dots, m$ and l, it holds

$$h_i(w^{k,*}, t_{n_l}^k) \leq 0. \tag{3.8}$$

Otherwise, there exist at least one i and a positive integer N such that $h_i(w^{k,*}, t_{n_N}^k) > 0$. Therefore, there is a positive $\bar{N} > N$ large enough such that for such i it holds $h_i(w^{k,n_{\bar{N}}}, t_{n_N}^k) > 0$. On the other hand, since $\bar{N} > N$ and the solution condition of the KKT system (3.4), we have $h_i(w^{k,n_{\bar{N}}}, t_{n_N}^k) \leq 0$ for all $i = 1, \dots, m$, which yields a contradiction. Thus (3.8) holds. Let $t_{n_l}^k \to t^*$ as $l \to \infty$. Taking limit for such inequality with $l \to \infty$ yields

$$h_i(w^{k,*}, t^*) \leq 0. \tag{3.9}$$

On the other hand, the assumption, i.e, the iteration is infinite between Step 2 and Step 3, implies that at least one index i it holds $h_i(w^{k,n_l}, t_{n_l}^k) > \delta_k$ when $l \to \infty$.

Without loss of the generality, taking limit for such inequality yields

$$h_i(w^{k,*},t^*) \geq \delta_k > 0.$$

This is a contradiction. We prove the finite iteration between Step 2 and Step 3.

Note that for each k, $t \in \Omega$ and $i \in \{1,\dots,m\}$

$$h_i(w^k,t) \leq \delta_k, \quad \text{and} \quad \delta_k \to 0. \tag{3.10}$$

(ii) We prove that any limit point of the infinite sequence $\{v^k\}$ generated by Algorithm 3.1 is a KKT point of the reformulated SIP problem, i.e., satisfying (3.2) or (3.6).

From the bounded assumption, there exists a subsequence of $\{v^k\}$, denoted by $\{v^{n_k}\}$, such that $v^{n_k} \to v^* = ((w^*)^T,(\lambda^*)^T,(\mu^*)^T,(\gamma^*)^T,(u^*)^T)^T$. We assume that the measure of discrete set holds $\sigma_{n_k} \xrightarrow{\text{weakly}} \sigma^*$. □

At the iterative point v^{n_k}, from (3.7) we have the following derivation:

$$\begin{aligned}
&\Bigg\|\nabla f(w^*) + \sum_{i=1}^{n_x}\lambda_i^*\nabla F_i^0(w^*) + \sum_{i=1}^{n_y}\mu_i^*\nabla G_i^0(w^*) + \sum_{i=1}^{m_0}\gamma_i^*\nabla H_i^0(w^*) \\
&\quad + \sum_{i=1}^{m}\int \nabla h_i(w^*,t)d\sigma^*\Bigg\| \\
&\quad = \Bigg\|(\nabla f(w^{n_k}) + \sum_{i=1}^{n_x}\lambda_i^{n_k}\nabla F_i^0(w^{n_k}) + \sum_{i=1}^{n_y}\mu_i^{n_k}\nabla G_i^0(w^{n_k}) + \sum_{i=1}^{m_0}\gamma_i^{n_k}\nabla H_i^0(w^{n_k}) \\
&\qquad + \sum_{i=1}^{m}\int \nabla h_i(w,t)d\sigma_{n_k}) - \Bigg(\nabla f(w^*) + \sum_{i=1}^{n_x}\lambda_i^*\nabla F_i^0(w^*) \\
&\qquad + \sum_{i=1}^{n_y}\mu_i^*\nabla G_i^0(w^*) + \sum_{i=1}^{m_0}\gamma_i^*\nabla H_i^0(w^*) + \sum_{i=1}^{m}\int \nabla h_i(w^*,t)d\sigma^*\Bigg)\Bigg\| \\
&\quad \leq \|\nabla f(w^{n_k}) - \nabla f(w^*)\| + \sum_{i=1}^{n_x}\|\lambda_i^{n_k}\nabla F_i^0(w^{n_k}) - \lambda_i^*\nabla F_i^0(w^*)\| \\
&\qquad + \sum_{i=1}^{n_y}\|\mu_i^{n_k}\nabla G_i^0(w^{n_k}) - \mu_i^*\nabla G_i^0(w^*)\| \\
&\qquad + \sum_{i=1}^{m_0}\|\gamma_i^{n_k}\nabla H_i^0(w^{n_k}) - \gamma_i^*\nabla H_i^0(w^*)\| \\
&\qquad + \sum_{i=1}^{m}\left\|\int \nabla h_i(w^{n_k},t)d\sigma_{n_k} - \int \nabla h_i(w^*,t)d\sigma^*\right\|.
\end{aligned} \tag{3.11}$$

Then from the weak convergence conditions and the triangle inequality, it is not difficult to derive that the right of (3.11) tends to zero. The first relationship of (3.2) is proved. Except the last relationship, i.e., the feasibility of infinite constraints, the other expressions in (3.2) can be proved by using the same way.

Next we prove that w^* is feasible for all $i = 1, 2, \dots, m$. The feasibility needs to prove $\max_{t\in\Omega} h_i(w^*, t) \le 0$. Since

$$h_i(w^{n_k}, t) \le \delta_{n_k}, \quad \text{for } i = 1, \dots, m,\ \forall t \in \Omega,$$

and

$$\delta_{n_k} \to 0 \quad \text{as } k \to \infty, \qquad h_i(w^*, t) \le 0 \quad \text{for } i = 1, \dots, m,\ \forall t \in \Omega.$$

Thus we have proved that w^* is feasible.

4 Quasi-Newton method for the subproblem

Since just the first derivate is available in the approximate KKT system (3.4) arising from OTS problem, we consider a quasi-Newton method for the subproblem (3.4) at each iteration k in this section.

