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The structure of parafermion vertex operator algebras

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ABSTRACT

It is proved that the parafermion vertex operator algebra associated to the irreducible highest weight module for the affine Kac–Moody algebra $A_1^{(1)}$ of level k coincides with a certain W-algebra. In particular, a set of generators for the parafermion vertex operator algebra is determined.

1. Introduction

The coset construction initiated in [9] is another major way besides the orbifold theory to construct new conformal field theories from given ones. The coset constructions have been used to establish the unitarity of the discrete series for the Virasoro algebra [9] and to produce many important conformal field theories associated to affine Kac–Moody algebras (see for example [4,8,12,19–21]). The coset construction in the language of vertex operator algebra can be stated as follows (see [7, Section 5], [14, Section 3.11]): Let $V = (V, Y, \mathbb{1}, \omega)$ be a vertex operator algebra and $U = (U, Y, \mathbb{1}, \omega^1)$ a vertex operator subalgebra of V. Then the commutant $U^c = \{v \in V \mid u_n v = 0 \text{ for } u \in U,\ n \geqslant 0\}$ of U in V is another vertex operator subalgebra of V with Virasoro vector $\omega^2 = \omega - \omega^1$ under a suitable assumption. The commutant U^c is called the coset vertex operator algebra associated to the pair $V \supset U$.

The parafermion algebras investigated in [21] are essentially the Z-algebras introduced and studied earlier in [15–17], as it was clarified in [4]. It was proved that the parafermion algebras are generalized vertex operator algebras [4]. A generalized vertex operator algebra as a vector space is a direct sum of a vertex operator algebra with some of its modules satisfying certain conditions. We call the

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vertex operator algebras in the parafermion algebras the parafermion vertex operator algebras, which are denoted by K_0 in [3].

Let $k \geqslant 2$ be an integer and $L(k,0)$ the level k irreducible highest weight module for the affine Kac–Moody algebra $A_1^{(1)}$. It turns out that K_0 is exactly the commutant of the Heisenberg vertex operator subalgebra in the vertex operator algebra $L(k,0)$. Certain W-algebras have been constructed and realized as vertex operator subalgebras of the commutant N_0 of the Heisenberg vertex operator subalgebra in the vertex operator algebra $V(k,0)$, where $V(k,0)$ denotes the level k Weyl module for the affine Kac–Moody algebra $A_1^{(1)}$ [1,10]. We remark that $L(k,0)$ is the simple quotient of $V(k,0)$ and K_0 is the simple quotient of N_0. For a better understanding of the moonshine vertex operator algebra [6], the vertex operator algebras K_0 and N_0 have been further investigated in [2,3]. In particular, several conjectures regarding the generators of N_0 and K_0, and the rationality and the C_2-cofiniteness of K_0 were made.

In this paper we will determine a set of generators for both N_0 and K_0, giving a positive answer to a conjecture in [3]. In particular, we obtain that N_0 is in fact equal to the W-algebra $W(2,3,4,5)$ [1]. We will also prove that the unique maximal ideal of N_0 is generated by a single vector and consequently determine the structure of the simple quotient K_0 of N_0. If $k \leqslant 6$, these results has been obtained in [3].

It has been conjectured that K_0 is a rational and C_2-cofinite vertex operator algebra. Again this has been established in the case $k \leqslant 6$ [3]. It is widely believed that if a vertex operator algebra V is rational and its subalgebra U is rational, then the commutant U^c of U in V is also rational. Although the Heisenberg vertex operator algebra is not rational, the commutant K_0 can be regraded as the commutant of a lattice vertex operator algebra associated to a rank one lattice inside $L(k,0)$ [3, Proposition 4.1]. Also, K_0 occurs as the commutant of a tensor product of vertex operator algebras associated to Virasoro algebras of discrete series in a lattice vertex operator algebra associated to a lattice of type $\sqrt{2}A_{k-1}$ [13, Theorem 4.2]. These facts should explain why the rationality of K_0 is expected. One can study the commutant of the Heisenberg vertex operator algebra inside the affine vertex operator algebra for any affine Kac–Moody algebra and obtain a class of rational vertex operator algebras [8].

