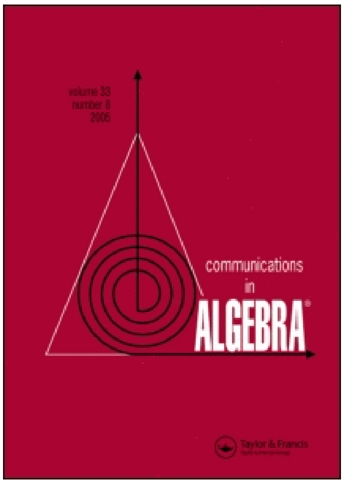

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Dade's Invariant Conjecture for the Symplectic Group $Sp_4(2^{<i>n</i>})$ and the Special Unitary Group $SU_4(2^{2<i>n</i>})$ in Defining Characteristic

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DADE'S INVARIANT CONJECTURE FOR THE SYMPLECTIC GROUP $Sp_4(2^n)$ AND THE SPECIAL UNITARY GROUP $SU_4(2^{2n})$ IN DEFINING CHARACTERISTIC

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In this article, we verify Dade's projective invariant conjecture for the symplectic group $\mathrm{Sp}_4(2^n)$ and the special unitary group $\mathrm{SU}_4(2^{2n})$ in the defining characteristic, that is, in characteristic 2. Furthermore, we show that the Isaacs–Malle–Navarro version of the McKay conjecture holds for $\mathrm{Sp}_4(2^n)$ and $\mathrm{SU}_4(2^{2n})$ in the defining characteristic, that is, $\mathrm{Sp}_4(2^n)$ and $\mathrm{SU}_4(2^{2n})$ are good for the prime 2 in the sense of Isaacs, Malle, and Navarro.

Key Words: Dade's conjecture; McKay's conjecture.

2000 Mathematics Subject Classification: Primary 20C20, 20C40.

1. INTRODUCTION

Let G be a finite group and p a prime dividing the order of G. There are several conjectures connecting the representation theory of G with the representation theory of certain p-local subgroups (i.e., the p-subgroups and their normalizers) of G. For example, it seems to be true, that if P is a Sylow p-subgroup of G, then the number of complex irreducible characters of G of degree coprime with p equals the same number for the normalizer $N_G(P)$.

This conjecture, called McKay conjecture [19], and its block-theoretic version due to Alperin [1] were generalized by various authors. In a series of articles [8–10], Dade developed several conjectures expressing the number of complex irreducible characters with a fixed defect in a given p-block of G in terms of an alternating sum of related values for p-blocks of certain p-local subgroups of G. In [9], Dade proved that his (projective) conjecture implies the McKay conjecture. In [15], Isaacs, Malle, and Navarro reduced the McKay conjecture to a question about finite simple

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groups. In particular, they showed that every finite group will satisfy the McKay conjecture if every finite non-abelian simple group is "good."

In this article, we show that Dade's invariant conjecture holds for the symplectic group $\mathrm{Sp}_4(2^n)$ and the special unitary group $\mathrm{SU}_4(2^{2n})$ in the defining characteristic, i.e., in characteristic 2. Since $\mathrm{Sp}_4(2^n)$ and $\mathrm{SU}_4(2^{2n})$ have a trivial Schur multiplier and a cyclic outer automorphism group except in the two cases $\mathrm{Sp}_4(2)$ and $\mathrm{SU}_4(4)$ (in which the exceptional Schur multiplier is 2), it follows that Dade's inductive conjecture is also true for $\mathrm{Sp}_4(2^n)$ and $\mathrm{SU}_4(2^{2n})$ in these cases. In the exceptional Schur multiplier case, we show that Dade's projective invariant conjecture holds for $\mathrm{Sp}_4(2)$ and $\mathrm{SU}_4(4)$. As an application of our results on characters fixed by certain outer automorphisms, we prove that $\mathrm{Sp}_4(2^n)$ and $\mathrm{SU}_4(2^{2n})$ are good for the prime 2.

The methods are similar to those in [2]. By a corollary of the Borel and Tits theorem [5], the normalizers of radical 2-chains of $\mathrm{Sp}_4(2^n)$ and $\mathrm{SU}_4(2^{2n})$ are exactly the parabolic subgroups. So we count characters of these chain normalizers which are fixed by certain outer automorphisms. Our calculations are based on the character table of $\mathrm{Sp}_4(2^n)$ in the character table library of the Maple [7] part of CHEVIE [12] and the character tables of the parabolic subgroups of $\mathrm{Sp}_4(2^n)$ and $\mathrm{GU}_4(2^{2n})$ which have been computed in [13] and [20]. Since $\mathrm{GU}_4(2^{2n}) = \mathbb{Z}_{2^n+1} \times \mathrm{SU}_4(2^{2n})$ and $\mathbb{Z}_{2^n+1}$ is a $2'$-group, it follows that the verifications of the conjectures for $\mathrm{SU}_4(2^{2n})$ is equivalent to that for $\mathrm{GU}_4(2^{2n})$ (see the remarks in Sections 7 and 9).

This article is organized as follows. In Section 2, we fix notation and state Dade's invariant and projective invariant conjectures in detail. In Section 3, we state and prove some lemmas from elementary number theory which we use to count fixed points of certain automorphisms of $\mathrm{Sp}_4(2^n)$ and $\mathrm{GU}_4(2^{2n})$. In Section 4, we compute the fixed points of the outer automorphisms of $\mathrm{Sp}_4(2^n)$ and $\mathrm{GU}_4(2^{2n})$ on the irreducible characters of the parabolic subgroups. In Sections 5 through 8, we verify Dade's conjecture for $\mathrm{Sp}_4(2^n)$ and $\mathrm{GU}_4(2^{2n})$ in the defining characteristic and in Section 9, we deal with the McKay conjecture for $\mathrm{Sp}_4(2^n)$ and $\mathrm{GU}_4(2^{2n})$ in the defining characteristic. Details on irreducible characters and conjugacy classes are summarized in tabular form in Appendices A, B, and C.

2. THE CONJECTURES

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) := N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and by $\mathrm{Blk}(G)$ the set of p-blocks. If $H \leq G$, $\widetilde{B} \in \mathrm{Blk}(G)$, and d is an integer, we denote by $\mathrm{Irr}(H, \widetilde{B}, d)$ the set of characters $\chi \in \mathrm{Irr}(H)$ satisfying $\mathrm{d}(\chi) = d$ and $b(\chi)^G = \widetilde{B}$ (in the sense of Brauer), where $\mathrm{d}(\chi) = \log_p(|H|_p) - \log_p(\chi(1)_p)$ is the p-defect of χ and $b(\chi)$ is the block of H containing χ.

Given a p-subgroup chain $C : P_0 < P_1 < \cdots < P_n$ of G, define the length $|C| := n$, $C_k : P_0 < P_1 < \cdots < P_k$ and

$$N(C) = N_G(C) := N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n).$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$; and
(b) $P_k = O_p(N(C_k))$ for $1 \le k \le n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Let Z be a cyclic group and $\widehat{G} = Z \cdot G$ a central extension of Z by G, and $C \in \mathcal{R}(G)$. Denote by $N_{\widehat{G}}(C)$ the preimage $\eta^{-1}(N(C))$ of $N(C)$ in $\widehat{G}$, where η is the natural group homomorphism from $\widehat{G}$ onto G with kernel Z. Let ρ be a faithful linear character of Z and $\widehat{B}$ a block of $\widehat{G}$ covering the block $B(\rho)$ of Z containing ρ. Denote by $\mathrm{Irr}(N_{\widehat{G}}(C), \widehat{B}, d, \rho)$ the set of irreducible characters ψ of $N_{\widehat{G}}(C)$ such that ψ lies over ρ, $\mathrm{d}(\psi) = d$ and $B(\psi)^{\widehat{G}} = \widehat{B}$ and set $\mathrm{k}(N_{\widehat{G}}(C), \widehat{B}, d, \rho) = |\mathrm{Irr}(N_{\widehat{G}}(C), \widehat{B}, d, \rho)|$.

If, moreover, $\widehat{E}$ is an extension of $\widehat{G}$ centralizing Z and $N_{\widehat{E}}(C, \psi)$ is the stabilizer of (C, ψ) in $\widehat{E}$, then $N_{\widehat{E}/\widehat{G}}(C, \psi) = N_{\widehat{E}}(C, \psi)/N_{\widehat{G}}(C, \psi)$ is a subgroup of $\widehat{E}/\widehat{G}$. For a subgroup $\widehat{U} \le \widehat{E}/\widehat{G}$, denote by $\mathrm{k}(N_{\widehat{G}}(C), \widehat{B}, d, \widehat{U}, \rho)$ the number of characters ψ in $\mathrm{Irr}(N_{\widehat{G}}(C), \widehat{B}, d, \rho)$ such that $N_{\widehat{E}/\widehat{G}}(C, \psi) = \widehat{U}$. In the notation above, Dade's projective invariant conjecture is stated as follows.

Dade's Projective Invariant Conjecture (See [10]). If $O_p(G) = 1$ and $\widehat{B}$ is a p-block of $\widehat{G}$ covering $B(\rho)$ with defect group $D(\widehat{B}) \neq O_p(Z)$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\widehat{G}}(C), \widehat{B}, d, \widehat{U}, \rho) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

In addition, if $\widehat{E}/\widehat{G}$ is cyclic and $u = |\widehat{U}|$, then we set

$$\mathrm{k}(N_{\widehat{G}}(C), \widehat{B}, d, u, \rho) = \mathrm{k}(N_{\widehat{G}}(C), \widehat{B}, d, \widehat{U}, \rho).$$

In particular, if $Z = 1$ and ρ is the trivial character of Z, then $\widehat{G} = G$ and $\widehat{B}$ is a block $\widetilde{B}$ of G; we set $U = \widehat{U}$ and

$$\mathrm{k}(N_G(C), \widetilde{B}, d, U) = \mathrm{k}(N_{\widehat{G}}(C), \widehat{B}, d, \widehat{U}, \rho).$$

Then the projective invariant conjecture is equivalent to the invariant conjecture.

Dade's Invariant Conjecture (See [10]). If $O_p(G) = 1$ and $\widetilde{B}$ is a p-block of G with defect $D(\widetilde{B}) \neq 1$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), \widetilde{B}, d, U) = 0.$$

Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ be the automorphism and outer automorphism groups of G, respectively. We may suppose $\widehat{E}/\widehat{G} = \mathrm{Out}(G)$. If moreover, $\mathrm{Out}(G)$ is cyclic, then we write

$$\mathrm{k}(N_G(C), \widetilde{B}, d, |U|) := \mathrm{k}(N_G(C), \widetilde{B}, d, U).$$

For $G \in \{\mathrm{Sp}_4(2^n), \mathrm{SU}_4(2^{2n})\}$, $\mathrm{Out}(G)$ is cyclic and the Schur multiplier of G is trivial except in the two cases $\mathrm{Sp}_4(2)$ and $\mathrm{SU}_4(4)$, in which the exceptional Schur multiplier is 2. So the invariant conjecture for G is equivalent to the inductive conjecture. In the exceptional Schur multiplier case, the projective invariant conjecture is equivalent to the inductive conjecture for $\mathrm{Sp}_4(2)$ and $\mathrm{SU}_4(4)$.

3. NOTATION AND LEMMAS FROM ELEMENTARY NUMBER THEORY

From now on, we assume that $p = 2$, n is a positive integer and $q = 2^n$. We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers including zero. In the next section, we will use the following lemmas, the first one is [2, Lemma 3.1].

Lemma 3.1. *Suppose* $m, n, a \in \mathbb{Z}$ *with* $m, n > 0$. *Then* $\gcd(a^m - 1, a^n - 1) = |a^d - 1|$ *where* $d := \gcd(m, n)$.

Lemma 3.2. *Let t be a positive integer with $t \mid n$. Then the following hold*:

(i) $\gcd(2^t - 1, q - 1) = 2^t - 1$;
(ii) $\gcd(2^t - 1, q + 1) = 1$;
(iii) $\gcd(2^t + 1, q - 1) = \begin{cases} 2^t + 1 & \text{if } 2t \mid n, \\ 1 & \text{if } 2t \nmid n; \end{cases}$
(iv) $\gcd(2^t + 1, q + 1) = \begin{cases} 1 & \text{if } 2t \mid n, \\ 2^t + 1 & \text{if } 2t \nmid n. \end{cases}$

Proof. (i) is clear by Lemma 3.1.

(ii) Suppose $d = \gcd(2^t - 1, q + 1)$. By (i), $d \mid q - 1$ and so $d \mid \gcd(q - 1, q + 1) = 1$.

(iii) Suppose $2t \mid n$. There are $k, t_u, n_u \in \mathbb{N}$ with $2 \nmid t_u$ and $2 \mid n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $2^t + 1 = -((-2^{2^k})^{t_u} - 1)$ and $q - 1 = (-2^{2^k})^{n_u} - 1$. So Lemma 3.1 implies $\gcd(2^t + 1, q - 1) = \gcd((-2^{2^k})^{t_u} - 1, (-2^{2^k})^{n_u} - 1) = |(-2^{2^k})^{t_u} - 1| = 2^t + 1$.

Suppose $2t \nmid n$. If $d \mid 2^t + 1, q - 1$, then $d \mid q + 1, q - 1$ and so $d \mid \gcd(q + 1, q - 1) = 1$.

(iv) Suppose $2t \mid n$. If $d \mid 2^t + 1, q + 1$, then $d \mid 2^{2t} - 1$ and so $d \mid 2^n - 1$ as $2t \mid n$. Thus $d \mid q - 1, q + 1$ and $d \mid \gcd(q + 1, q - 1) = 1$.

Suppose $2t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with odd $t_u n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. So we get $2^t + 1 = -((-2^{2^k})^{t_u} - 1)$ and $q + 1 = -((-2^{2^k})^{n_u} - 1)$. Now, Lemma 3.1 implies $\gcd(2^t + 1, q + 1) = \gcd((-2^{2^k})^{t_u} - 1, (-2^{2^k})^{n_u} - 1) = |(-2^{2^k})^{t_u} - 1| = 2^t + 1$. □

Lemma 3.3. *Let t be a positive integer with $t \mid 2n$. Then the following hold*:

(i) $\gcd(2^t - 1, q + 1) = \begin{cases} 1 & \text{if } t \mid n, \\ 2^{t/2} + 1 & \text{if } t \nmid n; \end{cases}$
(ii) $\gcd(2^t - 1, q^2 - 1) = 2^t - 1$;

(iii) $\gcd(2^t+1, q+1) = \begin{cases} 1 & \text{if } 2t \mid n \text{ or } t \nmid n, \\ 2^t+1 & \text{if } t \mid n \text{ and } 2t \nmid n; \end{cases}$

(iv) $\gcd(2^t+1, q^2-1) = \begin{cases} 2^t+1 & \text{if } t \mid n, \\ 1 & \text{if } t \nmid n; \end{cases}$

(v) $\gcd(2^t+1, q^2+1) = \begin{cases} 1 & \text{if } t \mid n, \\ 2^t+1 & \text{if } t \nmid n; \end{cases}$

Proof. (i) If $t \mid n$, then (i) follows by Lemma 3.2(ii).

Suppose $t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with $2 \mid t_u$ and $2 \nmid n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $2^t - 1 = (-2^{2^k})^{t_u} - 1$ and $q+1 = -((-2^{2^k})^{n_u} - 1)$. So Lemma 3.1 implies $\gcd(2^t-1, q+1) = \gcd((-2^{2^k})^{t_u} - 1, (-2^{2^k})^{n_u} - 1) = |(-2^{2^k})^{t_u/2} - 1| = 2^{t/2} + 1$.

(ii) is clear by Lemma 3.1.

(v) Suppose $t \mid n$. If $d \mid 2^t+1, q^2+1$, then $d \mid 2^{2t} - 1$ and $2^{2t} - 1 \mid 2^{2n} - 1 = q^2 - 1$, so that $d \mid \gcd(q^2-1, q^2+1) = 1$.

Suppose $t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with odd $t_u n_u$ such that $t = 2^{k+1} \cdot t_u$, $n = 2^k \cdot n_u$. By Lemma 3.1, $2^t + 1 = -((-2^{2^{k+1}})^{t_u} - 1) \mid (-2^{2^{k+1}})^{n_u} - 1$. So $2^t + 1 \mid q^2 + 1$.

(iii) Suppose $2t \mid n$ and $d \mid 2^t + 1, q+1$. Then, by Lemma 3.2(iii), $d \mid q - 1$ and so $d \mid \gcd(q-1, q+1) = 1$.

Suppose $t \mid n$ and $2t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with odd $t_u n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $2^t + 1 = -((-2^{2^k})^{t_u} - 1)$ and $q + 1 = -((-2^{2^k})^{n_u} - 1)$. So Lemma 3.1 implies $\gcd(2^t+1, q+1) = \gcd((-2^{2^k})^{t_u} - 1, (-2^{2^k})^{n_u} - 1) = |(-2^{2^k})^{t_u} - 1| = 2^t + 1$. Suppose $t \nmid n$. If $d \mid 2^t + 1, q+1$, then by (v), $d \mid q^2 + 1$ and so $d \mid \gcd(q^2+1, q^2-1) = 1$.

(iv) Suppose $t \mid n$. Then $2t \mid 2n$ and $2^t + 1 \mid 2^{2t} - 1$. Hence $2^t + 1 \mid 2^{2n} - 1 = q^2 - 1$.

Now suppose $t \nmid n$. If $d \mid 2^t + 1, q^2 - 1$, then by (v), $d \mid q^2 + 1$ and hence $d \mid \gcd(q^2+1, q^2-1) = 1$. □

Lemma 3.4. *Let $t \in \mathbb{N} \backslash \{0\}$ with $t \mid 2n$. Define $\delta := 1$ if $t|n$ and $\delta := \frac{1}{2}$ if $t \nmid n$. Then*

$$\gcd(2^{n-t}+1, q^2-1) = \begin{cases} 2^{\delta t}+1 & \text{if } 2t \mid n \text{ or } t \nmid n, \\ 1 & \text{if } t \mid n \text{ and } 2t \nmid n. \end{cases}$$

Proof. Suppose $2t \mid n$. There are $k, t_u, n_u \in \mathbb{N}$ with $2 \nmid t_u$ and $2 \mid n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $2^{n-t} + 1 = -((-2^{2^k})^{n_u - t_u} - 1)$ and $q^2 - 1 = (-2^{2^k})^{2n_u} - 1$. So Lemma 3.1 implies $\gcd(2^{n-t}+1, q^2-1) = \gcd((-2^{2^k})^{n_u - t_u} - 1, (-2^{2^k})^{2n_u} - 1) = |(-2^{2^k})^{t_u} - 1| = 2^t + 1$.

Suppose $t \mid n$ and $2t \nmid n$. Let $d = \gcd(2^{n-t}+1, q^2-1)$, so that $d \mid 2^{2(n-t)} - 1$. There is an odd $n_u \in \mathbb{N}$ such that $n = t \cdot n_u$, $n - t = t \cdot (n_u - 1) = 2t \cdot \frac{n_u - 1}{2}$. By Lemma 3.1, $\gcd(2^{2(n-t)} - 1, q^2 - 1) = 2^{2t} - 1$, so $d \mid 2^{2t} - 1$ and $2^{2t} \equiv 1 \pmod{d}$. Thus

$$0 \equiv 2^{n-t} + 1 = 2^{2t \cdot \frac{n_u - 1}{2}} + 1 \equiv 2 \pmod{d},$$

and $d = 1$ as d is odd.

