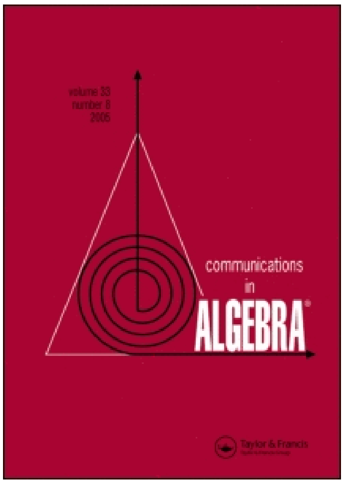

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Uno's Invariant Conjecture for the Finite Symplectic Group $Sp_4(q)$ in the Defining Characteristic

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UNO'S INVARIANT CONJECTURE FOR THE FINITE SYMPLECTIC GROUP $\mathrm{Sp}_4(q)$ IN THE DEFINING CHARACTERISTIC

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In this article, we verify Uno's invariant conjecture for the finite symplectic group $\mathrm{Sp}_4(q)$, q a power of an odd prime p, in the defining characteristic p. Uno's invariant conjecture is a refinement of Dade's invariant conjecture. Together with the results in [3], this completes the proof of Dade's invariant conjecture for the group $\mathrm{Sp}_4(q)$ in the defining characteristic.

Key Words: Dade's invariant conjecture; Defining characteristic; Symplectic groups; Uno's conjecture.

2000 Mathematics Subject Classification: Primary 20C20, 20C40.

1. INTRODUCTION

Let G be a finite group and p a prime dividing the order of G. There are several conjectures connecting the representation theory of G with the representation theory of certain p-local subgroups (i.e., the p-subgroups and their normalizers) of G. For example, it seems to be true, that if P is a Sylow p-subgroup of G, then the number of complex irreducible characters of G of degree coprime with p equals the same number for the normalizer $N_G(P)$.

This conjecture, called the McKay conjecture [14], and its block-theoretic version due to Alperin [1] were generalized by various authors. In [12], Isaacs and Navarro proposed a refinement of the McKay conjecture that deals with congruences of character degrees mod p. In a series of articles [7–9], Dade developed several conjectures expressing the number of complex irreducible characters with a fixed defect in a given p-block of G in terms of an alternating sum of related values for p-blocks of certain p-local subgroups of G. In [8], Dade proved that

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his (projective) conjecture implies the McKay conjecture. Motivated by the Isaacs–Navarro conjecture, Uno [18] suggested a further refinement of Dade's conjecture.

In this article, we show that Uno's invariant conjecture holds for the finite symplectic group $\mathrm{Sp}_4(q)$ with q a power of an odd prime p, in the defining characteristic p. This implies that Dade's invariant conjecture is true for $\mathrm{Sp}_4(q)$, q odd, in the defining characteristic. Together with the results in [3] this completes the proof of Dade's invariant conjecture for $\mathrm{Sp}_4(q)$ in the defining characteristic.

The methods are similar to those in [2]. By a corollary of the Borel and Tits theorem [5], the normalizers of radical p-chains of $\mathrm{Sp}_4(q)$ are exactly the parabolic subgroups. So we count characters of these chain normalizers which are fixed by certain outer automorphisms. Our calculations are based on the character tables of $\mathrm{Sp}_4(q)$ and their parabolic subgroups. The character tables of $\mathrm{Sp}_4(q)$ and the conformal symplectic groups $\mathrm{CSp}_4(q)$ have been computed by Srinivasan [17] and Shinoda [16], respectively, and that of parabolic subgroups of $\mathrm{Sp}_4(q)$ can be found in [19].

This article is organized as follows. In Section 2, we fix notation and state Dade's and Uno's invariant conjectures in detail. In Section 3, we state and prove some lemmas from elementary number theory which we use to count fixed points of certain automorphisms of $\mathrm{Sp}_4(q)$. In Section 4, we compute the fixed points of the outer automorphisms of $\mathrm{Sp}_4(q)$ on the irreducible characters of the parabolic subgroups. In Section 5, we verify Uno's invariant conjecture for $\mathrm{Sp}_4(q)$, $q = p^n$ odd, in the defining characteristic p. Details on irreducible characters and conjugacy classes of $\mathrm{Sp}_4(q)$ are summarized in tabular form in an Appendix A.

2. CONJECTURES OF DADE AND UNO

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) := N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and by $\mathrm{Blk}(G)$ the set of p-blocks. If $H \leq G$, $\widetilde{B} \in \mathrm{Blk}(G)$, and d is an integer, we denote by $\mathrm{Irr}(H, \widetilde{B}, d)$ the set of characters $\chi \in \mathrm{Irr}(H)$ satisfying $\mathrm{d}(\chi) = d$ and $b(\chi)^G = \widetilde{B}$ (in the sense of Brauer), where $\mathrm{d}(\chi) = \log_p(|H|_p) - \log_p(\chi(1)_p)$ is the p-defect of χ and $b(\chi)$ is the block of H containing χ.

Given a p-subgroup chain $C : P_0 < P_1 < \cdots < P_n$ of G, define the length $|C| := n$, $C_k : P_0 < P_1 < \cdots < P_k$ and

$$N(C) = N_G(C) := N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n).$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$; and
(b) $P_k = O_p(N(C_k))$ for $1 \leq k \leq n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Suppose $1 \to G \to E \to \overline{E} \to 1$ is an exact sequence, so that E is an extension of G by $\overline{E}$. Then E acts on $\mathcal{R}$ by conjugation. Given $C \in \mathcal{R}$ and $\psi \in \mathrm{Irr}(N_G(C))$, let $N_E(C, \psi)$ be the stabilizer of (C, ψ) in E, and

$$N_{\overline{E}}(C, \psi) := N_E(C, \psi)/N_G(C).$$

For $\widetilde{B} \in \mathrm{Blk}(G)$, an integer $d \geq 0$ and $U \leq \overline{E}$, let $\mathrm{k}(N_G(C), \widetilde{B}, d, U)$ be the number of characters in the set

$$\mathrm{Irr}(N_G(C), \widetilde{B}, d, U) := \{\psi \in \mathrm{Irr}(N_G(C), \widetilde{B}, d) \mid N_{\overline{E}}(C, \psi) = U\}.$$

Dade's invariant conjecture can be stated as follows.

Dade's Invariant Conjecture ([9]). If $O_p(G) = 1$ and $\widetilde{B} \in \mathrm{Blk}(G)$ with defect group $D(\widetilde{B}) \neq 1$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), \widetilde{B}, d, U) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Let H be a subgroup of G, $\varphi \in \mathrm{Irr}(H)$, and let $r(\varphi) = r_p(\varphi)$ be the integer $0 < r(\varphi) \leq (p-1)$ such that the p'-part $(|H|/\varphi(1))_{p'}$ of $|H|/\varphi(1)$ satisfies

$$\left(\frac{|H|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) \quad \mod p.$$

Given $1 \leq r < (p+1)/2$, let $\mathrm{Irr}(H, [r])$ be the subset of $\mathrm{Irr}(H)$ consisting of those characters φ with $r(\varphi) \equiv \pm r \mod p$. For $\widetilde{B} \in \mathrm{Blk}(G)$, $C \in \mathcal{R}$, an integer $d \geq 0$ and $U \leq \overline{E}$, we define

$$\mathrm{Irr}(N_G(C), \widetilde{B}, d, U, [r]) := \mathrm{Irr}(N_G(C), \widetilde{B}, d, U) \cap \mathrm{Irr}(N_G(C), [r])$$

and $\mathrm{k}(N_G(C), \widetilde{B}, d, U, [r]) := |\mathrm{Irr}(N_G(C), \widetilde{B}, d, U, [r])|$. The following refinement of Dade's conjecture is due to Uno.

Uno's Invariant Conjecture ([18], Conjecture 3.2). If $O_p(G) = 1$ and $\widetilde{B} \in \mathrm{Blk}(G)$ with defect group $D(\widetilde{B}) \neq 1$, then for all integers $d \geq 0$ and $1 \leq r < (p+1)/2$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), \widetilde{B}, d, U, [r]) = 0.$$

Note that if $p = 2$ or 3, then Uno's conjecture is equivalent to Dade's conjecture.

Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ be the automorphism and outer automorphism groups of G, respectively. We may suppose $\overline{E} = \mathrm{Out}(G)$. Let $G = \mathrm{Sp}_4(q)$, where $q = p^n$. By [13, Proposition 2.4.4], we have $\mathrm{Out}(G) = \langle \phi \rangle \times \langle \alpha \rangle$, where ϕ is a diagonal automorphism of order 2 and α is a field automorphism of order n.

3. NOTATION AND LEMMAS FROM ELEMENTARY NUMBER THEORY

From now on, we assume that p is an odd prime, n is a positive integer and $q = p^n$. We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers including zero. In the next section, we will use the following lemmas, the first one is [2, Lemma 3.1].

Lemma 3.1. *Suppose* $m, n, a \in \mathbb{Z}$ *with* $m, n > 0$. *Then* $\gcd(a^m - 1, a^n - 1) = |a^d - 1|$ *where* $d := \gcd(m, n)$.

Lemma 3.2. *Let* t *be a positive integer with* $t \mid n$. *Then the following hold.*

(i) $\gcd(p^t - 1, q - 1) = p^t - 1$.
(ii) $\gcd(p^t - 1, q + 1) = 2$.
(iii) $\gcd(p^t + 1, q - 1) = \begin{cases} p^t + 1 & \text{if } 2t|n, \\ 2 & \text{if } 2t \nmid n \end{cases}$;
(iv) $\gcd(p^t + 1, q + 1) = \begin{cases} 2 & \text{if } 2t \mid n, \\ p^t + 1 & \text{if } 2t \nmid n. \end{cases}$

Proof. (i) is clear by Lemma 3.1.

(ii) Suppose $d = \gcd(p^t - 1, q + 1)$. By (i), $d \mid q - 1$ and so $d \mid \gcd(q - 1, q + 1) = 2$.

(iv) Suppose $2t \mid n$. If $d \mid p^t + 1, q + 1$, then $d \mid p^{2t} - 1$ and so $d \mid p^n - 1$ as $2t|n$. Thus $d \mid q - 1, q + 1$ and $d \mid \gcd(q + 1, q - 1) = 2$.

Suppose $2t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with odd $t_u n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. So we get $p^t + 1 = -((-p^{2^k})^{t_u} - 1)$ and $q + 1 = -((-p^{2^k})^{n_u} - 1)$. Now, Lemma 3.1 implies $\gcd(p^t + 1, q + 1) = \gcd((-p^{2^k})^{t_u} - 1, (-p^{2^k})^{n_u} - 1) = |(-p^{2^k})^{t_u} - 1| = p^t + 1$.

(iii) Suppose $2t \mid n$. There are $k, t_u, n_u \in \mathbb{N}$ with $2 \not| \, t_u$ and $2 \mid n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $p^t + 1 = -((-p^{2^k})^{t_u} - 1)$ and $q - 1 = (-p^{2^k})^{n_u} - 1$. So Lemma 3.1 implies $\gcd(p^t + 1, q - 1) = \gcd((-p^{2^k})^{t_u} - 1, (-p^{2^k})^{n_u} - 1) = |(-p^{2^k})^{t_u} - 1| = p^t + 1$.