The paper is organized as follows. In Section 2, we recall the construction of the vertex operator algebra $V(k,0)$ associated to the affine Kac–Moody algebra $A_1^{(1)}$ from [7]. Moreover, we consider its subalgebra $V(k,0)(0)$, which is the kernel of the action of h on $V(k,0)$. Here we use the standard basis $\{h,e,f\}$ for the Lie algebra sl_2. We give a set of generators for $V(k,0)(0)$. This result is the foundation for the study of generators for both N_0 and K_0. In Section 3, we define vertex operator algebra N_0 and prove that N_0 coincides with its subalgebra $W(2,3,4,5)$ generated by the Virasoro vector and Virasoro primary vectors W^i of weight i for $i=3,4,5$ by showing that N_0 is in fact generated by the Virasoro vector and W^3. This solves a conjecture given in [3]. Section 4 is devoted to the unique maximal ideal of N_0 and its simple quotient K_0. The result in this section is that the maximal ideal is generated by $f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$. Furthermore, we prove an important property of the vector $f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$. These results settle down another conjecture in [3]. The main idea in proving the results in this section is to use the highest weight module theory for the finite dimensional simple Lie algebra sl_2.

2. Vertex operator algebras $V(k,0)$ and $V(k,0)(0)$

We are working in the setting of [3]. In particular, $\{h,e,f\}$ is a standard Chevalley basis of sl_2 with $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$ for the bracket, $\langle\,,\rangle$ is the normalized Killing form so that $\langle h,h\rangle=2$, $\langle e,f\rangle=1$, $\langle h,e\rangle=\langle h,f\rangle=\langle e,e\rangle=\langle f,f\rangle=0$, and $\widehat{sl}_2=sl_2\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ is the corresponding affine Lie algebra. Moreover, $k\geqslant 2$ is an integer and

$$V(k,0)=V_{\widehat{sl}_2}(k,0)=\mathrm{Ind}_{sl_2\otimes\mathbb{C}[t]\oplus\mathbb{C}C}^{\widehat{sl}_2}\mathbb{C}$$

is an induced $\widehat{sl}_2$-module such that $sl_2\otimes\mathbb{C}[t]$ acts as 0 and K acts as k on $\mathbb{1}=1$.

We denote by $a(n)$ the operator on $V(k,0)$ corresponding to the action of $a\otimes t^n$. Then

$$\bigl[a(m),b(n)\bigr]=[a,b](m+n)+m\langle a,b\rangle\delta_{m+n,0}k \tag{2.1}$$

for $a,b\in sl_2$ and $m,n\in\mathbb{Z}$. Note that $a(n)\mathbb{1}=0$ for $n\geqslant 0$. The vectors

$$h(-i_1)\cdots h(-i_p)e(-j_1)\cdots e(-j_q)f(-m_1)\cdots f(-m_r)\mathbb{1}, \tag{2.2}$$

$i_1\geqslant\cdots\geqslant i_p\geqslant 1$, $j_1\geqslant\cdots\geqslant j_q\geqslant 1$, $m_1\geqslant\cdots\geqslant m_r\geqslant 1$ and $p,q,r\geqslant 0$ form a basis of $V(k,0)$.