Suppose $t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with $2 \mid t_u$ and $2 \nmid n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $2^{n-t} + 1 = -((-2^{2^k})^{n_u - t_u} - 1)$ and $q^2 - 1 = (-2^{2^k})^{2n_u} - 1$. So Lemma 3.1 implies $\gcd(2^{n-t} + 1, q^2 - 1) = \gcd((-2^{2^k})^{n_u - t_u} - 1, (-2^{2^k})^{2n_u} - 1) = |(-2^{2^k})^{t_u/2} - 1| = 2^{t/2} + 1$. □

4. ACTION OF AUTOMORPHISMS ON IRREDUCIBLE CHARACTERS

Let $G \in \{\mathrm{Sp}_4(2^n), \mathrm{GU}_4(2^{2n})\}$, and let $O = \mathrm{Out}(G)$ and $A = \mathrm{Aut}(G)$. If $G = \mathrm{Sp}_4(2^n)$, then $O = \langle \beta \rangle$ with $\beta^2 = \alpha$, where α is a field automorphism of order n; and if $G = \mathrm{GU}_4(2^{2n})$, then $O = \langle \alpha \rangle$, where α is a field automorphism of order $2n$. We fix a Borel subgroup B and distinct maximal parabolic subgroups P and Q of G containing B as in [13, 20].

In this section, we determine the action of O on the irreducible characters of B, P, Q, and G. Our notation for the parameter sets of these groups is similar to that of CHEVIE and is given in Tables A.1 and A.6 in the Appendix A. The correspondence between the CHEVIE notation and that of Enomoto (respectively, Nozawa) is given in Tables A.2–A.4 (respectively Tables A.7–A.10).

The first column of Tables A.1 and A.6 defines a name for the parameter set which parameterizes those characters which are listed in the second column of the table. The list of parameters in the third column of Tables A.1 and A.6 in Appendix A is of the form

$$k = 0, \ldots, n_1 - 1 \qquad \text{or} \qquad \begin{array}{l} k = 0, \ldots, n_1 - 1 \\ l = 0, \ldots, n_2 - 1 \end{array}$$

where the n_j's are polynomials in q with integer coefficients. In the first case, the parameter k can be substituted by an element of $\mathbb{Z}$, but two parameters which differ by an element of $n_1\mathbb{Z}$ yield the same character. In the second case, the parameter vector (k, l) can be substituted by an element of $\mathbb{Z} \times \mathbb{Z}$, but two parameter vectors which differ by an element of $n_1\mathbb{Z} \times n_2\mathbb{Z}$ yield the same character. In other words, k can be taken to be an element of $\mathbb{Z}_{n_1}$ and (k, l) can be taken to be an element of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. The groups $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ are also called *character parameter groups* (see Section 3.7 of the CHEVIE [12] manual). The next lines of Tables A.1 and A.6 list elements which have to be excluded from the character parameter group. The remaining parameters are called *admissible* in the following. Different values of admissible parameters may give the same character. The fourth column of Tables A.1 and A.6 defines an equivalence relation on the set of admissible parameters. If no equivalence relation is listed we mean the identity relation. The parameter set is defined to be the set of these equivalence classes. Finally, the last column of Tables A.1 and A.6 gives the cardinality of the parameter set.

We consider the example ${}_PI_3$ in Table A.1. The character parameter group is $\mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$. The parameter vectors (k, l) and (l, k) yield the same character and the equivalence class of (k, l) is $\{(k, l), (l, k)\}$. Hence, the characters ${}_P\chi_3(k, l)$ are parameterized by the set

$${}_PI_3 = \{\{(k, l), (l, k)\} \mid (k, l) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}, q - 1 \nmid k - l\}.$$

If we want to emphasize the dependence of a parameter set, say ${}_PI_3$, from q we write ${}_PI_3(q)$. Tables A.1 and A.6 do not give any detailed information about the parameter sets ${}_G I_1$, ${}_G I_2$, ${}_G I_3$, ${}_G I_4$, ${}_G I_5$ and ${}_G I_2$, ${}_G I_5$, ${}_G I_6$, ${}_G I_9$, ${}_G I_{10}$, respectively, since we will not need an explicit knowledge of these sets (note that these parameter sets parameterize the regular semisimple irreducible characters of G). The data in Tables A.1 and A.6 is taken from [13] and [20].

The action of $O = \mathrm{Out}(G)$ on the conjugacy classes of elements of G, B, P and Q induces an action of O on the sets $\mathrm{Irr}(G)$, $\mathrm{Irr}(B)$, $\mathrm{Irr}(P)$ and $\mathrm{Irr}(Q)$ and then an action on the parameter sets. Using the values of the irreducible characters of G, B, P, and Q on the classes listed in the last column of Tables A.2–A.4 and A.7–A.10, we can describe the action of O on the parameter sets.

For an O-set I and each subgroup $H \leq O$ let $C_I(H)$ denote the set of fixed points of I under the action of H. In the following proposition we determine $|C_I(H)|$ where I runs through all (disjoint) unions of parameter sets which are listed in Table A.5 (respectively, Table A.11) except for ${}_G I_1 \cup {}_G I_2 \cup {}_G I_3 \cup {}_G I_4 \cup {}_G I_5$ (respectively, ${}_G I_2 \cup {}_G I_5 \cup {}_G I_6 \cup {}_G I_9 \cup {}_G I_{10}$). This last union of parameter sets will be treated separately since it requires different methods.

Proposition 4.1. *Let $G = \mathrm{Sp}_4(2^n)$, $t \mid n$, and let $I \neq {}_G I_1 \cup {}_G I_2 \cup {}_G I_3 \cup {}_G I_4 \cup {}_G I_5$ be one of the (disjoint) unions of parameter sets listed in Table* A.5. *If $H = \langle \alpha^t \rangle$ is a subgroup of O, then the second column of Table* A.5 *show the number of fixed points $|C_I(H)|$ of I under the action of H.*

Proof. We have to consider the following parameter sets I.

First let $I \in \{{}_G I_{14}, {}_G I_{15} \cup {}_G I_{16} \cup {}_G I_{17} \cup {}_G I_{19}, {}_B I_6, {}_B I_7 \cup {}_B I_8, {}_P I_8, {}_P I_9 \cup {}_P I_{10}, {}_Q I_8, {}_Q I_9 \cup {}_Q I_{10}\}$. The degrees and character values on the conjugacy classes listed in Tables A.3 and A.4 show $C_I(H) = I$ and hence $|C_I(H)| = |I|$. We demonstrate this for the parameter set $I = {}_P I_9 \cup {}_P I_{10}$. The degrees in Table A.3 show that ${}_P\theta_2(0)$ and ${}_P\theta_2(1)$ are the only irreducible characters of P of degree $\frac{1}{2}q(q^2-1)$. Furthermore, ${}_P\theta_3(0)$ and ${}_P\theta_3(1)$ are the only irreducible characters of P of degree $\frac{1}{2}q(q-1)^2$. Hence, ${}_P\theta_2(0)^\alpha \in \{{}_P\theta_2(0), {}_P\theta_2(1)\}$ and ${}_P\theta_3(0)^\alpha \in \{{}_P\theta_3(0), {}_P\theta_3(1)\}$. The class representatives in Table II-1 in [13] show that the conjugacy class A_{41} is fixed by α, and we can see from the character Table II-2 of P in [13] that the values of ${}_P\theta_2(0)$ and ${}_P\theta_2(1)$ on A_{41} are different. Similarly, the values of ${}_P\theta_3(0)$ and ${}_P\theta_3(1)$ on A_{41} are different. So ${}_P\theta_2(k)^\alpha = {}_P\theta_2(k)$ and ${}_P\theta_3(k)^\alpha = {}_P\theta_3(k)$ for $k = 0, 1$ and $|C_I(H)| = |I|$.

In each of the following cases, we have that the action of α on I is given by $x^\alpha = 2x$ for all $x \in I$ using the character values on the classes listed in the last column of Tables A.2–A.4. We demonstrate this for the parameter set $I = {}_P I_3 \cup {}_P I_7$. The degrees in Table A.3 show that the ${}_P\chi_3(k, l)$'s are the only irreducible characters of P of degree $q+1$, so ${}_P\chi_3(k, l)^\alpha = {}_P\chi_3(k', l')$ for some $\{(k', l'), \dots\} \in {}_P I_3$. We see from the class representatives in Table II-1 in [13] that α acts on the semisimple conjugacy classes of P like the 2nd power map which implies that the values of ${}_P\chi_3(k', l')$ and ${}_P\chi_3(2k, 2l)$ on the semisimple classes coincide. Then, the character values of ${}_P\chi_3(k, l)$ (see the character Table II-2 in [13]) imply that the values of ${}_P\chi_3(k', l')$ and ${}_P\chi_3(2k, 2l)$ coincide on all classes, hence ${}_P\chi_3(k', l') = {}_P\chi_3(2k, 2l)$ and therefore ${}_P\chi_3(k, l)^\alpha = {}_P\chi_3(2k, 2l)$. Similarly, ${}_P\chi_7(k)^\alpha = {}_P\chi_7(2k)$. Hence, $x^\alpha = 2x$ for all $x \in I$.

Let $I \in \{{}_G I_6 \cup {}_G I_8, {}_G I_7 \cup {}_G I_9, {}_G I_{10} \cup {}_G I_{12}, {}_G I_{11} \cup {}_G I_{13}, {}_P I_5 \cup {}_P I_6, {}_Q I_5 \cup {}_Q I_6\}$. So these unions of parameter sets are isomorphic H-sets, so that we can assume

$I = {}_GI_6 \cup {}_GI_8$. If $x = \{k, -k\} \in I$, then $x \in C_I(H)$ if and only if $(2^t - 1)k \equiv 0$ or $(2^t + 1)k \equiv 0$. Let

$$C_\pm := \big\{\{k, -k\} \in C_I(H) \mid (2^t \pm 1)k \equiv 0\big\},$$

so that $C_I(H) = C_- \cup C_+$ and $C_- \cap C_+ = \emptyset$. We claim

$$C_- = \left\{\{k, -k\} \in {}_GI_6 \mid k \text{ is a multiple of } \frac{q-1}{2^t - 1}\right\}.$$

The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_-$. If $x \in {}_GI_8$, then $(2^t - 1)k \equiv 0 \bmod q + 1$ and Lemma 3.2(ii) implies $k \equiv 0$, which is impossible. Hence $x \in {}_GI_6$ and $(2^t - 1)k \equiv 0 \bmod q - 1$. By Lemma 3.2(i), k is a multiple of $(q - 1)/(2^t - 1)$, proving the claim. Now we consider C_+.

If $2t \mid n$, we claim $C_+ = \{\{k, -k\} \in {}_GI_6 \mid k \text{ is a multiple of } (q - 1)/(2^t + 1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_+$. If $x \in {}_GI_8$, then $(2^t + 1)k \equiv 0 \bmod q + 1$ and Lemma 3.2(iv) implies $k \equiv 0$, which is impossible. Hence $x \in {}_GI_6$ and $(2^t + 1)k \equiv 0 \bmod q - 1$. By Lemma 3.2(iii), k is a multiple of $(q - 1)/(2^t + 1)$ and the claim holds.

If $2t \nmid n$, we claim $C_+ = \{\{k, -k\} \in {}_GI_8 \mid k \text{ is a multiple of } (q + 1)/(2^t + 1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_+$. If $x \in {}_GI_6$, then $(2^t + 1)k \equiv 0 \bmod q - 1$ and Lemma 3.2(iii) implies $k \equiv 0$, which is impossible. Hence $x \in {}_GI_8$ and $(2^t + 1)k \equiv 0 \bmod q + 1$. By Lemma 3.2(iv), k is a multiple of $(q + 1)/(2^t + 1)$ and the claim holds.

Thus in all cases, $|C_I(H)| = |C_-| + |C_+| = \frac{2^t - 2}{2} + \frac{2^t}{2} = 2^t - 1$.

Let $I = {}_BI_1$. If $(k, l) \in I$, then $(k, l) \in C_I(H)$ if and only if $(2^t - 1)k \equiv 0$ and $(2^t - 1)l \equiv 0 \bmod q - 1$. By Lemma 3.2(i), this is equivalent with k, l are multiples of $\frac{q-1}{2^t - 1}$. Thus, $|C_I(H)| = (2^t - 1)^2$.

Let $I \in \{{}_BI_2, {}_BI_3, {}_BI_4, {}_BI_5, {}_PI_1, {}_PI_2, {}_PI_4, {}_QI_1, {}_QI_2, {}_QI_4\}$. If $k \in I$, then $k \in C_I(H)$ if and only if $(2^t - 1)k \equiv 0 \bmod q - 1$. So we get $C_I(H) = \{k \in I \mid k \text{ is a multiple of } (q - 1)/(2^t - 1)\}$ and $|C_I(H)| = 2^t - 1$.

Let $I = {}_PI_3 \cup {}_PI_7$. First, we compute $|C_{{}_PI_3}(H)|$. Let

$$U_i := \begin{cases} \{\{(k, l), (l, k)\} \in C_{{}_PI_3}(H) \mid 2^t k \equiv k,\ 2^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k, l), (l, k)\} \in C_{{}_PI_3}(H) \mid 2^t k \equiv l,\ 2^t l \equiv k\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k, l), (l, k)\} \in {}_PI_3$, then $x \in U_1$ if and only if $(2^t - 1)k \equiv 0$ and $(2^t - 1)l \equiv 0 \bmod q - 1$. By Lemma 3.2(i), this is equivalent with that k, l are multiples of $\frac{q-1}{2^t - 1}$. Hence $|U_1| = (2^t - 2)(2^t - 1)/2$.

Suppose $2t \mid n$. If $x = \{(k, l), (l, k)\} \in {}_PI_3$, then $x \in U_2$ if and only if $2^t k \equiv l$ and $2^t l \equiv k \bmod q - 1$. So, we get $(2^{2t} - 1)k \equiv 0 \bmod q - 1$, and this is equivalent with k being a multiple of $(q - 1)/(2^{2t} - 1)$. Now exclude those solutions with $k \equiv l$, which means $(2^t - 1)k \equiv 0 \bmod q - 1$. This gives us $|U_2| = 2^t(2^t - 1)/2$.

Suppose $2t \nmid n$. If $\{(k, l), (l, k)\} \in U_2$, then $(2^{2t} - 1)k \equiv 0 \bmod q - 1$. By Lemma 3.2(i) and (iii), this is equivalent with $(2^t - 1)k \equiv 0 \bmod q - 1$. Then $k =$

$2^t k = l$, a contradiction to the definition of ${}_P I_3$. Hence, $U_2 = \emptyset$. So

$$|C_{{}_PI_3}(H)| = |U_1| + |U_2| = \begin{cases} (2^t-1)^2 & \text{if } 2t \mid n, \\ (2^t-2)(2^t-1)/2 & \text{if } 2t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_PI_7}(H)|$. If $x = \{k, qk\} \in {}_PI_7$, then $x \in C_{{}_PI_7}(H)$ if and only if $(2^t-1)k \equiv 0$ or $(2^t-q)k \equiv 0 \mod (q+1)(q-1)$. Suppose $(2^t-1)k \equiv 0$. By Lemma 3.2(i) and (ii), it follows that $\gcd(2^t-1, (q+1)(q-1)) = \gcd(2^t-1, q-1) = 2^t-1$. Thus $(q+1) \cdot \frac{(q-1)}{2^t-1} \mid k$. But then $(q+1) \mid k$, a contradiction to the definition of ${}_PI_7$. So we have proved that $x \in C_{{}_PI_7}(H)$ if and only if $(2^t-q)k \equiv 0 \mod (q+1)(q-1)$.

Suppose $2t \mid n$. If $\{k, qk\} \in C_{{}_PI_7}(H)$, then $(2^t-q)k \equiv 0 \mod (q+1)(q-1)$. Thus $(2^t-1)k \equiv 0 \mod q-1$ and $(2^t+1)k \equiv 0 \mod q+1$. Lemma 3.2(iv) implies $(q+1) \mid k$, a contradiction to the definition of ${}_PI_7$. Hence in this case $C_{{}_PI_7}(H) = \emptyset$.

Suppose $2t \nmid n$. We claim

$$C_{{}_PI_7}(H) = \left\{ \{k, qk\} \in {}_PI_7 \mid k \text{ is a multiple of } \frac{(q+1)(q-1)}{(2^t+1)(2^t-1)} \right\}.$$

Let $k = \frac{(q+1)(q-1)}{(2^t+1)(2^t-1)} \cdot m$ for some $m \in \mathbb{Z}$. Because $t \mid n$ and $2t \nmid n$ we have $2t \mid n-t$. Then we get $(2^t+1)(2^t-1) = 2^{2t}-1 \mid 2^{n-t}-1$. Thus $(2^{n-t}-1)k = \frac{2^{n-t}-1}{(2^t+1)(2^t-1)}(q+1)(q-1) \cdot m \equiv 0 \mod (q+1)(q-1)$. So $(2^t-q)k \equiv 0 \mod (q+1)(q-1)$ and $\{k, qk\} \in C_{{}_PI_7}(H)$.

Conversely, suppose $\{k, qk\} \in C_{{}_PI_7}(H)$. Then $(2^t-q)k \equiv 0 \mod (q+1)(q-1)$. Hence $(2^t+1)k \equiv 0 \mod q+1$ and $(2^t-1)k \equiv 0 \mod q-1$. By Lemma 3.2(i) and (iv), this is equivalent with $\frac{q+1}{2^t+1} \mid k$ and $\frac{q-1}{2^t-1} \mid k$. Since $\frac{q+1}{2^t+1} \mid q+1$ and $\frac{q-1}{2^t-1} \mid q-1$ and since $\gcd(q+1, q-1) = 1$, we have $\gcd\left(\frac{q+1}{2^t+1}, \frac{q-1}{2^t-1}\right) = 1$. Therefore, $\frac{(q+1)(q-1)}{(2^t+1)(2^t-1)} \mid k$, and the claim holds. So by the definition of ${}_PI_7$, we get $|C_{{}_PI_7}(H)| = 2^t(2^t-1)/2$. So in both cases, $|C_I(H)| = |C_{{}_PI_3}(H)| + |C_{{}_PI_7}(H)| = (2^t-1)^2$.

Let $I = {}_QI_3 \cup {}_QI_7$. First, we compute $|C_{{}_QI_3}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l),(k,-l)\} \in C_{{}_QI_3}(H) \mid 2^t k \equiv k, 2^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k,l),(k,-l)\} \in C_{{}_QI_3}(H) \mid 2^t k \equiv k, 2^t l \equiv -l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k,l),(k,-l)\} \in {}_QI_3$, then $x \in U_1$ if and only if $(2^t-1)k \equiv 0$ and $(2^t-1)l \equiv 0 \mod q-1$. By Lemma 3.2(i), this is equivalent with k, l are multiples of $\frac{q-1}{2^t-1}$. Hence $|U_1| = (2^t-2)(2^t-1)/2$.

Suppose $2t \mid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_3$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0$ and $(2^t+1)l \equiv 0 \mod q-1$. Hence

$$U_2 = \left\{ \{(k,l),(k,-l)\} \in {}_QI_7 \left| \frac{q-1}{2^t-1} \right| k \text{ and } \frac{q-1}{2^t+1} \mid l \right\}$$

and $|U_2| = 2^t(2^t-1)/2$.

Suppose $2t \nmid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_3$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0$ and $(2^t+1)l \equiv 0 \mod q-1$. By Lemma 3.2(iii), the second congruence is equivalent with $l \equiv 0 \bmod q-1$, a contradiction to the definition of ${}_QI_3$. Hence, $U_2 = \emptyset$. So

$$|C_{{}_QI_3}(H)| = |U_1| + |U_2| = \begin{cases} (2^t-1)^2 & \text{if } 2t \mid n, \\ (2^t-2)(2^t-1)/2 & \text{if } 2t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_QI_7}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l),(k,-l)\} \in C_{{}_QI_7}(H) \mid 2^tk \equiv k, 2^tl \equiv l\} & \text{if } i=1, \\ \{\{(k,l),(k,-l)\} \in C_{{}_QI_7}(H) \mid 2^tk \equiv k, 2^tl \equiv -l\} & \text{if } i=2. \end{cases}$$

If $x = \{(k,l),(k,-l)\} \in {}_QI_7$, then $x \in U_1$ if and only if $(2^t-1)k \equiv 0 \bmod q-1$ and $(2^t-1)l \equiv 0 \bmod q+1$. By Lemma 3.2(i) and (ii), this is equivalent with $\frac{q-1}{2^t-1} \mid k$, and $(q+1) \mid l$, a contradiction to the definition of ${}_QI_7$. Hence, $U_1 = \emptyset$.

