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Character tables of the maximal parabolic subgroups of the Ree groups ${}^2F_4(q^2)$

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Abstract

We compute the conjugacy classes of elements and the character tables of the maximal parabolic subgroups of the simple Ree groups ${}^2F_4(q^2)$. For one of the maximal parabolic subgroups, we find an irreducible character of the unipotent radical that does not extend to its inertia subgroup.

1. *Introduction*

Let ${}^2F_4(q^2)$ be the simple Ree group with $q^2 = 2^{2n+1}$ and n a positive integer. The (complex) irreducible characters of ${}^2F_4(q^2)$ were computed by Malle; see [**14**] and the CHEVIE [**6**] library. The character table of a Borel subgroup B of ${}^2F_4(q^2)$ was determined by the authors in [**10**].

In this paper we calculate the character tables of the maximal parabolic subgroups of ${}^2F_4(q^2)$. Our methods are similar to those in [**7**]. Up to conjugacy, ${}^2F_4(q^2)$ has two maximal parabolic subgroups, which we call P_a and P_b. A natural approach to construct the irreducible characters of P_a and P_b is to compute the orbits of P_a and P_b on the irreducible characters of their unipotent radicals, to determine the corresponding inertia subgroups and then to induce characters from these inertia subgroups. However, due to the complicated structure of the unipotent radicals, it seems to be difficult to determine all irreducible characters of the unipotent radicals and the orbits of P_a and P_b on these irreducible characters. Additionally, it turns out that there is an irreducible character of the unipotent radical of P_a that does not extend to its inertia subgroup in P_a.

Therefore, we induce characters only from the inertia subgroups of the linear characters of the unipotent radicals. In this way, we obtain all irreducible characters of P_a and P_b covering linear characters of the unipotent radicals. The remaining irreducible characters are obtained by inducing characters from the Borel subgroup B and by restricting unipotent irreducible characters from ${}^2F_4(q^2)$.

Up to conjugacy, ${}^2F_4(q^2)$, the Borel subgroup B and the maximal parabolic subgroups P_a and P_b are the only parabolic subgroups of ${}^2F_4(q^2)$. So, together with [**10**] and Malle's results [**14**], this paper completes the task of determining the character tables of the parabolic subgroups of the Ree groups ${}^2F_4(q^2)$.

We have at least three applications in mind. The first is the study of the decomposition numbers and the degrees of low-dimensional representations of ${}^2F_4(q^2)$ in non-defining characteristic along the same line as in [**9**] or [**18**]. The second is the verification of Dade's conjecture for ${}^2F_4(q^2)$ in defining characteristic using ideas from [**12**]. The third is the investigation of the restriction of irreducible modular representations of ${}^2F_4(q^2)$ to proper subgroups as in [**15**].

We have implemented the character tables of P_a and P_b as generic character tables in the MAPLE [**2**] part of CHEVIE [**6**] and we use MAPLE programs for restricting and inducing class functions. The use of CHEVIE allows us to easily compute scalar products of class functions

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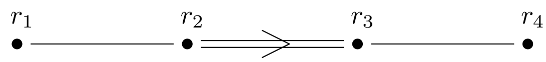
and provides tests for the obtained character tables. For calculations with elements of ${}^2F_4(q^2)$, we use computer programs written by Köhler and the first author in GAP [**5**].

This paper is organized as follows: in § 2, we fix some notation and give reference conjugacy classes and irreducible characters of ${}^2F_4(q^2)$ and the Borel subgroup B. In §§ 3 and 4, we determine the character tables of the maximal parabolic subgroups $P_a$and P_b, respectively. Details of conjugacy classes and character tables of P_a and P_b are given in an appendix. These character tables are available as PDF and CHEVIE files in the electronic appendix included with this paper.

2. *Notation and group theoretical properties of* ${}^2F_4(q^2)$

We use the same notation as in [**10**, § 2]. In particular, $n > 0$ is an integer and $q = \sqrt{2^{2n+1}}$. Let Φ be the root system of type F_4 described in [**10**], so Φ has simple roots r_1, r_2, r_3, r_4 and the following Dynkin diagram.



We fix a simply connected linear algebraic group $\mathbf{G}$ defined over $\mathbb{F}_{q^2}$ with root system Φ, a Frobenius map F such that $\mathbf{G}^F = {}^2F_4(q^2)$ and an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ as in [**10**]. Representatives for the conjugacy classes of the finite group $G := \mathbf{G}^F$ and of the Borel subgroup $B := \mathbf{B}^F$ are given in [**10**, Tables A.1–A.4], together with the corresponding class fusions. The character table of G is contained in the CHEVIE [**6**] library; the character table of B is given in [**10**, Tables A.5 and A.6].

3. *The character table of a maximal parabolic subgroup* P_a

Let $\mathbf{P}_a = \langle \mathbf{B}, n_{r_1}, n_{r_4}\rangle$ be the F-stable maximal parabolic subgroup of $\mathbf{G}$ corresponding to the set $\{r_1, r_4\}$ of simple roots and $P_a := \mathbf{P}_a^F$ be the corresponding maximal parabolic subgroup of $G = \mathbf{G}^F = {}^2F_4(q^2)$. Then, P_a is generated by B and n_a and $|P_a| = q^{24}(q^4 - 1)(q^2 - 1)$. In this section, we compute the conjugacy classes and the character table of P_a.

P_a is a semidirect product of the Levi complement $L_a = \langle \mathbf{T}^F, U_3, n_a\rangle$ and the unipotent radical $U_a := U_1U_2U^4 = U_1U_2U_4 \ldots U_{12}$. The conjugacy classes of P_a can be determined by investigating the splitting of the conjugacy classes of L_a when interpreting L_a as the factor group P_a/U_a. Since L_a is isomorphic with $\mathrm{GL}_2(\mathbb{F}_{q^2})$, we know the conjugacy classes of L_a. A parametrization of these classes adapted for P_a can be obtained as follows.

The Levi complement $\mathbf{L}_a = \langle \mathbf{T}, X_{\pm r_1}, X_{\pm r_4}\rangle$ of $\mathbf{P}_a$ is F-stable and $L_a = (\mathbf{L}_a)^F$. The Weyl group $\mathbf{W}_{\mathbf{L}_a} = \langle w_{r_1}, w_{r_4}\rangle$ of $\mathbf{L}_a$ is elementary abelian of order four and has exactly two F-conjugacy classes with representatives $w_1 = 1$ and $w_2 = w_{r_1}$. So, $\mathbf{L}_a$ has exactly two L_a-conjugacy classes of F-stable maximal tori. Using the notation in [**10**, §§ 2 and 3], we can choose representatives $\mathbf{T}_1 = \mathbf{T}$ and $\mathbf{T}_2$ such that the corresponding maximal tori of L_a are given by $T_1 := T := \mathbf{T}^F$ and $T_2 := \mathbf{T}^{(Fw_2^{-1})}$.

Representatives for the conjugacy classes of semisimple elements of L_a are given in Table 1. A set of representatives for all conjugacy classes of L_a is given in Table 2. A set of representatives for the semisimple conjugacy classes of L_a is also one for the semisimple conjugacy classes of P_a (this follows from [**3**, 0.12 and 3.16] and the proof of [**1**, Proposition 3.3.3] applied to $\mathbf{P}_a$). The unipotent conjugacy classes of P_a can be calculated using the relations in [**10**, Tables 1–4] and the fusions of the conjugacy classes of B in G in [**10**, Table A.4]. Representatives for the mixed classes can then be obtained by computing the non-trivial unipotent conjugacy classes of the centralizers $C_{P_a}(s)$, where s runs through a set of representatives for the non-trivial semisimple conjugacy classes of P_a. We get that P_a has exactly $q^4 + 12q^2 + 14$

conjugacy classes. Representatives and the class fusions are given in Tables A.1–A.3 in the appendix. The notation in these tables is the same as in [10].

The irreducible characters of P_a can be constructed as follows: ${}_{P_a}\chi_1(k)$, ${}_{P_a}\chi_2(k)$, ${}_{P_a}\chi_3(k,l)$ and ${}_{P_a}\chi_4(k)$ are the inflations of the irreducible characters of $L_a \cong \mathrm{GL}_2(\mathbb{F}_{q^2})$; for a construction of the irreducible characters of $\mathrm{GL}_2(\mathbb{F}_{q^2})$, see [3, 15.9].

We get most of the remaining irreducible characters of P_a by inducing irreducible characters from the Borel subgroup B. For every pair (i,j) in Table 3, we define ${}_{P_a}\chi_i := {}_B\chi_j^{P_a}$, where ${}_B\chi_j^{P_a}$ denotes the induced character (or ${}_{P_a}\chi_i(k) := {}_B\chi_j(k)^{P_a}$ if ${}_B\chi_j$ depends on some parameter k). The values of these induced characters can easily be computed using the character values in [10, Table A.6] and the class fusions in Table A.1 in the appendix.

To construct the remaining irreducible characters of P_a, we decompose the restrictions of several unipotent irreducible characters of $G = {}^2F_4(q^2)$. Our notation for the unipotent irreducible characters of G, which is given in the left-most column of Table 4, corresponds to the numbering of the irreducible characters of ${}^2F_4(q^2)$ in the CHEVIE library. The second and third columns describe the notation which is used in [14, Tabelle 1] and [1, p. 489], respectively. The ϕ_i occurring in the character degrees in Table 4 are polynomials in q, which are defined in [1, pp. 477 and 490]. Since not all of these unipotent irreducible characters of G are uniquely determined by their degree, we also provide some character values in the last two columns of Table 4. Here ε_4 is a complex fourth root of unity; see [10, Table 5].

Next, we construct the irreducible characters ${}_{P_a}\chi_{30}$, ${}_{P_a}\chi_{31}$, and ${}_{P_a}\chi_{33}(k)$. Using the character

$$\psi := ({}_G\chi_4)_B - {}_B\chi_1(0,0) - {}_B\chi_5(0) - {}_B\chi_{23}(0) - {}_B\chi_{39}(0) - {}_B\chi_{42}(0) - {}_B\chi_{56} - {}_B\chi_{57}$$

of B, we set

$${}_{P_a}\chi_{30} := ({}_G\chi_4)_{P_a} - {}_{P_a}\chi_2(0) - {}_{P_a}\chi_{26}(0) - \psi^{P_a}, \quad {}_{P_a}\chi_{31} := {}_{P_a}\chi_{32}(0) - {}_{P_a}\chi_{30},$$

TABLE 1. *Parametrization of the semisimple conjugacy classes of L_a.*

Representative	Parameters	Number of classes
$h_1(i) := h(\tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^{(4\theta^2-4\theta+1)i})$	$i = 0,\dots,q^2-2$	q^2-1
$h_2(i,j) := h(\tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^j, \tilde\zeta_2^{(2\theta-1)j})$	$i,j = 0,\dots,q^2-2$ $j \neq (\sqrt{2}q-1)i$	$\frac{q^4-3q^2+2}{2}$
$h_3(i) := h(\tilde\zeta_4^{(4\theta^3+2\theta^2+1)i}, \tilde\zeta_4^{(2\theta^2+2\theta-1)i},$ $\tilde\zeta_4^{(-2\theta^2+2\theta+1)i}, \tilde\zeta_4^{(-4\theta^3+2\theta^2+1)i})$	$i = 0,\dots,q^4-2$ $i \neq (q^2+1)l,$ $l = 0,\dots,q^2-2$	$\frac{q^4-q^2}{2}$

TABLE 2. *The conjugacy classes of L_a.*

Notation	Representative	$\lvert C_{L_a}\rvert$
$c_{1,0}(i)$	$h_1(i)$	$q^2(q^4-1)(q^2-1)$
$c_{1,1}(i)$	$h_1(i)\alpha_3(1)$	$q^2(q^2-1)$
$c_{2,0}(i,j)$	$h_2(i,j)$	$(q^2-1)^2$
$c_{3,0}(i)$	$h_3(i)$	q^4-1

TABLE 3. *Pairs (i,j) such that ${}_{P_a}\chi_i := {}_B\chi_j^{P_a}$ or ${}_{P_a}\chi_i(k) := {}_B\chi_j(k)^{P_a}$.*

i	5	6	7	8	9	10	11	12	13	14	15	16	17	21
j	2	8	3	4	6	7	10	11	14	17	18	21	22	15
i	22	26	27	28	29	32	34	35	36	37	38	39	40	
j	19	23	24	25	38	42	44	45	46	47	48	49	50	

and

$$ {}_{P_a}\chi_{33}(k) := {}_{P_a}\chi_4(k) \cdot {}_{P_a}\chi_{30} \quad \text{for } k = 0, 1, \dots, \frac{q^2}{2}. $$

We can now construct the characters ${}_{P_a}\chi_{18}$, ${}_{P_a}\chi_{19}$, ${}_{P_a}\chi_{23}$, ${}_{P_a}\chi_{24}$, and ${}_{P_a}\chi_{25}$. Let

$$ \begin{aligned} {}_{P_a}\chi_{18} := ({}_G\chi_{15})_{P_a} &- \frac{q^2+1}{3} \cdot {}_{P_a}\chi_{29} - \frac{q^2+1}{3} \cdot {}_{P_a}\chi_{31} - \frac{q^2+1}{3} \cdot \sum{}' \, {}_{P_a}\chi_{32}(k) \\ &- \frac{q^2-2}{3} \cdot \sum{}'' \, {}_{P_a}\chi_{33}(k) - \frac{q^2+1}{3} \cdot \sum{}''' \, {}_{P_a}\chi_{33}(k) - \frac{\sqrt{2}q(q^2+1)}{6} \cdot {}_{P_a}\chi_{36} \\ &- \frac{\sqrt{2}q(q^2+1)}{6} \cdot {}_{P_a}\chi_{37} - \frac{\sqrt{2}q(q^2+1)}{6} \cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{38}(k) \\ &- \frac{\sqrt{2}q(q^2+1)}{6} \cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{39}(k) - \frac{q^4-1}{3} \cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{40}(k), \end{aligned} $$

and ${}_{P_a}\chi_{19} := \overline{{}_{P_a}\chi_{18}}$ the complex-conjugate character. Furthermore, let

$$ \begin{aligned} {}_{P_a}\chi_{23} := ({}_G\chi_8)_{P_a} - ({}_G\chi_{13})_{P_a} &+ {}_{P_a}\chi_{14} + \frac{q^2-2}{6} \cdot {}_{P_a}\chi_{29} + \frac{q^2-2}{6} \cdot {}_{P_a}\chi_{31} \\ &+ \frac{q^2-2}{6} \cdot \sum{}' \, {}_{P_a}\chi_{32}(k) + \frac{q^2-2}{6} \cdot \sum{}'' \, {}_{P_a}\chi_{33}(k) \\ &+ \frac{q^2-8}{6} \cdot \sum{}''' \, {}_{P_a}\chi_{33}(k) + \frac{q}{\sqrt{2}} \cdot \sum_{k=0}^{q^2-2} {}_{P_a}\chi_{34}(k) + \frac{\sqrt{2}q(q^2-2)}{12} \cdot {}_{P_a}\chi_{36} \\ &+ \frac{\sqrt{2}q(q^2-2)}{12} \cdot {}_{P_a}\chi_{37} + \frac{\sqrt{2}q(q^2-2)}{12} \cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{38}(k) \\ &+ \frac{\sqrt{2}q(q^2-2)}{12} \cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{39}(k) + \frac{(q^2-1)(q^2-2)}{6} \cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{40}(k), \end{aligned} $$