Let $a(z)=\sum_{n\in\mathbb{Z}}a(n)z^{-n-1}$. Then $V(k,0)$ is a vertex operator algebra generated by $a(-1)\mathbb{1}$ for $a\in sl_2$ such that $Y(a(-1)\mathbb{1},z)=a(z)$ with the vacuum vector $\mathbb{1}$ and the Virasoro vector

$$\begin{aligned}\omega_{\mathrm{aff}}&=\frac{1}{2(k+2)}\left(\frac{1}{2}h(-1)^2\mathbb{1}+e(-1)f(-1)\mathbb{1}+f(-1)e(-1)\mathbb{1}\right)\\&=\frac{1}{2(k+2)}\left(-h(-2)\mathbb{1}+\frac{1}{2}h(-1)^2\mathbb{1}+2e(-1)f(-1)\mathbb{1}\right)\end{aligned}$$

of central charge $3k/(k+2)$ [7] (see [14, Section 6.2] also). The vector of the form (2.2) has weight $i_1+\cdots+i_p+j_1+\cdots+j_q+m_1+\cdots+m_r$. We also note that the vector of the form (2.2) is an eigenvector for $h(0)$ with eigenvalue $2(q-r)$.

For $\lambda\in 2\mathbb{Z}$, set

$$V(k,0)(\lambda)=\bigl\{v\in V(k,0)\bigm| h(0)v=\lambda v\bigr\}.$$

Then we have an eigenspace decomposition for $h(0)$:

$$V(k,0)=\bigoplus_{\lambda\in 2\mathbb{Z}}V(k,0)(\lambda). \tag{2.3}$$

Since $[h(0),Y(u,z)]=Y(h(0)u,z)$ for $u\in V(k,0)$ by the definition of affine vertex operator algebra, we see that $V(k,0)(0)$ is a vertex operator subalgebra of $V(k,0)$ with the same Virasoro vector ω_{aff} and each $V(k,0)(\lambda)$ is a module for $V(k,0)(0)$.

Our first theorem is on a set of generators for $V(k,0)(0)$, which will be fundamental in the study of generators of N_0 and K_0 later.

Theorem 2.1. *The vertex operator algebra $V(k,0)(0)$ is generated by two vectors $h(-1)\mathbb{1}$ and $f(-2)e(-1)\mathbb{1}$.*

Proof. First of all, note that $V(k,0)(0)$ is spanned by the vectors

$$h(-i_1)\cdots h(-i_p)f(-m_1)e(-n_1)\cdots f(-m_s)e(-n_s)\mathbb{1} \tag{2.4}$$

for $i_j,m_j,n_j>0$ and $p,s\geqslant 0$. Let U be the vertex operator subalgebra generated by $h(-1)\mathbb{1}$ and $f(-2)e(-1)\mathbb{1}$. Moreover, let $V(k,0)(0,t)$ be the subspace spanned by the vectors in (2.4) with $s\leqslant t$. We prove by induction on t that $V(k,0)(0,t)$ is a subspace of U. We first consider the case $t=1$.

Since $(h(-1)\mathbb{1})_n=h(n)$, we have $h(-i_1)\cdots h(-i_p)v\in U$ if $v\in U$. Thus, in order to show that $V(k,0)(0,1)$ is a subspace of U, it suffices to verify that $f(-m)e(-n)\mathbb{1}\in U$ for $m,n>0$. In fact, we prove by induction on n that $f(-n+i)e(-i)\mathbb{1}\in U$ for $n\geqslant 2$ and $1\leqslant i\leqslant n-1$.

We have $h(1)f(-2)e(-1)\mathbb{1} = -2f(-1)e(-1)\mathbb{1}$. Hence $f(-1)e(-1)\mathbb{1} \in U$ and so $\omega_{\text{aff}} \in U$. Set $L_{\text{aff}}(n) = (\omega_{\text{aff}})_{n+1}$, that is

$$Y(\omega_{\text{aff}}, z) = \sum_{n\in\mathbb{Z}} L_{\text{aff}}(n) z^{-n-2}.$$

Then

$$\left[L_{\text{aff}}(m), a(n)\right] = -na(m+n)$$

for $m, n \in \mathbb{Z}$ and $a \in sl_2$. Since $L_{\text{aff}}(-1)\mathbb{1} = 0$, it follows that

$$L_{\text{aff}}(-1)f(-m+i)e(-i)\mathbb{1} = (m-i)f(-m-1+i)e(-i)\mathbb{1} + if(-m+i)e(-i-1)\mathbb{1} \tag{2.5}$$

for any $m, i \in \mathbb{Z}$. In particular,

$$L_{\text{aff}}(-1)f(-1)e(-1)\mathbb{1} = f(-2)e(-1)\mathbb{1} + f(-1)e(-2)\mathbb{1}. \tag{2.6}$$

This implies that $f(-1)e(-2)\mathbb{1} \in U$.