Suppose $2t \mid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_7$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0 \bmod q-1$ and $(2^t+1)l \equiv 0 \bmod q+1$. By Lemma 3.2(i) and (iv), this is equivalent with $\frac{q-1}{2^t-1} \mid k$, and $l \equiv 0 \bmod q+1$, a contradiction to the definition of ${}_QI_7$. Hence, $U_2 = \emptyset$.

Suppose $2t \nmid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_7$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0 \bmod q-1$ and $(2^t+1)l \equiv 0 \bmod q+1$. Hence

$$U_2 = \left\{\{(k,l),(k,-l)\} \in {}_QI_7 \,\middle|\, \frac{q-1}{2^t-1} \,\middle|\, k \text{ and } \frac{q-1}{2^t+1} \,\middle|\, l\right\}$$

and $|U_2| = 2^t(2^t-1)/2$.

So

$$|C_{{}_QI_7}(H)| = |U_1| + |U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t(2^t-1)/2 & \text{if } 2t \nmid n. \end{cases}$$

So in both cases, $|C_I(H)| = |C_{{}_QI_3}(H)| + |C_{{}_QI_7}(H)| = (2^t-1)^2$. □

Proposition 4.2. *Let $G = \mathrm{GU}_4(2^{2n})$, $t \mid 2n$, and $I \neq {}_GI_2 \cup {}_GI_5 \cup {}_GI_6 \cup {}_GI_9 \cup {}_GI_{10}$ be one of the (disjoint) unions of parameter sets listed in Table* A.11. *If $H = \langle \alpha^t \rangle$ is a subgroup of O, then the second and third columns of Table* A.11 *show the number of fixed points $|C_I(H)|$ of I under the action of H.*

Proof. We have to consider the parameter sets in Table A.11.

In each of the following cases, we have that the action of α on I is given by $x^\alpha = 2x$ for all $x \in I$ using the character values on the classes listed in the last column of Tables A.7–A.10.

Suppose $I \in \{{}_GI_1, {}_GI_{12}, {}_GI_{13}, {}_GI_{14}, {}_BI_4, {}_BI_8, {}_PI_5, {}_PI_9, {}_PI_{10}, {}_QI_5\}$. If $k \in I$, then $k \in C_I(H)$ if and only if $(2^t-1)k \equiv 0 \bmod q+1$.

Suppose $t \mid n$. Then by Lemma 3.3(i), this is equivalent with $k \equiv 0 \bmod q+1$. So we get $|C_I(H)| = 1$.

Suppose $t \nmid n$. Then by Lemma 3.3(i), this is equivalent with $(2^{t/2}+1)k \equiv 0 \mod q+1$. So we get $C_I(H) = \{k \in I \mid k \text{ is a multiple of } (q+1)/(2^{t/2}+1)\}$ and $|C_I(H)| = 2^{t/2}+1$.

Let $I \in \{{}_GI_3 \cup {}_GI_{22}, {}_GI_4 \cup {}_GI_{20}, {}_PI_6 \cup {}_PI_7\}$. Then these unions of parameter sets are isomorphic H-sets, so that we can assume $I = {}_GI_3 \cup {}_GI_{22}$. First, we compute $|C_{{}_GI_3}(H)|$. Let

$$U_i := \begin{cases} \{\{k, -qk\} \in C_{{}_GI_3}(H) \mid 2^t k \equiv k\} & \text{if } i = 1, \\ \{\{k, -qk\} \in C_{{}_GI_3}(H) \mid 2^t k \equiv -qk\} & \text{if } i = 2. \end{cases}$$

If $x = \{k, -qk\} \in {}_GI_3$, then $x \in U_1$ if and only if $(2^t-1)k \equiv 0 \mod q^2-1$. By Lemma 3.3(ii), this is equivalent with k is a multiple of $\frac{q^2-1}{2^t-1}$. So

$$|U_1| = \begin{cases} (2^t-2)/2 & \text{if } t \mid n, \\ (2^t - 2^{t/2} - 2)/2 & \text{if } t \nmid n. \end{cases}$$

Suppose $t \mid n$. If $x = \{k, -qk\} \in {}_GI_3$, then $x \in U_2$ if and only if $(2^t+q)k \equiv 0 \mod q^2-1$. Then $(2^{n-t}+1)k \equiv 0 \mod q^2-1$. By Lemma 3.4, this is equivalent with $(2^t+1)k \equiv 0 \mod q^2-1$ if $2t \mid n$, and $k \equiv 0 \mod q^2-1$ if $2t \nmid n$. So by the definition of ${}_GI_3$, we get

$$|U_2| = \begin{cases} 2^t/2 & \text{if } 2t \mid n, \\ 0 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $x = \{k, -qk\} \in {}_GI_3$, then $x \in U_2$ if and only if $(2^t+q)k \equiv 0 \mod q^2-1$. Then $(2^{n-t}+1)k \equiv 0 \mod q^2-1$. By Lemma 3.4, this is equivalent with $(2^{t/2}+1)k \equiv 0 \mod q^2-1$. By Lemma 3.3(i), it follows that $\gcd(2^{t/2}+1, (q+1)(q-1)) = \gcd(2^{t/2}+1, q+1) = 2^{t/2}+1$. So $(q-1) \cdot \frac{(q+1)}{2^{t/2}+1} | k$. But then $q-1 \mid k$, a contradiction to the definition of ${}_GI_3$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_GI_3}(H)| = |U_1| + |U_2| = \begin{cases} 2^t - 1 & \text{if } 2t \mid n, \\ (2^t-2)/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ (2^t - 2^{t/2} - 2)/2 & \text{if } t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_GI_{22}}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l),(l,k)\} \in C_{{}_GI_{22}}(H) \mid 2^t k \equiv k, 2^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k,l),(l,k)\} \in C_{{}_GI_{22}}(H) \mid 2^t k \equiv l, 2^t l \equiv k\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k,l),(l,k)\} \in {}_GI_{22}$, then $x \in U_1$ if and only if $(2^t-1)k \equiv 0$ and $(2^t-1)l \equiv 0 \mod q+1$.

Suppose $t \mid n$. By Lemma 3.3(i), this implies $k \equiv l \mod q+1$, a contradiction to the definition of ${}_GI_{22}$. Hence, $U_1 = \emptyset$.

Suppose $t \nmid n$. By Lemma 3.3(i), this is equivalent with k, l are multiples of $\frac{q+1}{2^{t/2}+1}$. So we get $|U_1| = 2^{t/2}(2^{t/2}+1)/2$.

Suppose $t \mid n$. If $x = \{(k, l), (l, k)\} \in {}_GI_{22}$, then $x \in U_2$ if and only if $2^t k \equiv l$ and $2^t l \equiv k \mod q+1$. From these two congruences, we get $(2^{2t} - 1)k \equiv 0 \mod q+1$. By Lemma 3.3(i), this is equivalent with $(2^t + 1)k \equiv 0 \mod q+1$. By Lemma 3.3(iii), this is equivalent with $k \equiv 0 \mod q+1$ if $2t \mid n$, and $(2^t + 1)k \equiv 0 \mod q+1$ if $2t \nmid n$. Thus, we have

$$|U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t/2 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $x = \{(k, l), (l, k)\} \in {}_GI_{22}$, then $x \in U_2$ implies $(2^{2t} - 1)k \equiv 0$ and $(2^{2t} - 1)l \equiv 0 \mod q+1$. By Lemma 3.3(iii), this is equivalent with $(2^t - 1)k \equiv 0 \mod q+1$. Then $k \equiv 2^t k \equiv l \mod q+1$, a contradiction to the definition of ${}_GI_{22}$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_GI_{22}}(H)| = |U_1| + |U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ 2^{t/2}(2^{t/2}+1)/2 & \text{if } t \nmid n. \end{cases}$$

So, in all cases, $|C_I(H)| = |C_{{}_GI_3}(H)| + |C_{{}_GI_{22}}(H)| = 2^t - 1$.

Let $I \in \{{}_GI_7 \cup {}_GI_{16}, {}_GI_8 \cup {}_GI_{15}\}$. Then these unions of parameter sets are isomorphic H-sets, so that we can assume $I = {}_GI_7 \cup {}_GI_{16}$. First, we compute $|C_{{}_GI_7}(H)|$. Let

$$U_i := \begin{cases} \{\{(k, l), (-qk, l)\} \in C_{{}_GI_7}(H) \mid 2^t k \equiv k, 2^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k, l), (-qk, l)\} \in C_{{}_GI_7}(H) \mid 2^t k \equiv -qk, 2^t l \equiv l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k, l), (-qk, l)\} \in {}_GI_7$, then $x \in U_1$ if and only if $(2^t - 1)k \equiv 0 \mod q^2 - 1$ and $(2^t - 1)l \equiv 0 \mod q+1$.

Suppose $t \mid n$. By Lemma 3.3(i) and (ii), this is equivalent with $\frac{q^2-1}{2^t-1} \mid k$ and $l \equiv 0 \mod q+1$. Hence, $|U_1| = (2^t - 2)/2$.

Suppose $t \nmid n$. By Lemma 3.3(i) and (ii), this is equivalent with $\frac{q^2-1}{2^t-1} \mid k$ and $\frac{q+1}{2^{t/2}+1} \mid l$. So by the definition of ${}_GI_7$, we get $|U_1| = (2^t - 2^{t/2} - 2)(2^{t/2} + 1)/2$.

Suppose $t \mid n$. If $x = \{(k, l), (-qk, l)\} \in {}_GI_7$, then $x \in U_2$ if and only if $(2^t + q)k \equiv 0 \mod q^2 - 1$ and $(2^t - 1)l \equiv 0 \mod q+1$. Then $(2^{n-t} + 1)k \equiv 0 \mod q^2 - 1$ and $l \equiv 0 \mod q+1$. By Lemma 3.4, the first congruence is equivalent with $(2^t + 1)k \equiv 0 \mod q^2 - 1$ if $2t \mid n$, and $k \equiv 0 \mod q^2 - 1$ if $2t \nmid n$. So by the definition of ${}_GI_7$, we get

$$|U_2| = \begin{cases} 2^t/2 & \text{if } 2t \mid n, \\ 0 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $x = \{(k, l), (-qk, l)\} \in {}_GI_7$, then $x \in U_2$ if and only if $(2^t + q)k \equiv 0 \mod q^2 - 1$ and $(2^t - 1)l \equiv 0 \mod q+1$. Then the first congruence is equivalent with $(2^{n-t} + 1)k \equiv 0 \mod q^2 - 1$. By Lemma 3.4, this is equivalent with $(2^{t/2} + 1)k \equiv 0 \mod q^2 - 1$. By Lemma 3.3(i), it follows that $\gcd(2^{t/2} + 1,$

$(q+1)(q-1)) = \gcd(2^{t/2}+1, q+1) = 2^{t/2}+1$. So $(q-1)\cdot\frac{(q+1)}{2^{t/2}+1} \mid k$. But then $q-1 \mid k$, a contradiction to the definition of ${}_GI_7$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_GI_7}(H)| = |U_1| + |U_2| = \begin{cases} 2^t - 1 & \text{if } 2t \mid n, \\ (2^t-2)/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ (2^t - 2^{t/2} - 2)(2^{t/2}+1)/2 & \text{if } t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_GI_{16}}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l,m),(k,m,l)\} \in C_{{}_GI_{16}}(H) \mid 2^t k \equiv k, 2^t l \equiv l, 2^t m \equiv m\} & \text{if } i = 1, \\ \{\{(k,l,m),(k,m,l)\} \in C_{{}_GI_{16}}(H) \mid 2^t k \equiv k, 2^t l \equiv m, 2^t m \equiv l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k,l,m),(k,m,l)\} \in {}_GI_{16}$, then $x \in U_1$ if and only if $(2^t-1)k \equiv 0$, $(2^t-1)l \equiv 0$ and $(2^t-1)m \equiv 0 \bmod q+1$.

Suppose $t \mid n$. By Lemma 3.3(i), this is equivalent with $k \equiv l$ and $l \equiv m \bmod q+1$, a contradiction to the definition of ${}_GI_{16}$. Hence, $U_1 = \emptyset$.

Suppose $t \nmid n$. By Lemma 3.3(i), this is equivalent with k, l and m are multiples of $\frac{q+1}{2^{t/2}+1}$. So we get $|U_1| = 2^{t/2}(2^t-1)/2$.

Suppose $t \mid n$. If $x = \{(k,l,m),(k,m,l)\} \in {}_GI_{16}$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0$, $2^t l \equiv m$ and $2^t m \equiv l \bmod q+1$. From the last two congruences we get $(2^t-1)k \equiv 0$ and $(2^{2t}-1)l \equiv 0 \bmod q+1$. By Lemma 3.3(i), this is equivalent with $k \equiv 0$ and $(2^t+1)l \equiv 0 \bmod q+1$. By Lemma 3.3(iii), the second congruence is equivalent with $l \equiv 0 \bmod q+1$ if $2t \mid n$, and $(2^t+1)l \equiv 0 \bmod q+1$ if $2t \nmid n$. So by the definition of ${}_GI_{16}$, we get

$$|U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t/2 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $x = \{(k,l,m),(k,m,l)\} \in {}_GI_{16}$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0$, $2^t l \equiv m$ and $2^t m \equiv l \bmod q+1$. From the last two congruences we get $(2^{2t}-1)l \equiv 0 \bmod q+1$. By Lemma 3.3(iii), the second congruence is equivalent with $(2^t-1)l \equiv 0 \bmod q+1$. Then $l \equiv 2^t l \equiv m \bmod q+1$, a contradiction to the definition of ${}_GI_{16}$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_GI_{16}}(H)| = |U_1| + |U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ 2^{t/2}(2^t-1)/2 & \text{if } t \nmid n. \end{cases}$$

Thus,

$$|C_I(H)| = |C_{GI_7}(H)| + |C_{GI_{16}}(H)| = \begin{cases} 2^t - 1 & \text{if } t \mid n, \\ (2^{t/2}+1)(2^t - 2^{t/2} - 1) & \text{if } t \nmid n. \end{cases}$$

Let $I \in \{{}_GI_{17}, {}_GI_{18}, {}_GI_{19}, {}_GI_{21}\}$. If $(k,l) \in I$, then $(k,l) \in C_I(H)$ if and only if $(2^t-1)k \equiv 0$ and $(2^t-1)l \equiv 0 \bmod q+1$.

Suppose $t \mid n$. By Lemma 3.3(i), this is equivalent with k, l are multiples of $q+1$. Then $k \equiv l \bmod q+1$, a contradiction to the definition of I. So we get $|C_I(H)| = 0$.

Suppose $t \nmid n$. By Lemma 3.3(i), this is equivalent with k, l are multiples of $\frac{q+1}{2^{t/2}+1}$. So we get $|C_I(H)| = 2^{t/2}(2^{t/2}+1)$.

Let $I = {}_BI_1$. If $(k,l) \in I$, then $(k,l) \in C_I(H)$ if and only if $(2^t-1)k \equiv 0$ and $(2^t-1)l \equiv 0 \bmod q^2-1$. By Lemma 3.3(ii), this is equivalent with k, l are multiples of $\frac{q^2-1}{2^t-1}$. Thus, $|C_I(H)| = (2^t-1)^2$.

Let $I \in \{{}_BI_2, {}_PI_4, {}_QI_1, {}_QI_2\}$. If $(k,l) \in I$, then $(k,l) \in C_I(H)$ if and only if $(2^t-1)k \equiv 0 \bmod q^2-1$ and $(2^t-1)l \equiv 0 \bmod q+1$.

Suppose $t \mid n$. By Lemma 3.3(i) and (ii), this is equivalent with $\frac{q^2-1}{2^t-1} \mid k$ and $l \equiv 0 \bmod q+1$. So we get $C_I(H) = \{(k,l) \in I \mid \frac{q^2-1}{2^t-1} \mid k \text{ and } q+1 \mid l\}$ and $|C_I(H)| = 2^t-1$.

Suppose $t \nmid n$. By Lemma 3.3(i) and (ii), this is equivalent with $\frac{q^2-1}{2^t-1} \mid k$ and $(2^{t/2}+1)l \equiv 0 \bmod q+1$. So we get $C_I(H) = \{(k,l) \in I \mid \frac{q^2-1}{2^t-1} \mid k \text{ and } \frac{q+1}{2^{t/2}+1} \mid l\}$ and $|C_I(H)| = (2^t-1)(2^{t/2}+1)$.

Let $I \in \{{}_BI_3, {}_BI_6, {}_PI_1, {}_PI_2, {}_QI_4\}$. If $k \in I$, then $k \in C_I(H)$ if and only if $(2^t-1)k \equiv 0 \bmod q^2-1$. So we get $C_I(H) = \{k \in I \mid k \text{ is a multiple of } (q^2-1)/(2^t-1)\}$ and $|C_I(H)| = 2^t-1$.

Let $I = {}_BI_5$. If $(k,l) \in I$, then $(k,l) \in C_I(H)$ if and only if $(2^t-1)k \equiv 0 \bmod q+1$ and $(2^t-1)l \equiv 0 \bmod q^2-1$.

Suppose $t \mid n$. By Lemma 3.3(i) and (ii), this is equivalent with $k \equiv 0 \bmod q+1$ and $\frac{q^2-1}{2^t-1} \mid l$. So we get $C_I(H) = \{(k,l) \in I \mid q+1 \mid k \text{ and } \frac{q^2-1}{2^t-1} \mid l\}$ and $|C_I(H)| = 2^t-1$.

Suppose $t \nmid n$. By Lemma 3.3(i) and (ii), this is equivalent with $(2^{t/2}+1)k \equiv 0 \bmod q+1$ and $\frac{q^2-1}{2^t-1} \mid l$. So we get $C_I(H) = \{(k,l) \in I \mid \frac{q+1}{2^{t/2}+1} \mid k \text{ and } \frac{q^2-1}{2^t-1} \mid l\}$ and $|C_I(H)| = (2^{t/2}+1)(2^t-1)$.

Let $I \in \{{}_BI_7, {}_QI_7, {}_QI_9, {}_QI_{10}\}$. If $(k,l) \in I$, then $(k,l) \in C_I(H)$ if and only if $(2^t-1)k \equiv 0$ and $(2^t-1)l \equiv 0 \bmod q+1$.

Suppose $t \mid n$. By Lemma 3.3(i), this is equivalent with $k \equiv 0$ and $l \equiv 0 \bmod q+1$. So we get $C_I(H) = \{(k,l) \in I \mid q+1 \mid k, l\}$ and $|C_I(H)| = 1$.

Suppose $t \nmid n$. By Lemma 3.3(i), this is equivalent with k, l are multiples of $\frac{q+1}{2^{t/2}+1}$. So we get $|C_I(H)| = (2^{t/2}+1)^2$.

Let $I = {}_PI_3 \cup {}_PI_8$. First, we compute $|C_{{}_PI_3}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l),(l,k)\} \in C_{{}_PI_3}(H) \mid 2^tk \equiv k,\ 2^tl \equiv l\} & \text{if } i = 1, \\ \{\{(k,l),(l,k)\} \in C_{{}_PI_3}(H) \mid 2^tk \equiv l,\ 2^tl \equiv k\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k,l),(l,k)\} \in {}_PI_3$, then $x \in U_1$ if and only if $(2^t-1)k \equiv 0$ and $(2^t-1)l \equiv 0 \bmod q^2-1$. By Lemma 3.3(ii), this is equivalent with k, l are multiples of $\frac{q^2-1}{2^t-1}$. Hence $|U_1| = (2^t-2)(2^t-1)/2$.

Suppose $t \mid n$. If $x = \{(k,l),(l,k)\} \in {}_PI_3$, then $x \in U_2$ if and only if $2^tk \equiv l$ and $2^tl \equiv k \bmod q^2-1$. From these two congruences we get $(2^{2t}-1)k \equiv 0 \bmod q^2-1$, and this is equivalent to k being a multiple of $(q^2-1)/(2^{2t}-1)$. Excluding those solutions with $k \equiv l \bmod q^2-1$, we get $|U_2| = 2^t(2^t-1)/2$.