Let $\psi : R^2 \to R$ be the Fischer-Burmeister function [5] defined by

$$\psi(a, b) = \sqrt{a^2 + b^2} - (a + b).$$

This function has the following typical property

$$\psi(a, b) = 0 \quad \Longleftrightarrow \quad a \ge 0,\ b \ge 0 \text{ and } ab = 0$$

and is called a *complementarity function* in literature [4]. Moreover, ϕ is semismooth and its generalized Jacobian in the sense of Clarke [2] has the following result:

$$\frac{\partial \psi(a, b)}{\partial a} = \begin{cases} \frac{a}{\sqrt{a^2+b^2}} - 1, & \text{if } a^2 + b^2 \neq 0 \\ \alpha - 1, & \text{if } a^2 + b^2 = 0 \end{cases}$$

$$\frac{\partial \psi(a, b)}{\partial b} = \begin{cases} \frac{b}{\sqrt{a^2+b^2}} - 1, & \text{if } a^2 + b^2 \neq 0 \\ \beta - 1, & \text{if } a^2 + b^2 = 0 \end{cases}$$

with $\|(\alpha, \beta)\| \le 1$.

Denote $m_k = |\Omega_k|$ and

$$h^k_{(i,j)}(w) \equiv h_i(w, t_j), \quad i = 1, \dots, m,\ t_j \in \Omega_k. \tag{4.1}$$

Based on the function ψ_{FB}, the system (3.4) is reformulated equivalently to a system of semismooth equations for each iteration k as follows:

$$\begin{pmatrix} \nabla f(w)+\sum_{i=1}^{n_x}\lambda_i\nabla F_i^0(w)+\sum_{i=1}^{n_y}\mu_i\nabla G_i^0(w)+\sum_{i=1}^{m_0}\gamma_i\nabla H_i^0(w)+\sum_{i=1}^{m}\left(\sum_{j=1}^{m_k}u_i^j\nabla h_{(i,j)}^k(w)\right) \\ F^0(w) \\ G^0(w) \\ \psi(-H_i^0(w),\gamma_i),\quad (i=1,\dots,m_0) \\ \psi(-h_{(i,j)}^k(w),u_i^j),\quad (i=1,\dots,m;\ j=1,\dots,m_k) \end{pmatrix}=0. \tag{4.2}$$

Denote (4.2) as

$$\Upsilon^k(w,\lambda,\mu,\gamma,u)=\begin{pmatrix} L^k(w,\lambda,\mu,\gamma,u) \\ F^0(w) \\ G^0(w) \\ \Phi^k(w,\gamma,u) \end{pmatrix}=0 \tag{4.3}$$

with

$$\begin{aligned} L^k(w,\lambda,\mu,\gamma,u)&=\nabla f(w)+\sum_{i=1}^{n_x}\lambda_i\nabla F_i^0(w)+\sum_{i=1}^{n_y}\mu_i\nabla G_i^0(w) \\ &\quad+\sum_{i=1}^{m_0}\gamma_i\nabla H_i^0(w)+\sum_{i=1}^{m}\left(\sum_{j=1}^{m_k}u_i^j\nabla h_{(i,j)}^k(w)\right), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \Phi^k(w,\gamma,u)&=\left(\Phi_i^k(w,\gamma,u)\right)_{i=1,\dots,m_0+mm_k}^T \\ &\equiv\begin{pmatrix} \psi(-H_i^0(w),\gamma_i),\ (i=1,\dots,m_0) \\ \psi(-h_{(i,j)}^k(w),u_i^j),\ (i=1,\dots,m,\ j=1,\dots,m_k) \end{pmatrix}. \end{aligned} \tag{4.5}$$

Consider that just the first derivative of h is available in the reformulated SIP problem arising from power systems, we will provide a global quasi-Newton algorithm for (4.3) proposed in [10]. To this end, we denote the following notations.

For a given constant $\epsilon>0$, define

$$\psi^\epsilon(a,b)=\begin{cases} \sqrt{a^2+b^2}-(a+b), & \text{if } \sqrt{a^2+b^2}\ge\epsilon \\ (1/2\epsilon)[(a^2+b^2)-2\epsilon(a+b)+\epsilon^2], & \text{if } \sqrt{a^2+b^2}<\epsilon, \end{cases}$$

which is a smoothing function of $\psi(a,b)$. Then the smoothing function of Φ^k defined in (4.5) has the following version

$$\begin{aligned} \Phi^{k,\epsilon}(w,\gamma,u)&=\left(\Phi_i^{k,\epsilon}(w,\gamma,u)\right)_{i=1,\dots,m_0+mm_k}^T \\ &\equiv\begin{pmatrix} \psi^\epsilon(-H_i^0(w),\gamma_i),\ (i=1,\dots,m_0) \\ \psi^\epsilon(-h_{(i,j)}^k(w),u_i^j),\ (i=1,\dots,m;\ j=1,\dots,m_k) \end{pmatrix}. \end{aligned} \tag{4.6}$$

The function $\Upsilon^k(w,\lambda,\mu,\gamma,u)$ in (4.3) can be rewritten by a splitting form as

$$\Upsilon^k(w,\lambda,\mu,\gamma,u)\equiv\Upsilon^{k,\epsilon}(w,\lambda,\mu,\gamma,u)+\Gamma^{k,\epsilon}(w,\gamma,u), \tag{4.7}$$

where

$$\Upsilon^{k,\epsilon}(w,\lambda,\mu,\gamma,u)=\begin{pmatrix} L^k(w,\lambda,\mu,\gamma,u)\\ F^0(w)\\ G^0(w)\\ \Phi^{k,\epsilon}(w,\gamma,u)\end{pmatrix},$$
$$\Gamma^{k,\epsilon}(w,\gamma,u)=\begin{pmatrix} 0\\ 0\\ 0\\ (\Phi^k(w,\gamma,u)-\Phi^{k,\epsilon}(w,\gamma,u))\end{pmatrix}. \tag{4.8}$$