We have shown that $f(-n+i)e(-i)\mathbb{1} \in U$ for $1 \leqslant i \leqslant n-1$ in the cases $n = 2, 3$. Now, let $n \geqslant 3$ and assume that $f(-m+i)e(-i)\mathbb{1} \in U$ for $2 \leqslant m \leqslant n$ and $1 \leqslant i \leqslant m-1$. We want to show that $f(-n-1+i)e(-i)\mathbb{1} \in U$ for $1 \leqslant i \leqslant n$.

We need the following identity

$$(u_l v)_m = \sum_{j\geqslant 0}(-1)^j \binom{l}{j} u_{l-j} v_{m+j} - \sum_{j\geqslant 0}(-1)^{l+j}\binom{l}{j} v_{m+l-j} u_j \tag{2.7}$$

for $u, v \in V(k, 0)$ and $l, m \in \mathbb{Z}$, which is a consequence of the Jacobi identity of vertex operator algebra. Applying (2.7) to $(f(-2)e(-1)\mathbb{1})_1 = ((f(-1)\mathbb{1})_{-2}e(-1)\mathbb{1})_1$, we have

$$\left(f(-2)e(-1)\mathbb{1}\right)_1 = \sum_{j\geqslant 0}(j+1)f(-2-j)e(1+j) - \sum_{j\geqslant 0}(j+1)e(-1-j)f(j).$$

Notice that $e(1+j)f(-n+1)e(-1)\mathbb{1} = 0$ if $j \geqslant n-1$ and $f(j)f(-n+1)e(-1)\mathbb{1} = 0$ if $j \geqslant 2$. Thus we have

$$\begin{aligned}\left(f(-2)e(-1)\mathbb{1}\right)_1 f(-n+1)e(-1)\mathbb{1} = {} & \sum_{1\leqslant j\leqslant n-1} jf(-1-j)e(j)f(-n+1)e(-1)\mathbb{1} \\ & - \sum_{j=0,1}(j+1)e(-1-j)f(j)f(-n+1)e(-1)\mathbb{1}.\end{aligned} \tag{2.8}$$

We consider the first summation of the right-hand side of (2.8). Since $n \geqslant 3$, we have

$$\begin{aligned}f(-1-j)e(j)f(-n+1)e(-1)\mathbb{1} &= f(-1-j)h(-n+1+j)e(-1)\mathbb{1} \\ &= 2f(-n)e(-1)\mathbb{1} + h(-n+1+j)f(-1-j)e(-1)\mathbb{1}\end{aligned}$$

for $1 \leqslant j \leqslant n-2$ and

$$\begin{aligned} f(-n)e(n-1)f(-n+1)e(-1)\mathbb{1} &= f(-n)\big(h(0)+(n-1)k+f(-n+1)e(n-1)\big)e(-1)\mathbb{1} \\ &= f(-n)h(0)e(-1)\mathbb{1}+(n-1)kf(-n)e(-1)\mathbb{1} \\ &= \big(2+(n-1)k\big)f(-n)e(-1)\mathbb{1}. \end{aligned}$$