TABLE 4. *Notation for some unipotent irreducible characters of G.*

Notation					
CHEVIE	[14]	[1]	Degree	Conjugacy class	Value
${}_G\chi_2$	χ_5	${}^2B_2[a], 1$	$\frac{q}{\sqrt{2}}\phi_1\phi_2\phi_4^2\phi_{12}$	$c_{1,11}$	$-\frac{q}{\sqrt{2}} + \varepsilon_4 q^2$
${}_G\chi_4$	χ_2	ε'	$q^2\phi_{12}\phi_{24}$		
${}_G\chi_5$	χ_9	ρ_2'	$\frac{q^4}{4}\phi_4^2\phi_8''^2\phi_{12}\phi_{24}'$		
${}_G\chi_6$	χ_{10}	ρ_2''	$\frac{q^4}{4}\phi_4^2\phi_8'^2\phi_{12}\phi_{24}''$		
${}_G\chi_7$	χ_{11}	ρ_2	$\frac{q^4}{2}\phi_8^2\phi_{24}$		
${}_G\chi_8$	χ_{12}	cusp	$\frac{q^4}{12}\phi_1^2\phi_2^2\phi_8'^2\phi_{12}\phi_{24}'$		
${}_G\chi_9$	χ_{13}	cusp	$\frac{q^4}{12}\phi_1^2\phi_2^2\phi_8''^2\phi_{12}\phi_{24}''$		
${}_G\chi_{10}$	χ_{14}	cusp	$\frac{q^4}{6}\phi_1^2\phi_2^2\phi_4^2\phi_{24}$		
${}_G\chi_{11}$	χ_{15}	cusp	$\frac{q^4}{4}\phi_1^2\phi_2^2\phi_4^2\phi_{12}\phi_{24}''$	$c_{1,11}$	$-\frac{q^4}{4} - \varepsilon_4\frac{q^3}{\sqrt{2}}$
${}_G\chi_{13}$	χ_{17}	cusp	$\frac{q^4}{4}\phi_1^2\phi_2^2\phi_4^2\phi_{12}\phi_{24}'$	$c_{1,11}$	$-\frac{q^4}{4} - \varepsilon_4\frac{q^3}{\sqrt{2}}$
${}_G\chi_{15}$	χ_{19}	cusp	$\frac{q^4}{3}\phi_1^2\phi_2^2\phi_4^2\phi_8^2$	$c_{5,3}$	$\frac{q^2}{6} - \frac{1}{3} + \varepsilon_4\frac{\sqrt{3}q^2}{2}$

$$\begin{aligned}
{}_{P_a}\chi_{24} := {} & ({}_G\chi_9)_{P_a} - ({}_G\chi_{11})_{P_a} + {}_{P_a}\chi_{16} + \frac{q^2-2}{6}\cdot {}_{P_a}\chi_{29} + \frac{q^2-2}{6}\cdot {}_{P_a}\chi_{31}\\
& + \frac{q^2-2}{6}\cdot \sum{}' {}_{P_a}\chi_{32}(k) + \frac{q^2-2}{6}\cdot \sum{}'' {}_{P_a}\chi_{33}(k)\\
& + \frac{q^2-8}{6}\cdot \sum{}''' {}_{P_a}\chi_{33}(k) + \frac{q}{\sqrt{2}}\cdot \sum_{k=0}^{q^2-2} {}_{P_a}\chi_{34}(k) + \frac{\sqrt{2}q(q^2-2)}{12}\cdot {}_{P_a}\chi_{36}\\
& + \frac{\sqrt{2}q(q^2-2)}{12}\cdot {}_{P_a}\chi_{37} + \frac{\sqrt{2}q(q^2-2)}{12}\cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{38}(k)\\
& + \frac{\sqrt{2}q(q^2-2)}{12}\cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{39}(k) + \frac{(q^2-1)(q^2-2)}{6}\cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{40}(k),\\
{}_{P_a}\chi_{25} := {} & ({}_G\chi_7)_{P_a} + ({}_G\chi_{10})_{P_a} - {}_{P_a}\chi_1(0) - {}_{P_a}\chi_2(0) - {}_{P_a}\chi_5(0)\\
& - \frac{q}{\sqrt{2}}\cdot {}_{P_a}\chi_7(0) - \frac{q}{\sqrt{2}}\cdot {}_{P_a}\chi_8(0) - {}_{P_a}\chi_{12}(0) - {}_{P_a}\chi_{13} - {}_{P_a}\chi_{26}(0) - {}_{P_a}\chi_{27}\\
& - \sum_{k=1}^{q^2} {}_{P_a}\chi_{28}(k) - \frac{2q^2-1}{3}\cdot {}_{P_a}\chi_{29} - {}_{P_a}\chi_{30} - \frac{2q^2+2}{3}\cdot {}_{P_a}\chi_{31}\\
& - \frac{2q^2+2}{3}\cdot \sum{}' {}_{P_a}\chi_{32}(k) - \frac{2q^2-1}{3}\cdot \sum{}'' {}_{P_a}\chi_{33}(k)\\
& - \frac{2q^2-4}{3}\cdot \sum{}''' {}_{P_a}\chi_{33}(k) - \frac{q}{\sqrt{2}}\cdot \sum_{k=1}^{q^2-2} {}_{P_a}\chi_{34}(k) - \sqrt{2}q\cdot {}_{P_a}\chi_{34}(0)\\
& - \frac{q}{\sqrt{2}}\cdot \sum_{k=1}^{q^2-2} {}_{P_a}\chi_{35}(k) - \sqrt{2}q\cdot {}_{P_a}\chi_{35}(0) - \frac{\sqrt{2}q(2q^2-1)}{6}\cdot {}_{P_a}\chi_{36}\\
& - \frac{\sqrt{2}q(2q^2-1)}{6}\cdot {}_{P_a}\chi_{37} - \frac{\sqrt{2}q(2q^2-1)}{6}\cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{38}(k)\\
& - \frac{\sqrt{2}q(2q^2-1)}{6}\cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{39}(k) - \frac{2q^4+1}{3}\cdot \sum_{k=1}^{q^2} {}_{P_a}\chi_{40}(k),
\end{aligned}$$

where $\sum'$ is the sum over all *different* ${}_{P_a}\chi_{32}(k)$, $k \neq 0$, $\sum''$ is the sum over all *different* ${}_{P_a}\chi_{33}(k)$ with $3 \nmid k$, and $\sum'''$ is the sum over all *different* ${}_{P_a}\chi_{33}(k)$ with $3|k$, $k \neq 0$.

THEOREM 3.1. *The character table of the maximal parabolic subgroup P_a is given by Tables* A.4 *and* A.5 *in the appendix.*

Proof. Computing scalar products with CHEVIE, we see that we have constructed $q^4 + 12q^2 + 13$ different irreducible characters of P_a, so there is only one irreducible character missing, ${}_{P_a}\chi_{20}$. The values of this character can be calculated using orthogonality relations (applied to the factor group $P_a/U_7U_9U_{10}U_{11}U_{12}$). □

Let N be a normal subgroup of some finite group H. As in [**16**], we say that *maximal extensibility* holds with respect to $N \trianglelefteq H$ if every $\psi \in \mathrm{Irr}(N)$ extends to its inertia subgroup $I_H(\psi) = \{x \in H \mid {}^x\chi = \chi\}$. It is shown in [**7**] and [**8**] that for every prime power q and every parabolic subgroup P of Steinberg's simple triality group ${}^3D_4(q)$, maximal extensibility holds with respect to $U_P \trianglelefteq P$, where U_P is the unipotent radical of P. The analogous statement for the Ree groups ${}^2F_4(q^2)$ with $q^2 = 2^{2n+1}$ is *not* true, as shown by the following remark.

REMARK 3.2. *There is an irreducible character* $\psi \in \mathrm{Irr}(U_a)$ *of degree* $q/\sqrt{2}$ *such that* ψ *does not extend to its inertia subgroup* $I_{P_a}(\psi)$.

Proof. Using the notation in [10] and the commutator relations [10, Table 1], we see that the factor group $U_a/U_5U_6\dots U_{12}$ is isomorphic with the direct product of U_1U_2 and $\overline{U_4} := U_4U^5/U^5$. The group U_1U_2 is isomorphic with a Sylow 2-subgroup of the Suzuki group $\mathrm{Sz}(q^2)$; so there is $\psi_1' \in \mathrm{Irr}(U_1U_2)$ of degree $q/\sqrt{2}$. The group $\overline{U_4}$ is elementary abelian of order q^2 and we can choose a non-trivial character $\psi_2' \in \mathrm{Irr}(\overline{U_4})$. We define an irreducible character ψ' of $U_a/U_5U_6\dots U_{12}$ by $\psi'(x) := \psi_1(x_1)\psi_2(x_2)$, where $x = x_1x_2$ with $x_1 \in U_1U_2$ and $x_2 \in \overline{U_4}$. Let $\psi \in \mathrm{Irr}(U_a)$ be the inflation of ψ'.

Using the relations [10, Tables 1 and 2] and the class fusions of the Borel subgroup B described in [10], we can easily compute the values of the induced character ψ^{P_a} and see that $\psi^{P_a} = (q/\sqrt{2}) \cdot {}_{P_a}\chi_{11}$, so ${}_{P_a}\chi_{11}$ is the only irreducible character of P_a covering ψ. Suppose that ψ extends to $I_{P_a}(\psi)$. From Clifford theory [13, Corollary (6.17)], we then get $I_{P_a}(\psi) = U_a$. So, [13, Theorem (6.11)] implies that ψ^{P_a} is irreducible, a contradiction. □

4. *The character table of a maximal parabolic subgroup* P_b

Let $\mathbf{P}_b = \langle \mathbf{B}, n_{r_2}, n_{r_3}\rangle$ be the F-stable maximal parabolic subgroup of $\mathbf{G}$ corresponding to the set $\{r_2, r_3\}$ of simple roots and $P_b := \mathbf{P}_b^F$ be the corresponding maximal parabolic subgroup of $G = {}^2F_4(q^2)$. Then, P_b is generated by B and n_b and $|P_b| = q^{24}(q^4+1)(q^2-1)^2$. In this section, we compute the conjugacy classes and the character table of P_b.

P_b is the semidirect product of the Levi complement $L_b = \langle \mathbf{T}^F, U_1, U_2, n_b\rangle$ and the unipotent radical $U_b := U^3 = U_3U_4\dots U_{12}$. The conjugacy classes of P_b can be computed by the same methods as those that we have used for the conjugacy classes of P_a. The Levi complement $\mathbf{L}_b = \langle \mathbf{T}, X_{\pm r_2}, X_{\pm r_3}, X_{\pm r_6}, X_{\pm r_9}\rangle$ of $\mathbf{P}_b$ is F-stable and $L_b = (\mathbf{L}_b)^F$. The Weyl group $\mathbf{W}_{\mathbf{L}_b} = \langle w_{r_2}, w_{r_3}\rangle$ of $\mathbf{L}_b$ is dihedral of order eight and has exactly three F-conjugacy classes with representatives $w_1 = 1$, $w_3 = w_{r_3}$ and $w_4 = w_{r_2}w_{r_3}w_{r_2}$. So, $\mathbf{L}_b$ has exactly three L_b-conjugacy classes of F-stable maximal tori. Using the notation in [10, §§ 2 and 3], we can choose representatives $\mathbf{T}_1 = \mathbf{T}$, $\mathbf{T}_3$ and $\mathbf{T}_4$ such that the corresponding maximal tori of L_b are given by $T_1 := T := \mathbf{T}^F$, $T_3 := \mathbf{T}^{(Fw_3^{-1})}$ and $T_4 := \mathbf{T}^{(Fw_4^{-1})}$. The conjugacy classes of P_b can be determined analogously to those of P_a by investigating the splitting of the conjugacy classes of L_b when interpreting L_b as the factor group P_b/U_b. The parabolic subgroup P_b has exactly $q^4 + 14q^2 + 15$ conjugacy classes. Representatives are given in Tables A.6 and A.7.