Suppose $t \nmid n$. If $x = \{(k,l),(l,k)\} \in {}_PI_3$, then $x \in U_2$ if and only if $2^tk \equiv l$ and $2^tl \equiv k \bmod q^2-1$. From these two congruences we get $(2^{2t}-1)k \equiv 0 \bmod q^2-1$.

By Lemma 3.3(iv), this is equivalent with $(2^t - 1)k \equiv 0 \mod q^2 - 1$. Then $k \equiv 2^t k \equiv l$, a contradiction to the definition of ${}_P I_3$. Hence, $U_2 = \emptyset$. So

$$|C_{{}_P I_3}(H)| = |U_1| + |U_2| = \begin{cases} (2^t - 1)^2 & \text{if } t \mid n, \\ (2^t - 2)(2^t - 1)/2 & \text{if } t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_P I_8}(H)|$. If $x = \{k, q^2 k\} \in {}_P I_8$, then $x \in C_{{}_P I_8}(H)$ if and only if $(2^t - 1)k \equiv 0$ or $(2^t - q^2)k \equiv 0 \mod (q^2 + 1)(q^2 - 1)$. Suppose $(2^t - 1)k \equiv 0$. By Lemma 3.3(i), it follows that $\gcd(2^t - 1, (q^2 + 1)(q^2 - 1)) = \gcd(2^t - 1, q^2 - 1) = 2^t - 1$. Thus $(q^2 + 1) \cdot \frac{(q^2-1)}{2^t-1} \mid k$. But then $(q^2 + 1) \mid k$, a contradiction to the definition of ${}_P I_8$. So we have proved that $x \in C_{{}_P I_8}(H)$ if and only if $(2^t - q^2)k \equiv 0 \mod (q^2 + 1)(q^2 - 1)$.

Suppose $t \mid n$. If $\{k, q^2 k\} \in C_{{}_P I_8}(H)$, then $(2^t - q^2)k \equiv 0 \mod (q^2 + 1)(q^2 - 1)$. Thus $(2^t - 1)k \equiv 0 \mod q^2 - 1$ and $(2^t + 1)k \equiv 0 \mod (q^2 + 1)$. By Lemma 3.3(ii) and (v), we get $\frac{q^2-1}{2^t-1} \mid k$, and $(q^2 + 1) \mid k$, a contradiction to the definition of ${}_P I_8$. Hence in this case $C_{{}_P I_8}(H) = \emptyset$.

Suppose $t \nmid n$. We claim

$$C_{{}_P I_8}(H) = \left\{ \{k, q^2 k\} \in {}_P I_8 \mid k \text{ is a multiple of } \frac{(q^2 + 1)(q^2 - 1)}{(2^t + 1)(2^t - 1)} \right\}.$$

Let $k = \frac{(q^2+1)(q^2-1)}{(2^t+1)(2^t-1)} \cdot m$ for some $m \in \mathbb{Z}$. Because $t \mid 2n$ and $t \nmid n$, we have $2t \mid 2n - t$. Since $(2^t - 1)(2^t + 1) = 2^{2t} - 1 \mid 2^{2n-t} - 1$, we then get $(2^{2n-t} - 1)k = \frac{2^{2n-t}-1}{(2^t+1)(2^t-1)}(q^2 + 1)(q^2 - 1) \cdot m \equiv 0 \mod (q^2 + 1)(q^2 - 1)$. So $(2^t - q^2)k \equiv 0 \mod (q^2 + 1)(q^2 - 1)$ and $\{k, q^2 k\} \in C_{{}_P I_8}(H)$.

Conversely, suppose $\{k, q^2 k\} \in C_{{}_P I_8}(H)$. Then $(2^t - q^2)k \equiv 0 \mod (q^2 + 1)(q^2 - 1)$. Hence $(2^t + 1)k \equiv 0 \mod q^2 + 1$ and $(2^t - 1)k \equiv 0 \mod q^2 - 1$. By Lemma 3.3(ii) and (v), this is equivalent with $\frac{q^2+1}{2^t+1} \mid k$ and $\frac{q^2-1}{2^t-1} \mid k$. Since $\frac{q^2+1}{2^t+1} \mid q^2 + 1$ and $\frac{q^2-1}{2^t-1} \mid q^2 - 1$ and since $\gcd(q^2 + 1, q^2 - 1) = 1$, we have $\gcd\left(\frac{q^2+1}{2^t+1}, \frac{q^2-1}{2^t-1}\right) = 1$. Therefore, $\frac{(q^2+1)(q^2-1)}{(2^t+1)(2^t-1)} \mid k$, and the claim holds. So by the definition of ${}_P I_8$, we get $|C_{{}_P I_8}(H)| = 2^t(2^t - 1)/2$.

So in both cases, $|C_I(H)| = |C_{{}_P I_3}(H)| + |C_{{}_P I_8}(H)| = (2^t - 1)^2$.

Let $I = {}_Q I_3 \cup {}_Q I_8$. First, we compute $|C_{{}_Q I_3}(H)|$. Let

$$U_i := \begin{cases} \{\{(k, l), (k, -ql)\} \in C_{{}_Q I_3}(H) \mid 2^t k \equiv k, 2^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k, l), (k, -ql)\} \in C_{{}_Q I_3}(H) \mid 2^t k \equiv k, 2^t l \equiv -ql\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k, l), (k, -ql)\} \in {}_Q I_3$, then $x \in U_1$ if and only if $(2^t - 1)k \equiv 0$ and $(2^t - 1)l \equiv 0 \mod q^2 - 1$. By Lemma 3.3(ii), this is equivalent with k, l being multiples of $\frac{q^2-1}{2^t-1}$. So by the definition of ${}_Q I_3$, we get

$$|U_1| = \begin{cases} (2^t - 1)(2^t - 2)/2 & \text{if } t \mid n, \\ (2^t - 1)(2^t - 2^{t/2} - 2)/2 & \text{if } t \nmid n. \end{cases}$$

Suppose $t \mid n$. If $x = \{(k, l), (k, -ql)\} \in {}_QI_3$, then $x \in U_2$ if and only if $(2^t - 1)k \equiv 0$ and $(2^t + q)l \equiv 0 \bmod q^2 - 1$. Then $\frac{q^2-1}{2^t-1} \mid k$ and $(2^{n-t} + 1)l \equiv 0 \bmod q^2 - 1$. By Lemma 3.4, the second congruence is equivalent with $(2^t + 1)l \equiv 0 \bmod q^2 - 1$ if $2t \mid n$, and $l \equiv 0 \bmod q^2 - 1$ if $2t \nmid n$. So by the definition of ${}_QI_3$, we get

$$|U_2| = \begin{cases} 2^t(2^t - 1)/2 & \text{if } 2t \mid n, \\ 0 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $x = \{(k, l), (k, -ql)\} \in {}_QI_3$, then$\{(k, l), (k, -ql)\} \in U_2$ if and only if $(2^t - 1)k \equiv 0$ and $(2^t + q)l \equiv 0 \bmod q^2 - 1$. Then $\frac{q^2-1}{2^t-1} \mid k$ and $(2^{n-t} + 1)l \equiv 0 \bmod q^2 - 1$. By Lemma 3.4, the second congruence is equivalent with $(2^{t/2} + 1)l \equiv 0 \bmod q^2 - 1$. By Lemma 3.3(i), it follows that $\gcd(2^{t/2} + 1, (q + 1)(q - 1)) = \gcd(2^{t/2} + 1, q + 1) = 2^{t/2} + 1$. So $(q - 1) \cdot \frac{(q+1)}{2^{t/2}+1} \mid l$. But then $q - 1 \mid l$, a contradiction to the definition of ${}_QI_3$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_QI_3}(H)| = |U_1| + |U_2| = \begin{cases} (2^t - 1)^2 & \text{if } 2t \mid n, \\ (2^t - 1)(2^t - 2)/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ (2^t - 1)(2^t - 2^{t/2} - 2)/2 & \text{if } t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_QI_8}(H)|$. Let

$$U_i := \begin{cases} \{\{(k, l, m), (k, m, l)\} \in C_{{}_QI_8}(H) \mid 2^t k \equiv k, 2^t l \equiv l, 2^t m \equiv m\} & \text{if } i = 1, \\ \{\{(k, l, m), (k, m, l)\} \in C_{{}_QI_8}(H) \mid 2^t k \equiv k, 2^t l \equiv m, 2^t m \equiv l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k, l, m), (k, m, l)\} \in {}_QI_8$, then $x \in U_1$ if and only if $(2^t - 1)k \equiv 0 \bmod q^2 - 1$, and $(2^t - 1)l \equiv 0$ and $(2^t - 1)m \equiv 0 \bmod q + 1$.

Suppose $t \mid n$. Then $(2^t - 1)k \equiv 0 \bmod q^2 - 1$, and $l \equiv 0$ and $m \equiv 0 \bmod q + 1$. This is equivalent with $\frac{q^2-1}{2^t-1} \mid k$, and $l \equiv m \bmod q + 1$, a contradiction to the definition of ${}_QI_8$. Hence, $U_1 = \emptyset$.

Suppose $t \nmid n$. Then $(2^t - 1)k \equiv 0 \bmod q^2 - 1$, and $(2^{t/2} + 1)l \equiv 0$ and $(2^{t/2} + 1)m \equiv 0 \bmod q + 1$. Hence

$$U_1 = \left\{ \{(k, l, m), (k, m, l)\} \in {}_QI_8 \,\middle|\, \frac{q^2 - 1}{2^t - 1} \,\middle|\, k \text{ and } \frac{q + 1}{2^{t/2} + 1} \,\middle|\, l, m \right\}$$

and $|U_1| = 2^{t/2}(2^{t/2} + 1)(2^t - 1)/2$.

Suppose $t \mid n$. If $x = \{(k, l, m), (k, m, l)\} \in {}_QI_8$, then $x \in U_2$ if and only if $(2^t - 1)k \equiv 0 \bmod q^2 - 1$, $2^t l \equiv m$ and $2^t m \equiv l \bmod q + 1$. By Lemma 3.3(i), we get from the last two congruences, $\frac{q^2-1}{2^t-1} \mid k$ and $(2^{2t} - 1)l \equiv 0 \bmod q + 1$, $(2^t + 1)l \equiv 0 \bmod q + 1$. By Lemma 3.3(i) and (iii), the second congruence is equivalent with $l \equiv 0 \bmod q + 1$ if $2t \mid n$, and $(2^t + 1)l \equiv 0 \bmod q + 1$ if $2t \nmid n$. So by the definition of ${}_QI_8$, we get

$$|U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t(2^t - 1)/2 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $x = \{(k, l, m), (k, m, l)\} \in {}_QI_8$, then $x \in U_2$ if and only if $(2^t - 1)k \equiv 0 \mod q^2 - 1$, and $2^t l \equiv m$ and $2^t m \equiv l \mod q + 1$. By Lemma 3.3(i) we get from the last two congruences, we get $\frac{q^2-1}{2^t-1} \mid k$ and $(2^{2t} - 1)l \equiv 0 \mod q + 1$. By Lemma 3.3(iii), the second congruence is equivalent with $(2^t - 1)l \equiv 0 \mod q + 1$. Then $l \equiv 2^t l \equiv m \mod q + 1$, a contradiction to the definition of ${}_QI_8$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_QI_8}(H)| = |U_1| + |U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t(2^t - 1)/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ 2^{t/2}(2^{t/2} + 1)(2^t - 1)/2 & \text{if } t \nmid n. \end{cases}$$

So in both cases, $|C_I(H)| = |C_{{}_QI_3}(H)| + |C_{{}_QI_8}(H)| = (2^t - 1)^2$.

Let $I = {}_QI_6 \cup {}_QI_{11}$. First, we compute $|C_{{}_QI_6}(H)|$. Let

$$U_i := \begin{cases} \{\{(k, l), (k, -ql)\} \in C_{{}_QI_6}(H) \mid 2^t k \equiv k, 2^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k, l), (k, -ql)\} \in C_{{}_QI_6}(H) \mid 2^t k \equiv k, 2^t l \equiv -ql\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k, l), (k, -ql)\} \in {}_QI_6$, then $x \in U_1$ if and only if $(2^t - 1)k \equiv 0 \mod q + 1$ and $(2^t - 1)l \equiv 0 \mod q^2 - 1$.

Suppose $t \mid n$. By Lemma 3.3(i) and (ii), this is equivalent with $q + 1 \mid k$ and $\frac{q^2-1}{2^t-1} \mid l$. So we get $|U_1| = (2^t - 2)/2$.

Suppose $t \nmid n$. Then $\frac{q+1}{2^{t/2}+1} \mid k$ and $\frac{q^2-1}{2^t-1} \mid l$. So we get $|U_1| = (2^{t/2} + 1)(2^t - 2^{t/2} - 2)/2$.

Suppose $t \mid n$. If $x = \{(k, l), (k, -ql)\} \in {}_QI_6$, then $x \in U_2$ if and only if $(2^t - 1)k \equiv 0 \mod q + 1$ and $(2^t + q)l \equiv 0 \mod q^2 - 1$. Then $q + 1 \mid k$ and $(2^{n-t} + 1)l \equiv 0 \mod q^2 - 1$. By Lemma 3.4, the second congruence is equivalent with $(2^t + 1)l \equiv 0 \mod q^2 - 1$ if $2t \mid n$, and $l \equiv 0 \mod q^2 - 1$ if $2t \nmid n$. So by the definition of ${}_QI_6$, we get

$$|U_2| = \begin{cases} 2^t/2 & \text{if } 2t \mid n, \\ 0 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $\{(k, l), (k, -ql)\} \in U_2$, then $(2^t - 1)k \equiv 0 \mod q + 1$ and $(2^t + q)l \equiv 0 \mod q^2 - 1$. Then $\frac{q+1}{2^{t/2}+1} \mid k$ and $(2^{n-t} + 1)l \equiv 0 \mod q^2 - 1$. By Lemma 3.4, the second congruence is equivalent with $(2^{t/2} + 1)l \equiv 0 \mod q^2 - 1$. By Lemma 3.3(i), it follows that $\gcd(2^{t/2} + 1, (q + 1)(q - 1)) = \gcd(2^{t/2} + 1, q + 1) = 2^{t/2} + 1$. So $(q - 1) \cdot \frac{(q+1)}{2^{t/2}+1} \mid l$. But then $q - 1 \mid l$, a contradiction to the definition of ${}_QI_6$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_QI_6}(H)| = |U_1| + |U_2| = \begin{cases} 2^t - 1 & \text{if } 2t \mid n, \\ (2^t - 2)/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ (2^{t/2} + 1)(2^t - 2^{t/2} - 2)/2 & \text{if } t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_QI_{11}}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l,m),(k,m,l)\} \in C_{{}_QI_{11}}(H) \mid 2^t k \equiv k,\ 2^t l \equiv l,\ 2^t m \equiv m\} & \text{if } i = 1, \\ \{\{(k,l,m),(k,m,l)\} \in C_{{}_QI_{11}}(H) \mid 2^t k \equiv k,,\ 2^t l \equiv m,\ 2^t m \equiv l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k,l,m),(k,m,l)\} \in {}_QI_{11}$, then $x \in U_1$ if and only if $(2^t-1)k \equiv 0$, $(2^t-1)l \equiv 0$ and $(2^t-1)m \equiv 0 \bmod q+1$.

Suppose $t \mid n$. Then $k \equiv 0$, $l \equiv 0$ and $m \equiv 0 \bmod q+1$. This is equivalent with $q+1 \mid k$, and $l \equiv m \bmod q+1$, a contradiction to the definition of ${}_QI_{11}$. Hence, $U_1 = \emptyset$.

Suppose $t \nmid n$. Then $(2^{t/2}+1)k \equiv 0$, $(2^{t/2}+1)l \equiv 0$ and $(2^{t/2}+1)m \equiv 0 \bmod q+1$. Hence

$$U_1 = \left\{\{(k,l,m),(k,m,l)\} \in {}_QI_{11} \,\middle|\, \frac{q+1}{2^{t/2}+1} \,\middle|\, k,l,m\right\}$$

and $|U_1| = 2^{t/2}(2^{t/2}+1)^2/2$.

Suppose $t \mid n$. If $x = \{(k,l,m),(k,m,l)\} \in {}_QI_{11}$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0$, $2^t l \equiv m$ and $2^t m \equiv l \bmod q+1$. From Lemma 3.3(i) and the last two congruences, we get $q+1|k$ and $(2^{2t}-1)l \equiv 0 \bmod q+1$. By Lemma 3.3(i) and (iii), the second congruence is equivalent with $l \equiv 0 \bmod q+1$ if $2t \mid n$, and $(2^t+1)l \equiv 0 \bmod q+1$ if $2t \nmid n$. So by the definition of ${}_QI_{11}$, we get

$$|U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t/2 & \text{if } 2t \nmid n. \end{cases}$$

Suppose $t \nmid n$. If $x = \{(k,l,m),(k,m,l)\} \in {}_QI_{11}$, then $x \in U_2$ if and only if $(2^t-1)k \equiv 0$, $2^t l \equiv m$ and $2^t m \equiv l \bmod q+1$. From Lemma 3.3(i) and the last two congruences, we $(2^{2t}-1)l \equiv 0 \bmod q+1$. By Lemma 3.3 (iii), the second congruence is equivalent with $(2^t-1)l \equiv 0 \bmod q+1$. Then $l \equiv 2^t l \equiv m \bmod q+1$, a contradiction to the definition of ${}_QI_{11}$. Hence in this case $U_2 = \emptyset$. So

$$|C_{{}_QI_{11}}(H)| = |U_1| + |U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ 2^t/2 & \text{if } t \mid n \text{ and } 2t \nmid n, \\ 2^{t/2}(2^{t/2}+1)^2/2 & \text{if } t \nmid n. \end{cases}$$

Thus,

$$|C_I(H)| = |C_{{}_QI_6}(H)| + |C_{{}_QI_{11}}(H)| = \begin{cases} 2^t - 1 & \text{if } t \mid n, \\ (2^{t/2}+1)(2^t-1) & \text{if } t \nmid n. \end{cases} \qquad \square$$

Next, we deal with the regular semisimple irreducible characters of G.

Proposition 4.3. *Let* $G = \mathrm{Sp}_4(2^n)$, $t \mid n$, $I(2^n) := {}_GI_1(2^n) \cup {}_GI_2(2^n) \cup {}_GI_3(2^n) \cup {}_GI_4(2^n) \cup {}_GI_5(2^n)$ *and* $H = \langle \alpha^t \rangle$ *a subgroup of O. Then*

$$|C_I(H)| = (2^t - 1)^2.$$

Proof. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$, and let $\overline{T}$ be the maximal torus of $\overline{G} = \mathrm{Sp}_4(\mathbb{F})$ and $\overline{W} = N_{\overline{G}}(\overline{T})/\overline{T}$ the Weyl group.

Let $\chi = \chi(x)$ be an irreducible character of $\mathrm{Sp}_4(2^n)$ labeled by the parameter x given in [13]. Then $\chi(x)^\alpha = \chi(2x)$. (Using the character values on the classes listed in the last column of Tables A.3, we know that the action of α on the parameter sets is given by $x^\alpha = 2x$.) In addition, let (s, μ) be the semisimple and unipotent labels of $\chi(x)$. Then $x \in I(2^n)$ if and only if $(s, \mu) = (s, 1)$ with s regular, so that $C_{\overline{G}}(s) = \overline{T}$. Here and in the following, we identify $\mathrm{Sp}_4(2^n)$ with its dual group since these groups are isomorphic to each other. Thus $\chi^{\alpha^t} = \chi$ if and only if $(s)_G^{\alpha^t} = (s)_G$; namely, $s^{\alpha^t} = s^w$ for some $w \in \overline{W}$, where $(s)_G$ is the conjugacy class of G containing s. Thus $\chi^{\alpha^t} = \chi$ if and only if $s \in C_{\overline{T}}(\alpha^t w^{-1})$; namely, s is a regular element of $\mathrm{Sp}_4(2^t)$, since $C_{\overline{T}}(\alpha^t w^{-1})$ is a maximal torus of $\mathrm{Sp}_4(2^t)$. But a regular element s of $\mathrm{Sp}_4(2^t)$ labels an irreducible character $\psi = \psi_{s,1}$ of $\mathrm{Sp}_4(2^t)$ such that its parameter y (see [13]) lies in $I(2^t)$. It follows that

$$C_{I(2^n)}(H) \simeq I(2^t)$$

as H-sets, and $|C_{I(2^n)}(H)| = |I(2^t)| = (2^t - 1)^2$. □

Proposition 4.4. *Let* $G = \mathrm{GU}_4(q^2)$, $t \mid 2n$, $I := {}_GI_2 \cup {}_GI_5 \cup {}_GI_6 \cup {}_GI_9 \cup {}_GI_{10}$ *and* $H = \langle \alpha^t \rangle$ *a subgroup of O. Then*:

(a) $|C_I(H)|$ *is equal to the number of those regular semisimple conjugacy classes of* G *which are stabilized by* α^t;
(b) *If* $t \nmid n$ (*resp.*, $t \mid n$), *then* $|C_I(H)|$ *is equal to the number of regular semisimple conjugacy classes of* $\mathrm{GU}_4(2^t)$ (*resp.*, $\mathrm{Sp}_4(2^t)$);
(c) $|C_I(H)| = (2^t - 1)^2$.