Suppose $2t \nmid n$. If $d \mid p^t + 1, q - 1$, then by (iv), $d \mid q + 1, q - 1$ and so $d \mid \gcd(q + 1, q - 1) = 2$. $\square$

The following lemma follows from [2, Lemma 3.3] (by replacing δ by 1).

Lemma 3.3. *Let* t, m *be positive integers. Suppose* $t \mid n$ *and* $2t \not| \, n$. *If* $2^m \mid q - 1$, *then* $2^m \mid p^t - 1$.

4. ACTION OF AUTOMORPHISMS ON IRREDUCIBLE CHARACTERS

Let $G = \mathrm{Sp}_4(q)$ be the four dimensional symplectic group defined over a finite field with $q = p^n$ elements (always assuming that p is odd). Let $O = \mathrm{Out}(G)$ and $A = G \rtimes O$, so that $O = \langle \phi \rangle \times \langle \alpha \rangle$, where ϕ is a diagonal automorphism of order 2 and α is a field automorphism of order n. We fix a Borel subgroup B and distinct maximal parabolic subgroups P and Q of G containing B as in [19].

In this section, we determine the action of the field automorphism $\langle \alpha \rangle$ on the irreducible characters of B, P, Q and G. Our notation for the parameter sets of these groups is similar to that of CHEVIE notation and is given in Table A.1 in the Appendix. The correspondence between the CHEVIE notation and that of Yamada is given in Tables A.2–A.4.

The first column of Table A.1 defines a name for the parameter set which parameterizes those characters which are listed in the second column of the table.

The list of parameters in the third column of Table A.1 in the Appendix is of the form

$$k = 0, \ldots, n_1 - 1 \qquad \text{or} \qquad \begin{array}{l} k = 0, \ldots, n_1 - 1 \\ l = 0, \ldots, n_2 - 1 \end{array},$$

where the n_j's are polynomials in q with integer coefficients. In the first case, the parameter k can be substituted by an element of $\mathbb{Z}$, but two parameters which differ by an element of $n_1\mathbb{Z}$ yield the same character. In the second case, the parameter vector (k, l) can be substituted by an element of $\mathbb{Z} \times \mathbb{Z}$, but two parameter vectors which differ by an element of $n_1\mathbb{Z} \times n_2\mathbb{Z}$ yield the same character. In other words, k can be taken to be an element of $\mathbb{Z}_{n_1}$ and (k, l) can be taken to be an element of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. The groups $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ are also called *character parameter groups* (see Section 3.7 of the CHEVIE [10] manual). The next lines of Table A.1 list elements which have to be excluded from the character parameter group. The remaining parameters are called *admissible* in the following. Different values of admissible parameters may give the same character. The fourth column of Table A.1 defines an equivalence relation on the set of admissible parameters. If no equivalence relation is listed we mean the identity relation. The parameter set is defined to be the set of these equivalence classes. Finally, the last column of Table A.1 gives the cardinality of the parameter set.

We consider the example ${}_PI_3$ in Table A.1. The character parameter group is $\mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$. The parameter vectors (k, l) and (l, k) yield the same character and the equivalence class of (k, l) is $\{(k, l), (l, k)\}$. Hence, the characters ${}_P\chi_3(k, l)$ are parameterized by the set

$${}_PI_3 = \{\{(k, l), (l, k)\} \mid (k, l) \in \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1} q - 1 \nmid k - l\}.$$

If we want to emphasize the dependence of a parameter set, say ${}_PI_3$, from q we write ${}_PI_3(q)$. Table A.1 does not give any detailed information about the parameter sets ${}_GI_7$, ${}_GI_9$, ${}_GI_{16}$, ${}_GI_{18}$, ${}_GI_{19}$, ${}_GI_{20}$, ${}_GI_{21}$, since we will not need an explicit knowledge of these sets (note that these parameter sets parameterize the regular semisimple irreducible characters of G). The data in Table A.1 is taken from [19].

The action of $O = \mathrm{Out}(G)$ on the conjugacy classes of elements of G, B, P, and Q induces an action of O on the sets $\mathrm{Irr}(G)$, $\mathrm{Irr}(B)$, $\mathrm{Irr}(P)$, and $\mathrm{Irr}(Q)$, and then an action on the parameter sets. Using the values of the irreducible characters of G, B, P, and Q on the classes listed in the last column of Tables A.2–A.4, we can describe the action of O on the parameter sets.

For an O-set I and each subgroup $H \leq O$, let $C_I(H)$ denote the set of fixed points of I under the action of H. In the following proposition, we determine $|C_I(H)|$, where I runs through all (disjoint) unions of parameter sets which are listed in Table A.5 except for ${}_GI_7 \cup_G I_9 \cup_G I_{16} \cup {}_GI_{18} \cup {}_GI_{19} \cup {}_GI_{20} \cup {}_GI_{21}$. This last union of parameter sets will be treated separately since it requires different methods.

Proposition 4.1. *Let* $G = \mathrm{Sp}_4(p^n)$, $t \mid n$ *and* $I \neq_G I_7 \cup_G I_9 \cup_G I_{16} \cup {}_GI_{18} \cup {}_GI_{19} \cup {}_GI_{20} \cup {}_GI_{21}$ *be one of the (disjoint) unions of parameter sets listed in Table* A.5. *If* $H = \langle \alpha^t \rangle$ *is a subgroup of* O, *then the second column of Table* A.5 *shows the number of fixed points* $|C_I(H)|$ *of* I *under the action of* H.

Proof. We have to consider the following parameter sets I.

First let

$$I \in \big\{{}_GI_{22}, {}_GI_{24}, {}_GI_{25} \cup {}_GI_{26} \cup {}_GI_{27} \cup {}_GI_{28} \cup {}_GI_{39} \cup {}_GI_{40} \cup {}_GI_{43} \cup {}_GI_{44}, {}_GI_{29} \cup {}_GI_{30} \cup {}_GI_{33} \cup {}_GI_{34} \cup {}_GI_{37} \cup {}_GI_{38} \cup {}_GI_{41} \cup {}_GI_{42}, {}_GI_{31} \cup {}_GI_{32} \cup {}_GI_{35} \cup {}_GI_{36}, {}_BI_8 \cup {}_BI_9, {}_BI_{10} \cup {}_BI_{11} \cup {}_BI_{12} \cup {}_BI_{13}, {}_PI_9 \cup {}_PI_{10}, {}_PI_{11} \cup {}_PI_{12} \cup {}_PI_{13} \cup {}_PI_{14}, {}_QI_{14} \cup {}_QI_{15}, {}_QI_{16} \cup {}_QI_{17} \cup {}_QI_{18} \cup {}_QI_{19} \cup {}_QI_{24} \cup {}_QI_{25} \cup {}_QI_{26} \cup {}_QI_{27}, {}_QI_{20} \cup {}_QI_{21}, {}_QI_{22} \cup {}_QI_{23}\big\}.$$

The degrees and character values on the conjugacy classes listed in Tables A.2–A.4 show $C_I(H) = I$ and hence $|C_I(H)| = |I|$. We demonstrate this for the parameter set $I = {}_PI_9 \cup {}_PI_{10}$. The degrees in Table A.4 show that ${}_P\theta_1(0)$, ${}_P\theta_1(1)$, ${}_P\theta_2(0)$, and ${}_P\theta_2(1)$ are the only irreducible characters of P of degree $\frac{1}{2}(q-1)(q^2-1)$. The class representatives in Table II-1 in [19] show that the conjugacy class $A_{11}(\delta)$ ($\delta = \pm 1$) is fixed by α, and we can see from the character Table II-2 of P in [19] that the values of ${}_P\theta_1(0)$, ${}_P\theta_1(1)$, ${}_P\theta_2(0)$, and ${}_P\theta_2(1)$ on $A_{11}(-1)$ are different. So ${}_P\theta_1(k)^\alpha = {}_P\theta_1(k)$ and ${}_P\theta_2(k)^\alpha = {}_P\theta_2(k)$ for $k = 0, 1$ and $|C_I(H)| = |I|$.

In each of the following cases, we have that the action of α on I is given by $x^\alpha = px$ for all $x \in I$ using the character values on the classes listed in the last column of Tables A.2–A.4. We demonstrate this for the parameter set $I = {}_PI_3 \cup {}_PI_4$. The degrees in Table A.4 show that the ${}_P\chi_3(k, l)$'s are the only irreducible characters of P of degree $q + 1$, so ${}_P\chi_3(k, l)^\alpha = {}_P\chi_3(k', l')$ for some $\{(k', l'), \dots\} \in {}_PI_3$. We see from the class representatives in Table II-1 in [19] that α acts on the semisimple conjugacy classes of P like the pth power map which implies that the values of ${}_P\chi_3(k', l')$ and ${}_P\chi_3(pk, pl)$ on the semisimple classes coincide. Then, the character values of ${}_P\chi_3(k, l)$ (see the character Table II-2 in [19]) imply that the values of ${}_P\chi_3(k', l')$ and ${}_P\chi_3(pk, pl)$ coincide on all classes, hence ${}_P\chi_3(k', l') = {}_P\chi_3(pk, pl)$ and therefore ${}_P\chi_3(k, l)^\alpha = {}_P\chi_3(pk, pl)$. Similarly, ${}_P\chi_4(k)^\alpha = {}_P\chi_4(pk)$. Hence, $x^\alpha = px$ for all $x \in I$.

Let $I \in \{{}_GI_1 \cup {}_GI_{10}, {}_GI_2 \cup {}_GI_{11}, {}_GI_3 \cup {}_GI_{12}, {}_GI_4 \cup {}_GI_{13}, {}_GI_5 \cup {}_GI_{14}, {}_GI_6 \cup {}_GI_{15}, {}_GI_8 \cup {}_GI_{17}, {}_PI_5 \cup {}_PI_8\}$. So these unions of parameter sets are isomorphic H-sets, so that we can assume $I = {}_GI_1 \cup {}_GI_{10}$. If $x = \{k, -k\} \in I$, then $x \in C_I(H)$ if and only if $(p^t - 1)k \equiv 0$ or $(p^t + 1)k \equiv 0$. Let

$$C_\pm := \big\{\{k, -k\} \in C_I(H) \mid (p^t \pm 1)k \equiv 0\big\},$$

so that $C_I(H) = C_- \cup C_+$ and $C_- \cap C_+ = \emptyset$. We claim

$$C_- = \left\{\{k, -k\} \in {}_GI_1 \mid k \text{ is a multiple of } \frac{q-1}{p^t - 1}\right\}.$$

The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_-$. If $x \in {}_GI_{10}$, then $(p^t - 1)k \equiv 0 \bmod q + 1$ and Lemma 3.2(ii) implies $2 \cdot k \equiv 0$, which is impossible. Hence $x \in {}_GI_1$ and $(p^t - 1)k \equiv 0 \bmod q - 1$. By Lemma 3.2(i), k is a multiple of $(q-1)/(p^t - 1)$, proving the claim. Now we consider C_+.