By using this splitting version, it is clearly that for $\epsilon>0$, the function $\Upsilon^k(w,\lambda,\mu,\gamma,u)$ is composed of the differentiable function $\Upsilon^{k,\epsilon}$ and the non-differentiable function $\Gamma^{k,\epsilon}$. Moreover, it is not difficult to derive that the non-differentiable function $\Gamma^{k,\epsilon}$ is uniformly restricted by the small parameter ϵ and satisfies

$$\|\Gamma^{k,\epsilon}(w,\gamma,u)\|\le(\sqrt{n+m_0+mm_k}/2)\epsilon.$$

Denote

$$I^{\epsilon,0}(w,\gamma)=\Big\{i\,|\,\sqrt{(H_i^0(w))^2+\gamma_i^2}\ge\epsilon\Big\},$$
$$J^{\epsilon,0}=\Big\{i\,|\,\sqrt{(H_i^0(w))^2+\gamma_i^2}<\epsilon\Big\}; \tag{4.9}$$

$$I^{\epsilon,k}(w,u)=\Big\{(i,j)\,|\,\sqrt{(h_{(i,j)}^k(w))^2+(u_i^j)^2}\ge\epsilon\Big\},$$
$$J^{\epsilon,k}=\Big\{(i,j)\,|\,\sqrt{(h_{(i,j)}^k(w))^2+(u_i^j)^2}<\epsilon\Big\}. \tag{4.10}$$

Put

$$\alpha_i^{\epsilon,0}=\begin{cases}\frac{\gamma_i}{\sqrt{(H_i^0(w))^2+\gamma_i^2}}-1, & \text{if } i\in I^{\epsilon,0}(w,\gamma),\\ \frac{\gamma_i}{\epsilon}-1, & \text{if } i\in J^{\epsilon,0}(w,\gamma);\end{cases}$$
$$\beta_i^{\epsilon,0}=\begin{cases}\frac{-H_i^0(w)}{\sqrt{(H_i^0(w))^2+\gamma_i^2}}-1, & \text{if } i\in I^{\epsilon,0}(w,\gamma),\\ \frac{-H_i^0(w)}{\epsilon}-1, & \text{if } i\in J^{\epsilon,0}(w,\gamma);\end{cases} \tag{4.11}$$

$$\alpha_{(i,j)}^{\epsilon,k}=\begin{cases}\frac{u_i^j}{\sqrt{(h_{(i,j)}^k(w))^2+(u_i^j)^2}}-1, & \text{if } (i,j)\in I^{\epsilon,k}(w,u),\\ \frac{u_i^j}{\epsilon}-1, & \text{if } (i,j)\in J^{\epsilon,k}(w,u);\end{cases} \tag{4.12}$$

$$\beta_{(i,j)}^{\epsilon,k}=\begin{cases}\frac{-h_{(i,j)}^k(w)}{\sqrt{(h_{(i,j)}^k(w))^2+(u_i^j)^2}}-1, & \text{if } (i,j)\in I^{\epsilon,k}(w,u),\\ \frac{-h_{(i,j)}^k(w)}{\epsilon}-1, & \text{if } (i,j)\in J^{\epsilon,k}(w,u).\end{cases} \tag{4.13}$$

Define a matrix as

$$Q^k(w,\gamma,u)=\begin{pmatrix} 0 & \nabla F^0(w) & \nabla G^0(w) & \nabla H^0(w)\mathrm{diag}(\beta_i^{\epsilon,0}(w,\gamma)) & \nabla h^k(w)(\beta_i^{\epsilon,k}(w,u)) \\ \nabla F^0(w)^T & 0 & 0 & 0 & 0 \\ \nabla G^0(w)^T & 0 & 0 & 0 & 0 \\ \nabla H^0(w)^T & 0 & 0 & \mathrm{diag}(\alpha^{\epsilon,0}(w,\gamma)) & 0 \\ \nabla h^k(w)^T & 0 & 0 & 0 & \mathrm{diag}(\alpha^{\epsilon,k}(w,u)) \end{pmatrix}. \tag{4.14}$$

Let the iterative index for the k-th subproblem (3.4) be l. Throughout this section, the index (l,k) means the l-th subproblem iteration with the outer iteration k. Denote $V=(w,\lambda,\mu,\gamma,u)$. Select a positive η_l satisfies

$$\sum_{l=0}^{\infty}\eta_l\le\eta<\infty \tag{4.15}$$

with a positive constant $\eta>0$.

Algorithm 4.1 (Quasi-Newton algorithm)

Step 0. Choose constant $\rho\in(0,1),\kappa\in(0,1),0<\nu<2/\sqrt{m_0+mm_k}$, $\sigma_1>0$, $\sigma_2>0$, the initial multiplier of inequality $(\gamma_{-1},u_{-1})>0$. η_l satisfies (4.15). Choose an initial point $V_0=((w^0)^T,(\lambda^0)^T,(\mu^0)^T,(\gamma^0)^T,(u^0)^T)^T\in R^{n+n_x+n_y+m_0+mm_k}$, a symmetric positive definite matrix $B_0\in R^{(n+n_x+n_y+m_0+mm_k)\times(n+n_x+n_y+m_0+mm_k)}$, and $\epsilon_0\le(\nu/2)\|\Upsilon^k(V_0)\|$, the initial step-length of line search $\tau_{-1}=1$. Let $l:=0$.

Step 1. Let

$$q_l=\tau_{l-1}^{-1}\begin{pmatrix} L^k(w_l+\tau_{l-1}L^k(V_l),\lambda^l,\mu^l,\gamma^l,u^l)-L^k(V_l) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}+Q^{(k,l)}\Upsilon^{k,\epsilon_l}(V_l),$$

where $Q^{(k,l)}$ is defined in (4.14) at point (w^l,γ^l,u^l). Solve the linear equation

$$B_lp+q_l=0 \tag{4.16}$$

to get p_l.

Step 2. If

$$\|\Upsilon^k(V_l+p_l)\|\le\kappa\|\Upsilon^k(V_l)\|, \tag{4.17}$$

then let $\tau_l=1$ and go to Step 4. Otherwise, go to Step 3.

Step 3. Let j_l be the smallest nonnegative integer j such that $\tau=\rho^j$ satisfies the follows

$$\|\Upsilon^{k,\epsilon_l}(V_l+\tau p_l)\|^2-\|\Upsilon^{k,\epsilon_l}(V_l)\|^2\le\sigma_1\|\tau\Upsilon^{k,\epsilon_l}(V_l)\|^2-\sigma_2\|\tau p_l\|^2+\eta_l\|\Upsilon^{k,\epsilon_l}(V_l)\|^2, \tag{4.18}$$

and let $\tau_l=\rho^{j_l}$.