A natural way to construct the irreducible characters of P_b is to apply Clifford theory with respect to the decomposition $P_b = L_b \ltimes U_b$. Therefore, we collect some information on U_b. Every $u \in U_b$ can be written uniquely as

$$u = \alpha_3(d_3)\alpha_4(d_4)\dots\alpha_{12}(d_{12})$$

with $d_3, \dots, d_{12} \in \mathbb{F}_{q^2}$. Using the commutator relations [10, Table 1], we see that the center of U_b is $Z(U_b) = U_{12}$ and the derived subgroup is $U_b' = U_8U_9\dots U_{12}$. So, U_b has exactly q^{10} linear characters, namely $\lambda_{a,b,c,d,e} : U_b \to \mathbb{C}^\times$ defined by

$$\lambda_{a,b,c,d,e} : \alpha_3(d_3)\alpha_4(d_4)\dots\alpha_{12}(d_{12}) \mapsto \phi(a\cdot d_3 + b\cdot d_4 + c\cdot d_5 + d\cdot d_6 + e\cdot d_7),$$

where $a, b, c, d, e \in \mathbb{F}_{q^2}$ and ϕ is a fixed non-trivial linear character of the additive group $(\mathbb{F}_{q^2}, +)$ such that $\phi(x^2) = \phi(x)$ for all $x \in \mathbb{F}_{q^2}$ and $\phi(1) = -1$; see [10, § 4]. Due to the complicated structure of U_b, it seems to be difficult to determine all irreducible characters of U_b and the action of L_b on $\mathrm{Irr}(U_b)$. Therefore, we use Clifford theory only to construct those irreducible characters of P_b covering linear characters of U_b, that is, those $\chi \in \mathrm{Irr}(U_b)$ such that $U_b' \subseteq \ker(\chi)$.

The group P_b acts on $\mathrm{Irr}(U_b)$ by conjugation. For $\psi \in \mathrm{Irr}(U_b)$, we write $I_{P_b}(\psi) := \{x \in P_b \mid {}^x\psi = \psi\}$ for the inertia subgroup and we usually identify the inertia factor group

$\bar{I}_{P_b}(\psi) := I_{P_b}(\psi)/U_b$ with the corresponding subgroup of L_b. Choose linear characters of U_b as follows: $\lambda_0 := \lambda_{0,0,0,0,0}$, $\lambda_1 := \lambda_{1,0,0,0,0}$, $\lambda_2 := \lambda_{0,1,0,0,0}$, and $\lambda_3 := \lambda_{0,0,1,0,0}$.

PROPOSITION 4.1. *There are linear characters λ_4, λ_5 of U_b such that* (a), (b), (c) *are true:*

(a) *$\{\lambda_0, \ldots, \lambda_4, \lambda_5\}$ is a set of representatives for the orbits of P_b on the set of linear characters of U_b;*

(b) *the inertia factor groups $\bar{I}_j := \bar{I}_{P_b}(\lambda_j)$, $j = 0, 1, \ldots, 5$ have the orders:*

$$\begin{array}{lll} |\bar{I}_0| = |L_b|, & |\bar{I}_1| = q^4(q^2-1), & |\bar{I}_2| = q^2(q^2-1), \\ |\bar{I}_3| = 2(q^2-1), & |\bar{I}_4| = 4(q^2-\sqrt{2}q+1), & |\bar{I}_5| = 4(q^2+\sqrt{q}+1); \end{array}$$

(c) *the inertia factor groups are given by:*

(0) $\bar{I}_0 = L_b$;

(1) $\bar{I}_1 = \{h(z, z^{2\theta-1}, z^{2\theta-1}, z^{4\theta^2-4\theta+1})\alpha_1(r)\alpha_2(s) \mid z \in \mathbb{F}_{q^2}^\times, r, s \in \mathbb{F}_{q^2}\}$;

(2) $\bar{I}_2 = \{h(z, z^{2\theta-1}, z, z^{2\theta-1})\alpha_2(s) \mid z \in \mathbb{F}_{q^2}^\times, s \in \mathbb{F}_{q^2}\}$;

(3) $\bar{I}_3 = \langle n_b, h(1, 1, z, z^{2\theta-1}) \mid z \in \mathbb{F}_{q^2}^\times\rangle$;

(4) $\bar{I}_4 = \langle n_4\rangle \ltimes \{h(1, 1, z, z^{-q^2}) \mid z \in \mathbb{F}, z^{q^2-\sqrt{2}q+1} = 1\}$, *where the normal subgroup* $\{h(1, 1, z, z^{-q^2}) | z \in \mathbb{F}, z^{q^2-\sqrt{2}q+1} = 1\}$ *is cyclic of order $q^2-\sqrt{2}q+1$ and n_4 is an element of order four with $C_{\bar{I}_4}(n_4^2) = \langle n_4\rangle$;*

(5) $\bar{I}_5 = \langle n_5\rangle \ltimes \{h(1, 1, z, z^{-q^2}) \mid z \in \mathbb{F}, z^{q^2+\sqrt{2}q+1} = 1\}$, *where the normal subgroup* $\{h(1, 1, z, z^{-q^2}) | z \in \mathbb{F}, z^{q^2+\sqrt{2}q+1} = 1\}$ *is cyclic of order $q^2+\sqrt{2}q+1$ and n_5 is an element of order four with $C_{\bar{I}_5}(n_5^2) = \langle n_5\rangle$.*

Proof. It is not difficult to prove the assertions about $\bar{I}_0$, $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$. We demonstrate the computations for $\bar{I}_3$; the calculations for $\bar{I}_0$, $\bar{I}_1$, and $\bar{I}_2$ are similar. Each $u \in U_b$ can be written uniquely as $u = \alpha_3(d_3)\alpha_4(d_4)\ldots\alpha_{12}(d_{12})$ with $d_i \in \mathbb{F}_{q^2}$. By the Bruhat decomposition, each element $y \in L_b$ can be written uniquely as $y = h(z_1, z_2)\alpha_1(r)\alpha_2(s)$ or $y = \alpha_1(r')\alpha_2(s')h(z_1, z_2)n_b\alpha_1(r)\alpha_2(s)$ with $r, s, r', s' \in \mathbb{F}_{q^2}$ and $z_i \in \mathbb{F}_{q^2}^\times$. Let us first assume $y = h(z_1, z_2)\alpha_1(r)\alpha_2(s) \in \bar{I}_3$. Then, in particular, $\lambda_3(y^{-1}uy) = \lambda_3(u)$ for all $u \in U_b$, which is equivalent to

$$\phi(z_1^{2\theta-2}d_5 + s(z_1^{1-2\theta}z_2d_3)^{2\theta} + r(z_1^{1-2\theta}z_2^{2\theta-1}d_4)^{2\theta} + r^{2\theta+1}(z_1^{1-2\theta}z_2d_3)^{2\theta}) = \phi(d_5)$$

for all $d_3, d_4, \ldots, d_7 \in \mathbb{F}_{q^2}$. Using $\phi(x^2) = \phi(x)$ and $x^{2\theta^2} = x$ for all $x \in \mathbb{F}_{q^2}$, we get

$$(s^\theta z_1^{\theta-1}z_2^\theta + r^{\theta+1}z_1^{\theta-1}z_2^\theta)d_3 + r^\theta z_1^{\theta-1}z_2^{1-\theta}d_4 + (z_1^{2\theta-2}+1)d_5 \in \ker(\phi) \tag{1}$$

for all $d_3, d_4, \ldots, d_7 \in \mathbb{F}_{q^2}$. Inserting $d_3 = d_4 = 0$ implies that $(z_1^{2\theta-2}+1)d_5 \in \ker(\phi)$ for all $d_5 \in \mathbb{F}_{q^2}$ and thus $z_1 = 1$. Then, inserting $d_3 = d_5 = 0$ in ((1)) implies $r^\theta z_2^{1-\theta}d_4 \in \ker(\phi)$ for all $d_4 \in \mathbb{F}_{q^2}$. So, $r = 0$ and (1) becomes $s^\theta z_2^\theta d_3 \in \ker(\phi)$ for all $d_3 \in \mathbb{F}_{q^2}$, which implies $s = 0$. On the other hand, from the relations in [**10**, Table 2], it is clear that $y = h(1, z) \in \bar{I}_3$ for all $z \in \mathbb{F}_{q^2}$. The elements $y = \alpha_1(r')\alpha_2(s')h(z_1, z_2)n_b\alpha_1(r)\alpha_2(s)$ can be treated similarly. This proves the assertions about $\bar{I}_3$ in (b) and (c).

Finally, we prove the assertions about $\bar{I}_4$ and $\bar{I}_5$. The group L_b acts on U/U_b' by conjugation and, using the relations in [**10**, Tables 1–4], we see that L_b has exactly six orbits on U_b/U_b'. A set of representatives is given by the cosets of

$$1, \alpha_7(1), \alpha_6(1), \alpha_5(1), \alpha_5(1)\alpha_7(1), \alpha_5(1)\alpha_6(1).$$

Brauer's permutation lemma [**13**, Corollary (6.33)] implies that L_b has exactly six orbits on the set of linear characters of U_b. Since $\bar{I}_0, \ldots, \bar{I}_3$ have different orders, $\lambda_0, \ldots, \lambda_3$ are in different orbits under the action of L_b. Let $\bar{I}_4$ and $\bar{I}_5$ be the inertia factor groups corresponding to the missing two orbits. Consider the linear characters $\mu := \lambda_{0,1,1,1,0}$, $\mu' := \lambda_{1,1,1,1,0} \in \mathrm{Irr}(U_b)$. Using the relations [**10**, Tables 1–4], it is not difficult to see that μ, μ' are not conjugate to each other and not conjugate to $\lambda_0, \ldots, \lambda_3$ under the action of L_b and so $\{\lambda_0, \ldots, \lambda_3, \mu, \mu'\}$ is a set of representatives for the orbits of L_b on the set of linear characters of U_b.

We were not able to determine the order and structure of $\bar{I}_4$ and $\bar{I}_5$ by a direct calculation, since we were not able to solve the occurring systems of polynomial equations. Instead, we prove the assertions about $\bar{I}_4$ and $\bar{I}_5$ in several steps:

Step 1: $|\bar{I}_4|, |\bar{I}_5| \leq 8q^2 + 4$.
Let $\mathrm{Stab}_{L_b}(\mu) := \{y \in L_b \mid {}^y\mu = \mu\}$. Suppose $y = h(z_1, z_2)\alpha_1(r)\alpha_2(s) \in \mathrm{Stab}_{L_b}(\mu)$. So, $\mu(y^{-1}uy) = \mu(u)$ for all $u \in U_b$. Inserting $u = \alpha_5(d)$, $u = \alpha_6(d)$, $u = \alpha_4(d)$, $u = \alpha_3(d)$ for $d \in \mathbb{F}_{q^2}$ implies

$$z_1 = z_2 = 1 \text{ and } r, s \in \{0, 1\}.$$

Now suppose $y = \alpha_1(r')\alpha_2(s')h(z_1, z_2)n_b\alpha_1(r)\alpha_2(s) \in L_b$ such that ${}^y\mu = \mu$. So, $\mu(y^{-1}uy) = \mu(u)$ for all $u \in U_b$. Inserting $u = \alpha_5(d)$, $u = \alpha_7(d)$, $u = \alpha_6(d)$, $u = \alpha_4(d)$, $u = \alpha_3(d)$ for $d \in \mathbb{F}_{q^2}$ implies

$$\begin{aligned} z_1 &= 1, \\ s^\theta + s &= r^{\theta+1} + r, \\ z_2^{2\theta-1} &= 1 + r^\theta + r^{2\theta}, \\ r' &\in \{r, r+1\}, \\ s'^\theta + s' &= r' z_2^{2\theta-1} + r'^{\theta+1}. \end{aligned} \tag{2}$$

Since $\mathbb{F}_{q^2} \to \mathbb{F}_{q^2}, x \mapsto x^\theta + x$ is an $\mathbb{F}_2$-linear map with kernel $\{0, 1\}$ and $2\theta - 1$ is relatively prime with $|\mathbb{F}_{q^2}^\times|$, the number of solutions of the system of equations (2) is at most $8q^2$. Thus, $|\mathrm{Stab}_{L_b}(\mu)| \leq 4 + 8q^2$. Analogously, $|\mathrm{Stab}_{L_b}(\mu')| \leq 4 + 8q^2$. So, we get $|\bar{I}_4|, |\bar{I}_5| \leq 8q^2 + 4$.

Step 2: $\bar{I}_4$ contains an element n_4 of order four, $\bar{I}_5$ contains an element n_5 of order four.
Using the commutator relations [**10**, Table 1], we can verify that the element $\alpha_1(1)$ of order four stabilizes μ and μ'.

Step 3: $\bar{I}_4$ contains an element of order $q^2 - \sqrt{2}q + 1$, $\bar{I}_5$ contains an element of order $q^2 + \sqrt{2}q + 1$.
By Tables A.6 and A.7, the Levi complement L_b has a cyclic subgroup

$$\{h(1, 1, z, z^{-q^2}) \mid z \in \mathbb{F}, z^{q^2 - \sqrt{2}q + 1} = 1\}$$

of order $q^2 - \sqrt{2}q + 1$. Let h be a generator of this subgroup. By the orders of the centralizers in Table A.7, there is a non-trivial element of U_b/U_b' which is fixed by $\langle h \rangle$; see class $c_{1,9}$. By Brauer's permutation lemma, there is a non-trivial linear character λ_4 of U_b which is fixed by h. Similarly, there is a non-trivial linear character λ_5 of U_b which is fixed by the cyclic subgroup $\{h(1, 1, z, z^{-q^2}) \mid z \in \mathbb{F}, z^{q^2 + \sqrt{2}q + 1} = 1\}$ of order $q^2 + \sqrt{2}q + 1$. Since $|\bar{I}_1|, |\bar{I}_2|, |\bar{I}_3|$ are relatively prime with $q^2 \pm \sqrt{2}q + 1$, the linear characters λ_4 and λ_5 are not conjugate to $\lambda_0, \ldots, \lambda_3$ under the action of L_b. So, λ_4, λ_5 are contained in the two missing orbits of the action of L_b on the set of linear characters of U_b. Suppose that λ_4, λ_5 are conjugate to each other under the action of L_b. Then, $|\bar{I}_4|$ or $|\bar{I}_5|$ is a multiple of $4(q^4 + 1) = 4(q^2 - \sqrt{2}q + 1)(q^2 + \sqrt{2}q + 1)$, contradicting Step 1. So, $\{\lambda_0, \ldots, \lambda_5\}$ is a set of representatives for the orbits of L_b (or P_b) on the set of linear characters of U_b and $\bar{I}_4$ contains an element of order $q^2 - \sqrt{2}q + 1$ and $\bar{I}_5$ contains an element of order $q^2 + \sqrt{2}q + 1$.