As to the second summation of the right-hand side of (2.8), we have

$$\begin{aligned} & e(-1)f(0)f(-n+1)e(-1)\mathbb{1} \\ &\quad = -e(-1)f(-n+1)h(-1)\mathbb{1} \\ &\quad = -h(-n)h(-1)\mathbb{1} - f(-n+1)e(-1)h(-1)\mathbb{1} \\ &\quad = -h(-n)h(-1)\mathbb{1} + 2f(-n+1)e(-2)\mathbb{1} - f(-n+1)h(-1)e(-1)\mathbb{1} \\ &\quad = -h(-n)h(-1)\mathbb{1} + 2f(-n+1)e(-2)\mathbb{1} - 2f(-n)e(-1)\mathbb{1} - h(-1)f(-n+1)e(-1)\mathbb{1}, \\ & e(-2)f(1)f(-n+1)e(-1)\mathbb{1} = ke(-2)f(-n+1)\mathbb{1} = kh(-n-1)\mathbb{1} + kf(-n+1)e(-2)\mathbb{1}. \end{aligned}$$

Thus the identity (2.8) becomes

$$\begin{aligned} & \big(f(-2)e(-1)\mathbb{1}\big)_1 f(-n+1)e(-1)\mathbb{1} \\ &\quad = \big((n-1)\big(n+(n-1)k\big)+2\big)f(-n)e(-1)\mathbb{1} - 2(k+1)f(-n+1)e(-2)\mathbb{1} + u \end{aligned}$$

for some $u \in U$ by the induction assumption. Furthermore,

$$f(-n+1)e(-2)\mathbb{1} = L_{\mathrm{aff}}(-1)f(-n+1)e(-1)\mathbb{1} - (n-1)f(-n)e(-1)\mathbb{1}$$

by (2.5). Therefore,

$$\begin{aligned} & \big(f(-2)e(-1)\mathbb{1}\big)_1 f(-n+1)e(-1)\mathbb{1} \\ &\quad = \big((n-1)(nk+n+k+2)+2\big)f(-n)e(-1)\mathbb{1} - 2(k+1)L_{\mathrm{aff}}(-1)f(-n+1)e(-1)\mathbb{1} + u. \end{aligned}$$

Since $f(-n+1)e(-1)\mathbb{1} \in U$ by the induction assumption, this implies that $f(-n)e(-1)\mathbb{1}$ lies in U. As a result $f(-n-1+i)e(-i)\mathbb{1} \in U$ for $1 \leqslant i \leqslant n$ by (2.5) and the induction on n is complete. Thus $V(k,0)(0,1)$ is a subspace of U.

We now assume that $V(k,0)(0,t)$ is a subspace of U and show that $V(k,0)(0,t+1)$ is also a subspace of U. We use the expression

$$(u_{-m-1}v_{-n-1}\mathbb{1})_{-1} = u_{-m-1}v_{-n-1} + \sum_{i\geqslant 0} c_i u_{-m-n-2-i}v_i + \sum_{i\geqslant 0} d_i v_{-m-n-2-i}u_i \tag{2.9}$$

for $u, v \in V(k,0)$ and $m, n \geqslant 0$, where c_i, d_i are some constants. Indeed,

$$\begin{aligned} (u_{-m-1}v_{-n-1}\mathbb{1})_{-1} &= \sum_{i\geqslant 0}(-1)^i \binom{-m-1}{i} u_{-m-1-i}(v_{-n-1}\mathbb{1})_{-1+i} \\ &\quad - \sum_{i\geqslant 0}(-1)^{-m-1+i}\binom{-m-1}{i}(v_{-n-1}\mathbb{1})_{-m-2-i}u_i \end{aligned} \tag{2.10}$$

by (2.7). Similarly,

$$(v_{-n-1}\mathbb{1})_{-1+i}=\sum_{j\geqslant 0}(-1)^j\binom{-n-1}{j}v_{-n-1-j}\mathbb{1}_{-1+i+j}$$
$$-\sum_{j\geqslant 0}(-1)^{-n-1+j}\binom{-n-1}{j}\mathbb{1}_{-n-2+i-j}v_j.$$