If $2t \mid n$, we claim $C_+ = \{\{k, -k\} \in {}_GI_1 \mid k \text{ is a multiple of } (q-1)/(p^t+1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_+$. If $x \in {}_GI_{10}$, then $(p^t+1)k \equiv 0 \bmod q+1$, and Lemma 3.2(iv) implies $2 \cdot k \equiv 0$, which is impossible. Hence $x \in {}_GI_1$ and $(p^t+1)k \equiv 0 \bmod q-1$. By Lemma 3.2(iii), k is a multiple of $(q-1)/(p^t+1)$, and the claim holds.

If $2t \nmid n$, we claim $C_+ = \{\{k, -k\} \in {}_GI_{10} \mid k \text{ is a multiple of } (q+1)/(p^t+1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_+$. If $x \in {}_GI_1$, then $(p^t+1)k \equiv 0 \bmod q-1$, and Lemma 3.2(iii) implies $2 \cdot k \equiv 0$, which is impossible. Hence $x \in {}_GI_{10}$ and $(p^t+1)k \equiv 0 \bmod q+1$. By Lemma 3.2(iv), k is a multiple of $(q+1)/(p^t+1)$, and the claim holds.

Thus in all cases, $|C_I(H)| = |C_-| + |C_+| = \frac{p^t-3}{2} + \frac{p^t-1}{2} = p^t - 2$.

Let $I = {}_BI_1$. If $(k, l) \in I$, then $(k, l) \in C_I(H)$ if and only if $(p^t-1)k \equiv 0$ and $(p^t-1)l \equiv 0 \bmod q-1$. By Lemma 3.2(i), this is equivalent with k, l are multiples of $\frac{q-1}{p^t-1}$. Thus, $|C_I(H)| = (p^t-1)^2$.

Let $I \in \{{}_BI_2, {}_BI_3, {}_PI_1, {}_PI_2, {}_QI_1, {}_QI_2, {}_QI_5, {}_QI_6, {}_QI_7, {}_QI_8, {}_QI_9\}$. If $k \in I$, then $k \in C_I(H)$ if and only if $(p^t-1)k \equiv 0 \bmod q-1$. So we get $C_I(H) = \{k \in I \mid k \text{ is a multiple of } (q-1)/(p^t-1)\}$ and $|C_I(H)| = p^t - 1$.

Let $I \in \{{}_BI_4, {}_BI_5, {}_BI_6, {}_BI_7, {}_PI_6, {}_PI_7\}$. Then these parameter sets are isomorphic H-sets, so that we can assume $I = {}_BI_4$. Let ${}_BI'_4$ (respectively, ${}_BI''_4$) be the parameter set of ${}_B\chi_4(k, 0)$ (respectively, ${}_B\chi_4(k, 1)$). Then $I = {}_BI'_4 \cup {}_BI''_4$. Since ${}_BI'_4$ and ${}_BI''_4$ are isomorphic H-set, so it is suffices to consider ${}_BI'_4$. If $x \in {}_BI'_4$, then $x \in C_{{}_BI'_4}(H)$ if and only if $(p^t-1)k \equiv 0 \bmod q-1$ and $l = 0$. So we get $C_{{}_BI'_4}(H) = \{(k, 0) \in {}_BI'_4 \mid k \text{ is a multiple of } (q-1)/(p^t-1)\}$ and $|C_I(H)| = 2(p^t-1)$.

Let $I = {}_PI_3 \cup {}_PI_4$. First, we compute $|C_{{}_PI_3}(H)|$. Let

$$U_i := \begin{cases} \{\{(k, l), (l, k)\} \in C_{{}_PI_3}(H) \mid p^tk \equiv k, p^tl \equiv l\} & \text{if } i = 1, \\ \{\{(k, l), (l, k)\} \in C_{{}_PI_3}(H) \mid p^tk \equiv l, p^tl \equiv k\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k, l), (l, k)\} \in {}_PI_3$, then $x \in U_1$ if and only if $(p^t-1)k \equiv 0$ and $(p^t-1)l \equiv 0 \bmod q-1$. By Lemma 3.2(i), this is equivalent with that k, l are multiples of $\frac{q-1}{p^t-1}$. Hence $|U_1| = (p^t-1)(p^t-2)/2$.

Suppose $2t \mid n$. If $x = \{(k, l), (l, k)\} \in {}_PI_3$, then $x \in U_2$ if and only if $p^tk \equiv l$ and $p^tl \equiv k \bmod q-1$. So we get $(p^{2t}-1)k \equiv 0 \bmod q-1$, and this is equivalent with k is a multiple $(q-1)/(p^{2t}-1)$. Now exclude those solutions with $k \equiv l$, which means $(p^t-1)k \equiv 0 \bmod q-1$. This gives us $|U_2| = p^t(p^t-1)/2$.

Suppose $2t \nmid n$. If $\{(k, l), (l, k)\} \in U_2$, then $(p^{2t}-1)k \equiv 0 \bmod q-1$. By Lemma 3.2(iii), this is equivalent with $2(p^t-1)k \equiv 0 \bmod q-1$. By Lemma 3.3, this is equivalent with $(p^t-1)k \equiv 0 \bmod q-1$. Then $k = p^tk = l$, a contradiction to the definition of ${}_PI_3$. Hence, $U_2 = \emptyset$. So

$$|C_{{}_PI_3}(H)| = |U_1| + |U_2| = \begin{cases} (p^t-1)^2 & \text{if } 2t \mid n, \\ (p^t-1)(p^t-2)/2 & \text{if } 2t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_PI_4}(H)|$. If $x = \{k, qk\} \in {}_PI_4$, then $x \in C_{{}_PI_4}(H)$ if and only if $(p^t-1)k \equiv 0$ or $(p^t-q)k \equiv 0 \bmod (q+1)(q-1)$. Suppose $(p^t-1)k \equiv 0$. By Lemma 3.2(i) and (ii), it follows that $\gcd(p^t-1, (q+1)(q-1)) = \gcd(p^t-1, q-1) = p^t-1$. Thus $(q+1) \cdot \frac{(q-1)}{p^t-1} \mid k$. But then $(q+1) \mid k$, a contradiction to the definition of ${}_PI_4$. So we have proved that $x \in C_{{}_PI_4}(H)$ if and only if $(p^t - q)k \equiv 0 \mod (q+1)(q-1)$.

Suppose $2t \mid n$. If $\{k, qk\} \in C_{{}_PI_4}(H)$, then $(p^t - q)k \equiv 0 \mod (q+1)(q-1)$. Thus $(p^t+1)k \equiv 0 \mod q+1$ and $(p^t-1)k \equiv 0 \mod q-1$. By Lemma 3.2(i) and (iv), we get $\frac{q+1}{2} \mid k$ and $\frac{q-1}{p^t-1} \mid k$. Since $\frac{q+1}{2} \mid q+1$ and $\frac{q-1}{p^t-1} \mid q-1$ and since $p^t - 1$ is even, we have $\gcd(\frac{q+1}{2}, \frac{q-1}{p^t-1}) = 1$ and so $\frac{q+1}{2} \cdot \frac{q-1}{p^t-1} \mid k$. The condition $2t \mid n$ implies $(p^t-1)(p^t+1) \mid p^n - 1 = q-1$, so that $\frac{q-1}{p^t-1}$ is even. Thus $q+1 \mid k$, a contradiction to the definition of ${}_PI_4$. Hence in this case $C_{{}_PI_4}(H) = \emptyset$.

Suppose $2t \nmid n$. We claim

$$C_{{}_PI_4}(H) = \left\{ \{k, qk\} \in {}_PI_4 \mid k \text{ is a multiple of } \frac{(q+1)(q-1)}{(p^t+1)(p^t-1)} \right\}.$$

Let $k = \frac{(q+1)(q-1)}{(p^t+1)(p^t-1)} \cdot m$ for some $m \in \mathbb{Z}$. Because $t \mid n$ and $2t \nmid n$, we have $2t \mid n - t$. Since $(p^t+1)(p^t-1) = p^{2t} - 1 \mid p^{n-t} - 1$, we then get $(p^{n-t} - 1)k = \frac{p^{n-t}-1}{(p^t+1)(p^t-1)}(q+1)(q-1) \cdot m \equiv 0 \mod (q+1)(q-1)$. So $(p^t - q)k \equiv 0 \mod (q+1)(q-1)$ and $\{k, qk\} \in C_{{}_PI_4}(H)$.

Conversely, suppose $\{k, qk\} \in C_{{}_PI_4}(H)$. Then $(p^t - q)k \equiv 0 \mod (q+1)(q-1)$. Hence $(p^t+1)k \equiv 0 \mod q+1$ and $(p^t-1)k \equiv 0 \mod q-1$. By Lemma 3.2(i) and (iv), this is equivalent with $\frac{q+1}{p^t+1} \mid k$ and $\frac{q-1}{p^t-1} \mid k$. Since $\frac{q+1}{p^t+1} \mid q+1$ and $\frac{q-1}{p^t-1} \mid q-1$ and since $\frac{q-1}{p^t-1}$ is odd by Lemma 3.3, we have $\gcd\left(\frac{q+1}{p^t+1}, \frac{q-1}{p^t-1}\right) = 1$. Therefore, $\frac{(q+1)(q-1)}{(p^t+1)(p^t-1)} \mid k$, and the claim holds. So by the definition of ${}_PI_4$, we get $|C_{{}_PI_4}(H)| = p^t(p^t-1)/2$.

So in both cases, $|C_I(H)| = |C_{{}_PI_3}(H)| + |C_{{}_PI_4}(H)| = (p^t-1)^2$.

Let $I = {}_QI_3 \cup {}_QI_4$. First, we compute $|C_{{}_QI_3}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l),(k,-l)\} \in C_{{}_QI_3}(H) \mid p^t k \equiv k, p^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k,l),(k,-l)\} \in C_{{}_QI_3}(H) \mid p^t k \equiv k, p^t l \equiv -l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k,l),(k,-l)\} \in {}_QI_3$, then $x \in U_1$ if and only if $(p^t-1)k \equiv 0$ and $(p^t-1)l \equiv 0 \mod q-1$. By Lemma 3.2(i), this is equivalent with k, l are multiples of $\frac{q-1}{p^t-1}$. Hence $|U_1| = (p^t-1)(p^t-3)/2$.

Suppose $2t \mid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_3$, then $x \in U_2$ if and only if $(p^t-1)k \equiv 0$ and $(p^t+1)l \equiv 0 \mod q-1$. Hence

$$U_2 = \left\{ \{(k,l),(k,-l)\} \in {}_QI_3 \,\middle|\, \frac{q-1}{p^t-1} \,\middle|\, k \text{ and } \frac{q-1}{p^t+1} \,\middle|\, l \right\}$$

and $|U_2| = (p^t-1)^2/2$.