Proof. We use an argument similar to the one in [2, Proposition 4.2]. The set I parameterizes the regular semisimple irreducible characters of G. We fix some notation. Let $\mathbb{F}_{q^2}$ be a finite field with q^2 elements, $\mathbb{F}$ an algebraic closure of $\mathbb{F}_{q^2}$ and $\overline{G} = \mathrm{GL}_4(\mathbb{F})$. Let $\gamma : \overline{G} \to \overline{G}$ be a graph automorphism of order 2 and a field automorphism $\overline{\alpha} : \overline{G} \to \overline{G}$ obtained from the map $\mathbb{F} \to \mathbb{F}$, $x \mapsto x^2$. Setting $F := \overline{\alpha}^n \circ \gamma = \gamma \circ \overline{\alpha}^n$ we get $G = \mathrm{GU}_4(q^2) = \overline{G}^F = \mathrm{GL}_4(q^2)^F = \{g \in \overline{G} \mid F(g) = g\}$. Since the restriction $\overline{\alpha}|_G : G \to G$ of $\overline{\alpha}$ to G generates $\mathrm{Out}(G)$, we can assume $\overline{\alpha}|_G = \alpha$.

For $L \in \{G, \overline{G}\}$, let $\mathcal{S}_{reg}(L)$ be the set of all regular semisimple conjugacy classes of L. If ρ is an endomorphism of L, then let $\mathcal{S}_{reg}(L)^\rho := \{C \in \mathcal{S}_{reg}(L) \mid C^\rho = C\}$ be the set of ρ-stable regular semisimple conjugacy classes of L. Finally, let $\mathrm{Irr}^{ss}_{reg}(G)$ be the set of regular semisimple irreducible characters of G.

By [4, Corollary 3.10, p. 197] of Springer–Steinberg, the map $C \mapsto C \cap \overline{G}^F$ is a bijection from $\mathcal{S}_{reg}(\overline{G})^F$ onto $\mathcal{S}_{reg}(G)$ and this bijection induces a bijection between the set of regular semisimple conjugacy classes of G fixed by α^t and the set of F-stable regular semisimple conjugacy classes of $\overline{G}$ fixed by $\overline{\alpha}^t$. It follows that, since $\overline{\alpha}^t$ raises every element of a maximally split torus of $\overline{G}$ to its 2^t-power, the automorphism α^t maps each regular semisimple conjugacy class $(g)_G$ of G to the class $(g^{2^t})_G$. In other words, α^t acts on the regular semisimple conjugacy classes of G like the 2^tth power map (this does not mean, that α^t maps every regular semisimple element of G to its 2^tth power).

(a) Since $G = \mathrm{GU}_4(q^2)$ is isomorphic to its dual group (in the sense of [6, Section 4.4, p. 120]), the number $|\mathcal{S}_{reg}(G)^{\alpha^t}|$ of fixed points of α^t on $\mathcal{S}_{reg}(G)$ is equal to the number of fixed points of α^t on $\mathrm{Irr}^{ss}_{reg}(G)$. By definition, the latter equals $|C_I(H)|$.

(b) In this part of the proof, we imitate an argument which is used in the proof of [3, Lemma 4.1]. As we have seen at the beginning of this proof, there is a bijection from the set of regular semisimple conjugacy classes of G fixed by α^t onto $\mathcal{S}_{reg}(\overline{G})^{\langle F, \overline{\alpha}^t\rangle}$, the set of fixed points of $\mathcal{S}_{reg}(\overline{G})$ under the action of the group $\langle F, \overline{\alpha}^t\rangle$. So by (a), we have $|C_I(H)| = |\mathcal{S}_{reg}(G)^{\alpha^t}| = |\mathcal{S}_{reg}(\overline{G})^{\langle F, \overline{\alpha}^t\rangle}|$.

Case 1. Suppose $t \nmid n$, then $\langle F, \overline{\alpha}^t\rangle = \langle \overline{\alpha}^n \circ \gamma, \overline{\alpha}^t\rangle = \langle \overline{\alpha}^{t/2} \circ \gamma, \overline{\alpha}^t\rangle = \langle \overline{\alpha}^{t/2} \circ \gamma\rangle$. Thus $|C_I(H)| = |\mathcal{S}_{reg}(\overline{G})^{\overline{\alpha}^{t/2}\circ\gamma}| = |\mathcal{S}_{reg}(\overline{G}^{\overline{\alpha}^{t/2}\circ\gamma})|$. Since $\overline{G}^{\overline{\alpha}^{t/2}\circ\gamma} \cong \mathrm{GU}_4(2^t)$, we get $|C_I(H)| = |\mathcal{S}_{reg}(\mathrm{GU}_4(2^t))|$, proving (b) in this case.

Case 2. Suppose $t \mid n$, then $\langle F, \overline{\alpha}^t\rangle = \langle \overline{\alpha}^n \circ \gamma, \overline{\alpha}^t\rangle = \langle \gamma, \overline{\alpha}^t\rangle = \langle \overline{\alpha}^t \circ \gamma, \overline{\alpha}^t\rangle$. Thus $|C_I(H)| = |\mathcal{S}_{reg}(\overline{G})^{\langle F, \overline{\alpha}^t\rangle}| = |\mathcal{S}_{reg}(\overline{G})^{\overline{\alpha}^t\circ\gamma} \cap \mathcal{S}_{reg}(\overline{G})^{\overline{\alpha}^t}| = |\mathcal{S}_{reg}(\overline{G}^{\overline{\alpha}^t\circ\gamma}) \cap \mathcal{S}_{reg}(\overline{G}^{\overline{\alpha}^t})|$. But

$$(\overline{G})^{\overline{\alpha}^t\circ\gamma} \cap \overline{G}^{\overline{\alpha}^t} = \mathrm{GU}_4(2^{2t}) \cap \mathrm{GL}_4(2^t).$$

By [16, Table 3.5.B], $\mathrm{GU}_4(2^{2t})$ has a maximal subgroup isomorphic to $\mathrm{Sp}_4(2^t)$ and as shown in the proof of [16, Proposition 4.5.6] this maximal subgroup is contained in $\mathrm{GL}_4(2^t)$. Thus $\mathrm{GU}_4(2^{2t}) \cap \mathrm{GL}_4(2^t) \cong \mathrm{Sp}_4(2^t)$ and $|C_I(H)| = |\mathcal{S}_{reg}(\mathrm{Sp}_4(2^t))|$.

(c) From the character tables of $\mathrm{GU}_4(2^t)$ and $\mathrm{Sp}_4(2^t)$ in [20] and the CHEVIE library we get that if $t \nmid n$, then the number of regular semisimple conjugacy classes of $\mathrm{GU}_4(2^t)$ is equal to $(2^t - 1)^2$. Similarly, if $t \mid n$, then the number of regular semisimple conjugacy classes of $\mathrm{Sp}_4(2^t)$ is equal to $(2^t - 1)^2$. □

5. DADE'S INVARIANT CONJECTURE FOR $\mathrm{Sp}_4(2^n)$

In this section, we prove Dade's invariant conjecture for $G = \mathrm{Sp}_4(2^n)$ in the defining characteristic. By [14, p. 152], G has only two 2-blocks, the principal block $B_0 = B_0(G)$ and one defect-0-block (consisting of the Steinberg character). Hence we have to verify Dade's conjecture only for B_0.

As in the previous section, let $O = \mathrm{Out}(G) = \langle\beta\rangle$, and α is the field automorphism of G with order n, where $\alpha = \beta^2$. Fix a Borel subgroup B and maximal parabolic subgroups P and Q of G containing B as in [13]. In particular, we may assume that β stabilizes B, β permutes P and Q, so α stabilizes P and Q.

According to a corollary of the Borel–Tits theorem [5], the normalizers of radical 2-subgroups are parabolic subgroups. The radical 2-chains of G (up to G-conjugacy) are given in Table 1, where $A = \mathrm{Aut}(G) = G \rtimes \langle\beta\rangle$.

Table 1 Radical 2-chains of G

C		$N_G(C)$	$N_A(C)$
C_1	$\{1\}$	G	A
C_2	$\{1\} < O_2(P)$	P	$P \rtimes \langle\alpha\rangle$
C_3	$\{1\} < O_2(P) < O_2(B)$	B	$B \rtimes \langle\alpha\rangle$
C_4	$\{1\} < O_2(Q)$	Q	$Q \rtimes \langle\alpha\rangle$
C_5	$\{1\} < O_2(Q) < O_2(B)$	B	$B \rtimes \langle\alpha\rangle$
C_6	$\{1\} < O_2(B)$	B	$B \rtimes \langle\beta\rangle$

Thus, Dade's invariant conjecture for G is equivalent to

$$\sum_{i\in\{1,3,5\}} \mathrm{k}(N_G(C(i)), B_0, d, u) = \sum_{i\in\{2,4,6\}} \mathrm{k}(N_G(C(i)), B_0, d, u) \qquad (5.1)$$

for all $d \in \mathbb{N}$ and $u \mid 2n$.

Theorem 5.1. *Let n be a positive integer and $\widetilde{B}$ a 2-block of $G = \mathrm{Sp}_4(2^n)$ with positive defect. Then $\widetilde{B}$ satisfies Dade's invariant conjecture.*

Proof. By the proceeding remarks, we can suppose $\widetilde{B} = B_0$ and prove (5.1). Suppose $u \mid 2n$ and set $t := \frac{2n}{u}$ and $H := \langle\alpha^t\rangle$. Let $S \in \{G, B, P, Q\}$. By the character tables in [13], we have $\mathrm{k}(S, B_0, d, u) = 0$ when $d \notin \{3n, 3n+1, 4n\}$.

Case 1. Suppose $u \mid n$. Since a subgroup of order u in O is also a subgroup of $\langle\alpha\rangle$, it follows that (5.1) is equivalent to

$$\mathrm{k}(G, B_0, d, u) + \mathrm{k}(B, B_0, d, u) = \mathrm{k}(P, B_0, d, u) + \mathrm{k}(Q, B_0, d, u). \qquad (5.2)$$

(i) If $d = 3n$, then Table A.2 implies, that (5.2) is equivalent to

$$\sum_{j\in J_G} |C_{{}_G I_j}(H)| + \sum_{j\in J_B} |C_{{}_B I_j}(H)| = \sum_{j\in J_P} |C_{{}_P I_j}(H)| + \sum_{j\in J_Q} |C_{{}_Q I_j}(H)|$$

with the index sets $J_G := \{10, 11, 12, 13\}$, $J_B := \{4, 5\}$, $J_P := \{2, 5, 6\}$ and $J_Q := \{2, 5, 6\}$. By Tables A.2 and A.5, we have

$$\mathrm{k}(G, B_0, d, u) + \mathrm{k}(B, B_0, d, u) = \sum_{j\in J_G} |C_{{}_G I_j}(H)| + \sum_{j\in J_B} |C_{{}_B I_j}(H)| = 4(2^t - 1)$$

and

$$\mathrm{k}(P, B_0, d, u) + \mathrm{k}(Q, B_0, d, u) = \sum_{j\in J_P} |C_{{}_P I_j}(H)| + \sum_{j\in J_Q} |C_{{}_Q I_j}(H)| = 4(2^t - 1).$$

Thus (5.2) holds in this case.

(ii) If $d = 3n + 1$, then Table A.3 implies, that (5.2) is equivalent to

$$\sum_{j\in J_G} |C_{{}_G I_j}(H)| + \sum_{j\in J_B} |C_{{}_B I_j}(H)| = \sum_{j\in J_P} |C_{{}_P I_j}(H)| + \sum_{j\in J_Q} |C_{{}_Q I_j}(H)|$$

with the index sets $J_G := \{15, 16, 17, 19\}$, $J_B := \{7, 8\}$, $J_P := \{9, 10\}$ and $J_Q := \{9, 10\}$. By Tables A.3 and A.5, we have

$$\mathrm{k}(G, B_0, d, u) + \mathrm{k}(B, B_0, d, u) = \sum_{j\in J_G} |C_{{}_G I_j}(H)| + \sum_{j\in J_B} |C_{{}_B I_j}(H)| = 8$$

and

$$\mathrm{k}(P, B_0, d, u) + \mathrm{k}(Q, B_0, d, u) = \sum_{j\in J_P} |C_{{}_P I_j}(H)| + \sum_{j\in J_Q} |C_{{}_Q I_j}(H)| = 8.$$

Thus (5.2) holds in this case.

(iii) If $d = 4n$, then Table A.4 implies, that (5.2) is equivalent to

$$\sum_{j\in J_G} |C_{{}_G I_j}(H)| + \sum_{j\in J_B} |C_{{}_B I_j}(H)| = \sum_{j\in J_P} |C_{{}_P I_j}(H)| + \sum_{j\in J_Q} |C_{{}_Q I_j}(H)|$$

with the index sets $J_G := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 14\}$, $J_B := \{1, 2, 3, 6\}$, $J_P := \{1, 3, 4, 7, 8\}$ and $J_Q := \{1, 3, 4, 7, 8\}$. By Tables A.4 and A.5, we have

$$\mathrm{k}(G, B_0, d, u) + \mathrm{k}(B, B_0, d, u) = \sum_{j\in J_G} |C_{{}_G I_j}(H)| + \sum_{j\in J_B} |C_{{}_B I_j}(H)| = 2^{2t+1}$$

and

$$\mathrm{k}(P, B_0, d, u) + \mathrm{k}(Q, B_0, d, u) = \sum_{j\in J_P} |C_{{}_P I_j}(H)| + \sum_{j\in J_Q} |C_{{}_Q I_j}(H)| = 2^{2t+1}.$$

Thus (5.2) also holds in this case.

Case 2. Suppose $u \nmid n$, so that $t := \frac{2n}{u}$ is odd. Equation (5.1) is now equivalent to

$$\mathrm{k}(G, B_0, d, u) = \mathrm{k}(B, B_0, d, u) \tag{5.3}$$

for any $u|2n$ with $t = \frac{2n}{u}$ odd.

(i) Suppose $d = 3n$. Let $\chi(k) = {}_B\chi_4(k)$, $\chi'(k') = {}_B\chi_5(k') \in \mathrm{Irr}(B, B_0, d)$, where $k \in {}_BI_4$, $k' \in {}_BI_5$. Since $A_2^{\beta} = A_{31}$ as B-classes, it follows by the values of $\chi(k)$ and $\chi'(k')$ on A_2 and A_{31} that β swaps the two families $\{\chi(k)\}$ and $\{\chi'(k')\}$. Since α stabilizes each family $\{\chi(k)\}$ and $\{\chi'(k')\}$ and since $\beta^t = \beta\alpha^{\frac{t-1}{2}}$, it follows that β^t swaps $\{\chi(x)\}$ and $\{\chi'(x)\}$. In particular,

$$\chi(k), \chi'(k') \notin C_{\mathrm{Irr}(B,B_0,d)}(\beta^t) \tag{5.4}$$

for $k \in {}_BI_4$, $k' \in {}_BI_5$. Thus, β^t does not fix any character in $\{{}_B\chi_4(k), {}_B\chi_5(k)\}$, and so $C_{\mathrm{Irr}(B,B_0,3n)}(\beta^t) = \emptyset$.

Since β swaps the G-classes A_2 and A_{31}, it follows from the values of characters of $\mathrm{Irr}(G, B_0, d)$ on A_2 and A_{31} that β acts on $\mathrm{Irr}(G, B_0, d)$ as

$$(\chi_{10}(k), \chi_{11}(k'))(\chi_{12}(k), \chi_{13}(k')).$$

Thus, β^t does not fix any character in $\{\chi_{10}(k), \chi_{11}(k), \chi_{12}(k), \chi_{13}(k)\}$, and so

$$C_{\mathrm{Irr}(G,B_0,3n)}(\beta^t) = \emptyset.$$

Thus (5.3) holds in this case.

(ii) Suppose $d = 3n + 1$. The values of characters of $\mathrm{Irr}(G, B_0, 3n + 1)$ on the G-classes A_2 and A_{31} imply that β acts on $\mathrm{Irr}(G, B_0, 3n + 1)$ as

$$(\theta_1)(\theta_2, \theta_3)(\theta_5).$$

Note that there is a typo in the character table of G in [13]: the degrees of θ_1, θ_2 have to be swapped. So $C_{\mathrm{Irr}(G,B_0,3n+1)}(\beta^t) = \{\theta_1, \theta_5\}$.

Similarly, the values of characters of $\mathrm{Irr}(B, B_0, 3n + 1)$ on the B-classes A_{41} and A_{51} imply that β acts on $\mathrm{Irr}(B, B_0, 3n + 1)$ as

$$({}_B\theta_2(0))({}_B\theta_2(1), {}_B\theta_3(0))({}_B\theta_3(1)).$$

It follows that

$$C_{\mathrm{Irr}(B,B_0,3n+1)}(\beta^t) = \{{}_B\theta_2(0), {}_B\theta_3(1)\}.$$

Thus (5.3) holds in this case.

(iii) Suppose $d = 4n$. Using the degrees or values of characters on the G-classes A_2, A_{31} and B-classes A_{41}, A_{51}, we get the action of β on $\mathrm{Irr}(G, B_0, d)$ by

$$(\chi_1(i, j), \chi_1(i', j'))(\chi_2(a), \chi_2(a'))(\chi_3(k, l), \chi_3(k', l'))(\chi_4(m, n), \chi_4(m', n'))$$
$$(\chi_5(b), \chi_5(b'))(\chi_6(c), \chi_7(c'))(\chi_8(d), \chi_9(d'))(\theta_0)$$

and on $\mathrm{Irr}(B, B_0, d)$ by $({}_B\chi_1(k, l), {}_B\chi_1(k', l'))({}_B\chi_2(i), {}_B\chi_3(i'))({}_B\theta_1)$. In particular,

$$\theta_0 \in C_{\mathrm{Irr}(G,B_0,4n)}(\beta^t), \quad {}_B\theta_1 \in C_{\mathrm{Irr}(B,B_0,4n)}(\beta^t), \tag{5.5}$$

$\chi_6(k), \chi_7(k), \chi_8(k), \chi_9(k) \notin C_{\mathrm{Irr}(G,B_0,4n)}(\beta^t)$ and $\chi_2(k), \chi_3(k) \notin C_{\mathrm{Irr}(B,B_0,4n)}(\beta^t)$ for any integer k. Let

$$\Omega_G = \{\chi_1(k, l), \chi_2(k), \chi_3(k, l), \chi_4(k, l), \chi_5(k) \mid k, l \in \mathbb{N}\}$$

be a subset of $\mathrm{Irr}(G, B_0, 4n)$, and let $\chi \in \Omega_G$. Then $\chi = \chi_{s,1}$ and s is regular, where s and 1 are the semisimple and unipotent labels of χ, respectively. Thus $\chi \in C_{\Omega_G}(\beta^t)$

if and only if $(s)_G^{\beta^t} = (s)_G$, which is equivalent to $s \in C_{\overline{T}}(\beta^t w)$, where $w \in \overline{W}$ with $\overline{W}$ and $\overline{T}$ given in the proof of Lemma 4.3 and $s \in \overline{T}$. Since $\beta^{2t} = \alpha^t$ and β^t acts as an involution on $C_G(\alpha^t) = \mathrm{Sp}_4(2^t)$, it follows that $\chi \in C_{\Omega_G}(\beta^t)$ if and only if s can be regarded as a regular element of the Suzuki group ${}^2B_2(2^t)$. Moreover, if regular elements s and s' of ${}^2B_2(2^t)$ are ${}^2B_2(2^t)$-conjugate, then they are G-conjugate. Conversely, if $(s)_G = (s')_G$ for some regular elements s, s' of G and $(s)_G^{\beta^t} = (s)_G$, then we may suppose $s, s' \in {}^2B_2(2^t)$ and so $(s)_{{}^2B_2(2^t)} = (s')_{{}^2B_2(2^t)}$. By the character table of ${}^2B_2(2^t)$ in the CHEVIE library, the number of regular semisimple conjugacy classes of ${}^2B_2(2^t)$ is $2^t - 1$. Thus $|C_{\Omega_G}(\beta^t)| = 2^t - 1$, and it follows by (5.5) that $|C_{\mathrm{Irr}(G,B_0,4n)}(\beta^t)| = 2^t$.