Since $i,j\geqslant 0$ and $\mathbb{1}_r=\delta_{r,-1}$, we have that $\mathbb{1}_{-1+i+j}=0$ unless $i=j=0$. Then the first summation of the right-hand side of (2.10) is $u_{-m-1}v_{-n-1}+\sum_{i\geqslant 0}c_iu_{-m-n-2-i}v_i$ for some constants c_i. By a similar argument, we see that the second summation of the right-hand side of (2.10) is $\sum_{i\geqslant 0}d_iv_{-m-n-2-i}u_i$ for some constants d_i. Thus (2.9) holds.

Now, take $u=f(-1)\mathbb{1}$ and $v=e(-1)\mathbb{1}$ to obtain

$$\begin{aligned}&\big(f(-m-1)e(-n-1)\mathbb{1}\big)_{-1}\\&\quad=f(-m-1)e(-n-1)+\sum_{i\geqslant 0}c_if(-m-n-2-i)e(i)+\sum_{i\geqslant 0}d_ie(-m-n-2-i)f(i).\end{aligned}\tag{2.11}$$

Let $w=f(-m_1)e(-n_1)\cdots f(-m_t)e(-n_t)\mathbb{1}\in V(k,0)(0,t)$. Using the commutation relation (2.1) and the property that $e(i)\mathbb{1}=f(i)\mathbb{1}=0$ for $i\geqslant 0$, we can show that if we express each $f(-m-n-2-i)e(i)w$ and $e(-m-n-2-i)f(i)w$ for $i\geqslant 0$ as linear combinations of vectors in (2.4), then these vectors are contained in $V(k,0)(0,t)$. Recall that $V(k,0)(0,t)\subset U$ by the induction assumption. Then $(f(-m-1)e(-n-1)\mathbb{1})_{-1}w\in U$ and it follows from (2.11) that $f(-m-1)e(-n-1)w\in U$. Thus $V(k,0)(0,t+1)\subset U$, as desired. □

Remark 2.2. We can replace $f(-2)e(-1)\mathbb{1}$ by $f(-1)e(-2)\mathbb{1}-f(-2)e(-1)\mathbb{1}$ in Theorem 2.1. Indeed,

$$h(1)\big(f(-1)e(-2)\mathbb{1}-f(-2)e(-1)\mathbb{1}\big)=2h(-2)\mathbb{1}+4f(-1)e(-1)\mathbb{1}.$$

Hence by (2.6), $h(-1)\mathbb{1}$ and $f(-1)e(-2)\mathbb{1}-f(-2)e(-1)\mathbb{1}$ generate the vertex operator algebra $V(k,0)(0)$. In Section 4, we will consider an automorphism θ of the vertex operator algebra $V(k,0)$ of order 2, which leaves $V(k,0)(0)$ invariant. The above set of generators for $V(k,0)(0)$ is suitable to the action of the automorphism θ, since θ acts as -1 on those generators. The vector $f(-1)e(-2)\mathbb{1}-f(-2)e(-1)\mathbb{1}$ is also closely related to W^3 (see the proof of Theorem 3.1 below).

3. Vertex operator algebra N_0 and W-algebra

There are two subalgebras $\widehat{\mathfrak{h}}=\mathbb{C}h\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}C$ and $\widehat{\mathfrak{h}}_*=(\bigoplus_{n\neq 0}\mathbb{C}h\otimes t^n)\oplus\mathbb{C}C$ of $\widehat{sl}_2$. The subspace $V_{\widehat{\mathfrak{h}}}(k,0)$ spanned by $h(-i_1)\cdots h(-i_p)\mathbb{1}$ for $i_1\geqslant\cdots\geqslant i_p\geqslant 1$ and $p\geqslant 0$ is a vertex operator subalgebra of $V(k,0)$ associated to the Heisenberg algebra $\widehat{\mathfrak{h}}_*$ of level k with the Virasoro vector

$$\omega_\gamma=\frac{1}{4k}h(-1)^2\mathbb{1}\tag{3.1}$$

of central charge 1.