Step 4: $|\bar{I}_4| = 4(q^2 - \sqrt{2}q + 1)$ and $\bar{I}_5 = 4(q^2 + \sqrt{2}q + 1)$.
From Steps 2 and 3, we know that $|\bar{I}_4|$ is a multiple of $4(q^2 - \sqrt{2}q + 1)$ and $|\bar{I}_5|$ is a multiple of $4(q^2 + \sqrt{2}q + 1)$. Summation over the sizes of the orbits shows that equality holds.

Step 5: The cyclic subgroups of order $q^2 \pm \sqrt{2}q + 1$ are normal in $\bar{I}_4$, $\bar{I}_5$, respectively.
By Sylow's theorem, the Sylow subgroups of the cyclic subgroup of order $q^2 - \sqrt{2}q + 1$ are normal in $\bar{I}_4$; note that the indices of the normalizers of these Sylow subgroups are 1, 2 or 4

and $3 \nmid q^2 - \sqrt{2}q + 1$. It follows that the cyclic subgroup of order $q^2 - \sqrt{2}q + 1$ is normal in $\bar{I}_4$. The proof for $\bar{I}_5$ is similar.

Step 6: $C_{\bar{I}_4}(n_4^2) = \langle n_4 \rangle$ and $C_{\bar{I}_5}(n_5^2) = \langle n_5 \rangle$.
By the orders of the centralizers in Table A.7, there is no involution in L_b centralizing a non-identity element whose order is a divisor of $q^2 \pm \sqrt{2}q + 1$. The claims of Step 6 and Proposition 4.1 follow. □

Now, we start to construct the irreducible characters of P_b. In a first step, we construct all irreducible characters of P_b covering the linear characters of U_b. In a second step, we compute all nonlinear $\chi \in \mathrm{Irr}(P_b)$ such that $Z(U_b) \subseteq \ker(\chi)$. Finally, we determine those $\chi \in \mathrm{Irr}(P_b)$ such that $Z(U_b) \nsubseteq \ker(\chi)$.

The irreducible characters of the finite group of Lie type L_b can be obtained by Deligne–Lusztig theory and then be inflated to P_b. The characters ${}_{P_b}\chi_5(k,l)$, ${}_{P_b}\chi_6(k)$, ${}_{P_b}\chi_7(k)$ correspond (up to sign) to Deligne–Lusztig characters of L_b; the characters ${}_{P_b}\chi_1(k)$, ${}_{P_b}\chi_2(k)$, ${}_{P_b}\chi_3(k)$, ${}_{P_b}\chi_4(k)$ are the inflations of the unipotent irreducible characters of ${}^2B_2(q^2)$ multiplied by the linear characters of L_b. For the character table of ${}^2B_2(q^2)$, see the CHEVIE library or [**17**]. So, we have constructed all irreducible characters of P_b covering the trivial character λ_0 of U_b.

The irreducible characters of P_b covering λ_1 are obtained by induction from the Borel subgroup B: let ${}_{P_b}\chi_8(k) := {}_B\chi_5(k)^{P_b}$ for $k = 0, \ldots, q^2-2$, ${}_{P_b}\chi_9 := {}_B\chi_6^{P_b}$, ${}_{P_b}\chi_{10} := {}_B\chi_7^{P_b}$ and ${}_{P_b}\chi_{11} := {}_B\chi_8^{P_b}$. To see that these characters are the only characters covering λ_1, we compute the induced character $\lambda_1^{P_b}$ and see via scalar products that ${}_{P_b}\chi_8(k)$, ${}_{P_b}\chi_9$, ${}_{P_b}\chi_{10}$, ${}_{P_b}\chi_{11}$ are the only irreducible constituents.

Analogously, we get the irreducible characters of P_b covering λ_2 by induction from the Borel subgroup: ${}_{P_b}\chi_{12}(k) := {}_B\chi_9(k)^{P_b}$ for $k = 0, \ldots, q^2-2$, ${}_{P_b}\chi_{13} := {}_B\chi_{10}^{P_b}$.

Next, we construct the irreducible characters of P_b covering λ_3. We use the notation from [**7**, Lemma 5.4]. By Proposition 4.1, $\bar{I}_3$ is the semidirect product of $K := \langle n_b \rangle$ and $H_1 := \{h(1,1,z,z^{2\theta-1}) \mid z \in \mathbb{F}_{q^2}^\times\}$. By [**7**, Lemma 5.4], $\bar{I}_3$ has $(q^2-2)/2+2$ irreducible characters: two linear characters and $(q^2-2)/2$ irreducible characters of degree two. Let $\mathbf{1}_{\bar{I}_3}$ be the trivial and $\varepsilon_{\bar{I}_3}$ the non-trivial linear character of $\bar{I}_3$. So, again by [**7**, Lemma 5.4], P_b has exactly $(q^2-2)/2+2$ irreducible characters covering λ_3: the two irreducible characters $(\mathbf{1}_{\bar{I}_3} \ltimes \lambda_3)^{P_b}$ and $(\varepsilon_{\bar{I}_3} \ltimes \lambda_3)^{P_b}$ of degree $q^4(q^4+1)(q^2-1)/2$, say ${}_{P_b}\chi_{14}$ and ${}_{P_b}\chi_{15}$, and the $(q^2-2)/2$ irreducible characters $(\lambda \ltimes \lambda_3)^{P_b}$ of degree $q^4(q^4+1)(q^2-1)$, where λ runs through a set $\mathcal{S}$ for representatives of the orbits of K on $\mathrm{Irr}(H_1)$ not containing $\mathbf{1}_{H_1}$.

By the definition of ${}_B\chi_{11}(0)$, we have ${}_B\chi_{11}(0)^{P_b} = {}_{P_b}\chi_{14} + {}_{P_b}\chi_{15}$. By construction, ${}_{P_b}\chi_{14}$, ${}_{P_b}\chi_{15}$ coincide on all conjugacy classes of P_b, except for the classes $c_{1,14}, \ldots, c_{1,29}$, where the values of ${}_{P_b}\chi_{14}$, ${}_{P_b}\chi_{15}$ only differ in the sign. Hence, ${}_{P_b}\chi_{14}(x) = {}_{P_b}\chi_{15}(x) = \frac{1}{2}({}_B\chi_{11}(0)^{P_b}(x))$ for all $x \in P_b$ with $x \notin c_{1,14}, \ldots, c_{1,29}$. Let $x_j \in c_{1,j}$ for $j = 14, \ldots, 29$. We have ${}_{P_b}\chi_{14}(x_{14}) = {}_{P_b}\chi_{14}(x_j)$ for $j = 15, 16, 17$ and ${}_{P_b}\chi_{14}(x_{18}) = {}_{P_b}\chi_{14}(x_j)$ for $j = 19, 20, \ldots, 24$ and ${}_{P_b}\chi_{14}(x_{26}) = {}_{P_b}\chi_{14}(x_{27})$. Note that the values of ${}_{P_b}\chi_{14}$ on the classes $c_{1,21}(a), \ldots, c_{1,24}(a)$ do not depend on the parameter a since $\alpha_8(1) \in \ker({}_{P_b}\chi_{14})$. So, we have reduced the computation of the missing values of ${}_{P_b}\chi_{14}, {}_{P_b}\chi_{15}$ to six unknown character values and these can be determined from the conditions $({}_{P_b}\chi_{14}, {}_{P_b}\chi_1(0))_{P_b} = ({}_{P_b}\chi_{14}, {}_{P_b}\chi_9)_{P_b} = ({}_{P_b}\chi_{14}, {}_{P_b}\chi_{13})_{P_b} = 0$, $({}_{P_b}\chi_i, ({}_G\chi_{10})_{P_b})_{P_b}$, $({}_{P_b}\chi_i, ({}_G\chi_8)_{P_b})_{P_b} \in \mathbb{Z}_{\geq 0}$ for $i = 14, 15$, and $({}_{P_b}\chi_{14}, {}_{P_b}\chi_{14})_{P_b} = 1$. Here $({}_G\chi_8)_{P_b}$, $({}_G\chi_{10})_{P_b}$ are the restrictions of the unipotent irreducible characters ${}_G\chi_8, {}_G\chi_{10}$ of $G = {}^2F_4(q^2)$ to P_b (see Table 4). For $k = 1, \ldots, q^2-2$, we define ${}_{P_b}\chi_{16}(k) := {}_B\chi_{11}(k)^{P_b}$ and we have

$$\{{}_{P_b}\chi_{16}(k) \mid k = 1, \ldots, q^2-2\} = \{(\lambda \ltimes \lambda_3)^{P_b} \mid \lambda \in \mathcal{S}\}.$$

The characters ${}_{P_b}\chi_{14}$, ${}_{P_b}\chi_{15}$, ${}_{P_b}\chi_{16}(k)$ are the irreducible characters of P_b covering $\lambda_3 \in \mathrm{Irr}(U_{P_b})$.

Next, we construct the irreducible characters of P_b covering λ_4 or λ_5. By Proposition 4.1, $\bar{I}_4$ is the semidirect product of the group $K := \langle n_4 \rangle$ and the normal subgroup $H_1 := \{h(1, 1, z, z^{-q^2}) \mid z \in \mathbb{F}, z^{q^2-\sqrt{2}q+1} = 1\}$. So, by [**7**, Lemma 5.4], $\bar{I}_4$ has $(q^2 - \sqrt{2}q)/4 + 4$ irreducible characters: four linear characters and $(q^2 - \sqrt{2}q)/4$ irreducible characters of degree four.

Let $\varepsilon_{\bar{I}_4}, \varepsilon'_{\bar{I}_4}, \varepsilon''_{\bar{I}_4}, \varepsilon'''_{\bar{I}_4}$ be the linear characters of $\bar{I}_4$ and, for $k \in \mathbb{Z}$, let μ_k be the linear character of H_1 mapping $h(1, 1, \tilde{\varphi}_8''^{i}, \tilde{\varphi}_8''^{-q^2 i}) \mapsto (\varphi_8'')^{ik} \in \mathbb{C}$, where we use the notation from [**10**, Table 5]. So, again by [**7**, Lemma 5.4], P_b has exactly $(q^2 - \sqrt{2}q)/4 + 4$ irreducible characters covering λ_4: the irreducible characters $(\varepsilon_{\bar{I}_4} \ltimes \lambda_4)^{P_b}$, $(\varepsilon'_{\bar{I}_4} \ltimes \lambda_4)^{P_b}$, $(\varepsilon''_{\bar{I}_4} \ltimes \lambda_4)^{P_b}$, $(\varepsilon'''_{\bar{I}_4} \ltimes \lambda_4)^{P_b}$ of degree $q^4(q^2-1)^2(q^2+\sqrt{2}q+1)/4$, say ${}_{P_b}\chi_{17}, \ldots, {}_{P_b}\chi_{20}$, and the $(q^2 - \sqrt{2}q)/4$ irreducible characters ${}_{P_b}\chi_{21}(k) := (\mu_k \ltimes \lambda_4)^{P_b}$ of degree $q^4(q^2-1)^2(q^2+\sqrt{2}q+1)$ for $k = 1, \ldots, q^2 - \sqrt{2}q$, where certain values of k give the same character.

Analogously, P_b has $(q^2 + \sqrt{2}q)/4 + 4$ irreducible characters covering λ_5: four irreducible characters of degree $q^4(q^2-1)^2(q^2-\sqrt{2}+1)/4$, say ${}_{P_b}\chi_{22}, \ldots, {}_{P_b}\chi_{25}$, and $(q^2 + \sqrt{2}q)/4$ irreducible characters ${}_{P_b}\chi_{26}(k)$ of degree $q^4(q^2-1)^2(q^2-\sqrt{2}q+1)$.

The values of ${}_{P_b}\chi_{21}(k)$ can be computed as follows: by construction, ${}_{P_b}\chi_{21}(k)$ vanishes on all conjugacy classes of P_b except possibly for the classes $c_{1,0}, \ldots, c_{1,13}$ and $c_{8,0}(i), \ldots, c_{8,3}(i)$. Let $x_j \in c_{1,j}$ for $j = 14, \ldots, 13$ and $x_{8,j}(i) \in c_{8,j}(i)$ for $j = 0, 1, 2, 3$. We consider the induced character $\psi_{21} := {}_B\chi_{15}^{P_b} + {}_B\chi_{16}^{P_b} + {}_B\chi_{17}^{P_b} + {}_B\chi_{18}^{P_b}$. By construction, ψ_{21} is induced by a linear character of U_b, and with CHEVIE we can verify $(\psi_{21}, {}_{P_b}\chi_1(k))_{P_b} = (\psi_{21}, {}_{P_b}\chi_2(k))_{P_b} = \ldots = (\psi_{21}, {}_{P_b}\chi_{16}(k))_{P_b} = 0$ and $(\psi_{21}, \psi_{21})_{P_b} = 4(q^2 - \sqrt{2} + 1)$. So, Proposition 4.1 and Clifford theory imply that ${}_{P_b}\chi_{21}(k)(x_j) = (1/(q^2 - \sqrt{2}q + 1))\psi_{21}(x_j)$ for $j = 0, 1, \ldots, 13$. By the definition of induced characters, we have

$$ {}_{P_b}\chi_{21}(k)(x_{8,0}(i)) = {}_{P_b}\chi_{21}(k)(x_{8,1}(i)) = (q^2-1)(\varphi_8''^{ik} + \varphi_8''^{-ik} + \varphi_8''^{q^2 ik} + \varphi_8''^{-q^2 ik}) $$

and

$$ {}_{P_b}\chi_{21}(k)(x_{8,2}(i)) = {}_{P_b}\chi_{21}(k)(x_{8,3}(i)) = A \cdot (\varphi_8''^{ik} + \varphi_8''^{-ik} + \varphi_8''^{q^2 ik} + \varphi_8''^{-q^2 ik}) $$

for some $A \in \mathbb{C}$ independent of i and k. The constant A can now be determined from the fact that the scalar product of ${}_{P_b}\chi_{21}(k)$ with the trivial character is zero. The values of ${}_{P_b}\chi_{26}(k)$ can be obtained analogously using the induced character $\psi_{26} := {}_B\chi_{19}^{P_b} + {}_B\chi_{20}^{P_b} + {}_B\chi_{21}^{P_b} + {}_B\chi_{22}^{P_b}$.