Suppose $2t \nmid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_3$, then $x \in U_2$, then $(p^t-1)k \equiv 0$ and $(p^t+1)l \equiv 0 \mod q-1$. By Lemma 3.2(iii), the second congruence is equivalent with $2l \equiv 0 \mod q-1$, a contradiction to the definition of ${}_QI_3$. Hence, $U_2 = \emptyset$. So

$$|C_{{}_QI_3}(H)| = |U_1| + |U_2| = \begin{cases} (p^t-1)(p^t-2) & \text{if } 2t \mid n, \\ (p^t-1)(p^t-3)/2 & \text{if } 2t \nmid n. \end{cases}$$

Next we calculate $|C_{_QI_4}(H)|$. Let

$$U_i := \begin{cases} \{\{(k,l),(k,-l)\} \in C_{_QI_4}(H) \mid p^t k \equiv k, p^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k,l),(k,-l)\} \in C_{_QI_4}(H) \mid p^t k \equiv k, p^t l \equiv -l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k,l),(k,-l)\} \in {}_QI_4$, then $x \in U_1$ if and only if $(p^t-1)k \equiv 0 \bmod q-1$ and $(p^t-1)l \equiv 0 \bmod q+1$. By Lemma 3.2(i) and (ii), this is equivalent with $\frac{q-1}{p^t-1} \mid k$, and $\frac{q+1}{2} \mid l$, a contradiction to the definition of ${}_QI_4$. Hence, $U_1 = \emptyset$.

Suppose $2t \mid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_4$, then $x \in U_2$ if and only if $(p^t-1)k \equiv 0 \bmod q-1$ and $(p^t+1)l \equiv 0 \bmod q+1$. By Lemma 3.2(i) and (iv), this is equivalent with $\frac{q-1}{p^t-1} \mid k$, and $2l \equiv 0 \bmod q+1$, a contradiction to the definition of ${}_QI_4$. Hence, $U_2 = \emptyset$.

Suppose $2t \nmid n$. If $x = \{(k,l),(k,-l)\} \in {}_QI_4$, then $x \in U_2$ if and only if $(p^t-1)k \equiv 0 \bmod q-1$ and $(p^t+1)l \equiv 0 \bmod q+1$. Hence

$$U_2 = \left\{\{(k,l),(k,-l)\} \in {}_QI_4 \,\middle|\, \frac{q-1}{p^t-1} \,\middle|\, k \text{ and } \frac{q-1}{p^t+1} \,\middle|\, l\right\}$$

and $|U_2| = (p^t-1)^2/2$.

So

$$|C_{_QI_4}(H)| = |U_1| + |U_2| = \begin{cases} 0 & \text{if } 2t \mid n, \\ (p^t-1)^2/2 & \text{if } 2t \nmid n. \end{cases}$$

So in both cases, $|C_I(H)| = |C_{_QI_3}(H)| + |C_{_QI_4}(H)| = (p^t-1)(p^t-2)$.

Let $I \in \{{}_QI_{10} \cup {}_QI_{12}, {}_QI_{11} \cup {}_QI_{13}\}$. Then these unions of parameter sets are isomorphic H-sets, so that we can assume $I = {}_QI_{10} \cup {}_QI_{12}$. Let ${}_QI'_{10}$ and ${}_QI''_{10}$ (respectively, ${}_QI'_{12}$ and ${}_QI''_{12}$) be the parameter sets of ${}_Q\chi_{10}(k,0)$ and ${}_Q\chi_{10}(k,1)$ (respectively, ${}_Q\chi_{12}(0,l)$ and ${}_Q\chi_{12}(1,l)$). Then $I = {}_QI'_{10} \cup {}_QI''_{10} \cup {}_QI'_{12} \cup {}_QI''_{12}$. Since ${}_QI'_{10} \cup {}_QI'_{12} \simeq {}_QI''_{10} \cup {}_QI''_{12}$ as H-set, so it suffices to consider ${}_QI'_{10} \cup {}_QI'_{12}$. By construction and the definition of character parameter groups, ${}_QI'_{10} \cup {}_QI'_{12} \simeq {}_GI_1 \cup {}_GI_{10}$ as H-sets, so that we can identify ${}_QI'_{10} \cup {}_QI'_{12} = {}_GI_1 \cup {}_GI_{10}$. By the calculation above, we get

$$|C_{_QI'_{10} \cup {}_QI'_{12}}(H)| = |C_{_GI_1 \cup {}_QI_{10}}(H)| = p^t - 2$$

and

$$|C_I(H)| = |C_{_QI'_{10} \cup {}_QI'_{12} \cup {}_QI''_{10} \cup {}_QI''_{12}}(H)| = 2(p^t-2).$$

□

Now we deal with the regular semisimple irreducible characters of G.

Proposition 4.2. *Let* $G = \mathrm{Sp}_4(p^n)$, $t \mid n$, $I(p^n) := {}_GI_7(p^n) \cup {}_GI_9(p^n) \cup {}_GI_{16}(p^n) \cup {}_GI_{18}(p^n) \cup {}_GI_{19}(p^n) \cup {}_GI_{20}(p^n) \cup {}_GI_{21}(p^n)$ *and* $H = \langle \alpha^t \rangle$ *a subgroup of* O. *Then*

$$|C_I(H)| = (p^t-1)^2.$$

Proof. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$, $G^* = \mathrm{SO}_5(q)$ the dual group of G, and let $\overline{T}$ be the maximal torus of $\overline{G^*} = \mathrm{SO}_5(\mathbb{F})$ and $\overline{W} = N_{\overline{G^*}}(\overline{T})/\overline{T}$ the Weyl group.

Let $\chi = \chi(x)$ be an irreducible character of $\mathrm{Sp}_4(p^n)$ labelled by the parameter x given in [19]. Then $\chi(x)^\alpha = \chi(px)$ (using the character values on the classes listed in the last column of Table A.4, we known that the action of α on the parameter sets is given by $x^\alpha = px$). In addition, let T be a maximal torus of G^* and $s \in T$ such that χ is a constituent of the Deligne–Lusztig generalized character $R_T^{G^*}(s)$ (see [6, Corollary 7.5.8]). It follows by the degree of χ that $\chi = \pm R_T^{G^*}(s)$ for some sign $\pm$ and $C_{G^*}(s) = T$. Thus $C_{\overline{G^*}}(s) = \overline{T}$ and $\chi^{\alpha^t} = \chi$ if and only if $(s)_{G^*}^{\alpha^t} = (s)_{G^*}$; namely, $s^{\alpha^t} = s^w$ for some $w \in \overline{W}$, where $(s)_{G^*}$ is the conjugacy class of G^* containing s and α is also viewed as the standard field automorphism of G^*. Thus $\chi^{\alpha^t} = \chi$ if and only if $s \in C_{\overline{T}}(\alpha^t w^{-1})$; namely, s is a regular element of $\mathrm{SO}_5(p^t)$, since $C_{\overline{T}}(\alpha^t w^{-1})$ is conjugate to a maximal torus of $\mathrm{SO}_5(p^t)$. But a regular element s of $\mathrm{SO}_5(p^t)$ corresponds to the irreducible character $\psi = \pm R^{\mathrm{SO}_5(p^t)}_{C_{\mathrm{SO}_5(p^t)}(s)}(s)$ of $\mathrm{Sp}_4(p^t)$, and its parameter y (see [19]) lies in $I(p^t)$. It follows that

$$C_{I(p^n)}(H) \simeq I(p^t)$$

as H-sets, and $|C_{I(p^n)}(H)| = |I(p^t)| = (p^t - 1)^2$. □

5. UNO'S INVARIANT CONJECTURE FOR $\mathrm{Sp}_4(q)$, q ODD

In this section, we prove Uno's invariant conjecture for $G = \mathrm{Sp}_4(q)$ in the defining characteristic p, where $q = p^n$ with an odd prime p. By [11, p. 152], G has only two p-blocks, the principal block $B_0 = B_0(G)$ and one defect-0-block consisting of the Steinberg character. Hence we have to verify Uno's conjecture only for B_0.

As in the previous section, let $O = \mathrm{Out}(G) = \langle \phi \rangle \times \langle \alpha \rangle$ and $A = G \rtimes O$, where ϕ is a diagonal automorphism of order 2 and α is a field automorphism of G with order n. Fix a Borel subgroup B and maximal parabolic subgroups P and Q of G containing B as in [19]. In particular, we may assume that O stabilizes B, P, and Q.

By a corollary of the Borel–Tits theorem [5], the normalizers of radical p-subgroups are parabolic subgroups. The radical p-chains of G (up to G-conjugacy) are given in Table 1.

Since C_5 and C_6 have the same normalizers $N_G(C_5) = N_G(C_6) = B$ and $N_A(C_5) = N_A(C_6) = B \rtimes O$, it follows that

$$\mathrm{k}(N_G(C_5), B_0, d, U, [r]) = \mathrm{k}(N_G(C_6), B_0, d, U, [r])$$

Table 1 Radical p-chains of G

C		$N_G(C)$	$N_A(C)$
C_1	$\{1\}$	G	A
C_2	$\{1\} < O_p(P)$	P	$P \rtimes O$
C_3	$\{1\} < O_p(P) < O_p(B)$	B	$B \rtimes O$
C_4	$\{1\} < O_p(Q)$	Q	$Q \rtimes O$
C_5	$\{1\} < O_p(Q) < O_p(B)$	B	$B \rtimes O$
C_6	$\{1\} < O_p(B)$	B	$B \rtimes O$

for all $d \in \mathbb{N}$, $U \leq O$ and $1 \leq r < (p+1)/2$. Thus the contribution of C_5 and C_6 in the alternating sum of Uno's invariant conjecture is zero. So Uno's invariant conjecture for G is equivalent to

$$\mathrm{k}(G, B_0, d, U, [r]) + \mathrm{k}(B, B_0, d, U, [r]) = \mathrm{k}(P, B_0, d, U, [r]) + \mathrm{k}(Q, B_0, d, U, [r]) \tag{1}$$

for all $d \in \mathbb{N}$, $U \leq O$ and $1 \leq r < (p+1)/2$.

Theorem 5.1. *Let $p > 2$ be a prime and $\widetilde{B}$ a p-block of $G = \mathrm{Sp}_4(p^n)$ of positive defect. Then $\widetilde{B}$ satisfies Uno's invariant conjecture.*

Proof. By the proceeding remarks, we can suppose $\widetilde{B} = B_0$. Let $S \in \{G, B, P, Q\}$. By the character tables in [19], we have $\mathrm{k}(S, B_0, d, U, [r]) = 0$ when $d \notin \{2n, 3n, 4n\}$ or $[r] \notin \{[1], [2], [3], [4]\}$. Let U be a subgroup of $O = \mathrm{Out}(G)$.

(A) Suppose $U \leq \langle\alpha\rangle$ with $|U| = u$.
Suppose, moreover that $u \mid n$ and set $t := \frac{n}{u}$ and $H := \langle\alpha^t\rangle$.

(i) If $d = 2n$ and $[r] = [2]$, then by Tables A.2 and A.5, we have $\mathrm{k}(B, B_0, d, U, [2]) = \mathrm{k}(P, B_0, d, U, [2]) = 0$ and

$$\mathrm{k}(G, B_0, d, U, [2]) = \sum_{j \in J_G} |C_{{}_G I_j}(H)| = \mathrm{k}(Q, B_0, d, U, [2]) = \sum_{j \in J_Q} |C_{{}_Q I_j}(H)| = 4$$

with the index sets $J_G := \{31, 32, 35, 36\}$, $J_Q := \{22, 23\}$. Thus (1) holds in this case.