Now, $V(k,0)$ and each $V(k,0)(\lambda)$, $\lambda\in 2\mathbb{Z}$ are completely reducible as a $V_{\widehat{\mathfrak{h}}}(k,0)$-module. More precisely,

$$V(k,0)=\bigoplus_{\lambda\in 2\mathbb{Z}} M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes N_\lambda, \tag{3.2}$$

$$V(k,0)(\lambda)=M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes N_\lambda, \tag{3.3}$$

where $M_{\widehat{\mathfrak{h}}}(k,\lambda)$ denotes an irreducible highest weight module for $\widehat{\mathfrak{h}}$ with a highest weight vector v_λ such that $h(0)v_\lambda=\lambda v_\lambda$ and

$$N_\lambda=\bigl\{v\in V(k,0)\bigm| h(m)v=\lambda\delta_{m,0}v \text{ for } m\geqslant 0\bigr\}.$$

Note that $M_{\widehat{\mathfrak{h}}}(k,0)$ can be identified with $V_{\widehat{\mathfrak{h}}}(k,0)$ and N_0 is the commutant [7, Theorem 5.1] of $V_{\widehat{\mathfrak{h}}}(k,0)$ in $V(k,0)$. The commutant N_0 is a vertex operator algebra with the Virasoro vector $\omega=\omega_{\mathrm{aff}}-\omega_\gamma$,

$$\omega=\frac{1}{2k(k+2)}\bigl(-kh(-2)\mathbb{1}-h(-1)^2\mathbb{1}+2ke(-1)f(-1)\mathbb{1}\bigr), \tag{3.4}$$

whose central charge is $3k/(k+2)-1=2(k-1)/(k+2)$. Since the Virasoro vector of $V_{\widehat{\mathfrak{h}}}(k,0)$ is ω_γ, we have $N_0=\{v\in V(k,0)\mid (\omega_\gamma)_0v=0\}$ [7, Theorem 5.2]. It is clear that the weight of v in N_0 agrees with that in $V(k,0)$, since $\omega_1 v=(\omega_{\mathrm{aff}})_1 v$ for $v\in N_0$.

The dimension of weight i subspace $(N_0)_{(i)}$ is 2, 4 and 6 for $i=3$, 4 and 5, respectively. It is known that there is up to a scalar multiple, a unique Virasoro primary vector W^i in $(N_0)_{(i)}$ for $i=3,4,5$ [3, Section 2]. Here a Virasoro primary vector of weight i means that $\omega_2W^i=\omega_3W^i=0$ and $\omega_1W^i=iW^i$. As in [3], we take

$$\begin{aligned}W^3={}&k^2h(-3)\mathbb{1}+3kh(-2)h(-1)\mathbb{1}+2h(-1)^3\mathbb{1}-6kh(-1)e(-1)f(-1)\mathbb{1}\\&+3k^2e(-2)f(-1)\mathbb{1}-3k^2e(-1)f(-2)\mathbb{1},\end{aligned}$$

$$\begin{aligned}W^4={}&-2k^2\bigl(k^2+k+1\bigr)h(-4)\mathbb{1}-8k\bigl(k^2+k+1\bigr)h(-3)h(-1)\mathbb{1}-k\bigl(5k^2-6\bigr)h(-2)^2\mathbb{1}\\&-2k(11k+6)h(-2)h(-1)^2\mathbb{1}-(11k+6)h(-1)^4\mathbb{1}+4k^2(6k-5)h(-2)e(-1)f(-1)\mathbb{1}\\&+4k(11k+6)h(-1)^2e(-1)f(-1)\mathbb{1}-4k^2(5k+11)h(-1)e(-2)f(-1)\mathbb{1}\\&+4k^2(5k+11)h(-1)e(-1)f(-2)\mathbb{1}+8k^2(k-3)(k-2)e(-3)f(-1)\mathbb{1}\\&-4k^2\bigl(3k^2-3k+8\bigr)e(-2)f(-2)\mathbb{1}-2k^2(6k-5)e(-1)^2f(-1)^2\mathbb{1}\\&+8k^2\bigl(k^2+k+1\bigr)e(-1)f(-3)\mathbb{1},\end{aligned}$$