The values of the irreducible characters ${}_{P_b}\chi_{17}, {}_{P_b}\chi_{18}, {}_{P_b}\chi_{19}, {}_{P_b}\chi_{20}, {}_{P_b}\chi_{22}, {}_{P_b}\chi_{23}, {}_{P_b}\chi_{24}, {}_{P_b}\chi_{25}$ can be computed analogously to the characters ${}_{P_b}\chi_{14}, {}_{P_b}\chi_{15}$ using orthogonality relations with the irreducible characters ${}_{P_b}\chi_1(k), \ldots, {}_{P_b}\chi_{16}(k)$ and with the restrictions of the unipotent irreducible characters ${}_G\chi_5$, ${}_G\chi_6$, ${}_G\chi_8$, ${}_G\chi_9$, ${}_G\chi_{11}$, $\overline{{}_G\chi_{11}}$, ${}_G\chi_{13}$, $\overline{{}_G\chi_{13}}$ of G; see Table 4. The characters $\overline{{}_G\chi_{11}}$, $\overline{{}_G\chi_{13}}$ are the complex-conjugate characters of ${}_G\chi_{11}$, ${}_G\chi_{13}$, respectively.

The characters ${}_{P_b}\chi_{17}, \ldots, {}_{P_b}\chi_{21}(k)$ are the only irreducible characters of P_b covering $\lambda_4 \in \mathrm{Irr}(U_b)$ and the characters ${}_{P_b}\chi_{22}, \ldots, {}_{P_b}\chi_{26}(k)$ are the only irreducible characters of P_b covering $\lambda_5 \in \mathrm{Irr}(U_b)$. So, we have computed all irreducible characters of P_b covering the linear characters of U_b.

Next, we compute all nonlinear characters $\chi \in \mathrm{Irr}(P_b)$ such that $Z(U_b) \subseteq \ker(\chi)$. We get all of these characters by inducing irreducible characters from the Borel subgroup B. For $i = 27, 28, \ldots, 42$, we define ${}_{P_b}\chi_i := {}_B\chi_{i-1}^{P_b}$, where ${}_B\chi_{i-1}^{P_b}$ denotes the induced character (or ${}_{P_b}\chi_i(k) := {}_B\chi_{i-1}(k)^{P_b}$ if ${}_B\chi_{i-1}$ depends on some parameter k). The values of these induced characters can easily be computed using the character values in [**10**, Table A.6] and the class fusions in Table A.1 in the appendix. Computing scalar products with CHEVIE, we see that the characters ${}_{P_b}\chi_{27}(k), \ldots, {}_{P_b}\chi_{42}(k)$ are irreducible and pairwise different. Summing up the squares of the degrees, we see that we have constructed all $\chi \in \mathrm{Irr}(P_b)$ such that $Z(U_b) = U_{12} \subseteq \ker(\chi)$.

Finally, we compute all characters $\chi \in \mathrm{Irr}(P_b)$ such that $Z(U_b) \not\subseteq \ker(\chi)$. We use the restriction $({}_G\chi_2)_{P_b}$ of the unipotent irreducible character ${}_G\chi_2$ of G to P_b; see Table 4. We set

$$ {}_{P_b}\chi_{43} := ({}_G\chi_2)_{P_b} - {}_{P_b}\chi_2(0) - {}_{P_b}\chi_{27}(0) $$

and let ${}_{P_b}\chi_{50} := \overline{{}_{P_b}\chi_{43}}$ be its complex conjugate. The characters ${}_{P_b}\chi_{44}, \ldots, {}_{P_b}\chi_{49}(k)$ and ${}_{P_b}\chi_{51}, \ldots, {}_{P_b}\chi_{56}(k)$ are then obtained by tensoring ${}_{P_b}\chi_{43}$ and ${}_{P_b}\chi_{50}$ with the irreducible characters ${}_{P_b}\chi_2(0), {}_{P_b}\chi_3(0), \ldots, {}_{P_b}\chi_7(k)$.

Theorem 4.2. *The character table of the maximal parabolic subgroup P_b is given by Tables A.8 and A.9 in the appendix.*

Proof. Computing scalar products with CHEVIE, we see that we have constructed $q^4 + 14q^2 + 15$ irreducible and pairwise different characters of P_b. □

We point out that we are not able to describe all values of all irreducible characters of P_a and P_b. This is due to the fact that we do not have generic descriptions of certain unipotent conjugacy classes, for example the classes $c_{1,21}(a), \ldots, c_{1,24}(a)$ of P_b. This seems to be a usual phenomenon for generic character tables of parabolic subgroups (see for example [**4**]). However, even for those characters where we do not know *all* character values it is possible to compute the values on *some* unipotent classes. The following lemma, which is used in [**11**], demonstrates this for the faithful characters of P_a.

Lemma 4.3. *For $k = 1, 2, \ldots, q^2$, the characters ${}_{P_a}\chi_{38}(k), {}_{P_a}\chi_{39}(k), {}_{P_a}\chi_{40}(k)$ have the following values:*

(a) ${}_{P_a}\chi_{38}(k)(\alpha_{12}(1)) = {}_{P_a}\chi_{39}(k)(\alpha_{12}(1)) = -\dfrac{q^7}{\sqrt{2}}(q^2-1)$;

(b) ${}_{P_a}\chi_{38}(k)(\alpha_8(1)) = {}_{P_a}\chi_{39}(k)(\alpha_8(1)) = -\dfrac{q^5}{\sqrt{2}}(q^2-1)^2$;

(c) ${}_{P_a}\chi_{40}(k)(\alpha_{12}(1)) = -q^8(q^2-1)$;

(d) ${}_{P_a}\chi_{40}(k)(\alpha_8(1)) = q^6(q^2-1)$.

Proof. The irreducible characters ${}_{P_a}\chi_i(k)$, $i = 38, 39, 40$, are induced from the Borel subgroup B; more precisely, ${}_{P_a}\chi_i(k) = {}_B\chi_{i+10}(k)^{P_a}$ for $i = 38, 39, 40$ and $k = 1, 2, \ldots, q^2$. The values of the sums $\sum_{k=1}^{q^2} {}_{P_a}\chi_i(k)$, $i = 38, 39, 40$, are given in Table A.5.

By the construction in [**10**, § 4], the values of ${}_B\chi_{48}(k)$ and ${}_B\chi_{49}(k)$ on the elements $\alpha_{12}(1)$, $\alpha_{11}(1)$, $\alpha_8(1)$ do not depend on k and all of these characters vanish on $\alpha_2(1)$. So, the class fusions in Table A.1 imply that the values of the induced characters ${}_{P_a}\chi_{38}(k) = {}_B\chi_{48}(k)^{P_a}$, ${}_{P_a}\chi_{39}(k) = {}_B\chi_{49}(k)^{P_a}$ on $\alpha_{12}(1)$ and $\alpha_8(1)$ do not depend on k. So, we can compute the values of ${}_{P_a}\chi_{38}(k)$, ${}_{P_a}\chi_{39}(k)$ on $\alpha_{12}(1)$ and $\alpha_8(1)$ from Table A.5 by dividing by q^2.

The characters ${}_{P_a}\chi_{40}(k)$ can be treated similarly. As above, we see that the values of ${}_B\chi_{50}(k)$ on the elements $\alpha_i(1)$, $i = 12, 11, 8, 2$, do not depend on k. So, we can compute the values of ${}_{P_a}\chi_{40}(k) = {}_B\chi_{50}(k)^{P_a}$ on $\alpha_{12}(1)$, $\alpha_8(1)$ from Table A.5 by dividing by q^2. □

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Appendix

TABLE A.1. *Fusion of the conjugacy classes of the Borel subgroup B in the maximal parabolic subgroups P_a and P_b.*

Conjugacy class of B	Fusion in P_a	Fusion in P_b	Conjugacy class of B	Fusion in P_a	Fusion in P_b
$c_{1,0}$	$c_{1,0}$	$c_{1,0}$	$c_{1,50}$	$c_{1,35}$	$c_{1,26}$
$c_{1,1}$	$c_{1,1}$	$c_{1,1}$	$c_{1,51}$	$c_{1,36}$	$c_{1,27}$
$c_{1,2}$	$c_{1,1}$	$c_{1,2}$	$c_{1,52}$	$c_{1,37}$	$c_{1,28}$
$c_{1,3}$	$c_{1,2}$	$c_{1,3}$	$c_{1,53}$	$c_{1,38}$	$c_{1,29}$
$c_{1,4}$	$c_{1,3}$	$c_{1,3}$	$c_{1,54}$	$c_{1,12}$	$c_{1,30}$
$c_{1,5}$	$c_{1,4}$	$c_{1,2}$	$c_{1,55}$	$c_{1,13}$	$c_{1,31}$
$c_{1,6}$	$c_{1,5}$	$c_{1,3}$	$c_{1,56}$	$c_{1,34}$	$c_{1,32}$
$c_{1,7}$	$c_{1,3}$	$c_{1,4}$	$c_{1,57}$	$c_{1,16}$	$c_{1,33}$
$c_{1,8}$	$c_{1,6}$	$c_{1,5}$	$c_{1,58}$	$c_{1,17}$	$c_{1,34}$
$c_{1,9}$	$c_{1,7}$	$c_{1,6}$	$c_{1,59}(a)$	$c_{1,34}$	$c_{1,35}(a)$
$c_{1,10}$	$c_{1,8}$	$c_{1,7}$	$c_{1,60}(a)$	$c_{1,35}$	$c_{1,36}(a)$
$c_{1,11}$	$c_{1,9}$	$c_{1,8}$	$c_{1,61}(a)$	$c_{1,36}$	$c_{1,37}(a)$
$c_{1,12}$	$c_{1,7}$	$c_{1,9}$	$c_{1,62}$	$c_{1,14}$	$c_{1,38}$
$c_{1,13}$	$c_{1,8}$	$c_{1,10}$	$c_{1,63}$	$c_{1,15}$	$c_{1,39}$
$c_{1,14}$	$c_{1,9}$	$c_{1,11}$	$c_{1,64}$	$c_{1,34}$	$c_{1,40}$
$c_{1,15}$	$c_{1,10}$	$c_{1,12}$	$c_{1,65}(a)$	$c_{1,34}$	$c_{1,41}(a)$
$c_{1,16}$	$c_{1,11}$	$c_{1,13}$	$c_{1,66}(a)$	$c_{1,35}$	$c_{1,42}(a)$
$c_{1,17}$	$c_{1,12}$	$c_{1,6}$	$c_{1,67}(a)$	$c_{1,36}$	$c_{1,43}(a)$
$c_{1,18}$	$c_{1,13}$	$c_{1,8}$	$c_{1,68}$	$c_{1,16}$	$c_{1,44}$
$c_{1,19}$	$c_{1,14}$	$c_{1,7}$	$c_{1,69}$	$c_{1,17}$	$c_{1,45}$
$c_{1,20}$	$c_{1,15}$	$c_{1,8}$	$c_{1,70}$	$c_{1,39}$	$c_{1,46}$
$c_{1,21}$	$c_{1,16}$	$c_{1,12}$	$c_{1,71}$	$c_{1,40}$	$c_{1,47}$
$c_{1,22}$	$c_{1,17}$	$c_{1,13}$	$c_{1,72}$	$c_{1,41}$	$c_{1,48}$
$c_{1,23}$	$c_{1,18}$	$c_{1,4}$	$c_{1,73}$	$c_{1,42}$	$c_{1,49}$
$c_{1,24}$	$c_{1,19}$	$c_{1,5}$	$c_{2,0}(i)$	$c_{3,0}(\theta i)$	$c_{2,0}(i)$
$c_{1,25}$	$c_{1,20}$	$c_{1,8}$	$c_{2,1}(i)$	$c_{3,1}(\theta i)$	$c_{2,1}(i)$
$c_{1,26}$	$c_{1,21}$	$c_{1,8}$	$c_{2,2}(i)$	$c_{3,2}(\theta i)$	$c_{2,2}(i)$
$c_{1,27}$	$c_{1,22}$	$c_{1,6}$	$c_{2,3}(i)$	$c_{3,3}(\theta i)$	$c_{2,3}(i)$
$c_{1,28}$	$c_{1,23}$	$c_{1,7}$	$c_{3,0}(i)$	$c_{2,0}(i)$	$c_{3,0}(i)$
$c_{1,29}(a)$	$c_{1,24}(a)$	$c_{1,8}$	$c_{3,1}(i)$	$c_{2,1}(i)$	$c_{3,1}(i)$
$c_{1,30}(a)$	$c_{1,25}(a)$	$c_{1,8}$	$c_{4,0}(i)$	$c_{3,0}(i)$	$c_{4,0}(i)$
$c_{1,31}$	$c_{1,26}$	$c_{1,11}$	$c_{4,1}(i)$	$c_{3,1}(i)$	$c_{4,1}(i)$
$c_{1,32}$	$c_{1,27}$	$c_{1,13}$	$c_{4,2}(i)$	$c_{3,2}(i)$	$c_{4,2}(i)$
$c_{1,33}$	$c_{1,28}$	$c_{1,12}$	$c_{4,3}(i)$	$c_{3,3}(i)$	$c_{4,3}(i)$
$c_{1,34}(a)$	$c_{1,29}(a)$	$c_{1,12}$	$c_{5,0}(i)$	$c_{4,0}(i)$	$c_{5,0}(i)$
$c_{1,35}(a)$	$c_{1,30}(a)$	$c_{1,13}$	$c_{5,1}(i)$	$c_{4,1}(i)$	$c_{5,1}(i)$
$c_{1,36}(a)$	$c_{1,31}(a)$	$c_{1,12}$	$c_{5,2}(i)$	$c_{4,2}(i)$	$c_{5,2}(i)$
$c_{1,37}(a)$	$c_{1,32}(a)$	$c_{1,13}$	$c_{5,3}(i)$	$c_{4,3}(i)$	$c_{5,3}(i)$
$c_{1,38}$	$c_{1,4}$	$c_{1,14}$	$c_{6,0}(i)$	$c_{4,0}(2\theta i)$	$c_{4,0}(i)$
$c_{1,39}$	$c_{1,5}$	$c_{1,15}$	$c_{6,1}(i)$	$c_{4,1}(2\theta i)$	$c_{4,1}(i)$
$c_{1,40}$	$c_{1,6}$	$c_{1,16}$	$c_{6,2}(i)$	$c_{4,2}(2\theta i)$	$c_{4,2}(i)$
$c_{1,41}$	$c_{1,11}$	$c_{1,17}$	$c_{6,3}(i)$	$c_{4,3}(2\theta i)$	$c_{4,3}(i)$
$c_{1,42}$	$c_{1,7}$	$c_{1,18}$	$c_{7,0}(i)$	$c_{5,0}(i)$	$c_{3,0}(i)$
$c_{1,43}$	$c_{1,8}$	$c_{1,19}$	$c_{7,1}(i)$	$c_{5,1}(i)$	$c_{3,1}(i)$
$c_{1,44}$	$c_{1,9}$	$c_{1,20}$	$c_{8,0}(i)$	$c_{5,0}(i)$	$c_{6,0}(i)$
$c_{1,45}(a)$	$c_{1,9}$	$c_{1,21}(a)$	$c_{8,1}(i)$	$c_{5,1}(i)$	$c_{6,1}(i)$
$c_{1,46}(a)$	$c_{1,10}$	$c_{1,22}(a)$	$c_{9,0}(i)$	$c_{6,0}(i)$	$c_{6,0}(-i)$
$c_{1,47}(a)$	$c_{1,11}$	$c_{1,23}(a)$	$c_{9,1}(i)$	$c_{6,1}(i)$	$c_{6,1}(-i)$
$c_{1,48}(a)$	$c_{1,33}$	$c_{1,24}(a)$	$c_{10,0}(i,j)$	$c_{7,0}(i,j)$	$c_{7,0}(i,j)$
$c_{1,49}$	$c_{1,34}$	$c_{1,25}$			