Similarly, let $\Omega_B = \{{}_B\chi_1(k,l) \mid (k,l) \in {}_BI_1\}$ and $\chi \in \Omega_B$. Since $O_2(B)$ is a subgroup of the kernel of each $\chi \in \Omega_B$, we may suppose $\Omega_B = \mathrm{Irr}(T)$, where $T = B/O_2(B) \simeq \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$. Now β^t acts on both T and $\mathrm{Irr}(T)$. By Brauer's permutation Lemma, [21, Chapter 3, Lemma 2.19], $|C_{\mathrm{Irr}(T)}(\beta^t)| = |C_T(\beta^t)|$. But $C_T(\beta^t) \simeq \mathbb{Z}_{2^t-1}$ and by (5.5), it follows that $C_{\mathrm{Irr}(B,B_0,4n)}(\beta^t) = 2^t$. Thus (5.3) holds in this case. This completes the proof the Theorem 5.1. □

6. DADE'S PROJECTIVE INVARIANT CONJECTURE FOR $2.\mathrm{Sp}_4(2)$

Let C be a radical 2-chain of $\mathrm{Sp}_4(2)$. The character table of $N_{2.\mathrm{Sp}_4(2)}(C)$ and $N_{2.\mathrm{Sp}_4(2).2}(C)$ can easily be computed using GAP [11] or MAGMA [17].

Let $H \in \{N_{2.\mathrm{Sp}_4(2)}(C), N_{2.\mathrm{Sp}_4(2).2}(C)\}$, and let ξ be a linear character of $Z := Z(2.\mathrm{Sp}_4(2))$. Denote by $\mathrm{Irr}(H \mid \xi)$ the subset of $\mathrm{Irr}(H)$ of all characters covering ξ. Then $\mathrm{Irr}(H) = \mathrm{Irr}(H \mid 1) \cup \mathrm{Irr}(H \mid \rho)$, where $\rho \in \mathrm{Irr}(Z)\backslash\{1\}$. But since we can identify $\mathrm{Irr}(H \mid 1)$ with $\mathrm{Irr}(H/Z)$ and have $H/Z \in \{N_{\mathrm{Sp}_4(2)}(C), N_{\mathrm{Sp}_4(2).2}(C)\}$, we see that Dade's projective invariant conjecture for $2.\mathrm{Sp}_4(2)$ is equivalent with Dade's invariant conjecture for $\mathrm{Sp}_4(2)$ and $2.\mathrm{Sp}_4(2)$. By Theorem 5.1, Dade's invariant conjecture holds for $\mathrm{Sp}_4(2)$, so it suffices to show that the conjecture holds for $2.\mathrm{Sp}_4(2)$.

Theorem 6.1. *Let $\widetilde{B}$ be a 2-block of $G = 2.\mathrm{Sp}_4(2)$ with defect group $D(\widetilde{B}) \neq O_2(G)$. Then $\widetilde{B}$ satisfies Dade's projective invariant conjecture.*

Proof. By the remarks in Section 5, we can assume $\widetilde{B} = B_0(2.\mathrm{Sp}_4(2))$. Let $N(C) = N_G(C)$ for $C \in \mathcal{R}(\mathrm{Sp}_4(2))$ and $E = G.2 = 2.\mathrm{Sp}_4(2).2$. Set $\mathrm{k}(j,d,u) := \mathrm{k}(N(C_j), \widetilde{B}, d, u)$ for $j = 1, \ldots, 6$. The values $\mathrm{k}(j,d,u)$ in Table 2 can be derived from the Tables in Appendix B.

Hence $\sum_{i=1}^{6} (-1)^{|C_j|}\mathrm{k}(N(C_j), \widetilde{B}, d, u) = 0$ and the theorem follows. □

7. DADE'S INVARIANT CONJECTURE FOR $\mathrm{SU}_4(2^{2n})$

In this section, we prove Dade's invariant conjecture for $K = \mathrm{SU}_4(2^{2n})$ in the defining characteristic. By [14, p. 152], K has only two 2-blocks, the principal block $B_0(K)$ and one defect-0-block (consisting of the Steinberg character). Hence we have to verify Dade's conjecture only for $B_0(K)$.

Let $G = \mathrm{GU}_4(2^{2n})$ and $O = \mathrm{Out}(G) = \langle\alpha\rangle$, where α is a field automorphism of G of order $2n$. Fix a Borel subgroup B and maximal parabolic subgroups P and Q

Table 2 Values of $\mathrm{k}(j, d, u)$

Defect d Value u	5 2	5 1	4 2	4 1	Otherwise Otherwise
$\mathrm{k}(1, d, u)$	4	12	2	2	0
$\mathrm{k}(2, d, u)$	0	16	0	4	0
$\mathrm{k}(3, d, u)$	0	16	0	4	0
$\mathrm{k}(4, d, u)$	0	16	0	4	0
$\mathrm{k}(5, d, u)$	0	16	0	4	0
$\mathrm{k}(6, d, u)$	4	12	2	2	0

of G containing B as in [20]. In particular, we may assume that α stabilizes B, P, and Q.

Suppose $G = \mathrm{GU}_4(2^{2n})$ and $L \in \{G, P, Q, B\}$. Then

$$L = \mathbb{Z}_{2^n+1} \times L_S$$

for a unique parabolic subgroup L_S of $\mathrm{SU}_4(2^{2n})$. Let $\widehat{B} \in \mathrm{Blk}(G)$ with $D(\widehat{B}) \neq 1$, so that $\widehat{B} = B_\rho \times B_0$, where $B_0 = B_0(\mathrm{SU}_4(2^{2n}))$ and B_ρ is a block of $\mathbb{Z}_{2^n+1}$ such that $\mathrm{Irr}(B_\rho) = \{\rho\}$ for some $\rho \in \mathrm{Irr}(\mathbb{Z}_{2^n+1})$. If, moreover, $\widehat{B}_L \in \mathrm{Blk}(L)$ such that $\widehat{B}_L^G = \widehat{B}$, then $\widehat{B}_L = B_\rho \times B_0(L_S)$. Let $\chi \in \mathrm{Irr}(\widehat{B}_L)$ and $\sigma \in O$, so that $\chi = \rho \times \chi_2$ for some $\chi_2 \in \mathrm{Irr}(B_0(L_S))$ and $\mathrm{d}(\chi) = \mathrm{d}(\chi_2)$. Thus $\chi^\sigma = \rho^\sigma \times \chi_2^\sigma$ and so

$$\chi^\sigma = \chi \quad \text{if and only if } \rho^\sigma = \rho \quad \text{and} \quad \chi_2^\sigma = \chi_2.$$

It follows that for any d and $\widehat{U} \leq \mathrm{Out}(\mathrm{GU}_4(2^{2n})) = \mathrm{Out}(\mathrm{SU}_4(2^{2n}))$

$$\mathrm{k}(L, \widehat{B}, d, \widehat{U}, \rho) = \mathrm{k}(L_S, B_0, d, \widehat{U}) \quad \text{or} \quad 0 \tag{7.1}$$

according as $\rho \in C_{\mathrm{Irr}(\mathbb{Z}_{2^n+1})}(\widehat{U})$ or $\rho \notin C_{\mathrm{Irr}(\mathbb{Z}_{2^n+1})}(\widehat{U})$.

Let $\widetilde{B}_0$ be the union of 2-blocks $\widehat{B} \in \mathrm{Blk}(\mathrm{GU}_4(2^{2n}))$ such that $D(\widehat{B}) \neq 1$, and $t_{\widehat{U}} = |C_{\mathrm{Irr}(\mathbb{Z}_{q^n+1})}(\widehat{U})|$. Then

$$\mathrm{k}(L, \widetilde{B}_0, d, \widehat{U}) = t_{\widehat{U}}\mathrm{k}(L_S, B_0, d, \widehat{U}). \tag{7.2}$$

Since $t_{\widehat{U}}$ is independent of L, it follows that the invariant conjecture for both $\mathrm{SU}_4(2^{2n})$ and $\mathrm{GU}_4(2^{2n})$ is equivalent to

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\mathrm{k}(N_G(C), \widetilde{B}_0, d, \widehat{U}) = 0. \tag{7.3}$$

By a corollary of the Borel–Tits theorem [5], the normalizers of radical 2-subgroups are parabolic subgroups. The radical 2-chains of G (up to G-conjugacy) are given in Table 3.

Since C_5 and C_6 have the same normalizers $N_G(C_5) = N_G(C_6) = B$ and $N_A(C_5) = N_A(C_6) = B \rtimes O$, it follows that for all $d, u \in \mathbb{N}$ and $\widetilde{B} \in \mathrm{Blk}(G)$

$$\mathrm{k}(N_G(C_5), \widetilde{B}_0, d, u) = \mathrm{k}(N_G(C_6), \widetilde{B}_0, d, u).$$

Table 3 Radical 2-chains of G

C		$N_G(C)$	$N_A(C)$
C_1	$\{1\}$	G	A
C_2	$\{1\} < O_2(P)$	P	$P \rtimes \langle\alpha\rangle$
C_3	$\{1\} < O_2(P) < O_2(B)$	B	$B \rtimes \langle\alpha\rangle$
C_4	$\{1\} < O_2(Q)$	Q	$Q \rtimes \langle\alpha\rangle$
C_5	$\{1\} < O_2(Q) < O_2(B)$	B	$B \rtimes \langle\alpha\rangle$
C_6	$\{1\} < O_2(B)$	B	$B \rtimes \langle\alpha\rangle$

Thus the contribution of C_5 and C_6 in the alternating sum of Dade's invariant conjecture is zero. It follows by (7.3) that Dade's invariant conjecture for both G and K is equivalent to

$$\mathrm{k}(G, \widetilde{B}_0, d, u) + \mathrm{k}(B, \widetilde{B}_0, d, u) = \mathrm{k}(P, \widetilde{B}_0, d, u) + \mathrm{k}(Q, \widetilde{B}_0, d, u) \tag{7.4}$$

for all $d \in \mathbb{N}$ and $u \mid 2n$.

Theorem 7.1. *Let n be a positive integer and $\widetilde{B}$ a 2-block of $\mathrm{SU}_4(2^{2n})$ with positive defect. Then $\widetilde{B}$ satisfies Dade's invariant conjecture.*

Proof. By the proceeding remarks, we can assume $\widetilde{B} = \widetilde{B}_0$ and $G = \mathrm{GU}_4(q^{2n})$. Suppose $u \mid 2n$ and set $t := \frac{2n}{u}$ and $H := \langle\alpha^t\rangle$. Let $S \in \{G, B, P, Q\}$. By the character tables in [20], we have $\mathrm{k}(S, \widetilde{B}_0, d, u) = 0$ when $d \notin \{3n, 4n, 5n, 6n\}$.

(i) If $d = 3n$, then we have

$$\mathrm{k}(G, \widetilde{B}_0, d, u) = \sum_{j \in \{13,18\}} |C_{_G I_j}(H)| = \begin{cases} 1 & \text{if } t \mid n, \\ (2^{t/2}+1)^2 & \text{if } t \nmid n, \end{cases}$$

and

$$\mathrm{k}(Q, \widetilde{B}_0, d, u) = |C_{_Q I_9}(H)| = \begin{cases} 1 & \text{if } t \mid n, \\ (2^{t/2}+1)^2 & \text{if } t \nmid n, \end{cases}$$

by Tables A.7 and A.11. Thus (7.4) holds in this case.

(ii) If $d = 4n$, then Table A.8 implies, that (7.4) is equivalent to

$$\sum_{j \in J_G} |C_{_G I_j}(H)| + \sum_{j \in J_B} |C_{_B I_j}(H)| = \sum_{j \in J_P} |C_{_P I_j}(H)| + \sum_{j \in J_Q} |C_{_Q I_j}(H)|$$

with the index sets $J_G := \{4, 12, 20\}$, $J_B := \{5, 7\}$, $J_P := \{2, 10\}$ and $J_Q := \{6, 10, 11\}$. By Tables A.8 and A.11, we have

$$\sum_{j \in J_G} |C_{_G I_j}(H)| + \sum_{j \in J_B} |C_{_B I_j}(H)| = \begin{cases} 2^{t+1} & \text{if } t \mid n, \\ 2^{t/2}(2^{t/2}+1)(2^{t/2}+2) & \text{if } t \nmid n, \end{cases}$$

and

$$\sum_{j\in J_P}|C_{_PI_j}(H)|+\sum_{j\in J_Q}|C_{_QI_j}(H)|=\begin{cases}2^{t+1} & \text{if } t\mid n,\\ 2^{t/2}(2^{t/2}+1)(2^{t/2}+2) & \text{if } t\nmid n.\end{cases}$$

Thus (7.4) also holds in this case.

(iii) If $d=5n$, then Table A.9 implies that (7.4) is equivalent to

$$\sum_{j\in J_G}|C_{_GI_j}(H)|+\sum_{j\in J_B}|C_{_BI_j}(H)|=\sum_{j\in J_P}|C_{_PI_j}(H)|+\sum_{j\in J_Q}|C_{_QI_j}(H)|$$

with the index sets $J_G:=\{8,14,15,17,21\}$, $J_B:=\{6,8\}$, $J_P:=\{6,7,9\}$ and $J_Q:=\{2,7\}$. By Tables A.9 and A.11, we have

$$\sum_{j\in J_G}|C_{_GI_j}(H)|+\sum_{j\in J_B}|C_{_BI_j}(H)|=\begin{cases}2^{t+1} & \text{if } t\mid n,\\ 2^{t/2}(2^{t/2}+1)(2^{t/2}+2) & \text{if } t\nmid n,\end{cases}$$

and

$$\sum_{j\in J_P}|C_{_PI_j}(H)|+\sum_{j\in J_Q}|C_{_QI_j}(H)|=\begin{cases}2^{t+1} & \text{if } t\mid n,\\ 2^{t/2}(2^{t/2}+1)(2^{t/2}+2) & \text{if } t\nmid n.\end{cases}$$

Thus (7.4) also holds in this case.

(iv) If $d=6n$, then Table A.10 implies, that (7.4) is equivalent to

$$\sum_{j\in J_G}|C_{_GI_j}(H)|+\sum_{j\in J_B}|C_{_BI_j}(H)|=\sum_{j\in J_P}|C_{_PI_j}(H)|+\sum_{j\in J_Q}|C_{_QI_j}(H)|$$

with the index sets $J_G:=\{1,2,3,5,6,7,9,10,16,19,22\}$, $J_B:=\{1,2,3,4\}$, $J_P:=\{1,3,4,5,8\}$ and $J_Q:=\{1,3,4,5,8\}$. By Tables A.10 and A.11, we have

$$\sum_{j\in J_G}|C_{_GI_j}(H)|+\sum_{j\in J_B}|C_{_BI_j}(H)|=\begin{cases}2^{2t+1} & \text{if } t\mid n,\\ 2\cdot 2^{3t/2}(2^{t/2}+1) & \text{if } t\nmid n,\end{cases}$$

and

$$\sum_{j\in J_P}|C_{_PI_j}(H)|+\sum_{j\in J_Q}|C_{_QI_j}(H)|=\begin{cases}2^{2t+1} & \text{if } t\mid n,\\ 2\cdot 2^{3t/2}(2^{t/2}+1) & \text{if } t\nmid n.\end{cases}$$

Thus (7.4) also holds in this case. □

Note that it follows by (7.1) and Theorem 7.1 that Dade's projective invariant conjecture also holds for any block $\widetilde{B}\in \mathrm{Blk}(\mathrm{GU}_4(2^{2n}))$ with positive defect.

Table 4 Values of $\mathrm{k}(j, d, u)$

Defect d Value u	7 2	7 1	6 2	6 1	5 2	5 1	4 2	4 1	3 2	Otherwise Otherwise
$\mathrm{k}(1, d, u)$	4	4	2	4	4	10	1	2	1	0
$\mathrm{k}(2, d, u)$	4	4	2	0	4	2	0	0	1	0
$\mathrm{k}(3, d, u)$	4	4	2	0	4	6	0	0	0	0
$\mathrm{k}(4, d, u)$	4	4	2	4	4	14	1	2	0	0

8. DADE'S PROJECTIVE INVARIANT CONJECTURE FOR $2.\mathrm{SU}_4(4)$

Let C be a radical 2-chain of $\mathrm{SU}_4(4)$. The character table of $N_{2.\mathrm{SU}_4(4)}(C)$ and $N_{2.\mathrm{SU}_4(4).2}(C)$ can either be found in the library of character tables distributed with GAP [11] or computed directly using MAGMA [17].

Let $H = N_{2.\mathrm{SU}_4(4)}(C)$ or $N_{2.\mathrm{SU}_4(4).2}(C)$, and let ξ be a linear character of $Z := Z(2.\mathrm{SU}_4(4))$. Denote by $\mathrm{Irr}(H \mid \xi)$ the subset of $\mathrm{Irr}(H)$ consisting of characters covering ξ. Then $\mathrm{Irr}(H) = \mathrm{Irr}(H \mid 1) \cup \mathrm{Irr}(H \mid \rho)$, where $\rho \in \mathrm{Irr}(Z)\backslash\{1\}$. But $\mathrm{Irr}(H \mid 1) = \mathrm{Irr}(H/Z)$ and $H/Z = N_{\mathrm{SU}_4(4)}(C)$ or $N_{\mathrm{SU}_4(4).2}(C)$, so Dade's projective invariant conjecture for $2.\mathrm{SU}_4(4)$ is equivalent to Dade's invariant conjecture for $\mathrm{SU}_4(4)$ and $2.\mathrm{SU}_4(4)$. By Theorem 7.1, Dade's invariant conjecture holds for $\mathrm{SU}_4(4)$, so it suffices to show that the conjecture holds for $2.\mathrm{SU}_4(4)$.

Theorem 8.1. *Let $\widetilde{B}$ be a 2-block of $G = 2.\mathrm{SU}_4(4)$ with defect group $D(\widetilde{B}) \neq O_2(G)$. Then $\widetilde{B}$ satisfies Dade's projective invariant conjecture.*

Proof. By the proceeding remarks in Section 7, we can assume $\widetilde{B} = B_0(2.\mathrm{SU}_4(4))$. Let $N(C) = N_G(C)$ for $C \in \mathcal{R}(\mathrm{SU}_4(4))$ and $E = G.2 = 2.\mathrm{SU}_4(4).2$.

Set $\mathrm{k}(j, d, u) := \mathrm{k}(N(C_j), \widetilde{B}, d, u)$ for $j = 1, \ldots, 6$. Note that the contributions of the chains C_5 and C_6 to the alternating sum cancel. The values $\mathrm{k}(j, d, u)$ for $j = 1, \ldots, 4$ in Table 4 can be derived from the Tables in Appendix C.