Step 4. Let $V_{l+1} = V_l + \tau_l p_l$.
Step 5. Update B_l by the BFGS formula

$$B_{l+1} = B_l - \frac{B_l s_l s_l^T B_l}{s_l B_l s_l} + \frac{\zeta_l \zeta_l^T}{\zeta_l^T s_l}, \tag{4.19}$$

where

$$s_l = V_{l+1} - V_l, \tag{4.20}$$

and ζ_l is determined by the follows:

$$\zeta_l = r_l + \chi_l \|\Upsilon^{k,\epsilon_l}(V_l)\| s_l \tag{4.21}$$

with

$$\begin{aligned} r_l = &\begin{pmatrix} L^k(w_l + L^k(V_{l+1}) - L^k(V_l), \lambda^l, \mu^l, \gamma^l, u^l) - L^k(V_l) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &+ Q^{k,l}\Big(\Upsilon^{k,\epsilon_l}(V_{l+1}) - \Upsilon^{k,\epsilon_l}(V_l)\Big) \\ \chi_l = 1 &- \frac{1}{\|\Upsilon^{k,\epsilon_l}(V_l)\|} \min\left\{\frac{r_l^T s_l}{\|s_l\|^2}, 0\right\}. \end{aligned}$$

Step 6. If $\epsilon_l < \nu\|\Upsilon^k(V_{l+1})\|$, take $\epsilon_{l+1} = \epsilon_l$. Otherwise, determine ϵ_{l+1} by

$$\epsilon_{l+1} \le \min\left\{\frac{\nu}{2}\|\Upsilon^k(V_{l+1})\|, \frac{1}{2}\epsilon_l\right\}. \tag{4.22}$$

Step 7. Let $l := l + 1$. Go to Step 1.

Under some mild conditions, the global and locally superlinear convergence of Algorithm 4.1 can be proved, see the reference [10] for details.

5 Numerical examples

In this section, two OTS problems arising from power systems are tested by Algorithm 3.1. One is an Available Transfer Capability (ATC) calculation for a single-machine infinite-bus model system, and other is the Economic Dispatch (ED) problem for a system of two generators with six buses. Both of examples are chosen from [1] with some modifications. Algorithm 3.1 described in Sect. 3 has been implemented in MATLAB 7.0. The tests were conducted on a Pentium IV 2.50 GHz computer with 256 MB of RAM, running on Microsoft Windows XP Professional operating system.

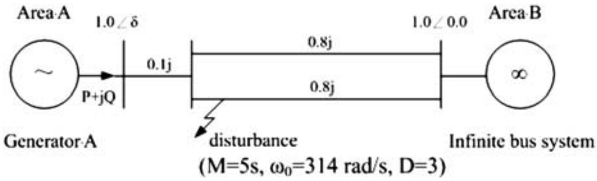

Fig. 1 Single-machine infinite-bus model system

5.1 Available transfer capability calculation with transient stability constraints

A simple model of one-machine infinite-bus system with a single disturbance is considered (see Fig. 1). For this example, all variables are fixed except rotor angel $\delta(t)$ of the generator-A. The dynamics of generator-A is described by the second-order swing equation. The relationships of real and reactive power (P, Q) during the disturbance process are expressed as follows:

- **Pre-disturbance** system at $t = 0$ with the initial variable $\bar{\delta}$:

$$\begin{cases} X = 0, \\ P(\bar{\delta}) = \frac{\sin\bar{\delta}}{0.5}, \\ Q(\bar{\delta}) = \frac{1-\cos\bar{\delta}}{0.5}. \end{cases}$$

- **During-disturbance** system for $t \in (0, t^1]$:

$$\begin{cases} X = \infty, \\ P(\bar{\delta}) = 0, \\ Q(\bar{\delta}) = 0. \end{cases}$$

- **Post-disturbance** system for $t \in [t^1, T]$:

$$\begin{cases} X = 0.9, \\ P(\delta(t)) = \frac{\sin\delta(t)}{0.9}, \\ Q(\delta(t)) = \frac{1-\cos\delta(t)}{0.9}, \end{cases}$$

where t^1 is the fault-clearing time, and T is the study period of the disturbance.

The ATC computation can be carried out by OPF models (see [1]). For the single-machine infinite-bus system, the ATC with transient stability constraints has the following version:

$$\begin{aligned} \max_{\bar{\delta}} \quad & P(\bar{\delta}) \\ \text{s.t.} \quad & 0 \le P(\bar{\delta}) \le 2 \\ & -2 \le Q(\bar{\delta}) \le 2 \\ & \begin{cases} \frac{d\delta(t)}{dt} = \omega(t) & \text{with } \delta(0) = \bar{\delta} \\ \frac{d\omega(t)}{dt} = \frac{\omega_0}{M}(P(\bar{\delta}) - P(\delta(t))) - \frac{D}{M}\omega(t) & \text{with } \omega(0) = 0 \end{cases} \\ & \delta(t) \le \delta_{\max}, \quad t \in [0, T], \end{aligned} \tag{5.1}$$

where the system parameters are: $D = 3$, $M = 5s$, $\omega_0 = 314$ rad/s, $\delta_{\max} = 2.5(\approx 0.8\pi)$, and with various choices of t^1 and $T = 2s$. The last three expressions in (5.1) are called *transient stability constraints* (for abbreviation TSCs). In order to test the effect of TSCs for the system under the disturbance, we solve ATC problem for two cases, i.e., involving TSCs and without TSCs respectively. The later one is the ordinary OPF problem.

Case-I: ATC Calculation Involving TSCs. We choose the parameters in Algorithm 3.1 as

$$\text{Error} = 1.0e^{-6}, \qquad \delta_0 = 1.0e^{-5}, \qquad \epsilon = 0.01, \qquad \bar{\delta}^0 = 0.314,$$

where Error is the stopping-test; $\bar{\delta}^0$ is the initial iteration. The differential equations and the subproblem of finite nonlinear programming (3.4) in Algorithm 3.1 are solved by ode45 code and "fmincon" code in MATLAB, respectively. The computing results with the different fault clearing times are reported in Table 1. where $\bar{\delta}^*$ is the optimal solution of OTS at the initial point $t = 0$; *Out-Iter* indicates the outer iteration number in Algorithm 3.1; *Total-Iter* means the total iteration number involving the subproblem calculation; CPU is the computing time.