(ii) If $p \geq 5$, $d = 3n$ and $r = 1$, then Table A.3 implies that (1) is equivalent to

$$\sum_{j \in J_G} |C_{{}_G I_j}(H)| + \sum_{j \in J_B} |C_{{}_B I_j}(H)| = \sum_{j \in J_P} |C_{{}_P I_j}(H)| + \sum_{j \in J_Q} |C_{{}_Q I_j}(H)|, \tag{2}$$

where the index sets $J_G := \{2, 8, 11, 17, 24\}$, $J_P := \{2, 5, 8\}$,

$$J_B := \begin{cases} \{3, 10, 11, 12, 13\} & \text{if } p = 5, \\ \{3\} & \text{if } p \geq 7, \end{cases}$$

and

$$J_Q := \begin{cases} \{2, 16, 17, 18, 19, 24, 25, 26, 27\} & \text{if } p = 5, \\ \{2\} & \text{if } p \geq 7. \end{cases}$$

By Table A.5, we have

$$\sum_{j \in J_G} |C_{{}_G I_j}(H)| + \sum_{j \in J_B} |C_{{}_B I_j}(H)| = \begin{cases} 3p^t + 12 & \text{if } p = 5, \\ 3p^t - 4 & \text{if } p \geq 7, \end{cases}$$

and

$$\sum_{j\in J_P}|C_{PI_j}(H)|+\sum_{j\in J_Q}|C_{QI_j}(H)|=\begin{cases}3p^t+12 & \text{if } p=5,\\ 3p^t-4 & \text{if } p\geq 7.\end{cases}$$

Thus (1) holds in this case.

If $p\geq 5$, $d=3n$, and $r=2$, then (1) is also equivalent to (2), where by Table A.3, the index sets $J_G:=\{25,26,27,28,39,40,43,44\}$, $J_B:=\{6,7\}$, $J_P:=\{11,12,13,14\}$, and $J_Q:=\{10,11,12,13,20,21\}$. By Table A.5, the sums on both sides of the Eq. (2) are equal to $4p^t+4$. Thus (1) also holds in this case.

Suppose $d=3n$, $p=7$ or $p\geq 11$, and $r=3$ or 4 according as $p=7$ or $p\geq 11$. Then (1) is equivalent to (2) with $J_G=J_P=\emptyset$, $J_B:=\{10,11,12,13\}$, and $J_Q:=\{16,17,18,19,24,25,26,27\}$. By Table A.5, the sums on both sides of Eq. (2) are equal to 16. Thus (1) holds in this case.

If $p=3$ and $d=3n$, then $[1]=[2]=[4]$, $\mathrm{k}(K,B_0,3n,U,[1])=\mathrm{k}(K,B_0,3n,U)$ for $K\in\{G,B,P,Q\}$ and Eq. (1) (with $[r]=[1]$) is equivalent to (2), where by Table A.3, the index sets $J_G:=\{2,8,11,17,24,25,26,27,28,39,40,43,44\}$, $J_B:=\{3,6,7,10,11,12,13\}$, $J_P:=\{2,5,8,11,12,13,14\}$, and

$$J_Q:=\{2,10,11,12,13,16,17,18,19,20,21,24,25,26,27\}.$$

The same proof as above (with $r=1,2$, and 4) implies that (1) holds in this case.

(iii) If $p\geq 5$, $d=4n$, and $r=1$, then by Table A.4, (1) is equivalent to (2) with $J_G:=\{1,7,9,10,16,18,19,20,21,22\}$, $J_B:=\{1,2\}$, $J_P:=\{1,3,4\}$, and $J_Q:=\{1,3,4,9\}$. By Table A.5, the sums on both sides of Eq. (2) are equal to $2p^t(p^t-1)$.

If $p\geq 5$, $d=4n$, and $r=2$, then by Table A.4, (1) is equivalent to (2) with $J_G:=\{3,4,5,6,12,13,14,15,29,30,33,34,37,38,41,42\}$, $J_B:=\{4,5,8,9\}$, $J_P:=\{6,7,9,10\}$, and $J_Q:=\{5,6,7,8,14,15\}$. By Table A.5, the sums on both sides of Eq. (2) are equal to $8p^t$. Thus (1) also holds in this case.

If $p=3$ and $d=4n$, then $[1]=[2]$, $\mathrm{k}(K,B_0,4n,U,[1])=\mathrm{k}(K,B_0,4n,U)$ for $K\in\{G,B,P,Q\}$ and Eq. (1) (with $[r]=[1]$) is equivalent to (2), where by Table A.4, the index sets $J_G:=\{1,3,4,5,6,7,9,10,12,13,14,15,16,18,19,20,21,22,29,30,33,34,37,38,41,42\}$, $J_B:=\{1,2,4,5,8,9\}$, $J_P:=\{1,3,4,6,7,9,10\}$, and $J_Q:=\{1,3,4,5,6,7,8,9,14,15\}$. The same proof as above (with $r=1$ and 2) implies that (1) holds in this case.

(B) Suppose $U=\langle\phi\rangle$.

(i) In addition, suppose $d=2n$. By Table A.2, we have that $\mathrm{k}(B,B_0,d)=\mathrm{k}(P,B_0,d)=0$, $\mathrm{Irr}(G,B_0,d)=\mathrm{Irr}(G,B_0,d,[2])$, and $\mathrm{Irr}(B,B_0,d)=\mathrm{Irr}(B,B_0,d,[2])$, where

$$\mathrm{Irr}(K,B_0,d)=\mathrm{Irr}(K,B_0,d,1)\cup\mathrm{Irr}(K,B_0,d,U)$$

and $\mathrm{Irr}(K,B_0,d,[r])=\mathrm{Irr}(K,B_0,d)\cap\mathrm{Irr}(K,B_0,[r])$ for $K\in\{G,B,P,Q\}$ and $\mathrm{k}(K,B_0,d)=|\mathrm{Irr}(K,B_0,d)|$. Thus it suffices to consider the action of ϕ on $\mathrm{Irr}(G,B_0,d)$ and $\mathrm{Irr}(Q,B_0,d)$.

Since ϕ swaps the G-classes $A_{11}(1)$ and $A_{12}(1)$, it follows from the values of characters of $\mathrm{Irr}(G, B_0, d)$ on $A_{11}(1)$ and $A_{12}(1)$ that ϕ acts on $\mathrm{Irr}(G, B_0, d)$ as the permutation

$$(\theta_9, \theta_{10})(\theta_{13}, \theta_{14}).$$

Thus, ϕ does not fix any character in $\mathrm{Irr}(G, B_0, d) = \{\theta_9, \theta_{10}, \theta_{13}, \theta_{14}\}$, and so

$$C_{\mathrm{Irr}(G,B_0,2n)}(\phi) = \emptyset.$$

Since ϕ swaps the Q-classes $A_{11}(1)$ and $A_{12}(1)$, it follows from the values of characters of $\mathrm{Irr}(Q, B_0, d)$ on $A_{11}(1)$ and $A_{12}(1)$ that ϕ acts on $\mathrm{Irr}(Q, B_0, d)$ as

$$\left({}_Q\theta_9(0), {}_Q\theta_{10}(0)\right)\left({}_Q\theta_9(1), {}_Q\theta_{10}(1)\right).$$

Thus, ϕ does not fix any character in $\mathrm{Irr}(Q, B_0, d) = \{{}_Q\theta_9(0), {}_Q\theta_{10}(0), {}_Q\theta_9(1), {}_Q\theta_{10}(1)\}$, and so

$$C_{\mathrm{Irr}(Q,B_0,2n)}(\phi) = \emptyset.$$

Thus (1) holds in this case.

(ii) Suppose $d = 3n$. Using the degrees and values of characters on the G-classes $A_{11}(1)$, $A_{12}(1)$, B-classes $A_{11}(1)$, $A_{12}(1)$, P-classes B_{11}, B_{12}, and Q-classes $A_{11}(1)$, $A_{12}(1)$, we get the action of ϕ on $\mathrm{Irr}(G, B_0, d)$ by

$$\begin{aligned}&(\chi_2(k), \chi_2(k'))(\chi_8(l), \chi_8(l'))(\chi_{11}(m), \chi_{11}(m'))(\chi_{17}(n), \chi_{17}(n'))\\&(\theta_2)(\theta_3)(\theta_4)(\theta_5)(\theta_6)(\theta_{17}, \theta_{18})(\theta_{21}, \theta_{22}),\end{aligned}$$

on $\mathrm{Irr}(B, B_0, d)$ by

$$\begin{aligned}&({}_B\chi_3(k), {}_B\chi_3(k'))({}_B\chi_6(m, n), {}_B\chi_7(m', n'))\\&({}_B\theta_3(a, b), {}_B\theta_6(a', b'))({}_B\theta_4(c, d), {}_B\theta_5(c', d')),\end{aligned}$$

on $\mathrm{Irr}(P, B_0, d)$ by

$$\begin{aligned}&({}_P\chi_2(k), {}_P\chi_2(k'))({}_P\chi_5(l), {}_P\chi_5(l'))({}_P\chi_8(m), {}_P\chi_8(m'))({}_P\theta_3(0))({}_P\theta_3(1))\\&({}_P\theta_4(0), {}_P\theta_4(1))({}_P\theta_5(0))({}_P\theta_5(1))({}_P\theta_6(0), {}_P\theta_6(1)),\end{aligned}$$

and on $\mathrm{Irr}(Q, B_0, d)$ by

$$\begin{aligned}&({}_Q\chi_2(k), {}_Q\chi_2(k'))({}_Q\chi_{10}(m, n), {}_Q\chi_{11}(m', n'))({}_Q\chi_{12}(a, b), {}_Q\chi_{13}(a', b'))({}_Q\theta_3(0), {}_Q\theta_4(0))\\&({}_Q\theta_3(1), {}_Q\theta_4(1))({}_Q\theta_5(0), {}_Q\theta_6(0))({}_Q\theta_5(1), {}_Q\theta_6(1))({}_Q\theta_7(0), {}_Q\theta_8(0))({}_Q\theta_7(1), {}_Q\theta_8(1))\\&({}_Q\theta_{11}(0), {}_Q\theta_{12}(0))({}_Q\theta_{11}(1), {}_Q\theta_{12}(1))({}_Q\theta_{13}(0), {}_Q\theta_{14}(0))({}_Q\theta_{13}(1), {}_Q\theta_{14}(1)).\end{aligned}$$