Hence $\sum_{i=1}^{4}(-1)^{|C_j|}\mathrm{k}(N(C_j), \widetilde{B}, d, u) = 0$, and the theorem follows. □

9. McKAY'S CONJECTURE FOR $\mathrm{Sp}_4(2^n)$ AND $\mathrm{SU}_4(2^{2n})$

In [15], Isaacs, Malle, and Navarro reduced the McKay conjecture to a question about finite simple groups. They showed that every finite group will satisfy the McKay conjecture if every finite non-abelian simple group is good in the sense of [15, Section 10]. From Tables A.5 and A.11, we can derive the following theorem.

Theorem 9.1. *Let $n \in \mathbb{N}_{>0}$ and $K \in \{\mathrm{Sp}_4(2^n)', \mathrm{SU}_4(2^{2n})\}$. Then K is good for the prime 2 in the sense of* [15, *Section* 10].

Proof. We use the notation of [15, Section 10] and have to show that the conditions (1)–(8) in [15, Section 10] are satisfied for $X := K$ and $p = 2$. Note that K has trivial Schur multiplier except when $K = \mathrm{SU}_4(4)$ or $\mathrm{Sp}_4(2)'$, and in which case, 2 is good for $\mathrm{SU}_4(4)$ and $\mathrm{Sp}_4(2)'$ by [18]. Thus we only have to consider $S := X = K$, so that $\mathrm{Aut}(K)$ acts trivially on $Z := Z(X) = \{1\}$.

Let $T := B_S$ be our Borel subgroup of S and $A := N_{\mathrm{Aut}(K)}(B_S)$, so that T is the normalizer of the Sylow 2-subgroup $O_2(B_S)$ of S and T is stabilized by A. Since $T < S$, it follows that the conditions (1) and (2) of [15, Section 10] hold.

Since $A = B_S \rtimes O$ and $\mathrm{Aut}(S) = S \rtimes O$, it follows that the conditions (3) and (4) of [15, Section 10] are equivalent to that there exists an O-equivariant bijection $\theta \to \theta^*$ from $\mathrm{Irr}_{2'}(S)$ to $\mathrm{Irr}_{2'}(T)$, where for any $L \le S$, $\mathrm{Irr}_{2'}(L)$ is the subset of $\mathrm{Irr}(L)$ consisting of characters of odd degrees. Since O is cyclic, it follows that the conditions (3) and (4) of [15, Section 10] are then equivalent to

$$\mathrm{k}(S, B_0, d(B_0), u) = \mathrm{k}(T, B_0, d(B_0), u) \tag{9.1}$$

for any $u \in \mathbb{N}_{>0}$, where B_0 is the principal block of S and the defect $d(B_0)$ is the integer such that $|D(B_0)| = 2^{d(B_0)}$. It follows by (7.2) that (9.1) is equivalent to

$$\mathrm{k}(G, \widetilde{B}_0, d(B_0), u) = \mathrm{k}(B, \widetilde{B}_0, d(B_0), u), \tag{9.2}$$

where $G = \mathrm{Sp}_4(2^n)$ or $\mathrm{GU}_4(2^{2n})$, $\widetilde{B}_0 = B_0(S)$ or $\widetilde{B}_0 = \bigcup_{\mathrm{d}(\widetilde{B})\neq 0} \widetilde{B}$, $d(B_0) = 4n$ or $6n$ according as $S = \mathrm{Sp}_4(2^n)$ or $\mathrm{SU}_4(2^{2n})$, and B is a Borel subgroup of G and $u \in \mathbb{N}_{>0}$. If $G = \mathrm{Sp}_4(2^n)$, then the irreducible characters of G and B with defect $4n$ are given by Table A.4. If $G = \mathrm{GU}_4(2^{2n})$, then the irreducible characters of G and B with defect $6n$ are given by Table A.10. It follows from Tables A.5 and A.11 and the proof of Case 2(ii) of Theorem 5.1 that (9.2) holds, and so the conditions (3) and (4) of [15, Section 10] hold.

For $\theta \in \mathrm{Irr}_{2'}(S)$ we set $M(\theta) = A_\theta$ to be the stabilizer of θ in A and let $M = \langle S, M(\theta)\rangle \le \mathrm{Aut}(S)$. Then $N_M(T) = TM(\theta)$, $C_M(S) = Z(S) = \{1\}$, and so the conditions (5), (6), and (7) of [15, Section 10] hold. Since $\mathrm{Out}(K)$ is cyclic, it follows that the condition (8) of [15, Section 10] is satisfied automatically. This completes the proof. $\square$

Note that Theorem 9.1 implies that the McKay conjecture for $p = 2$ is true for $\mathrm{Sp}_4(2^n)$ and $\mathrm{SU}_4(2^{2n})$.

APPENDIX A

Table A.1 Parameter sets for the irreducible characters of the parabolic subgroups $G = \mathrm{Sp}_4(2^n)$, B, P, Q

Parameter set	Characters	Parameters	Equivalence relation	Number of characters
${}_GI_1$	$\chi_1(k,l)$	See the remarks in Section 4		$\frac{(q-2)(q-4)}{8}$
${}_GI_2$	$\chi_2(k)$	See the remarks in Section 4		$\frac{q(q-2)}{4}$
${}_GI_3$	$\chi_3(k,l)$	See the remarks in Section 4		$\frac{q(q-2)}{4}$
${}_GI_4$	$\chi_4(k,l)$	See the remarks in Section 4		$\frac{q(q-2)}{8}$
${}_GI_5$	$\chi_5(k)$	See the remarks in Section 4		$\frac{q^2}{4}$
${}_GI_6 = {}_GI_7$	$\chi_6(k), \chi_7(k)$	$k = 0, \ldots, q-2; k \neq 0$	$\{k \equiv -k\}$	$\frac{q-2}{2}$

(continued)

Table A.1 Continued

Parameter set	Characters	Parameters	Equivalence relation	Number of characters
${}_GI_8 = {}_GI_9$	$\chi_8(k), \chi_9(k)$	$k = 0, \ldots, q; k \neq 0$	$\{k \equiv -k\}$	$\frac{q}{2}$
${}_GI_{10} = {}_GI_{11}$	$\chi_{10}(k), \chi_{11}(k)$	$k = 0, \ldots, q-2; k \neq 0$	$\{k \equiv -k\}$	$\frac{q-2}{2}$
${}_GI_{12} = {}_GI_{13}$	$\chi_{12}(k), \chi_{13}(k)$	$k = 0, \ldots, q; k \neq 0$	$\{k \equiv -k\}$	$\frac{q}{2}$
${}_GI_{14} = \cdots = {}_GI_{19}$	$\theta_0, \ldots, \theta_5$			1
${}_BI_1$	${}_B\chi_1(k, l)$	$k, l = 0, \ldots, q-2$		$(q-1)^2$
${}_BI_2 = \cdots = {}_BI_5$	${}_B\chi_2(k), \ldots, {}_B\chi_5(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_BI_6$	${}_B\theta_1$			1
${}_BI_7 = {}_BI_8$	${}_B\theta_2(k), {}_B\theta_3(k)$	$k = 0, 1$		2
${}_PI_1 = {}_PI_2$	${}_P\chi_1(k), {}_P\chi_2(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_PI_3$	${}_P\chi_3(k, l)$	$k, l = 0, \ldots, q-2$ $q-1 \nmid k-l$	$\{(k, l) \equiv (l, k)\}$	$\frac{(q-1)(q-2)}{2}$
${}_PI_4$	${}_P\chi_4(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_PI_5$	${}_P\chi_5(k)$	$k = 0, \ldots, q-2; k \neq 0$	$\{k \equiv -k\}$	$\frac{q-2}{2}$
${}_PI_6$	${}_P\chi_6(k)$	$k = 0, \ldots, q; k \neq 0$	$\{k \equiv -k\}$	$\frac{q}{2}$
${}_PI_7$	${}_P\chi_7(k)$	$k = 0, \ldots, q^2-2$ $q+1 \nmid k$	$\{k \equiv qk\}$	$\frac{q(q-1)}{2}$
${}_PI_8$	${}_P\theta_1$			1
${}_PI_9 = {}_PI_{10}$	${}_P\theta_2(k), {}_P\theta_3(k)$	$k = 0, 1$		2
${}_QI_1 = {}_QI_2$	${}_Q\chi'_1(k), {}_Q\chi'_2(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_QI_3$	${}_Q\chi'_3(k, l)$	$k, l = 0, \ldots, q-2;$ $l \neq 0$	$\{(k, l) \equiv (k, -l)\}$	$\frac{(q-1)(q-2)}{2}$
${}_QI_4$	${}_Q\chi'_4(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_QI_5$	${}_Q\chi'_5(k)$	$k = 0, \ldots, q-2; k \neq 0$	$\{k \equiv -k\}$	$\frac{q-2}{2}$
${}_QI_6$	${}_Q\chi'_6(k)$	$k = 0, \ldots, q; k \neq 0$	$\{k \equiv -k\}$	$\frac{q}{2}$
${}_QI_7$	${}_Q\chi'_7(k, l)$	$k = 0, \ldots, q-2$ $l = 0, \ldots, q; l \neq 0$	$\{(k, l) \equiv (k, -l)\}$	$\frac{q(q-1)}{2}$
${}_QI_8$	${}_Q\theta'_1$			1
${}_QI_9 = {}_QI_{10}$	${}_Q\theta'_2(k), {}_Q\theta'_3(k)$	$k = 0, 1$		2

Table A.2 The irreducible characters of the chain normalizers in $\mathrm{Sp}_4(2^n)$ of defect $3n$

	Character	Degree	Param. set	Param. Nozawa	Number	Class
G	$\chi_{10}(k)$	$q(q+1)(q^2+1)$	${}_GI_{10}$	T_1	$(q-2)/2$	C_1, C_2
	$\chi_{11}(k)$	$q(q+1)(q^2+1)$	${}_GI_{11}$	T_1	$(q-2)/2$	C_1, C_2
	$\chi_{12}(k)$	$q(q-1)(q^2+1)$	${}_GI_{12}$	T_2	$q/2$	C_3, C_4
	$\chi_{13}(k)$	$q(q-1)(q^2+1)$	${}_GI_{13}$	T_2	$q/2$	C_3, C_4
B	${}_B\chi_4(k)$	$q(q-1)$	${}_BI_4$	T_0	$q-1$	C_2
	${}_B\chi_5(k)$	$q(q-1)$	${}_BI_5$	T_0	$q-1$	C_1
P	${}_P\chi_2(k)$	q	${}_PI_2$	T_0	$q-1$	C_1
	${}_P\chi_5(k)$	$q(q^2-1)$	${}_PI_5$	T_1	$(q-2)/2$	C_2
	${}_P\chi_6(k)$	$q(q-1)^2$	${}_PI_6$	T_2	$q/2$	C_4
Q	${}_Q\chi'_2(k)$	q	${}_QI_2$	T_0	$q-1$	C_2
	${}_Q\chi'_5(k)$	$q(q^2-1)$	${}_QI_5$	T_1	$(q-2)/2$	C_1
	${}_Q\chi'_6(k)$	$q(q-1)^2$	${}_QI_6$	T_2	$q/2$	C_4

Table A.3 The irreducible characters of the chain normalizers in $\mathrm{Sp}_4(2^n)$ of defect $3n+1$. Note that there is a typo in [13]: the degrees of $\theta_1, \theta_2 \in \mathrm{Irr}(G)$ have to be swapped

	Character	Degree	Param. set	Param. Nozawa	Number	Class
G	θ_1	$q(q+1)^2/2$	${}_GI_{15}$		1	A_{31}
	θ_2	$q(q^2+1)/2$	${}_GI_{16}$		1	
	θ_3	$q(q^2+1)/2$	${}_GI_{17}$		1	A_{31}
	θ_5	$q(q-1)^2/2$	${}_GI_{19}$		1	
B	${}_B\theta_2(k)$	$q(q-1)^2/2$	${}_BI_7$		2	A_{41}, A_{51}
	${}_B\theta_3(k)$	$q(q-1)^2/2$	${}_BI_8$		2	A_{41}, A_{51}
P	${}_P\theta_2(k)$	$q(q^2-1)/2$	${}_PI_9$		2	A_{41}
	${}_P\theta_3(k)$	$q(q-1)^2/2$	${}_PI_{10}$		2	A_{41}
Q	${}_Q\theta'_2(k)$	$q(q^2-1)/2$	${}_QI_9$		2	A_{41}
	${}_Q\theta'_3(k)$	$q(q-1)^2/2$	${}_QI_{10}$		2	A_{41}

Table A.4 The irreducible characters of the chain normalizers in $\mathrm{Sp}_4(2^n)$ of defect $4n$

	Character	Degree	Param. set	Param. Nozawa	Number	Class
G	$\chi_1(k,l)$	$(q+1)^2(q^2+1)$	${}_GI_1$	S_1	$(q-2)(q-4)/8$	B_1
	$\chi_2(k)$	q^4-1	${}_GI_2$	R_2	$q(q-2)/4$	C_2, C_4
	$\chi_3(k,l)$	q^4-1	${}_GI_3$	$T_1\times T_2$	$q(q-2)/4$	B_3
	$\chi_4(k,l)$	$(q-1)^2(q^2+1)$	${}_GI_4$	S_2	$q(q-2)/8$	B_4
	$\chi_5(k)$	$(q^2-1)^2$	${}_GI_5$	R_3	$q^2/4$	B_5
	$\chi_6(k)$	$(q+1)(q^2+1)$	${}_GI_6$	T_1	$(q-2)/2$	C_1
	$\chi_7(k)$	$(q+1)(q^2+1)$	${}_GI_7$	T_1	$(q-2)/2$	D_1
	$\chi_8(k)$	$(q-1)(q^2+1)$	${}_GI_8$	T_2	$q/2$	C_3
	$\chi_9(k)$	$(q-1)(q^2+1)$	${}_GI_9$	T_2	$q/2$	D_3
	θ_0	1	${}_GI_{14}$		1	
B	${}_B\chi_1(k,l)$	1	${}_BI_1$	$T_0\times T_0$	$(q-1)^2$	B
	${}_B\chi_2(k)$	$q-1$	${}_BI_2$	T_0	$q-1$	C_4
	${}_B\chi_3(k)$	$q-1$	${}_BI_3$	T_0	$q-1$	C_3
	${}_B\theta_1$	$(q-1)^2$	${}_BI_6$		1	
P	${}_P\chi_1(k)$	1	${}_PI_1$	T_0	$q-1$	C_1
	${}_P\chi_3(k,l)$	$q+1$	${}_PI_3$	S_0	$(q-1)(q-2)/2$	B_1
	${}_P\chi_4(k)$	q^2-1	${}_PI_4$	T_0	$q-1$	C_1
	${}_P\chi_7(k)$	$q-1$	${}_PI_7$	R_1	$q(q-1)/2$	C_3, C_4
	${}_P\theta_1$	$(q-1)(q^2-1)$	${}_PI_8$		1	
Q	${}_Q\chi'_1(k)$	1	${}_QI_1$	T_0	$q-1$	C_2
	${}_Q\chi'_3(k,l)$	$q+1$	${}_QI_3$	$T_0\times T_1$	$(q-1)(q-2)/2$	B_1
	${}_Q\chi'_4(k)$	q^2-1	${}_QI_4$	T_0	$q-1$	C_2
	${}_Q\chi'_7(k,l)$	$q-1$	${}_QI_7$	$T_0\times T_2$	$q(q-1)/2$	B_2
	${}_Q\theta'_1$	$(q-1)(q^2-1)$	${}_QI_8$		1	

Table A.5 Number of fixed points of $H = \langle\alpha^t\rangle$ on parameter sets of the irreducible characters of the parabolic subgroups of $\mathrm{Sp}_4(2^n)$. The unions of parameter sets in this table are disjoint unions

Parameter set I	Number of fixed points $\lvert C_I(H)\rvert$ if $t \mid n$
${}_GI_1 \cup {}_GI_2 \cup {}_GI_3 \cup {}_GI_4 \cup {}_GI_5$	$(2^t-1)^2$
${}_GI_6 \cup {}_GI_8$	2^t-1
${}_GI_7 \cup {}_GI_9$	2^t-1
${}_GI_{10} \cup {}_GI_{12}$	2^t-1
${}_GI_{11} \cup {}_GI_{13}$	2^t-1
${}_GI_{14}$	1
${}_GI_{15} \cup {}_GI_{16} \cup {}_GI_{17} \cup {}_GI_{19}$	4
${}_BI_1$	$(2^t-1)^2$
${}_BI_2$	2^t-1
${}_BI_3$	2^t-1
${}_BI_4$	2^t-1
${}_BI_5$	2^t-1
${}_BI_6$	1
${}_BI_7 \cup {}_BI_8$	4
${}_PI_1$	2^t-1
${}_PI_2$	2^t-1
${}_PI_3 \cup {}_PI_7$	$(2^t-1)^2$
${}_PI_4$	2^t-1
${}_PI_5 \cup {}_PI_6$	2^t-1
${}_PI_8$	1
${}_PI_9 \cup {}_PI_{10}$	4
${}_QI_1$	2^t-1
${}_QI_2$	2^t-1
${}_QI_3 \cup {}_QI_7$	$(2^t-1)^2$
${}_QI_4$	2^t-1
${}_QI_5 \cup {}_QI_6$	2^t-1
${}_QI_8$	1
${}_QI_9 \cup {}_QI_{10}$	4

Table A.6 Parameter sets for the irreducible characters of the parabolic subgroups $G = \mathrm{GU}_4(2^{2n})$, B, P, Q

Parameter set	Characters	Parameters	Equivalence relation	Number of characters
${}_GI_1$	$\chi_1(k)$	$k = 0, \ldots, q$		$q+1$
${}_GI_2$	$\chi_2(k, l)$	See the remarks in Section 4		$\frac{(q^2-q-2)(q^2-q-4)}{8}$
${}_GI_3 = {}_GI_4$	$\chi_3(k), \chi_4(k)$	$k = 0, \ldots, q^2-2$ $q-1 \nmid k$	$\{k \equiv -qk\}$	$\frac{q^2-q-2}{2}$
${}_GI_5$	$\chi_5(k)$	See the remarks in Section 4		$\frac{q^2(q^2-1)}{4}$
${}_GI_6$	$\chi_6(k, l, m)$	See the remarks in Section 4		$\frac{q(q+1)(q^2-q-2)}{4}$
${}_GI_7 = {}_GI_8$	$\chi_7(k, l), \chi_8(k, l)$	$k = 0, \ldots, q^2-2$ $l = 0, \ldots, q$ $q-1 \nmid k$	$\{(k, l) \equiv (-qk, l)\}$	$\frac{(q+1)(q^2-q-2)}{2}$
${}_GI_9$	$\chi_9(k, l)$	See the remarks in Section 4		$\frac{q(q+1)(q^2-1)}{3}$

(continued)