TABLE A.2. *Parametrization of the semisimple conjugacy classes of P_a.*

Representative	Parameters	Number of classes
$h_1 := h(1,1,1,1)$		1
$h_2(i) := h(\tilde{\zeta}_2^i, \tilde{\zeta}_2^{(2\theta-1)i}, \tilde{\zeta}_2^{(2\theta-1)i}, \tilde{\zeta}_2^{(4\theta^2-4\theta+1)i})$	$i = 0, \ldots, q^2-2$ $i \neq 0$	q^2-2
$h_3(i) := h(\tilde{\zeta}_2^i, \tilde{\zeta}_2^{(2\theta-1)i}, \tilde{\zeta}_2^i, \tilde{\zeta}_2^{(2\theta-1)i})$	$i = 0, \ldots, q^2-2$ $i \neq 0$	q^2-2
$h_4(i) := h(1, 1, \tilde{\zeta}_2^i, \tilde{\zeta}_2^{(2\theta-1)i})$	$i = 0, \ldots, q^2-2$ $i \neq 0$	q^2-2
$h_5(i) := h(\tilde{\zeta}_2^{(2\theta-1)i}, \tilde{\zeta}_2^{(4\theta^2-4\theta+1)i}, \tilde{\zeta}_2^i, \tilde{\zeta}_2^{(2\theta-1)i})$	$i = 0, \ldots, q^2-2$ $i \neq 0$	q^2-2
$h_6(i) := h(\tilde{\zeta}_2^{(1-2\theta)i}, \tilde{\zeta}_2^{(-4\theta^2+4\theta-1)i}, \tilde{\zeta}_2^i, \tilde{\zeta}_2^{(2\theta-1)i})$	$i = 0, \ldots, q^2-2$ $i \neq 0$	$\frac{q^2-2}{2}$
$h_7(i,j) := h(\tilde{\zeta}_2^i, \tilde{\zeta}_2^{(2\theta-1)i}, \tilde{\zeta}_2^j, \tilde{\zeta}_2^{(2\theta-1)j})$	$i, j = 0, \ldots, q^2-2$ $i, j \neq 0$ $j \neq \pm i, \pm(2\theta-1)i$ $i \neq \pm(2\theta-1)j$	$\frac{q^4-10q^2+16}{2}$
$h_8 := h(\tilde{\varepsilon}_3, \tilde{\varepsilon}_3^{1-\theta}, \tilde{\varepsilon}_3^{-1-\theta}, \tilde{\varepsilon}_3^{-\theta})$		1
$h_9(i) := h(\tilde{\xi}_2^i, \tilde{\xi}_2^{(1-\theta)i}, \tilde{\xi}_2^{(-1-\theta)i}, \tilde{\xi}_2^{-\theta i})$	$i = 0, \ldots, q^2$ $i \neq 0, \frac{q^2+1}{3}, \frac{2(q^2+1)}{3}$	$\frac{q^2-2}{2}$
$h_{10}(i) := h(\tilde{\zeta}_4^{(4\theta^3+2\theta^2+1)i}, \tilde{\zeta}_4^{(2\theta^2+2\theta-1)i}, \tilde{\zeta}_4^{(-2\theta^2+2\theta+1)i}, \tilde{\zeta}_4^{(-4\theta^3+2\theta^2+1)i})$	$i = 0, \ldots, q^4-2$ $i \neq (q^2-1)l, l = 0, \ldots, q^2$ $i \neq (q^2+1)l, l = 0, \ldots, q^2-2$	$\frac{q^4-2q^2}{2}$

TABLE A.3. *The conjugacy classes of P_a. (The parameter a in the representatives for the conjugacy classes of types $c_{1,24}$, $c_{1,25}$, $c_{1,29}$, …, $c_{1,32}$ runs through the sets $I_1, I_2, \ldots, I_6$ respectively with $|I_1| = |I_2| = q^2-2$ and $|I_3| = |I_4| = |I_5| = |I_6| = (q^2/2)-1$. The sets $I_1, \ldots, I_6$ are defined in [10, § 4]. The field element $\zeta \in \mathbb{F}_{q^2}$ and the unipotent elements x, x', x'' are defined in [10, § 3].)*

Notation	Representative	$\lvert C_{P_a}\rvert$	Fusion in G
$c_{1,0}$	1	$q^{24}(q^2+1)(q^2-1)^2$	$c_{1,0}$
$c_{1,1}$	$\alpha_{12}(1)$	$q^{24}(q^2-1)$	$c_{1,1}$
$c_{1,2}$	$\alpha_{10}(1)$	$q^{20}(q^2+1)(q^2-1)$	$c_{1,2}$
$c_{1,3}$	$\alpha_9(1)$	$q^{18}(q^2-1)$	$c_{1,2}$
$c_{1,4}$	$\alpha_8(1)$	$q^{18}(q^2-1)$	$c_{1,1}$
$c_{1,5}$	$\alpha_8(1)\alpha_{11}(1)$	q^{18}	$c_{1,2}$
$c_{1,6}$	$\alpha_7(1)\alpha_8(1)$	q^{16}	$c_{1,5}$
$c_{1,7}$	$\alpha_5(1)$	$2q^{14}(q^2-1)$	$c_{1,4}$
$c_{1,8}$	$\alpha_5(1)\alpha_{12}(1)$	$2q^{14}(q^2-1)$	$c_{1,3}$
$c_{1,9}$	$\alpha_5(1)\alpha_7(1)$	q^{14}	$c_{1,6}$
$c_{1,10}$	$\alpha_5(1)\alpha_6(1)$	$6q^{12}$	$c_{1,7}$
$c_{1,11}$	$\alpha_5(1)\alpha_6(1)\alpha_8(1)$	$2q^{12}$	$c_{1,8}$
$c_{1,12}$	$\alpha_4(1)$	$2q^{12}(q^2-1)$	$c_{1,4}$
$c_{1,13}$	$\alpha_4(1)\alpha_{11}(1)$	$2q^{12}$	$c_{1,6}$
$c_{1,14}$	$\alpha_4(1)\alpha_8(1)$	$2q^{12}(q^2-1)$	$c_{1,3}$
$c_{1,15}$	$\alpha_4(1)\alpha_8(1)\alpha_{11}(1)$	$2q^{12}$	$c_{1,6}$
$c_{1,16}$	$\alpha_4(1)\alpha_6(1)$	$2q^{10}$	$c_{1,7}$
$c_{1,17}$	$\alpha_4(1)\alpha_6(1)\alpha_{11}(1)$	$2q^{10}$	$c_{1,8}$
$c_{1,18}$	$\alpha_3(1)$	$q^{12}(q^2-1)$	$c_{1,2}$

TABLE A.3. (*Continued.*)

Notation	Representative	$\lvert C_{P_a}\rvert$	Fusion in G
$c_{1,19}$	$\alpha_3(1)\alpha_{11}(1)$	q^{12}	$c_{1,5}$
$c_{1,20}$	$\alpha_3(1)\alpha_6(1)$	$2q^{12}$	$c_{1,6}$
$c_{1,21}$	$\alpha_3(1)\alpha_6(1)\alpha_{11}(1)$	$2q^{12}$	$c_{1,6}$
$c_{1,22}$	$\alpha_3(1)\alpha_6(1)\alpha_9(1)$	$2q^{12}$	$c_{1,4}$
$c_{1,23}$	$\alpha_3(1)\alpha_6(1)\alpha_9(1)\alpha_{11}(1)$	$2q^{12}$	$c_{1,3}$
$c_{1,24}(a)$	$\alpha_3(1)\alpha_6(1)\alpha_9(a)$	$2q^{12}$	$c_{1,6}$
$c_{1,25}(a)$	$\alpha_3(1)\alpha_6(1)\alpha_9(a)\alpha_{11}(1)$	$2q^{12}$	$c_{1,6}$
$c_{1,26}$	$\alpha_3(1)\alpha_5(1)$	q^{10}	$c_{1,6}$
$c_{1,27}$	$\alpha_3(1)\alpha_5(1)\alpha_6(1)$	$2q^{10}$	$c_{1,8}$
$c_{1,28}$	$\alpha_3(1)\alpha_5(1)\alpha_6(1)\alpha_8(1)$	$2q^{10}$	$c_{1,7}$
$c_{1,29}(a)$	$\alpha_3(1)\alpha_5(a)\alpha_6(1)$	$2q^{10}$	$c_{1,7}$
$c_{1,30}(a)$	$\alpha_3(1)\alpha_5(a)\alpha_6(1)$	$2q^{10}$	$c_{1,8}$
$c_{1,31}(a)$	$\alpha_3(1)\alpha_5(a)\alpha_6(1)\alpha_8(t_a)$	$2q^{10}$	$c_{1,7}$
$c_{1,32}(a)$	$\alpha_3(1)\alpha_5(a)\alpha_6(1)\alpha_8(t_a)$	$2q^{10}$	$c_{1,8}$
$c_{1,33}$	$\alpha_2(1)\alpha_6(\zeta)\alpha_8(1)$	$3q^{12}$	$c_{1,9}$
$c_{1,34}$	$\alpha_2(1)\alpha_4(1)$	$2q^8$	$c_{1,10}$
$c_{1,35}$	$\alpha_2(1)\alpha_4(1)\alpha_5(1)$	$4q^8$	$c_{1,11}$
$c_{1,36}$	$\alpha_2(1)\alpha_4(1)\alpha_5(1)\alpha_8(1)$	$4q^8$	$c_{1,12}$
$c_{1,37}$	$\alpha_2(1)\alpha_3(1)$	$2q^6$	$c_{1,13}$
$c_{1,38}$	$\alpha_2(1)\alpha_3(1)\alpha_5(1)$	$2q^6$	$c_{1,14}$
$c_{1,39}$	$\alpha_1(1)\alpha_3(1)$	$4q^4$	$c_{1,15}$
$c_{1,40}$	$\alpha_1(1)\alpha_3(1)\alpha_5(1)$	$4q^4$	$c_{1,17}$
$c_{1,41}$	$\alpha_1(1)\alpha_2(1)\alpha_3(1)$	$4q^4$	$c_{1,16}$
$c_{1,42}$	$\alpha_1(1)\alpha_2(1)\alpha_3(1)\alpha_5(1)$	$4q^4$	$c_{1,18}$
$c_{2,0}(i)$	$h_2(i)$	$q^2(q^2+1)(q^2-1)^2$	$c_{3,0}(i)$
$c_{2,1}(i)$	$h_2(i)\alpha_3(1)$	$q^2(q^2-1)$	$c_{3,1}(i)$
$c_{3,0}(i)$	$h_3(i)$	$q^4(q^2-1)^2$	$c_{2,0}(2\theta i)$
$c_{3,1}(i)$	$h_3(i)\alpha_8(1)$	$q^4(q^2-1)$	$c_{2,1}(2\theta i)$
$c_{3,2}(i)$	$h_3(i)\alpha_4(1)$	$2q^2(q^2-1)$	$c_{2,2}(2\theta i)$
$c_{3,3}(i)$	$h_3(i)\alpha_4(1)\alpha_8(1)$	$2q^2(q^2-1)$	$c_{2,3}(2\theta i)$
$c_{4,0}(i)$	$h_4(i)$	$q^4(q^2-1)^2$	$c_{2,0}(i)$
$c_{4,1}(i)$	$h_4(i)\alpha_{12}(1)$	$q^4(q^2-1)$	$c_{2,1}(i)$
$c_{4,2}(i)$	$h_4(i)\alpha_5(1)$	$2q^2(q^2-1)$	$c_{2,2}(i)$
$c_{4,3}(i)$	$h_4(i)\alpha_5(1)\alpha_{12}(1)$	$2q^2(q^2-1)$	$c_{2,3}(i)$
$c_{5,0}(i)$	$h_5(i)$	$q^2(q^2-1)^2$	$c_{3,0}(i)$
$c_{5,1}(i)$	$h_5(i)\alpha_9(1)$	$q^2(q^2-1)$	$c_{3,1}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^2(q^2-1)^2$	$c_{3,0}(i)$
$c_{6,1}(i)$	$h_6(i)\alpha_{10}(1)$	$q^2(q^2-1)$	$c_{3,1}(i)$
$c_{7,0}(i,j)$	$h_7(i,j)$	$(q^2-1)^2$	$c_{4,0}(i,j)$
$c_{8,0}$	h_8	$q^6(q^4-1)$	$c_{5,0}$
$c_{8,1}$	$h_8x_{17}(1)x_{22}(1)$	$q^6(q^2+1)$	$c_{5,1}$
$c_{8,2}$	h_8x	$3q^4$	$c_{5,2}$
$c_{8,3}$	h_8x'	$3q^4$	$c_{5,3}$
$c_{8,4}$	h_8x''	$3q^4$	$c_{5,4}$
$c_{9,0}(i)$	$h_9(i)$	$q^2(q^4-1)$	$c_{6,0}(i)$
$c_{9,1}(i)$	$h_9(i)x_{17}(1)x_{22}(1)$	$q^2(q^2+1)$	$c_{6,1}(i)$
$c_{10,0}(i)$	$h_{10}(i)$	q^4-1	$c_{7,0}(i)$