(iii) Suppose $d = 4n$. Using the degrees and values of characters on the G-classes $A_{11}(1)$, $A_{12}(1)$, B-classes $A_{31}(1)$, $A_{32}(1)$, P-classes $A_{11}(1)$, $A_{12}(1)$, and Q-classes $A_{31}(1)$, $A_{32}(1)$, we get the action of ϕ on $\mathrm{Irr}(G, B_0, d)$ by

$$(\chi_1(k), \chi_1(k'))(\chi_3(l), \chi_4(l'))(\chi_5(m), \chi_6(m'))(\chi_7(n), \chi_7(n'))$$
$$(\chi_9(a, b), \chi_9(a', b'))(\chi_{10}(c), \chi_{10}(c'))(\chi_{12}(d), \chi_{13}(d'))(\chi_{14}(e), \chi_{15}(e'))$$
$$(\chi_{16}(f), \chi_{16}(f'))(\chi_{18}(g, h), \chi_{18}(g', h'))(\chi_{19}(i, j), \chi_{19}(i', j'))(\chi_{20}(r), \chi_{20}(r'))$$
$$(\chi_{21}(s), \chi_{21}(s'))(\theta_0)(\theta_7, \theta_8)(\theta_{11}, \theta_{12})(\theta_{15}, \theta_{16})(\theta_{19}, \theta_{20}),$$

on $\mathrm{Irr}(B, B_0, d)$ by

$$({}_B\chi_1(k, l), {}_B\chi_1(k', l'))({}_B\chi_2(m), {}_B\chi_2(m'))$$
$$({}_B\chi_4(a, b), {}_B\chi_5(a', b'))({}_B\theta_1(0), {}_B\theta_2(0))({}_B\theta_1(1), {}_B\theta_2(1)),$$

on $\mathrm{Irr}(P, B_0, d)$ by

$$({}_P\chi_1(k), {}_P\chi_1(k'))({}_P\chi_3(m, n), {}_P\chi_3(m', n'))({}_P\chi_4(a), {}_P\chi_4(a'))$$
$$({}_P\chi_6(c, d), {}_P\chi_7(c', d'))({}_P\theta_1(0), {}_P\theta_2(0))({}_P\theta_1(1), {}_P\theta_2(1)),$$

and on $\mathrm{Irr}(Q, B_0, d)$ by

$$({}_Q\chi_1(k), {}_Q\chi_1(k'))({}_Q\chi_3(m, n), {}_Q\chi_3(m', n'))$$
$$({}_Q\chi_4(a, b), {}_Q\chi_4(a', b'))({}_Q\chi_5(c), {}_Q\chi_6(c'))({}_Q\chi_7(d), {}_Q\chi_8(d'))$$
$$({}_Q\chi_9(e), {}_Q\chi_9(e'))({}_Q\theta_1(0), {}_Q\theta_2(0))({}_Q\theta_1(1), {}_Q\theta_2(1)).$$

(iv) By proofs (i)–(iii) above, we have that $\mathrm{Irr}(G, B_0, 2n)$ contains two orbits of length 2 (called 2-cycles) under the action of ϕ, $\mathrm{Irr}(G, B_0, 3n)$ contains at least two 2-cycles, and $\mathrm{Irr}(G, B_0, 4n)$ contains at least $4 + (q-3) + (q-1)$ 2-cycles. Thus $\mathrm{Irr}(G)$ contains at least $8 + (q-3) + (q-1)$ 2-cycles under the action of ϕ. Let $\mathscr{C}\ell(G)$ be the set of conjugacy classes of G. By [19, Table IV-1], $\mathscr{C}\ell(G)$ contains exactly $8 + (q-3) + (q-1)$ 2-cycles under the action of ϕ. It follows from Brauer's permutation lemma [15, Lemma 3.2.19] that $\mathrm{Irr}(G)$ has exactly $8 + (q-3) + (q-1)$ 2-cycles under the action of ϕ. It follows that

$$C_{\mathrm{Irr}(G,B_0,3n)}(\phi) = \{\chi_2(k), \chi_8(k), \chi_{11}(k), \chi_{17}(k), \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$$

and $C_{\mathrm{Irr}(G,B_0,4n)}(\phi) = \{\chi_1(k), \chi_7(k), \chi_9(k, l), \chi_{10}(k), \chi_{16}(k), \chi_{18}(k, l), \chi_{19}(k, l), \theta_0, \chi_{20}(k), \chi_{21}(k)\}$.

Similarly, $\mathrm{Irr}(B)$ contains at least $10 + 4(q-1)$ 2-cycles, and by [19, Table I-1], $\mathscr{C}\ell(B)$ has exactly $18 + 4(q-3)$ 2-cycles under the action of ϕ. So Brauer's permutation lemma implies that $\mathrm{Irr}(B)$ has exactly $10 + 4(q-1)$ 2-cycle orbits. Thus

$$C_{\mathrm{Irr}(B,B_0,d)}(\phi) = \begin{cases} \{{}_B\chi_3(k)\} & \text{if } d = 3n, \\ \{{}_B\chi_1(k, l), {}_B\chi_2(k)\} & \text{if } d = 4n. \end{cases}$$

By proofs (i), (ii), and (iii) and [19, Table II-1], $\mathrm{Irr}(P)$ contains at least $4+2(q-1)$ 2-cycles, $\mathscr{C}\ell(P)$ has exactly $8+2(q-3)$ 2-cycles under the action of ϕ, and by Brauer's permutation lemma, $\mathrm{Irr}(P)$ contains exactly $4+2(q-1)$ 2-cycles. Thus

$$C_{\mathrm{Irr}(P,B_0,d)}(\phi)=\begin{cases}\{{}_P\chi_2(k),\,{}_P\chi_5(k),\,{}_P\chi_8(k),\,{}_P\theta_3(k),\,{}_P\theta_5(k)\} & \text{if } d=3n,\\ \{{}_P\chi_1(k),\,{}_P\chi_3(k,l),\,{}_P\chi_4(k)\} & \text{if } d=4n.\end{cases}$$

Similarly, $\mathrm{Irr}(Q)$ contains at least $14+(q-3)+(q-1)+2(q-1)$ 2-cycles, $\mathscr{C}\ell(Q)$ has exactly $18+3(q-3)+(q-1)$ 2-cycles under the action of ϕ, and by the Brauer permutation lemma, $\mathrm{Irr}(Q)$ contains exactly $14+3(q-1)+(q-3)$ 2-cycles. Thus

$$C_{\mathrm{Irr}(Q,B_0,d)}(\phi)=\begin{cases}\{{}_Q\chi_2(k)\} & \text{if } d=3n,\\ \{{}_Q\chi_1(k),\,{}_Q\chi_3(k,l),\,{}_Q\chi_4(k,l),\,{}_Q\chi_9(k)\} & \text{if } d=4n.\end{cases}$$

By Table A.3,

$$\mathrm{k}(K,B_0,3n,\langle\phi\rangle,[r])+\mathrm{k}(H,B_0,3n,\langle\phi\rangle,[r])=\begin{cases}3q-4 & \text{if } r=1 \text{ and } p\geq 5,\\ 4 & \text{if } r=2 \text{ and } p\geq 5,\\ 3q & \text{if } r=1 \text{ and } p=3,\end{cases}$$

where $(K,H)=(G,B)$ or (P,Q). Thus (1) holds in this case.

Note that by Table A.4, $C_{\mathrm{Irr}(K,B_0,4n)}(\phi)=\mathrm{Irr}(K,B_0,4n,1,[1])$ for each $K\in\{G,B,P,Q\}$. By the proof (A) above,

$$\mathrm{k}(G,B_0,4n,1,[1])+\mathrm{k}(B,B_0,4n,1,[1])=\mathrm{k}(P,B_0,4n,1,[1])+\mathrm{k}(Q,B_0,4n,1,[1])$$

and so

$$\begin{aligned}&\mathrm{k}(G,B_0,4n,\langle\phi\rangle,[r])+\mathrm{k}(B,B_0,4n,\langle\phi\rangle,[r])\\ &\quad=\mathrm{k}(P,B_0,4n,\langle\phi\rangle,[r])+\mathrm{k}(Q,B_0,4n,\langle\phi\rangle,[r])\end{aligned}$$

for all $[r]$. Thus (1) holds in this case.

(C) Suppose $U\leq O=\langle\alpha\rangle\times\langle\phi\rangle$. By proofs **(A)** and **(B)**, there exists a bijection ψ from

$$\mathrm{Irr}(G,B_0,d,[r])\cup\mathrm{Irr}(B,B_0,d,[r]) \text{ onto } \mathrm{Irr}(P,B_0,d,[r])\cup\mathrm{Irr}(Q,B_0,d,[r]),$$

such that

$$\psi(\chi)^x=\psi(\chi^x),\quad \chi\in\mathrm{Irr}(G,B_0,d,[r])\cup\mathrm{Irr}(B,B_0,d,[r])$$

for any $x\in\langle\alpha\rangle$ or $x\in\langle\phi\rangle$.

Let $y\in U$, so that $y=y_1y_2$ with $y_1\in\langle\alpha\rangle$ and $y_2\in\langle\phi\rangle$. Thus $\psi(\chi)^y=\psi(\chi)^{y_1y_2}=\psi(\chi^{y_1})^{y_2}=\psi(\chi^y)$ and hence (1) holds. This completes the proof of Theorem 5.1.

APPENDIX

Table A.1 Parameter sets for the irreducible characters of the parabolic subgroups $G = Sp_4(q)$, B, P, Q

Parameter set	Characters	Parameters	Equivalence relation	Number of characters
${}_GI_1 = \ldots$ $= {}_GI_6$	$\chi_1(k), \ldots, \chi_6(k)$	$k = 0, \ldots, q-2$ $k \neq 0, \frac{q-1}{2}$	$\{k \equiv -k\}$	$\frac{q-3}{2}$
${}_GI_7$	$\chi_7(k)$	see the remarks in Section 4		$\frac{q-3}{2}$
${}_GI_8$	$\chi_8(k)$	$k = 0, \ldots, q-2$ $k \neq 0, \frac{q-1}{2}$	$\{k \equiv -k\}$	$\frac{q-3}{2}$
${}_GI_9$	$\chi_9(k, l)$	see the remarks in Section 4		$\frac{(q-3)(q-5)}{8}$
${}_GI_{10} = \ldots$ $= {}_GI_{15}$	$\chi_{10}(k), \ldots, \chi_{15}(k)$	$k = 0, \ldots, q$ $k \neq 0, \frac{q+1}{2}$	$\{k \equiv -k\}$	$\frac{q-1}{2}$
${}_GI_{16}$	$\chi_{16}(k)$	see the remarks in Section 4		$\frac{q-1}{2}$
${}_GI_{17}$	$\chi_{17}(k)$	$k = 0, \ldots, q$ $k \neq 0, \frac{q+1}{2}$	$\{k \equiv -k\}$	$\frac{q-1}{2}$
${}_GI_{18}$	$\chi_{18}(k, l)$	see the remarks in Section 4		$\frac{(q-1)(q-3)}{8}$
${}_GI_{19}$	$\chi_{19}(k, l)$	see the remarks in Section 4		$\frac{(q-1)(q-3)}{4}$
${}_GI_{20}$	$\chi_{20}(k)$	see the remarks in Section 4		$\frac{(q-1)^2}{4}$
${}_GI_{21}$	$\chi_{21}(k)$	see the remarks in Section 4		$\frac{q^2-1}{4}$
${}_GI_{22} = \ldots$ $= {}_GI_{44}$	$\theta_0, \ldots, \theta_{22}$			1
${}_BI_1$	${}_B\chi_1(k, l)$	$k, l = 0, \ldots,$ $q-2$		$(q-1)^2$
${}_BI_2 = {}_BI_3$	${}_B\chi_2(k), {}_B\chi_3(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_BI_4 = \ldots$ $= {}_BI_7$	${}_B\chi_4(k, l), \ldots,$ ${}_B\chi_7(k, l)$	$k = 0, \ldots, q-2$ $l = 0, 1$		$2(q-1)$
${}_BI_8 = {}_BI_9$	${}_B\theta_1(k), {}_B\theta_2(k)$	$k = 0, 1$		2
${}_BI_{10} = \ldots$ $= {}_BI_{13}$	${}_B\theta_3(k, l), \ldots,$ ${}_B\theta_6(k, l)$	$k, l = 0, 1$		4
${}_PI_1 = {}_PI_2$	${}_P\chi_1(k), {}_P\chi_2(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_PI_3$	${}_P\chi_3(k, l)$	$k, l = 0, \ldots, q-2$ $q-1 \nmid k-l$	$\{(k, l) \equiv (l, k)\}$	$\frac{(q-1)(q-2)}{2}$
${}_PI_4$	${}_P\chi_4(k)$	$k = 0, \ldots, q^2-2$ $q+1 \nmid k$	$\{k \equiv qk\}$	$\frac{q(q-1)}{2}$
${}_PI_5$	${}_P\chi_5(k)$	$k = 0, \ldots, q-2$ $k \neq 0, \frac{q-1}{2}$	$\{k \equiv -k\}$	$\frac{q-3}{2}$
${}_PI_6 = {}_PI_7$	${}_P\chi_6(k, l), {}_P\chi_7(k, l)$	$k = 0, \ldots, q-2$ $l = 0, 1$		$2(q-1)$
${}_PI_8$	${}_P\chi_8(k)$	$k = 0, \ldots, q$ $k \neq 0, \frac{q+1}{2}$	$\{k \equiv -k\}$	$\frac{q-1}{2}$
${}_PI_9 = \ldots = {}_PI_{14}$	${}_P\theta_1(k), \ldots, {}_P\theta_6(k)$	$k = 0, 1$		2
${}_QI_1 = {}_QI_2$	${}_Q\chi_1(k), {}_Q\chi_2(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_QI_3$	${}_Q\chi_3(k, l)$	$k, l = 0, \ldots, q-$ 2 $l \neq 0, \frac{q-1}{2}$	$\{(k, l) \equiv (k, -l)\}$	$\frac{(q-1)(q-3)}{2}$
${}_QI_4$	${}_Q\chi_4(k, l)$	$k = 0, \ldots, q-2$ $l = 0, \ldots, q$ $l \neq 0, \frac{q+1}{2}$	$\{(k, l) \equiv (k, -l)\}$	$\frac{(q-1)^2}{2}$