Table A.6 Continued

Parameter set	Characters	Parameters	Equivalence relation	Number of characters
${}_GI_{10}$	$\chi_{10}(k)$	See the remarks in Section 4		$\frac{q(q^2-1)(q-2)}{24}$
${}_GI_{11}$	$\chi_{11}(k)$	$k = 0, \ldots, q$		$q+1$
${}_GI_{12}$	$\chi_{12}(k)$	$k = 0, \ldots, q$		$q+1$
${}_GI_{13}$	$\chi_{13}(k)$	$k = 0, \ldots, q$		$q+1$
${}_GI_{14}$	$\chi_{14}(k)$	$k = 0, \ldots, q$		$q+1$
${}_GI_{15} = {}_GI_{16}$	$\chi_{15}(k, l, m)$, $\chi_{16}(k, l, m)$	$k, l, m = 0, \ldots, q$ $k \neq l, m; l \neq k, m;$ $m \neq k, l$	$\{(k, l, m) \equiv (k, m, l)\}$	$\frac{q(q^2-1)}{2}$
${}_GI_{17} = {}_GI_{18} = {}_GI_{19}$	$\chi_{17}(k, l)$, $\chi_{18}(k, l)$, $\chi_{19}(k, l)$	$k, l = 0, \ldots, q$ $k \neq l$		$q(q+1)$
${}_GI_{20}$	$\chi_{20}(k, l)$	$k, l = 0, \ldots, q$ $k \neq l$	$\{(k, l) \equiv (l, k)\}$	$\frac{q(q+1)}{2}$
${}_GI_{21}$	$\chi_{21}(k, l)$	$k, l = 0, \ldots, q$ $k \neq l$		$q(q+1)$
${}_GI_{22}$	$\chi_{22}(k, l)$	$k, l = 0, \ldots, q$ $k \neq l$	$\{(k, l) \equiv (l, k)\}$	$\frac{q(q+1)}{2}$
${}_BI_1$	${}_B\alpha_1(k, l)$	$k, l = 0, \ldots, q^2-2$		$(q^2-1)^2$
${}_BI_2$	${}_B\alpha_2(k, l)$	$k = 0, \ldots, q^2-2$ $l = 0, \ldots, q$		$(q+1)(q^2-1)$
${}_BI_3$	${}_B\alpha_3(k)$	$k = 0, \ldots, q^2-2$		q^2-1
${}_BI_4$	${}_B\alpha_4(k)$	$k = 0, \ldots, q$		$q+1$
${}_BI_5$	${}_B\alpha_5(k, l)$	$k = 0, \ldots, q$ $l = 0, \ldots, q^2-2$		$(q+1)(q^2-1)$
${}_BI_6$	${}_B\alpha_6(k)$	$k = 0, \ldots, q^2-2$		q^2-1
${}_BI_7$	${}_B\alpha_7(k)$	$k, l = 0, \ldots, q$		$(q+1)^2$
${}_BI_8$	${}_B\alpha_8(k)$	$k = 0, \ldots, q$		$q+1$
${}_PI_1 = {}_PI_2$	${}_P\beta_1(k)$, ${}_P\beta_2(k)$	$k = 0, \ldots, q^2-2$		q^2-1
${}_PI_3$	${}_P\beta_3(k, l)$	$k, l = 0, \ldots, q^2-2$ $k \neq l$	$\{(k, l) \equiv (l, k)\}$	$\frac{(q^2-1)(q^2-2)}{2}$
${}_PI_4$	${}_P\beta_4(k, l)$	$k = 0, \ldots, q^2-2$ $l = 0, \ldots, q$		$(q+1)(q^2-1)$
${}_PI_5$	${}_P\beta_5(k)$	$k = 0, \ldots, q$		$q+1$
${}_PI_6$	${}_P\beta_6(k)$	$k = 0, \ldots, q^2-2$ $q-1 \nmid k$	$\{k \equiv -qk\}$	$\frac{q^2-q-2}{2}$
${}_PI_7$	${}_P\beta_7(k, l)$	$k, l = 0, \ldots, q$ $k \neq l$	$\{(k, l) \equiv (l, k)\}$	$\frac{q(q+1)}{2}$
${}_PI_8$	${}_P\beta_8(k)$	$k = 0, \ldots, q^4-2$ $q^2+1 \nmid k$	$\{k \equiv q^2k\}$	$\frac{q^2(q^2-1)}{2}$
${}_PI_9 = {}_PI_{10}$	${}_P\beta_9(k)$, ${}_P\beta_{10}(k)$	$k = 0, \ldots, q$		$q+1$
${}_QI_1 = {}_QI_2$	${}_Q\gamma_1(k, l)$, ${}_Q\gamma_2(k, l)$	$k = 0, \ldots, q^2-2$ $l = 0, \ldots, q$		$(q+1)(q^2-1)$
${}_QI_3$	${}_Q\gamma_3(k, l)$	$k, l = 0, \ldots, q^2-2$ $q-1 \nmid l$	$\{(k, l) \equiv (k, -ql)\}$	$\frac{(q^2-1)(q^2-q-2)}{2}$
${}_QI_4$	${}_Q\gamma_4(k)$	$k = 0, \ldots, q^2-2$		q^2-1
${}_QI_5$	${}_Q\gamma_5(k)$	$k = 0, \ldots, q$		$q+1$

(continued)

Table A.6 Continued

Parameter set	Characters	Parameters	Equivalence relation	Number of characters
${}_QI_6$	${}_Q\gamma_6(k,l)$	$k=0,\ldots,q$ $l=0,\ldots,q^2-2$ $q-1\nmid l$	$\{(k,l)\equiv(k,-ql)\}$	$\frac{(q+1)(q^2-q-2)}{2}$
${}_QI_7$	${}_Q\gamma_7(k,l)$	$k,l=0,\ldots,q$		$(q+1)^2$
${}_QI_8$	${}_Q\gamma_8(k,l,m)$	$k=0,\ldots,q^2-2$ $l,m=0,\ldots,q$ $l\neq m$	$\{(k,l,m)\equiv(k,m,l)\}$	$\frac{q(q+1)(q^2-1)}{2}$
${}_QI_9={}_QI_{10}$	${}_Q\gamma_9(k,l)$, ${}_Q\gamma_{10}(k,l)$	$k,l=0,\ldots,q$		$(q+1)^2$
${}_QI_{11}$	${}_Q\gamma_{11}(k,l,m)$	$k,l,m=0,\ldots,q$ $l\neq m$	$\{(k,l,m)\equiv(k,m,l)\}$	$\frac{q(q+1)^2}{2}$

Table A.7 The irreducible characters of the chain normalizers in $\mathrm{GU}_4(2^{2n})$ of defect $3n$

	Character	Degree	Param. set	Param. Nozawa	Number	Class
G	$\chi_{13}(k)$	$q^3(q^2-q+1)$	${}_GI_{13}$	I_1	$q+1$	A_1
	$\chi_{18}(k,l)$	$q^3(q-1)(q^2+1)$	${}_GI_{18}$	I_2	$q(q+1)$	A_7
Q	${}_Q\gamma_9(k,l)$	$q^3(q-1)$	${}_QI_9$	$I_1\times I_1$	$(q+1)^2$	A_{11}

Table A.8 The irreducible characters of the chain normalizers in $\mathrm{GU}_4(2^{2n})$ of defect $4n$

	Character	Degree	Param. set	Param. Nozawa	Number	Class
G	$\chi_4(k)$	$q^2(q+1)(q^3+1)$	${}_GI_4$	J_1	$\frac{q^2-q-2}{2}$	C_1
	$\chi_{12}(k)$	$q^2(q^2+1)$	${}_GI_{12}$	I_1	$q+1$	A_1
	$\chi_{20}(k,l)$	$q^2(q^2+1)(q^2-q+1)$	${}_GI_{20}$	I_2	$\frac{q(q+1)}{2}$	A_{10}
B	${}_B\alpha_5(k,l)$	$q^2(q-1)$	${}_BI_5$	$I_1\times J_0$	$(q+1)(q^2-1)$	B_3
	${}_B\alpha_7(k,l)$	$q^2(q-1)^2$	${}_BI_7$	$I_1\times I_1$	$(q+1)^2$	A_9
P	${}_P\beta_2(k)$	q^2	${}_PI_2$	J_0	q^2-1	C_1
	${}_P\beta_{10}(k)$	$q^2(q-1)(q^2+1)$	${}_PI_{10}$	I_1	$q+1$	A_1
Q	${}_Q\gamma_6(k,l)$	$q^2(q^2-1)$	${}_QI_6$	$I_1\times J_1$	$\frac{(q+1)(q^2-q-2)}{2}$	B_1
	${}_Q\gamma_{10}(k,l)$	$q^2(q-1)$	${}_QI_{10}$	$I_1\times I_1$	$(q+1)^2$	A_{11}
	${}_Q\gamma_{11}(k,l,m)$	$q^2(q-1)^2$	${}_QI_{11}$	$I_1\times I_2$	$\frac{q(q+1)^2}{2}$	A_{15}

Table A.9 The irreducible characters of the chain normalizers in $\mathrm{GU}_4(2^{2n})$ of defect $5n$

	Character	Degree	Param. set	Param. Nozawa	Number	Class
G	$\chi_8(k,l)$	$q(q^2+1)(q^3+1)$	${}_GI_8$	J_2	$\frac{(q+1)(q^2-q-2)}{2}$	B_2
	$\chi_{14}(k)$	$q(q^2-q+1)$	${}_GI_{14}$	I_1	$q+1$	A_1
	$\chi_{15}(k,l,m)$	$q(q-1)(q^2+1)(q^2-q+1)$	${}_GI_{15}$	I_3	$\frac{q(q^2-1)}{2}$	A_{12}
	$\chi_{17}(k,l)$	$q(q-1)^2(q^2+1)$	${}_GI_{17}$	I_2	$q(q+1)$	A_8
	$\chi_{21}(k,l)$	$q(q^2+1)(q^2-q+1)$	${}_GI_{21}$	I_2	$q(q+1)$	A_8
B	${}_B\alpha_6(k)$	$q(q^2-1)$	${}_BI_6$	J_0	q^2-1	C_2
	${}_B\alpha_8(k)$	$q(q-1)(q^2-1)$	${}_BI_8$	I_1	$q+1$	A_1
P	${}_P\beta_6(k)$	$q(q^4-1)$	${}_PI_6$	J_1	$\frac{q^2-q-2}{2}$	C_2
	${}_P\beta_7(k,l)$	$q(q-1)^2(q^2+1)$	${}_PI_7$	I_2	$\frac{q(q+1)}{2}$	A_7
	${}_P\beta_9(k)$	$q(q-1)(q^2+1)$	${}_PI_9$	I_1	$q+1$	A_1
Q	${}_Q\gamma_2(k,l)$	q	${}_QI_2$	$J_0\times I_1$	$(q+1)(q^2-1)$	B_2
	${}_Q\gamma_7(k,l)$	$q(q-1)(q^2-1)$	${}_QI_7$	I_2	$(q+1)^2$	A_8

Table A.10 The irreducible characters of the chain normalizers in $\mathrm{GU}_4(2^{2n})$ of defect $6n$. We use the abbreviation $\eta := (q-1)(q^2+1)(q^2-q+1)$

	Character	Degree	Param. set	Param. Nozawa	Number	Class
G	$\chi_1(k)$	1	${}_GI_1$	I_1	$q+1$	A_1
	$\chi_2(k,l)$	$(q+1)(q^2+1)(q^3+1)$	${}_GI_2$	J_4	$\frac{(q^2-q-2)(q^2-q-4)}{8}$	C_3
	$\chi_3(k)$	$(q+1)(q^3+1)$	${}_GI_3$	J_1	$\frac{q^2-q-2}{2}$	C_1
	$\chi_5(k)$	$(q+1)(q^2-1)(q^3+1)$	${}_GI_5$	S_1	$\frac{q^2(q^2-1)}{4}$	E_1
	$\chi_6(k,l,m)$	$(q-1)(q^2+1)(q^3+1)$	${}_GI_6$	J_3	$\frac{q(q+1)(q^2-q-2)}{4}$	B_3
	$\chi_7(k,l)$	$(q^2+1)(q^3+1)$	${}_GI_7$	J_2	$\frac{(q+1)(q^2-q-2)}{2}$	C_1
	$\chi_9(k,l)$	$(q^2-1)(q^4-1)$	${}_GI_9$	R_2	$\frac{q(q+1)(q^2-1)}{3}$	D_1
	$\chi_{10}(k,l,m,n)$	$(q-1)\eta$	${}_GI_{10}$	I_4	$\frac{q(q^2-1)(q-2)}{24}$	A_{14}
	$\chi_{16}(k,l,m)$	η	${}_GI_{16}$	I_3	$\frac{q(q^2-1)}{2}$	A_{12}
	$\chi_{19}(k,l)$	$(q-1)(q^2+1)$	${}_GI_{19}$	I_2	$q(q+1)$	A_8
	$\chi_{22}(k,l)$	$(q^2+1)(q^2-q+1)$	${}_GI_{22}$	I_2	$\frac{q(q+1)}{2}$	A_{11}
B	${}_B\alpha_1(k,l)$	1	${}_BI_1$	$J_0\times J_0$	$(q^2-1)^2$	C_5
	${}_B\alpha_2(k,l)$	$q-1$	${}_BI_2$	$J_0\times I_1$	$(q+1)(q^2-1)$	B_2
	${}_B\alpha_3(k)$	q^2-1	${}_BI_3$	J_0	q^2-1	C_1
	${}_B\alpha_4(k)$	$(q-1)(q^2-1)$	${}_BI_4$	I_1	$q+1$	A_1
P	${}_P\beta_1(k)$	1	${}_PI_1$	J_0	q^2-1	C_1
	${}_P\beta_3(k,l)$	q^2+1	${}_PI_3$	$J_0\times J_0$	$\frac{(q^2-1)(q^2-2)}{2}$	C_5
	${}_P\beta_4(k,l)$	$(q-1)(q^2+1)$	${}_PI_4$	$J_0\times I_1$	$(q+1)(q^2-1)$	B_1
	${}_P\beta_5(k)$	$(q-1)(q^4-1)$	${}_PI_5$	I_1	$q+1$	A_1
	${}_P\beta_8(k)$	q^2-1	${}_PI_8$	S_1	$\frac{q^2(q^2-1)}{2}$	E_1
Q	${}_Q\gamma_1(k,l)$	1	${}_QI_1$	$J_0\times I_1$	$(q+1)(q^2-1)$	B_2
	${}_Q\gamma_3(k,l)$	$q+1$	${}_QI_3$	$J_0\times J_1$	$\frac{(q^2-1)(q^2-q-2)}{2}$	C_3
	${}_Q\gamma_4(k)$	$(q+1)(q^2-1)$	${}_QI_4$	J_0	q^2-1	C_1
	${}_Q\gamma_5(k)$	$(q^2-1)^2$	${}_QI_5$	I_1	$q+1$	A_1
	${}_Q\gamma_8(k,l,m)$	$q-1$	${}_QI_8$	$J_0\times I_2$	$\frac{q(q+1)(q^2-1)}{2}$	B_5

Table A.11 Number of fixed points of $H = \langle \alpha^t \rangle$ on parameter sets of the irreducible characters of the parabolic subgroups of $\mathrm{GU}_4(2^{2n})$. The unions of parameter sets in this table are disjoint unions

Parameter set I	Number of fixed points $\lvert C_I(H)\rvert$ if $t \mid n$	Number of fixed points $\lvert C_I(H)\rvert$ if $t \nmid n$
${}_GI_1$	1	$2^{t/2}+1$
${}_GI_2 \cup {}_GI_5 \cup {}_GI_6 \cup {}_GI_9 \cup {}_GI_{10}$	$(2^t-1)^2$	$(2^t-1)^2$
${}_GI_3 \cup {}_GI_{22}$	2^t-1	2^t-1
${}_GI_4 \cup {}_GI_{20}$	2^t-1	2^t-1
${}_GI_7 \cup {}_GI_{16}$	2^t-1	$(2^{t/2}+1)(2^t-2^{t/2}-1)$
${}_GI_8 \cup {}_GI_{15}$	2^t-1	$(2^{t/2}+1)(2^t-2^{t/2}-1)$
${}_GI_{12}$	1	$2^{t/2}+1$
${}_GI_{13}$	1	$2^{t/2}+1$
${}_GI_{14}$	1	$2^{t/2}+1$
${}_GI_{17}$	0	$2^{t/2}(2^{t/2}+1)$
${}_GI_{18}$	0	$2^{t/2}(2^{t/2}+1)$
${}_GI_{19}$	0	$2^{t/2}(2^{t/2}+1)$
${}_GI_{21}$	0	$2^{t/2}(2^{t/2}+1)$
${}_BI_1$	$(2^t-1)^2$	$(2^t-1)^2$
${}_BI_2$	2^t-1	$(2^t-1)(2^{t/2}+1)$
${}_BI_3$	2^t-1	2^t-1
${}_BI_4$	1	$2^{t/2}+1$
${}_BI_5$	2^t-1	$(2^{t/2}+1)(2^t-1)$
${}_BI_6$	2^t-1	2^t-1
${}_BI_7$	1	$(2^{t/2}+1)^2$
${}_BI_8$	1	$2^{t/2}+1$
${}_PI_1$	2^t-1	2^t-1
${}_PI_2$	2^t-1	2^t-1
${}_PI_3 \cup {}_PI_8$	$(2^t-1)^2$	$(2^t-1)^2$
${}_PI_4$	2^t-1	$(2^t-1)(2^{t/2}+1)$
${}_PI_5$	1	$2^{t/2}+1$
${}_PI_6 \cup {}_PI_7$	2^t-1	2^t-1
${}_PI_9$	1	$2^{t/2}+1$
${}_PI_{10}$	1	$2^{t/2}+1$
${}_QI_1$	2^t-1	$(2^t-1)(2^{t/2}+1)$
${}_QI_2$	2^t-1	$(2^t-1)(2^{t/2}+1)$
${}_QI_3 \cup {}_QI_8$	$(2^t-1)^2$	$(2^t-1)^2$
${}_QI_4$	2^t-1	2^t-1
${}_QI_5$	1	$2^{t/2}+1$
${}_QI_6 \cup {}_QI_{11}$	2^t-1	$(2^{t/2}+1)(2^t-1)$
${}_QI_7$	1	$(2^{t/2}+1)^2$
${}_QI_9$	1	$(2^{t/2}+1)^2$
${}_QI_{10}$	1	$(2^{t/2}+1)^2$

APPENDIX B

Let B be a Borel subgroup of $\mathrm{Sp}_4(2)$, and P, Q distinct maximal parabolic subgroups of $\mathrm{Sp}_4(2)$.

Table B.1 The degrees of characters in $\mathrm{Irr}(2.\mathrm{Sp}_4(2))$

Degree	1	5	9	10	16
Number	4	8	4	4	2

Table B.2 The degrees of characters in $\mathrm{Irr}(2.\mathrm{Sp}_4(2).2)$

Degree	1	2	9	10	16	18	20
Number	4	1	4	8	4	1	1

Table B.3 The degrees of characters in $\mathrm{Irr}(2.B)$

Degree	1	2
Number	16	4

Table B.4 The degrees of characters in $\mathrm{Irr}(2.B.2)$

Degree	1	2	4
Number	8	10	1

Table B.5 The degrees of characters in $\mathrm{Irr}(2.P)$

Degree	1	2	3
Number	8	4	8

Table B.6 The degrees of characters in $\mathrm{Irr}(2.Q)$

Degree	1	2	3
Number	8	4	8

APPENDIX C

Let B be a Borel subgroup of $SU_4(4)$, and P, Q distinct maximal parabolic subgroups of $SU_4(4)$. Then $2.SU_4(4) = Sp_4(3)$ and $2.SU_4(4).2 = CSp_4(3)$.

Table C.1 The degrees of characters in $\mathrm{Irr}(2.SU_4(4))$

Degree	1	4	5	6	10	15	20	24
Number	1	2	2	1	2	2	6	1
Degree	30	36	40	45	60	64	80	81
Number	3	2	2	2	4	2	1	1

Table C.2 The degrees of characters in $\mathrm{Irr}(2.SU_4(4).2)$

Degree	1	6	8	10	15	20	24	30
Number	2	2	1	1	4	5	2	2
Degree	40	60	64	72	80	81	90	120
Number	2	5	4	1	3	2	1	1

Table C.3 The degrees of characters in $\mathrm{Irr}(2.B)$

Degree	1	3	4	6	12
Number	6	2	9	2	1

Table C.4 The degrees of characters in $\mathrm{Irr}(2.B.2)$

Degree	1	2	3	4	6	8	12
Number	4	2	4	6	4	3	2

Table C.5 The degrees of characters in $\mathrm{Irr}(2.P)$

Degree	1	3	4	5	10	12	15	16	20
Number	1	2	2	4	2	2	1	1	2

Table C.6 The degrees of characters in $\mathrm{Irr}(2.P.2)$

Degree	1	4	5	6	10	15	16	20	24
Number	2	4	4	1	5	2	2	4	1

Table C.7 The degrees of characters in Irr(2.Q)

Degree	1	2	4	6	8	9	12
Number	6	3	15	3	3	2	3

Table C.8 The degrees of characters in Irr(2.Q.2)

Degree	1	2	4	6	8	9	12	16	24
Number	4	4	7	2	8	4	3	1	1

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