Table A.4. *Parametrization of the irreducible characters of P_a.*

Character	Degree	Parameters	Number of characters
${}_{P_a}\chi_1(k)$	1	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_2(k)$	q^2	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_3(k,l)$	q^2+1	$k,l=0,\dots,q^2-2$ $l\neq(\sqrt{2}q-1)k$	$\frac{q^4-3q^2+2}{2}$
${}_{P_a}\chi_4(k)$	q^2-1	$k=0,\dots,q^4-2$ $k\neq(q^2+1)m$, $m=0,\dots,q^2-2$	$\frac{q^4-q^2}{2}$
${}_{P_a}\chi_5(k)$	q^4-1	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_6$	$(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_7(k)$	$\frac{q}{\sqrt{2}}(q^4-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_8(k)$	$\frac{q}{\sqrt{2}}(q^4-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_9$	$\frac{q}{\sqrt{2}}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{10}$	$\frac{q}{\sqrt{2}}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{11}$	$q^2(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{12}(k)$	$q^4(q^4-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_{13}$	$\frac{q^4}{2}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{14}$	$\frac{q^4}{4}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{15}$	$\frac{q^4}{4}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{16}$	$\frac{q^4}{4}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{17}$	$\frac{q^4}{4}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{18}$	$\frac{q^4}{3}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{19}$	$\frac{q^4}{3}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{20}$	$\frac{q^4}{3}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{21}$	$\frac{q^4}{4}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{22}$	$\frac{q^4}{4}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{23}$	$\frac{q^4}{12}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{24}$	$\frac{q^4}{12}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{25}$	$\frac{q^4}{6}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{26}(k)$	$q^4(q^4-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_{27}$	$q^4(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{28}(k)$	$q^4(q^4-1)(q^2-1)$	$k=1,\dots,q^2$	q^2
${}_{P_a}\chi_{29}$	$q^6(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{30}$	$q^8(q^2-1)$		1
${}_{P_a}\chi_{31}$	$q^{10}(q^2-1)$		1
${}_{P_a}\chi_{32}(k)$	$q^8(q^4-1)$	$k=0,\dots,q^2-2;\ k\neq0$	$\frac{q^2-2}{2}$
${}_{P_a}\chi_{33}(k)$	$q^8(q^2-1)^2$	$k=0,\dots,q^2;\ k\neq0$	$\frac{q^2}{2}$
${}_{P_a}\chi_{34}(k)$	$\frac{q^7}{\sqrt{2}}(q^4-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_{35}(k)$	$\frac{q^7}{\sqrt{2}}(q^4-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_a}\chi_{36}$	$\frac{q^7}{\sqrt{2}}(q^4-1)(q^2-1)$		1

TABLE A.4. (*Continued.*)

Character	Degree	Parameters	Number of characters
${}_{P_a}\chi_{37}$	$\frac{q^7}{\sqrt{2}}(q^4-1)(q^2-1)$		1
${}_{P_a}\chi_{38}(k)$	$\frac{q^7}{\sqrt{2}}(q^4-1)(q^2-1)$	$k=1,\dots,q^2$	q^2
${}_{P_a}\chi_{39}(k)$	$\frac{q^7}{\sqrt{2}}(q^4-1)(q^2-1)$	$k=1,\dots,q^2$	q^2
${}_{P_a}\chi_{40}(k)$	$q^8(q^4-1)(q^2-1)$	$k=1,\dots,q^2$	q^2

TABLE A.5. *The character table of P_a.* (*Zeros are replaced by dots. See* [**10**, Table 5] *for notation for the irrational character values.*)

Due to its size, this table is stored in a separate file; see

`2f4maxparab_tables_5_and_9.pdf`

in the electronic appendix to this paper.

TABLE A.6. *Parametrization of the semisimple conjugacy classes of P_b.*

Representative	Parameters	Number of classes
$h_1 := h(1,1,1,1)$		1
$h_2(i) := h(\tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i}, 1, 1)$	$i=0,\dots,q^2-2$, $i\neq 0$	q^2-2
$h_3(i) := h(\tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^{(4\theta^2-4\theta+1)i})$	$i=0,\dots,q^2-2$, $i\neq 0$	q^2-2
$h_4(i) := h(\tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i})$	$i=0,\dots,q^2-2$, $i\neq 0$	q^2-2
$h_5(i) := h(1,1,\tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i})$	$i=0,\dots,q^2-2$, $i\neq 0$	$\frac{q^2-2}{2}$
$h_6(i) := h(\tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^{(4\theta^2-4\theta+1)i}, \tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i})$	$i=0,\dots,q^2-2$, $i\neq 0$	q^2-2
$h_7(i,j) := h(\tilde\zeta_2^i, \tilde\zeta_2^{(2\theta-1)i}, \tilde\zeta_2^j, \tilde\zeta_2^{(2\theta-1)j})$	$i,j=0,\dots,q^2-2$, $i,j\neq 0$, $j\neq \pm i, \pm(2\theta-1)i$, $i\neq\pm(2\theta-1)j$	$\frac{q^4-10q^2+16}{2}$
$h_8(i) := h(1,1,\tilde\varphi_8''^i, \tilde\varphi_8''^{-q^2 i})$	$i=0,\dots,q^2-\sqrt{2}q$, $i\neq 0$	$\frac{q^2-\sqrt{2}q}{4}$
$h_9(i) := h(\tilde\psi_8''^{(2\theta^2-2\theta+1)i}, \tilde\psi_8''^{(4\theta^3-6\theta^2+4\theta-1)i}, \tilde\psi_8''^{(2\theta^2-1)i}, \tilde\psi_8''^{(-4\theta^4+2\theta^2)i})$	$i=0,\dots,q^4-\sqrt{2}q^3+\sqrt{2}q-2$, $i\neq(q^2-1)l$, $l=0,\dots,q^2-\sqrt{2}q$, $i\neq(q^2-\sqrt{2}q+1)l$, $l=0,\dots,q^2-2$	$\frac14(q^4-\sqrt{2}q^3-2q^2+2\sqrt{2}q)$
$h_{10}(i) := h(1,1,\tilde\varphi_8'^i, \tilde\varphi_8'^{-q^2 i})$	$i=0,\dots,q^2+\sqrt{2}q$, $i\neq 0$	$\frac{q^2+\sqrt{2}q}{4}$
$h_{11}(i) := h(\tilde\psi_8'^{(2\theta^2+2\theta+1)i}, \tilde\psi_8'^{(4\theta^3+2\theta^2-1)i}, \tilde\psi_8'^{(2\theta^2-1)i}, \tilde\psi_8'^{(-4\theta^4+2\theta^2)i})$	$i=0,\dots,q^4+\sqrt{2}q^3-\sqrt{2}q-2$, $i\neq(q^2-1)l$, $l=0,\dots,q^2+\sqrt{2}q$, $i\neq(q^2+\sqrt{2}q+1)l$, $l=0,\dots,q^2-2$	$\frac14(q^4+\sqrt{2}q^3-2q^2-2\sqrt{2}q)$

Table A.7. *The conjugacy classes of* P_b. (*The parameter* a *in the representatives for the conjugacy classes of types* $c_{1,21}$, $c_{1,22}$, $c_{1,23}$, $c_{1,24}$, $c_{1,35}$, $c_{1,36}$, $c_{1,37}$, $c_{1,41}$, $c_{1,42}$, $c_{1,43}$ *runs through the sets* I_7, I_8, $\dots$, I_{16} *respectively with* $|I_7| = q^2-2$, $|I_8| = (q^2-2)/6$, $|I_9| = |I_{11}| = |I_{14}| = (q^2/2)-1$, $|I_{10}| = (q^2+1)/3$, $|I_{12}| = |I_{16}| = (q^2+\sqrt{2}q)/4$, $|I_{13}| = |I_{15}| = (q^2-\sqrt{2}q)/4$. *These parameter sets are defined in* [10, § 4].)

Notation	Representative	$\lvert C_{P_b}\rvert$	Fusion in G
$c_{1,0}$	1	$q^{24}(q^4+1)(q^2-1)^2$	$c_{1,0}$
$c_{1,1}$	$\alpha_{12}(1)$	$q^{24}(q^4+1)(q^2-1)$	$c_{1,1}$
$c_{1,2}$	$\alpha_{11}(1)$	$q^{22}(q^2-1)$	$c_{1,1}$
$c_{1,3}$	$\alpha_{10}(1)$	$q^{20}(q^2-1)$	$c_{1,2}$
$c_{1,4}$	$\alpha_7(1)$	$q^{16}(q^2-1)$	$c_{1,2}$
$c_{1,5}$	$\alpha_7(1)\alpha_8(1)$	q^{16}	$c_{1,5}$
$c_{1,6}$	$\alpha_6(1)$	$2q^{14}(q^2-1)$	$c_{1,4}$
$c_{1,7}$	$\alpha_6(1)\alpha_{11}(1)$	$2q^{14}(q^2-1)$	$c_{1,3}$
$c_{1,8}$	$\alpha_6(1)\alpha_9(1)$	q^{14}	$c_{1,6}$
$c_{1,9}$	$\alpha_5(1)$	$2q^{14}(q^4+1)(q^2-1)$	$c_{1,4}$
$c_{1,10}$	$\alpha_5(1)\alpha_{12}(1)$	$2q^{14}(q^4+1)(q^2-1)$	$c_{1,3}$
$c_{1,11}$	$\alpha_5(1)\alpha_7(1)$	q^{14}	$c_{1,6}$
$c_{1,12}$	$\alpha_5(1)\alpha_6(1)$	$2q^{12}$	$c_{1,7}$
$c_{1,13}$	$\alpha_5(1)\alpha_6(1)\alpha_8(1)$	$2q^{12}$	$c_{1,8}$
$c_{1,14}$	$\alpha_2(1)$	$q^{16}(q^2-1)$	$c_{1,1}$
$c_{1,15}$	$\alpha_2(1)\alpha_{12}(1)$	q^{16}	$c_{1,2}$
$c_{1,16}$	$\alpha_2(1)\alpha_9(1)$	q^{14}	$c_{1,5}$
$c_{1,17}$	$\alpha_2(1)\alpha_8(1)$	q^{12}	$c_{1,8}$
$c_{1,18}$	$\alpha_2(1)\alpha_6(1)$	$2q^{14}$	$c_{1,4}$
$c_{1,19}$	$\alpha_2(1)\alpha_6(1)\alpha_{11}(1)$	$2q^{14}$	$c_{1,3}$
$c_{1,20}$	$\alpha_2(1)\alpha_6(1)\alpha_9(1)$	q^{14}	$c_{1,6}$
$c_{1,21}(a)$	$\alpha_2(1)\alpha_6(a)\alpha_9(1)$	q^{14}	$c_{1,6}$
$c_{1,22}(a)$	$\alpha_2(1)\alpha_6(a)\alpha_8(1)$	q^{12}	$c_{1,7}$
$c_{1,23}(a)$	$\alpha_2(1)\alpha_6(a)\alpha_8(1)$	q^{12}	$c_{1,8}$
$c_{1,24}(a)$	$\alpha_2(1)\alpha_6(a)\alpha_8(1)$	q^{12}	$c_{1,9}$
$c_{1,25}$	$\alpha_2(1)\alpha_4(1)$	$2q^8$	$c_{1,10}$
$c_{1,26}$	$\alpha_2(1)\alpha_4(1)\alpha_5(1)$	$4q^8$	$c_{1,11}$
$c_{1,27}$	$\alpha_2(1)\alpha_4(1)\alpha_5(1)\alpha_8(1)$	$4q^8$	$c_{1,12}$
$c_{1,28}$	$\alpha_2(1)\alpha_3(1)$	$2q^6$	$c_{1,13}$
$c_{1,29}$	$\alpha_2(1)\alpha_3(1)\alpha_5(1)$	$2q^6$	$c_{1,14}$
$c_{1,30}$	$\alpha_1(1)$	$2q^{10}(q^2-1)$	$c_{1,4}$
$c_{1,31}$	$\alpha_1(1)\alpha_{12}(1)$	$2q^{10}$	$c_{1,6}$
$c_{1,32}$	$\alpha_1(1)\alpha_8(1)$	$2q^8$	$c_{1,10}$
$c_{1,33}$	$\alpha_1(1)\alpha_5(1)$	$4q^8$	$c_{1,7}$
$c_{1,34}$	$\alpha_1(1)\alpha_5(1)\alpha_{12}(1)$	$4q^8$	$c_{1,8}$
$c_{1,35}(a)$	$\alpha_1(1)\alpha_6(1)\alpha_8(a)$	$2q^8$	$c_{1,10}$
$c_{1,36}(a)$	$\alpha_1(1)\alpha_6(1)\alpha_8(a)$	$2q^8$	$c_{1,11}$
$c_{1,37}(a)$	$\alpha_1(1)\alpha_6(1)\alpha_8(a)$	$2q^8$	$c_{1,12}$
$c_{1,38}$	$\alpha_1(1)\alpha_2(1)$	$2q^{10}(q^2-1)$	$c_{1,3}$
$c_{1,39}$	$\alpha_1(1)\alpha_2(1)\alpha_{12}(1)$	$2q^{10}$	$c_{1,6}$
$c_{1,40}$	$\alpha_1(1)\alpha_2(1)\alpha_8(1)$	$2q^8$	$c_{1,10}$
$c_{1,41}(a)$	$\alpha_1(1)\alpha_2(1)\alpha_6(1)\alpha_8(a)$	$2q^8$	$c_{1,10}$
$c_{1,42}(a)$	$\alpha_1(1)\alpha_2(1)\alpha_6(1)\alpha_8(a)$	$2q^8$	$c_{1,11}$
$c_{1,43}(a)$	$\alpha_1(1)\alpha_2(1)\alpha_6(1)\alpha_8(a)$	$2q^8$	$c_{1,12}$
$c_{1,44}$	$\alpha_1(1)\alpha_2(1)\alpha_6(1)$	$4q^8$	$c_{1,7}$