According to the solution of (5.1) with $t^1 = 0.3$ and the given disturbance, the dynamic track of the differential equations with the initial point $\bar{\delta}^* = 0.2401$ is shown in Fig. 2 for the real-line curve. From the trajectory we can see that, the system tends to stability. Make a small variation to the initial point, say $\bar{\delta} = 0.25$/rad, then the system tends instability under the disturbance, see Fig. 2 for the dot-line curve with the same clearing time t^1.

Table 1 Results of OTS by Algorithm 3.1

t^1/s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\bar{\delta}^*$/rad	0.4589	0.3410	0.2401	0.1716	0.1271	0.0975	0.0770	0.0624	0.051
Out-Iter	1	1	3	2	2	2	2	3	2
Total-Iter	8	11	13	22	14	10	11	19	7
CPU/s	20.81	26.14	31.92	41.13	29.92	22.57	23.45	35.67	16.16

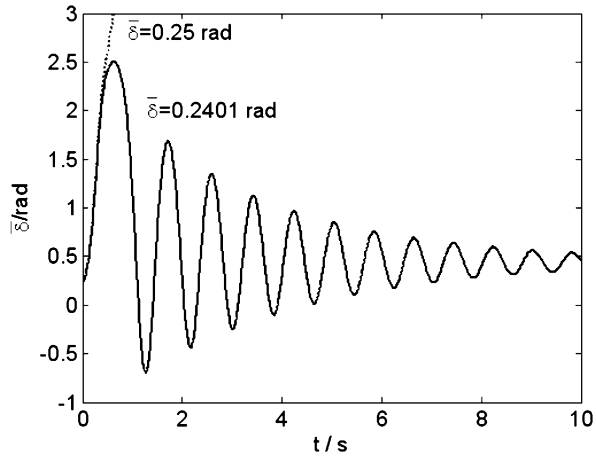

Fig. 2 The trajectory of the disturbance starting from $\bar{\delta} = 0.2401/\text{rad}$

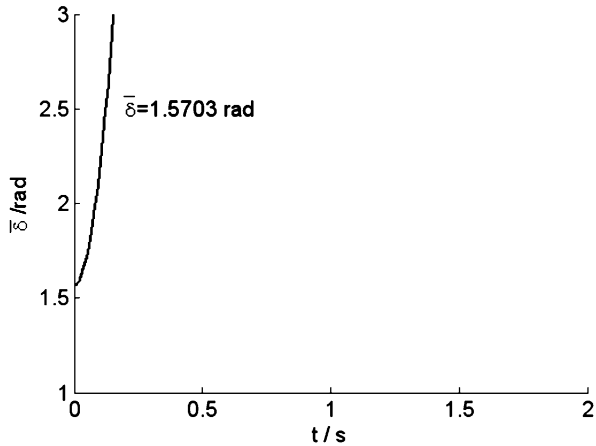

Fig. 3 The trajectory of the disturbance without TSC, $\bar{\delta} = 1.5789$

Case-II: ATC Calculation without TSCs.
Without the TSCs, the ATC problem is a normal OPF problem as

$$\begin{aligned} \max_{\bar{\delta}} \quad & P(\bar{\delta}) \\ \text{s.t.} \quad & 0 \le P(\bar{\delta}) \le 2 \\ & -2 \le Q(\bar{\delta}) \le 2. \end{aligned} \tag{5.2}$$

By using the "fmincon" code in MATLAB, we find the solution of (5.2) is $\delta(0) = \bar{\delta}^* = 1.5789$. Then under the disturbance with the clearing time $t^1 = 0.3$ and the initial point $\bar{\delta}^* = 1.5789$, the trajectory of the solution of the differential equations is shown in Fig. 3, which indicates that the system is instable under the disturbance.

5.2 Economic dispatch (ED) problem with transient stability constraints

Consider a system of two generators with six-buses and the economic dispatch (ED) problem, which is the modification of example in [1]. Suppose that the disturbance happens between 3-bus and 4-bus, see Fig. 4 for the system and the disturbance. We use the second generator as a reference bus, and set $\frac{D_1}{M_1} = \frac{D_2}{M_2} = d_0$, then the OTS problem is described as

$$
\begin{aligned}
\min_{(\bar{\delta}, \bar{V}, p_L)} \quad & \textstyle\sum_{i=1}^{2} [\alpha_i P_{gi}^2(\bar{\delta}, \bar{V}) + \beta_i P_{gi}(\bar{\delta}, \bar{V}) + \gamma_i] \\
\text{s.t.} \quad & G^0(\bar{\delta}, \bar{V}, p_L) = 0 \\
& H^0(\bar{\delta}, \bar{V}, p_L) \le 0 \\
& \begin{cases} \frac{d\delta_{g12}(t)}{dt} = \omega_{g12}(t) \\ \quad \text{with } \delta_{g12}(0) = \delta_{g1}(0) - \delta_{g2}(0) \\ \frac{d\omega_{g12}(t)}{dt} = \frac{\omega_0}{M}[P_{g1}(\bar{\delta}, \bar{V}) - P_{E1}(t)] - \frac{\omega_0}{M}[P_{g2}(\bar{\delta}, \bar{V}) - P_{E2}(t)] - d_0\omega_{g12}(t) \\ \quad \text{with } \omega_{g12}(0) = 0. \end{cases} \\
& |\delta_{gi}(t) - \delta_c(t)| \le \delta_{\max} \quad (i = 1, 2), \ \forall t \in [0, T],
\end{aligned}
\tag{5.3}
$$

where the details of the model (5.3) are given as follows:

(i)

$$
\bar{V} = (\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4)^T, \qquad \bar{\delta} = (\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3, \bar{\delta}_4)^T, \qquad p_L = (P_{L1}, P_{L2}, Q_{L1}, Q_{L2})^T.
$$

$$
\delta_{g12}(t) = \delta_{g1}(t) - \delta_{g2}(t), \qquad \omega_{g12} = \omega_{g1}(t) - \omega_{g2}(t).
$$

$$
\begin{cases} P_{g1} \equiv P_{g1}(\bar{V}, \bar{\delta}) = V_{g1}^2 G'_{11} + V_{g1}V_{g2}(G'_{12}\cos\bar{\delta}_{g12} + B'_{12}\sin\bar{\delta}_{g12}) \\ P_{g2} \equiv P_{g2}(\bar{V}, \bar{\delta}) = V_{g2}^2 G'_{22} + V_{g1}V_{g2}(G'_{12}\cos\bar{\delta}_{g12} - B'_{12}\sin\bar{\delta}_{g12}) \end{cases}
$$

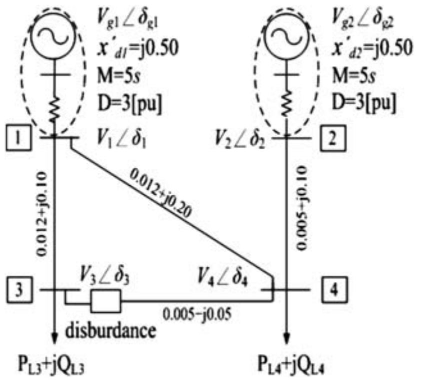

Fig. 4 Two-generators system with 4-buses

with

$$\begin{pmatrix} G'_{11}+jB'_{11} & G'_{12}+jB'_{12} & G'_{13}+jB'_{13} \\ G'_{21}+jB'_{21} & G'_{22}+jB'_{22} & G'_{23}+jB'_{23} \\ G'_{31}+jB'_{31} & G'_{32}+jB'_{32} & G'_{33}+jB'_{33} \end{pmatrix}$$
$$= \begin{pmatrix} 0.3780-j1.0985 & 0.3333+j0.6054 & -0.7113+j0.4931 \\ 0.333+j0.6054 & 0.3278-j1.0651 & -0.6611+j0.4597 \\ -0.7113+j0.4931 & -0.6611+j0.4597 & 1.3725-j0.9528 \end{pmatrix}$$

$$V_{gi}\cos\delta_{gi}+jV_{gi}\sin\delta_{gi}=\left(\bar{V}_i\cos\bar{\delta}_i+j\bar{V}_i\sin\bar{\delta}_i\right)+\frac{P_{Li}-jQ_{Li}}{\bar{V}_i\cos\bar{\delta}_i-j\bar{V}_i\sin\bar{\delta}_i}\chi'_{di},$$
$$(i=1,2);$$

(ii) $G^0(\bar{\delta},\bar{V},p_L)=0$ is the power flow equations with six-buses as

$$G^0(\bar{\delta},\bar{V},p_L)$$
$$=\begin{pmatrix} P_{L1}-\bar{V}_1[\bar{V}_1G_{11}+\bar{V}_3(G_{13}\cos\bar{\delta}_{13}+B_{13}\sin\bar{\delta}_{13})+\bar{V}_4(G_{14}\cos\bar{\delta}_{14}+B_{14}\sin\bar{\delta}_{14})] \\ P_{L2}-\bar{V}_2[\bar{V}_2G_{22}+\bar{V}_4(G_{24}\cos\bar{\delta}_{24}+B_{24}\sin\bar{\delta}_{24})] \\ P_{L3}+\bar{V}_3[\bar{V}_1(G_{31}\cos\bar{\delta}_{31}+B_{31}\sin\bar{\delta}_{31})+\bar{V}_3G_{33}+\bar{V}_4(G_{34}\cos\bar{\delta}_{34}+B_{34}\sin\bar{\delta}_{34})] \\ P_{L4}+\bar{V}_4[\bar{V}_1(G_{41}\cos\bar{\delta}_{41}+B_{41}\sin\bar{\delta}_{41})+\bar{V}_2(G_{42}\cos\bar{\delta}_{42}+B_{42}\sin\bar{\delta}_{42})+\bar{V}_3(G_{43}\cos\bar{\delta}_{43}+B_{43}\sin\bar{\delta}_{43})+\bar{V}_4G_{44}] \\ Q_{L1}-\bar{V}_1[-B_{11}\bar{V}_1+\bar{V}_3(G_{13}\sin\bar{\delta}_{13}-B_{13}\cos\bar{\delta}_{13})+\bar{V}_4(G_{14}\sin\bar{\delta}_{14}-B_{14}\cos\bar{\delta}_{14})] \\ Q_{L2}-\bar{V}_2[-B_{22}\bar{V}_2+\bar{V}_4(G_{24}\sin\bar{\delta}_{24}-B_{24}\cos\bar{\delta}_{24})] \\ Q_{L3}+\bar{V}_3[\bar{V}_1(G_{31}\sin\bar{\delta}_{31}-B_{31}\cos\bar{\delta}_{31})-B_{33}\bar{V}_3+\bar{V}_4(G_{34}\sin\bar{\delta}_{34}+B_{34}\cos\bar{\delta}_{34})] \\ Q_{L4}+\bar{V}_4[\bar{V}_1(G_{41}\sin\bar{\delta}_{41}-B_{41}\cos\bar{\delta}_{41})+\bar{V}_2(G_{42}\sin\bar{\delta}_{42}-B_{42}\cos\bar{\delta}_{42})+\bar{V}_3(G_{43}\sin\bar{\delta}_{43}-B_{43}\cos\bar{\delta}_{43})-B_{44}\bar{V}_4] \end{pmatrix}$$

where

$$\bar{\delta}_{ij}=\bar{\delta}_i-\bar{\delta}_j\quad(i,j=1,2,3,4);\qquad \bar{V}_2=1.0,\qquad \bar{\delta}_2=0\quad\text{(reference bus)}.$$
$$P_{L3}+jQ_{L3}=1.0+j0.1,\qquad P_{L4}+jQ_{L4}=1.0+j0.1.$$
$$\begin{pmatrix} G_{11}+jB_{11} & G_{12}+jB_{12} & G_{13}+jB_{13} & G_{14}+jB_{14} \\ G_{21}+jB_{21} & G_{22}+jB_{22} & G_{23}+jB_{23} & G_{24}+jB_{24} \\ G_{31}+jB_{31} & G_{32}+jB_{32} & G_{33}+jB_{33} & G_{34}+jB_{34} \\ G_{41}+jB_{41} & G_{42}+jB_{42} & G_{43}+jB_{43} & G_{44}+jB_{44} \end{pmatrix}$$
$$=\begin{pmatrix} 1.4819-j14.8401 & 0 & -1.183+j9.858 & -0.2989+j4.9821 \\ 0 & 0.4988-j9.9751 & 0 & -0.4988+j9.9751 \\ -1.183+j9.858 & 0 & 3.1632-j29.660 & -1.9802+j19.802 \\ -0.2989+j4.9821 & -0.4988+j9.9751 & -1.9802+j19.802 & 2.7779-j34.7591 \end{pmatrix};$$