(continued)

Table A.1 Continued

Parameter set	Characters	Parameters	Equivalence relation	Number of characters
${}_QI_5 = \cdots = {}_QI_9$	${}_Q\chi_5(k), \ldots, {}_Q\chi_9(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_QI_{10} = {}_QI_{11}$	${}_Q\chi_{10}(k,l), {}_Q\chi_{11}(k,l)$	$k = 0, \ldots, q-2$ $l = 0, 1$ $k \neq 0, \frac{q-1}{2}$	$\{(k,l) \equiv (-k,l)\}$	$q-3$
${}_QI_{12} = {}_QI_{13}$	${}_Q\chi_{12}(k,l), {}_Q\chi_{13}(k,l)$	$k = 0, 1$ $l = 0, \ldots, q$ $l \neq 0, \frac{q+1}{2}$	$\{(k,l) \equiv (k,-l)\}$	$q-1$
${}_QI_{14} = \cdots = {}_QI_{27}$	${}_Q\theta_1(k), \ldots, {}_Q\theta_{14}(k)$	$k = 0, 1$		2

Table A.2 The irreducible characters of the chain normalizers in $Sp_4(q)$ of defect $2n$

	Character	Degree	Param. set	Param. Yamada	Number	Class
G	θ_9	$q^2(q^2+1)/2$	${}_GI_{31}$		1	$A_{11}(1)$
	θ_{10}	$q^2(q^2+1)/2$	${}_GI_{32}$		1	$A_{11}(1)$
	θ_{13}	$q^2(q^2-1)/2$	${}_GI_{35}$		1	$A_{11}(1)$
	θ_{14}	$q^2(q^2-1)/2$	${}_GI_{36}$		1	$A_{11}(1)$
Q	${}_Q\theta_9(k)$	$q^2(q-1)/2$	${}_QI_{22}$		2	$A_{11}(-1)$
	${}_Q\theta_{10}(k)$	$q^2(q-1)/2$	${}_QI_{23}$		2	$A_{11}(-1)$

Table A.3 The irreducible characters of the chain normalizers in $Sp_4(q)$ of defect $3n$

	Character	Degree	Param. set	Param. Yamada	Number	Class
G	$\chi_2(k)$	$q(q+1)(q^2+1)$	${}_GI_2$	${}^2R_1$	$\frac{q-3}{2}$	$C_{11}(1,i)$
	$\chi_8(k)$	$q(q+1)(q^2+1)$	${}_GI_8$	${}^2R_1$	$\frac{q-3}{2}$	$C_{11}(1,i)$
	$\chi_{11}(k)$	$q(q-1)(q^2+1)$	${}_GI_{11}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{11}(1,i)$
	$\chi_{17}(k)$	$q(q-1)(q^2+1)$	${}_GI_{17}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{11}(1,i)$
	θ_2	$q(q^2+1)$	${}_GI_{24}$		1	
	θ_3	$q(q+1)^2/2$	${}_GI_{25}$		1	
	θ_4	$q(q-1)^2/2$	${}_GI_{26}$		1	
	θ_5	$q(q^2+1)/2$	${}_GI_{27}$		1	$A_{11}(1)$
	θ_6	$q(q^2+1)/2$	${}_GI_{28}$		1	$A_{11}(1)$
G	θ_{17}	$q(q+1)(q^2+1)/2$	${}_GI_{39}$		1	$A_{11}(1)$
	θ_{18}	$q(q+1)(q^2+1)/2$	${}_GI_{40}$		1	$A_{11}(1)$
	θ_{21}	$q(q-1)(q^2+1)/2$	${}_GI_{43}$		1	$A_{11}(1)$
	θ_{22}	$q(q-1)(q^2+1)/2$	${}_GI_{44}$		1	$A_{11}(1)$
B	${}_B\chi_3(k)$	$q(q-1)$	${}_BI_3$	R_0	$q-1$	$C_{31}(i)$
	${}_B\chi_6(k,l)$	$q(q-1)/2$	${}_BI_6$	$R_0 \times \mathbb{Z}_2$	$2(q-1)$	$C_{11}(-1,i), C_{12}(-1,i)$
	${}_B\chi_7(k,l)$	$q(q-1)/2$	${}_BI_7$	$R_0 \times \mathbb{Z}_2$	$2(q-1)$	$C_{11}(-1,i), C_{12}(-1,i)$
	${}_B\theta_3(k,l)$	$q(q-1)^2/4$	${}_BI_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	$A_{41}(-1), B_{33}(-1)$

(continued)

Table A.3 Continued

	Character	Degree	Param. set	Param. Yamada	Number	Class
	${}_B\theta_4(k,l)$	$q(q-1)^2/4$	${}_BI_{11}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	4	$A_{41}(-1), B_{33}(-1)$
	${}_B\theta_5(k,l)$	$q(q-1)^2/4$	${}_BI_{12}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	4	$A_{41}(-1), B_{33}(-1)$
	${}_B\theta_6(k,l)$	$q(q-1)^2/4$	${}_BI_{13}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	4	$A_{41}(-1), B_{33}(-1)$
P	${}_P\chi_2(k)$	q	${}_PI_2$	R_0	$q-1$	$C_{11}(1,i)$
	${}_P\chi_5(k)$	$q(q^2-1)$	${}_PI_5$	${}^2R_1$	$\frac{q-3}{2}$	$C_{21}(i)$
	${}_P\chi_8(k)$	$q(q-1)^2$	${}_PI_8$	${}^2S_1$	$\frac{q-1}{2}$	$D_{11}(i)$
	${}_P\theta_3(k)$	$q(q^2-1)/2$	${}_PI_{11}$	$\mathbb{Z}_2$	2	B_{31}
	${}_P\theta_4(k)$	$q(q^2-1)/2$	${}_PI_{12}$	$\mathbb{Z}_2$	2	B_{31}
	${}_P\theta_5(k)$	$q(q-1)^2/2$	${}_PI_{13}$	$\mathbb{Z}_2$	2	B_{31}
	${}_P\theta_6(k)$	$q(q-1)^2/2$	${}_PI_{14}$	$\mathbb{Z}_2$	2	B_{31}
Q	${}_Q\chi_2(k)$	q	${}_QI_2$	R_0	$q-1$	$C_{21}(1,i)$
	${}_Q\chi_{10}(k,l)$	$q(q^2-1)/2$	${}_QI_{10}$	${}^2R_1\times\mathbb{Z}_2$	$q-3$	$C_{11}(-1,i), C_{12}(-1,i)$
	${}_Q\chi_{11}(k,l)$	$q(q^2-1)/2$	${}_QI_{11}$	${}^2R_1\times\mathbb{Z}_2$	$q-3$	$C_{11}(-1,i), C_{12}(-1,i)$
	${}_Q\chi_{12}(k,l)$	$q(q-1)^2/2$	${}_QI_{12}$	$\mathbb{Z}_2\times{}^2S_1$	$q-1$	$D_{11}(-1,i), D_{12}(-1,i)$
	${}_Q\chi_{13}(k,l)$	$q(q-1)^2/2$	${}_QI_{13}$	$\mathbb{Z}_2\times{}^2S_1$	$q-1$	$D_{11}(-1,i), D_{12}(-1,i)$
	${}_Q\theta_3(k)$	$q(q^2-1)/4$	${}_QI_{16}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$
	${}_Q\theta_4(k)$	$q(q^2-1)/4$	${}_QI_{17}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$
	${}_Q\theta_5(k)$	$q(q^2-1)/4$	${}_QI_{18}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$
	${}_Q\theta_6(k)$	$q(q^2-1)/4$	${}_QI_{19}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$
	${}_Q\theta_7(k)$	$q(q-1)/2$	${}_QI_{20}$	$\mathbb{Z}_2$	2	$A_{11}(-1)$
	${}_Q\theta_8(k)$	$q(q-1)/2$	${}_QI_{21}$	$\mathbb{Z}_2$	2	$A_{11}(-1)$
	${}_Q\theta_{11}(k)$	$q(q-1)^2/4$	${}_QI_{24}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$
	${}_Q\theta_{12}(k)$	$q(q-1)^2/4$	${}_QI_{25}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$
	${}_Q\theta_{13}(k)$	$q(q-1)^2/4$	${}_QI_{26}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$
	${}_Q\theta_{14}(k)$	$q(q-1)^2/4$	${}_QI_{27}$	$\mathbb{Z}_2$	2	$A_{31}(-1), A_{41}(-1)$

Table A.4 The irreducible characters of the chain normalizers in $Sp_4(q)$ of defect $4n$