TABLE A.7. (*Continued.*)

Notation	Representative	$\lvert C_{P_b}\rvert$	Fusion in G
$c_{1,45}$	$\alpha_1(1)\alpha_2(1)\alpha_4(1)$	$4q^8$	$c_{1,8}$
$c_{1,46}$	$\alpha_1(1)\alpha_3(1)$	$4q^4$	$c_{1,15}$
$c_{1,47}$	$\alpha_1(1)\alpha_3(1)\alpha_5(1)$	$4q^4$	$c_{1,17}$
$c_{1,48}$	$\alpha_1(1)\alpha_2(1)\alpha_3(1)$	$4q^4$	$c_{1,16}$
$c_{1,49}$	$\alpha_1(1)\alpha_2(1)\alpha_3(1)\alpha_5(1)$	$4q^4$	$c_{1,18}$
$c_{2,0}(i)$	$h_2(i)$	$q^4(q^4+1)(q^2-1)^2$	$c_{2,0}(i)$
$c_{2,1}(i)$	$h_2(i)\alpha_2(1)$	$q^4(q^2-1)$	$c_{2,1}(i)$
$c_{2,2}(i)$	$h_2(i)\alpha_1(1)$	$2q^2(q^2-1)$	$c_{2,2}(i)$
$c_{2,3}(i)$	$h_2(i)\alpha_1(1)\alpha_2(1)$	$2q^2(q^2-1)$	$c_{2,3}(i)$
$c_{3,0}(i)$	$h_3(i)$	$q^2(q^2-1)^2$	$c_{3,0}(i)$
$c_{3,1}(i)$	$h_3(i)\alpha_3(1)$	$q^2(q^2-1)$	$c_{3,1}(i)$
$c_{4,0}(i)$	$h_4(i)$	$q^4(q^2-1)^2$	$c_{2,0}(2\theta i)$
$c_{4,1}(i)$	$h_4(i)\alpha_8(1)$	$q^4(q^2-1)$	$c_{2,1}(2\theta i)$
$c_{4,2}(i)$	$h_4(i)\alpha_4(1)$	$2q^2(q^2-1)$	$c_{2,2}(2\theta i)$
$c_{4,3}(i)$	$h_4(i)\alpha_4(1)\alpha_8(1)$	$2q^2(q^2-1)$	$c_{2,3}(2\theta i)$
$c_{5,0}(i)$	$h_5(i)$	$q^4(q^2-1)^2$	$c_{2,0}(i)$
$c_{5,1}(i)$	$h_5(i)\alpha_{12}(1)$	$q^4(q^2-1)$	$c_{2,1}(i)$
$c_{5,2}(i)$	$h_5(i)\alpha_5(1)$	$2q^2(q^2-1)$	$c_{2,2}(i)$
$c_{5,3}(i)$	$h_5(i)\alpha_5(1)\alpha_{12}(1)$	$2q^2(q^2-1)$	$c_{2,3}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^2(q^2-1)^2$	$c_{3,0}(i)$
$c_{6,1}(i)$	$h_6(i)\alpha_9(1)$	$q^2(q^2-1)$	$c_{3,1}(i)$
$c_{7,0}(i,j)$	$h_7(i,j)$	$(q^2-1)^2$	$c_{4,0}(i,j)$
$c_{8,0}(i)$	$h_8(i)$	$q^4(q^2-\sqrt{2}q+1)(q^2-1)$	$c_{8,0}(i)$
$c_{8,1}(i)$	$h_8(i)x_{21}(1)x_{24}(1)$	$q^4(q^2-\sqrt{2}q+1)$	$c_{8,1}(i)$
$c_{8,2}(i)$	$h_8(i)x_8(1)x_{16}(1)x_{21}(1)$	$2q^2(q^2-\sqrt{2}q+1)$	$c_{8,2}(i)$
$c_{8,3}(i)$	$h_8(i)x_8(1)x_{16}(1)x_{24}(1)$	$2q^2(q^2-\sqrt{2}q+1)$	$c_{8,3}(i)$
$c_{9,0}(i)$	$h_9(i)$	$(q^2-\sqrt{2}q+1)(q^2-1)$	$c_{9,0}(i)$
$c_{10,0}(i)$	$h_{10}(i)$	$q^4(q^2+\sqrt{2}q+1)(q^2-1)$	$c_{10,0}(i)$
$c_{10,1}(i)$	$h_{10}(i)x_{21}(1)x_{24}(1)$	$q^4(q^2+\sqrt{2}q+1)$	$c_{10,1}(i)$
$c_{10,2}(i)$	$h_{10}(i)x_8(1)x_{16}(1)x_{21}(1)$	$2q^2(q^2+\sqrt{2}q+1)$	$c_{10,2}(i)$
$c_{10,3}(i)$	$h_{10}(i)x_8(1)x_{16}(1)x_{24}(1)$	$2q^2(q^2+\sqrt{2}q+1)$	$c_{10,3}(i)$
$c_{11,0}(i)$	$h_{11}(i)$	$(q^2+\sqrt{2}q+1)(q^2-1)$	$c_{11,0}(i)$

TABLE A.8. *Parametrization of the irreducible characters of P_b.*

Character	Degree	Parameters	Number of characters
${}_{P_b}\chi_1(k)$	1	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_2(k)$	$\frac{q}{\sqrt{2}}(q^2-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_3(k)$	$\frac{q}{\sqrt{2}}(q^2-1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_4(k)$	q^4	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_5(k,l)$	q^4+1	$k,l=0,\dots,q^2-2; l\neq 0$	$\frac{q^4-3q^2+2}{2}$
${}_{P_b}\chi_6(k)$	$(q^2-\sqrt{2}q+1)(q^2-1)$	$k=0,\dots,q^4+\sqrt{2}q^3-\sqrt{2}q-2$ $k\neq(q^2+\sqrt{2}q+1)m$, $m=0,\dots,q^2-2$	$\frac{q^4+\sqrt{2}q^3-q^2-\sqrt{2}q}{4}$

TABLE A.8. (*Continued.*)

Character	Degree	Parameters	Number of characters
${}_{P_b}\chi_7(k)$	$(q^2+\sqrt{2}q+1)(q^2-1)$	$k=0,\dots,q^4-\sqrt{2}q^3+\sqrt{2}q-2$ $k\neq(q^2-\sqrt{2}q+1)m$, $m=0,\dots,q^2-2$	$\frac{q^4-\sqrt{2}q^3-q^2+\sqrt{2}q}{4}$
${}_{P_b}\chi_8(k)$	$(q^2-1)(q^4+1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_9$	$\frac{q}{\sqrt{2}}(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{10}$	$\frac{q}{\sqrt{2}}(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{11}$	$(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{12}(k)$	$q^2(q^2-1)(q^4+1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_{13}$	$q^2(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{14}$	$\frac{q^4}{2}(q^2-1)(q^4+1)$		1
${}_{P_b}\chi_{15}$	$\frac{q^4}{2}(q^2-1)(q^4+1)$		1
${}_{P_b}\chi_{16}(k)$	$q^4(q^2-1)(q^4+1)$	$k=0,\dots,q^2-2; k\neq 0$	$\frac{q^2-2}{2}$
${}_{P_b}\chi_{17}$	$\frac{q^4}{4}(q^2-1)^2(q^2+\sqrt{2}q+1)$		1
${}_{P_b}\chi_{18}$	$\frac{q^4}{4}(q^2-1)^2(q^2+\sqrt{2}q+1)$		1
${}_{P_b}\chi_{19}$	$\frac{q^4}{4}(q^2-1)^2(q^2+\sqrt{2}q+1)$		1
${}_{P_b}\chi_{20}$	$\frac{q^4}{4}(q^2-1)^2(q^2+\sqrt{2}q+1)$		1
${}_{P_b}\chi_{21}(k)$	$q^4(q^2-1)^2(q^2+\sqrt{2}q+1)$	$k=0,\dots,q^2-\sqrt{2}q; k\neq 0$	$\frac{q^2-\sqrt{2}q}{4}$
${}_{P_b}\chi_{22}$	$\frac{q^4}{4}(q^2-1)^2(q^2-\sqrt{2}q+1)$		1
${}_{P_b}\chi_{23}$	$\frac{q^4}{4}(q^2-1)^2(q^2-\sqrt{2}q+1)$		1
${}_{P_b}\chi_{24}$	$\frac{q^4}{4}(q^2-1)^2(q^2-\sqrt{2}q+1)$		1
${}_{P_b}\chi_{25}$	$\frac{q^4}{4}(q^2-1)^2(q^2-\sqrt{2}q+1)$		1
${}_{P_b}\chi_{26}(k)$	$q^4(q^2-1)^2(q^2-\sqrt{2}q+1)$	$k=0,\dots,q^2+\sqrt{2}q; k\neq 0$	$\frac{q^2+\sqrt{2}q}{4}$
${}_{P_b}\chi_{27}(k)$	$\frac{q^3}{\sqrt{2}}(q^2-1)(q^4+1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_{28}(k)$	$\frac{q^3}{\sqrt{2}}(q^2-1)(q^4+1)$	$k=0,\dots,q^2-2$	q^2-1
${}_{P_b}\chi_{29}$	$\frac{q^3}{\sqrt{2}}(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{30}$	$\frac{q^3}{\sqrt{2}}(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{31}$	$\frac{q^4}{2}(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{32}$	$\frac{q^4}{2}(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{33}$	$\frac{q^4}{2}(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{34}$	$\frac{q^4}{2}(q^2-1)^2(q^4+1)$		1

TABLE A.8. (*Continued.*)

Character	Degree	Parameters	Number of characters
${}_{P_b}\chi_{35}$	$q^4(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{36}(k)$	$\frac{q^4}{2}(q^2-1)^2(q^4+1)$	$k=1,\ldots,4\cdot\frac{q^2-2}{6}$	$4\cdot\frac{q^2-2}{6}$
${}_{P_b}\chi_{37}(k)$	$\frac{q^4}{2}(q^2-1)^2(q^4+1)$	$k=1,\ldots,4\cdot\frac{q^2-2}{2}$	$4\cdot\frac{q^2-2}{2}$
${}_{P_b}\chi_{38}(k)$	$q^4(q^2-1)^2(q^4+1)$	$k=1,\ldots,\frac{q^2+1}{3}$	$\frac{q^2+1}{3}$
${}_{P_b}\chi_{39}$	$q^6(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{40}(k)$	$q^6(q^2-1)(q^4+1)$	$k=0,\ldots,q^2-2$	q^2-1
${}_{P_b}\chi_{41}$	$q^6(q^2-1)^2(q^4+1)$		1
${}_{P_b}\chi_{42}(k)$	$q^6(q^2-1)^2(q^4+1)$	$k=1,\ldots,q^2$	q^2
${}_{P_b}\chi_{43}$	$\frac{q^9}{\sqrt{2}}(q^2-1)$		1
${}_{P_b}\chi_{44}$	$\frac{q^{10}}{2}(q^2-1)^2$		1
${}_{P_b}\chi_{45}$	$\frac{q^{10}}{2}(q^2-1)^2$		1
${}_{P_b}\chi_{46}$	$\frac{q^{13}}{\sqrt{2}}(q^2-1)$		1
${}_{P_b}\chi_{47}(k)$	$\frac{q^9}{\sqrt{2}}(q^2-1)(q^4+1)$	$k=0,\ldots,q^2-2;\, k\neq 0$	$\frac{q^2-2}{2}$
${}_{P_b}\chi_{48}(k)$	$\frac{q^9}{\sqrt{2}}(q^2-1)^2(q^2-\sqrt{2}q+1)$	$k=0,\ldots,q^2+\sqrt{2}q;\, k\neq 0$	$\frac{q^2+\sqrt{2}q}{4}$
${}_{P_b}\chi_{49}(k)$	$\frac{q^9}{\sqrt{2}}(q^2-1)^2(q^2+\sqrt{2}q+1)$	$k=0,\ldots,q^2-\sqrt{2}q;\, k\neq 0$	$\frac{q^2-\sqrt{2}q}{4}$
${}_{P_b}\chi_{50}$	$\frac{q^9}{\sqrt{2}}(q^2-1)$		1
${}_{P_b}\chi_{51}$	$\frac{q^{10}}{2}(q^2-1)^2$		1
${}_{P_b}\chi_{52}$	$\frac{q^{10}}{2}(q^2-1)^2$		1
${}_{P_b}\chi_{53}$	$\frac{q^{13}}{\sqrt{2}}(q^2-1)$		1
${}_{P_b}\chi_{54}(k)$	$\frac{q^9}{\sqrt{2}}(q^2-1)(q^4+1)$	$k=0,\ldots,q^2-2;\, k\neq 0$	$\frac{q^2-2}{2}$
${}_{P_b}\chi_{55}(k)$	$\frac{q^9}{\sqrt{2}}(q^2-1)^2(q^2-\sqrt{2}q+1)$	$k=0,\ldots,q^2+\sqrt{2}q;\, k\neq 0$	$\frac{q^2+\sqrt{2}q}{4}$
${}_{P_b}\chi_{56}(k)$	$\frac{q^9}{\sqrt{2}}(q^2-1)^2(q^2+\sqrt{2}q+1)$	$k=0,\ldots,q^2-\sqrt{2}q;\, k\neq 0$	$\frac{q^2-\sqrt{2}q}{4}$

TABLE A.9. *The character table of P_b. (Zeros are replaced by dots. See* [**10**, Table 5] *for notation for the irrational character values.)*

Due to its size, this table is stored in a separate file; see

`2f4maxparab_tables_5_and_9.pdf`

in the electronic appendix to this paper.

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