(iii) $H^0(\bar{\delta},\bar{V})\le 0$ is the system limits in the static operation station (or initial operation) with respect to the voltage upper and lower bounds, and two pairs of inequalities for generator outputs

$$1.0\le V_{gi}\le 1.5;\qquad 0.25\le P_{gi}(\bar{\delta},\bar{V})\le 2.0\quad(i=1,2);$$

(iv) For the differential equations involved in the model (5.3), it has the following relationship:

$$P_{E1}(t)=V_{g1}^2G'_{11}+V_{g1}V_{g2}(G'_{12}\cos\delta_{g12}(t)+B'_{12}\sin\delta_{g12}(t)),$$

$$P_{E2}(t) = V_{g2}^2 G'_{22} + V_{g1} V_{g2} (G'_{12} \cos \delta_{g12}(t) - B'_{12} \sin \delta_{g12}(t)),$$

where the difference of differential equations between the during-disturbance and the post-disturbance is the parameters in $P_{E1}(t)$ and $P_{E2}(t)$, which can be deduced and have the following results:

During-disturbance.

$$\begin{pmatrix} G'_{11} + jB'_{11} & G'_{12} + jB'_{12} & G'_{13} + jB'_{13} \\ G'_{21} + jB'_{21} & G'_{22} + jB'_{22} & G'_{23} + jB'_{23} \\ G'_{31} + jB'_{31} & G'_{32} + jB'_{32} & G'_{33} + jB'_{33} \end{pmatrix} = \begin{pmatrix} 0.0233 - j1.7507 & 0.0058 + j0.0385 & -0.0290 + j1.7123 \\ 0.0058 + j0.0385 & 0.0262 - j1.5574 & -0.0322 + j1.5189 \\ -0.0290 + j1.7123 & -0.0320 + j1.5189 & 0.0610 - j3.2312 \end{pmatrix}$$

Post-disturbance.

$$\begin{pmatrix} G'_{11} + jB'_{11} & G'_{12} + jB'_{12} & G'_{13} + jB'_{13} \\ G'_{21} + jB'_{21} & G'_{22} + jB'_{22} & G'_{23} + jB'_{23} \\ G'_{31} + jB'_{31} & G'_{32} + jB'_{32} & G'_{33} + jB'_{33} \end{pmatrix} = \begin{pmatrix} 0.4084 - j1.0545 & 0.3166 + j0.5363 & -0.7250 + j0.5183 \\ 0.3166 + j0.5363 & 0.3156 - j0.9714 & -0.6322 + j0.4351 \\ -0.7250 + j0.5182 & -0.6322 + j0.4351 & 1.3572 - j0.9533 \end{pmatrix};$$

(v) $\delta_c(t) = \frac{\sum_{i=1}^{2} M_i \delta_{gi}(t)}{\sum_{i=1}^{2} M_i}$ is called *the center of inertia* (denoted by COI), which is the ordinary expression of the transient stability in multi-machines systems. In details, the stability of systems is shown by the difference of $|\delta_{gi}(t) - \delta_c(t)|$. Then the last constraint in (5.3) is set $|\delta_{gi}(t) - \delta_c(t)| = |\delta_{g1}(t) - \delta_{g2}(t)| \le \delta_{\max}$ $(i = 1, 2)$ since $M = M_1 = M_2$ in this example;

(vi) Some parameters involved in (5.3) are given by

$$M = 5; \qquad D = 3; \qquad \omega_0 = 1; \qquad \delta_{\max} = 2.5 (\approx 0.8\pi).$$

The other system parameters in the OTS model (5.3) are listed in Table 2.

Let the disturbance clearing time be $t^1 = 5$ s and the study period be $T = 20$ s. The parameters of Algorithm 3.1 works same as the single-machine infinite-bus system (see the last subsection). The solution methods for differential equations and the subproblem in Algorithm 3.1 are used the same ways in the last numerical example. The computing results for the ED problem with TSCs are reported in Table 3. where *Fuel cost* indicates the optimal objective of ED problem; $\bar{V}^* = (V_{g1}, V_{g2}, V_1, V_2,$

Table 2 The system parameters

	α_i	β_i	γ_i
Generator-1	0.0	4250.0	12068.0
Generator-2	1.13	1304.5	18720.0

Table 3 Solution of ED with TSCs by Algorithm 3.1

Fuel cost	P_{g1}	P_{g2}	V_{g1}	V_{g2}	V_1	V_2	V_3	V_4
35512.2	0.7381	1.2156	1.0	1.4044	0.9538	1.011	0.9538	0.9512