	Character	Degree	Param. set	Param. Yamada	Number	Class
G	$\chi_1(k)$	$(q+1)(q^2+1)$	${}_GI_1$	${}^2R_1$	$\frac{q-3}{2}$	$C_{11}(1,i)$
	$\chi_3(k)$	$(q+1)^2(q^2+1)/2$	${}_GI_3$	${}^2R_1$	$\frac{q-3}{2}$	$C_{12}(1,i)$
	$\chi_4(k)$	$(q+1)^2(q^2+1)/2$	${}_GI_4$	${}^2R_1$	$\frac{q-3}{2}$	$C_{12}(1,i)$
	$\chi_5(k)$	$(q^4-1)/2$	${}_GI_5$	${}^2R_1$	$\frac{q-3}{2}$	$C_{12}(1,i)$
	$\chi_6(k)$	$(q^4-1)/2$	${}_GI_6$	${}^2R_1$	$\frac{q-3}{2}$	$C_{12}(1,i)$
	$\chi_7(k)$	$(q+1)(q^2+1)$	${}_GI_7$	${}^2R_1$	$\frac{q-3}{2}$	$C_{11}(1,i)$
	$\chi_9(k,l)$	$(q+1)^2(q^2+1)$	${}_GI_9$	${}^8R_3$	$\frac{(q-3)(q-5)}{8}$	$C_{11}(1,i)$
	$\chi_{10}(k)$	$(q-1)(q^2+1)$	${}_GI_{10}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{11}(1,i)$
	$\chi_{12}(k)$	$(q-1)^2(q^2+1)/2$	${}_GI_{12}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{12}(1,i)$
	$\chi_{13}(k)$	$(q-1)^2(q^2+1)/2$	${}_GI_{13}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{12}(1,i)$
	$\chi_{14}(k)$	$(q^4-1)/2$	${}_GI_{14}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{12}(1,i)$
	$\chi_{15}(k)$	$(q^4-1)/2$	${}_GI_{15}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{12}(1,i)$
	$\chi_{16}(k)$	$(q-1)(q^2+1)$	${}_GI_{16}$	${}^2S_1$	$\frac{q-1}{2}$	$D_{11}(1,i)$
	$\chi_{18}(k,l)$	$(q-1)^2(q^2+1)$	${}_GI_{18}$	${}^8S_3$	$\frac{(q-1)(q-3)}{8}$	$D_{11}(1,i)$

(continued)

Table A.4 Continued

	Character	Degree	Param. set	Param. Yamada	Number	Class
	$\chi_{19}(k, l)$	$q^4 - 1$	${}_G I_{19}$	${}^2R_1 \times {}^2S_1$	$\frac{(q-1)(q-3)}{4}$	$E(i, j)$
	$\chi_{20}(k)$	$q^4 - 1$	${}_G I_{20}$	${}^4T_3$	$\frac{(q-1)^2}{4}$	$E_1(i)$
	$\chi_{21}(k)$	$(q^2 - 1)^2$	${}_G I_{21}$	${}^4V_1$	$\frac{q^2-1}{4}$	$E_2(i)$
	θ_0	1	${}_G I_{22}$		1	
	θ_7	$(q^2+1)/2$	${}_G I_{29}$		1	$A_{11}(1)$
	θ_8	$(q^2+1)/2$	${}_G I_{30}$		1	$A_{11}(1)$
	θ_{11}	$(q^2-1)/2$	${}_G I_{33}$		1	$A_{11}(1)$
	θ_{12}	$(q^2-1)/2$	${}_G I_{34}$		1	$A_{11}(1)$
	θ_{15}	$(q+1)(q^2+1)/2$	${}_G I_{37}$		1	$A_{11}(1)$
	θ_{16}	$(q+1)(q^2+1)/2$	${}_G I_{38}$		1	$A_{11}(1)$
	θ_{19}	$(q-1)(q^2+1)/2$	${}_G I_{41}$		1	$A_{11}(1)$
	θ_{20}	$(q-1)(q^2+1)/2$	${}_G I_{42}$		1	$A_{11}(1)$
B	${}_B\chi_1(k, l)$	1	${}_B I_1$	$R_0 \times R_0$	$(q-1)^2$	$C(i, j)$
	${}_B\chi_2(k)$	$q-1$	${}_B I_2$	R_0	$q-1$	$C_{41}(i)$
	${}_B\chi_4(k, l)$	$(q-1)/2$	${}_B I_4$	$R_0 \times \mathbb{Z}_2$	$2(q-1)$	$C_{22}(-1, i)$
	${}_B\chi_5(k, l)$	$(q-1)/2$	${}_B I_5$	$R_0 \times \mathbb{Z}_2$	$2(q-1)$	$C_{22}(-1, i)$
	${}_B\theta_1(k)$	$(q-1)^2/2$	${}_B I_8$		2	$A_{31}(-1)$
	${}_B\theta_2(k)$	$(q-1)^2/2$	${}_B I_9$		2	$A_{31}(-1)$
P	${}_P\chi_1(k)$	1	${}_P I_1$	R_0	$q-1$	$C_{11}(1, i)$
	${}_P\chi_3(k, l)$	$q+1$	${}_P I_3$	${}^2R_2$	$\frac{(q-1)(q-2)}{2}$	$C(i, j)$
	${}_P\chi_4(k)$	$q-1$	${}_P I_4$	${}^2T_1$	$\frac{q(q-1)}{2}$	$E(i)$
	${}_P\chi_6(k, l)$	$(q^2-1)/2$	${}_P I_6$	$R_0 \times \mathbb{Z}_2$	$2(q-1)$	$C_{12}(-1, i)$
	${}_P\chi_7(k, l)$	$(q^2-1)/2$	${}_P I_7$	$R_0 \times \mathbb{Z}_2$	$2(q-1)$	$C_{12}(-1, i)$
	${}_P\theta_1(k)$	$(q-1)(q^2-1)/2$	${}_P I_9$	$\mathbb{Z}_2$	2	$A_{11}(-1)$
	${}_P\theta_2(k)$	$(q-1)(q^2-1)/2$	${}_P I_{10}$	$\mathbb{Z}_2$	2	$A_{11}(-1)$
Q	${}_Q\chi_1(k)$	1	${}_Q I_1$	R_0	$q-1$	$C_{21}(1, i)$
	${}_Q\chi_3(k, l)$	$q+1$	${}_Q I_3$	$R_0 \times {}^2R_1$	$\frac{(q-1)(q-3)}{2}$	$C(i, j)$
	${}_Q\chi_4(k, l)$	$q-1$	${}_Q I_4$	$R_0 \times {}^2S_1$	$\frac{(q-1)^2}{2}$	$E(i, j)$
	${}_Q\chi_5(k)$	$(q+1)/2$	${}_Q I_5$	R_0	$q-1$	$C_{22}(-1, i)$
	${}_Q\chi_6(k)$	$(q+1)/2$	${}_Q I_6$	R_0	$q-1$	$C_{22}(-1, i)$
	${}_Q\chi_7(k)$	$(q-1)/2$	${}_Q I_7$	R_0	$q-1$	$C_{22}(-1, i)$
	${}_Q\chi_8(k)$	$(q-1)/2$	${}_Q I_8$	R_0	$q-1$	$C_{22}(-1, i)$
	${}_Q\chi_9(k)$	q^2-1	${}_Q I_9$	R_0	$q-1$	$C_{31}(i)$
	${}_Q\theta_1(k)$	$(q-1)(q^2-1)/2$	${}_Q I_{14}$		2	$A_{31}(-1)$
	${}_Q\theta_2(k)$	$(q-1)(q^2-1)/2$	${}_Q I_{15}$		2	$A_{31}(-1)$

Table A.5 Number of fixed points of $H = \langle \alpha^t \rangle$ on parameter sets of the irreducible characters of the parabolic subgroups of $Sp_4(q)$. The unions of parameter sets in this table are disjoint unions

Parameter set I	Number of fixed points $\lvert C_I(H)\rvert$ if $t \mid n$
${}_G I_1 \cup {}_G I_{10}$	$p^t - 2$
${}_G I_2 \cup {}_G I_{11}$	$p^t - 2$
${}_G I_3 \cup {}_G I_{12}$	$p^t - 2$

(continued)

Table A.5 Continued

Parameter set I	Number of fixed points $\lvert C_I(H)\rvert$ if $t \mid n$
${}_GI_4 \cup {}_GI_{13}$	$p^t - 2$
${}_GI_5 \cup {}_GI_{14}$	$p^t - 2$
${}_GI_6 \cup {}_GI_{15}$	$p^t - 2$
${}_GI_8 \cup {}_GI_{17}$	$p^t - 2$
${}_GI_7 \cup {}_GI_9 \cup {}_GI_{16} \cup$ ${}_GI_{18} \cup {}_GI_{19} \cup$ ${}_GI_{20} \cup {}_GI_{21}$	$(p^t - 1)^2$
${}_GI_{22}$	1
${}_GI_{24}$	1
${}_GI_{25} \cup {}_GI_{26} \cup {}_GI_{27} \cup$ ${}_GI_{28} \cup {}_GI_{39} \cup {}_GI_{40} \cup$ ${}_GI_{43} \cup {}_GI_{44}$	8
${}_GI_{29} \cup {}_GI_{30} \cup {}_GI_{33} \cup$ ${}_GI_{34} \cup {}_GI_{37} \cup$ ${}_GI_{38} \cup {}_GI_{41} \cup {}_GI_{42}$	8
${}_GI_{31} \cup {}_GI_{32} \cup {}_GI_{35} \cup {}_GI_{36}$	4
${}_BI_1$	$(p^t - 1)^2$
${}_BI_2$	$p^t - 1$
${}_BI_3$	$p^t - 1$
${}_BI_4$	$2(p^t - 1)$
${}_BI_5$	$2(p^t - 1)$
${}_BI_6$	$2(p^t - 1)$
${}_BI_7$	$2(p^t - 1)$
${}_BI_8 \cup {}_BI_9$	4
${}_BI_{10} \cup {}_BI_{11} \cup {}_BI_{12} \cup {}_BI_{13}$	16
${}_PI_1$	$p^t - 1$
${}_PI_2$	$p^t - 1$
${}_PI_3 \cup {}_PI_4$	$(p^t - 1)^2$
${}_PI_5 \cup {}_PI_8$	$p^t - 2$
${}_PI_6$	$2(p^t - 1)$
${}_PI_7$	$2(p^t - 1)$
${}_PI_9 \cup {}_PI_{10}$	4
${}_PI_{11} \cup {}_PI_{12} \cup {}_PI_{13} \cup {}_PI_{14}$	8
${}_QI_1$	$p^t - 1$
${}_QI_2$	$p^t - 1$
${}_QI_3 \cup {}_QI_4$	$(p^t - 1)(p^t - 2)$
${}_QI_5$	$p^t - 1$
${}_QI_6$	$p^t - 1$
${}_QI_7$	$p^t - 1$
${}_QI_8$	$p^t - 1$
${}_QI_9$	$p^t - 1$
${}_QI_{10} \cup {}_QI_{12}$	$2(p^t - 2)$
${}_QI_{11} \cup {}_QI_{13}$	$2(p^t - 2)$
${}_QI_{14} \cup {}_QI_{15}$	4
${}_QI_{16} \cup {}_QI_{17} \cup {}_QI_{18} \cup$ ${}_QI_{19} \cup {}_QI_{24} \cup {}_QI_{25} \cup$ ${}_QI_{26} \cup {}_QI_{27}$	16
${}_QI_{20} \cup {}_QI_{21}$	4
${}_QI_{22} \cup {}_QI_{23}$	4

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