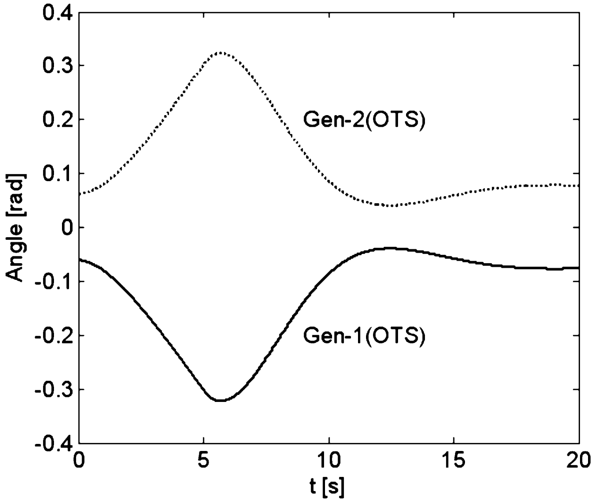

Fig. 5 The trajectories of two generators under TSCs

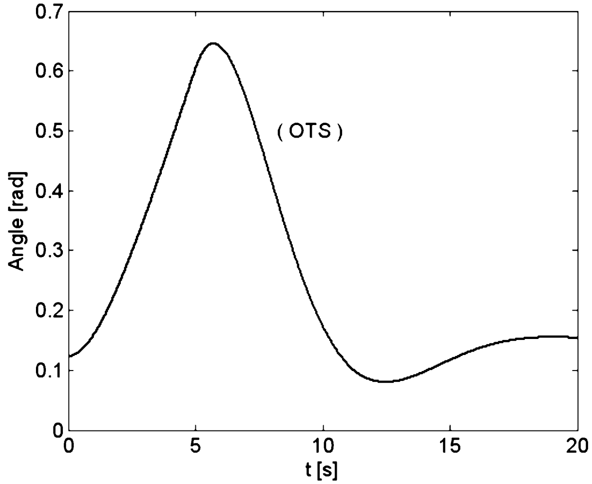

Fig. 6 The trajectory of the difference $|\delta_{g1}(t) - \delta_{g2}(t)|$ under TSCs

$V_3, V_4)^T$ is the optimal solution of ED with TSCs (initial operation in the disturbance). Here we just give the optimal voltage value $\bar{V}^*$. The optimal angle value $\bar{\delta}^*$ with respect to generators is involved in the following trajectory figures. From the optimal value $((\bar{V}^*)^T, (\bar{\delta}^*)^T)^T$ (i.e., the initial operation point), we draw the trajectory of angles for two generators and the difference of $|\delta_{gi}(t) - \delta_c(t)|$ $(i = 1, 2)$ in Figs. 5 and 6, respectively. Two figures show that for this case, the system is stable under the disturbance.

Table 4 Solution of ED without TSCs

Fuel cost	P_{g1}	P_{g2}	V_{g1}	V_{g2}	V_1	V_2	V_3	V_4
34112	0.2500	1.7310	1.0462	1.5000	0.9796	1.011	0.9680	0.9738

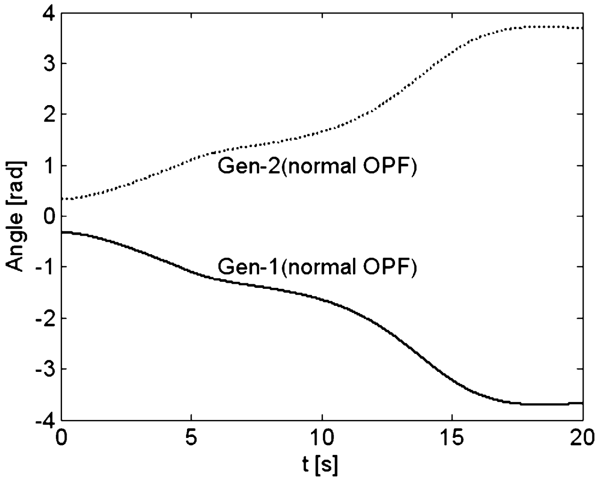

Fig. 7 The trajectories of two generators without TSCs

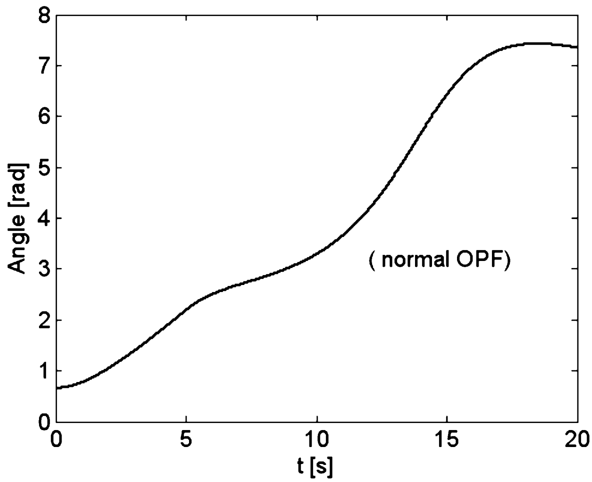

Fig. 8 The trajectory of the difference $|\delta_{g1}(t) - \delta_{g2}(t)|$ without TSCs

In order to compare the effect of TSCs, we also compute ED without TSCs, i.e., a normal OPF solution problem. The optimal solution $((\hat{V}^*)^T, (\hat{\delta}^*)^T)^T$ is reported in Table 4.

Similar to Figs. 5 and 6, we draw the trajectories of $\delta_{g1}(t)$, $\delta_{g2}(t)$ and $|\delta_{g1}(t) - \delta_{g2}(t)|$ under the disturbance with the clearing time $t^1 = 5$ s and the optimal point $((\hat{V}^*)^T, (\hat{\delta}^*)^T)^T$, respectively, see Figs. 7 and 8.

Two figures indicates that the system is instable for the optimal value without TSCs as an initial operation point. Therefore thought the optimal ED value without TSCs is better than one of ED with TSCs, the cost is the stability.

Two examples of power systems show that, on the one hand, the OPF with TSCs models is important in the stability operation of power systems; On the other hand, the approach proposed in this paper is valid.

6 Final comments

This paper develops some iterative methods in SIP problems to a class of complicated SIP problems arising from OTS in power systems. The convergence of the new algorithm is established. According to the specific construction of the reformulated SIP problem, a quasi-Newton type method is also presented for the subproblems. Suitable mathematical method and computing technology are used to save the computation cost. Two actual power systems are used to test the approach proposed in this paper. Numerical results show the validity of the reformulation and approach. From the actual background of power systems for stability analysis, there exist some general OTS problems to be studied, such as OTS problems with variable disturbance clearing time, OTS problems with critical disturbance clearing time (CCT). Both of these are interesting problems in power systems and are our further research topics.

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