$$\begin{aligned}W^5={}&-2k^3\bigl(k^2+3k+5\bigr)h(-5)\mathbb{1}-10k^2\bigl(k^2+3k+5\bigr)h(-4)h(-1)\mathbb{1}\\&-5k^2\bigl(3k^2-4\bigr)h(-3)h(-2)\mathbb{1}-5k\bigl(7k^2+12k+16\bigr)h(-3)h(-1)^2\mathbb{1}\\&-15k\bigl(3k^2-4\bigr)h(-2)^2h(-1)\mathbb{1}-5k(19k+12)h(-2)h(-1)^3\mathbb{1}-2(19k+12)h(-1)^5\mathbb{1}\\&+10k^2\bigl(4k^2-7k+8\bigr)h(-3)e(-1)f(-1)\mathbb{1}+20k^2(10k-7)h(-2)h(-1)e(-1)f(-1)\mathbb{1}\\&+10k(19k+12)h(-1)^3e(-1)f(-1)\mathbb{1}-5k^2\bigl(11k^2-14k+12\bigr)h(-2)e(-2)f(-1)\mathbb{1}\\&-5k^2(17k+64)h(-1)^2e(-2)f(-1)\mathbb{1}+15k^2\bigl(3k^2-4\bigr)h(-2)e(-1)f(-2)\mathbb{1}\\&+5k^2(17k+64)h(-1)^2e(-1)f(-2)\mathbb{1}+30k^2(k-4)(k-3)h(-1)e(-3)f(-1)\mathbb{1}\\&-40k^2\bigl(k^2+3k+5\bigr)h(-1)e(-2)f(-2)\mathbb{1}-10k^2(10k-7)h(-1)e(-1)^2f(-1)^2\mathbb{1}\end{aligned}$$

$$
\begin{aligned}
&+10k^2\big(3k^2+19k+8\big)h(-1)e(-1)f(-3)\mathbb{1}-10k^3(k-4)(k-3)e(-4)f(-1)\mathbb{1}\\
&+20k^3(k-4)(k-3)e(-3)f(-2)\mathbb{1}+5k^3(10k-7)e(-2)e(-1)f(-1)^2\mathbb{1}\\
&-10k^3\big(2k^2-4k+17\big)e(-2)f(-3)\mathbb{1}-5k^3(10k-7)e(-1)^2f(-2)f(-1)\mathbb{1}\\
&+10k^3\big(k^2+3k+5\big)e(-1)f(-4)\mathbb{1}.
\end{aligned}
$$

Denote by $\widetilde{\mathcal{W}}$ the subalgebra of N_0 generated by ω, W^3, W^4 and W^5. Then $\widetilde{\mathcal{W}}$ coincides with $W(2,3,4,5)$ of [1]. It is in fact generated by ω and W^3 [3].

The following theorem is essentially a conjecture in [3].

Theorem 3.1. *The vertex operator algebra N_0 is generated by ω and W^3. In particular, N_0 coincides with $\widetilde{\mathcal{W}}$ or $W(2,3,4,5)$.*

Proof. We first show that $V(k,0)(0)=M_{\widehat{\mathfrak{h}}}(k,0)\otimes N_0$ is generated by $h(-1)\mathbb{1}$, ω and W^3. Let U be the vertex operator subalgebra generated by $h(-1)\mathbb{1}$, ω and W^3. Then U contains $f(-1)e(-1)\mathbb{1}$ and ω_{aff}. Moreover, we see from the expression of W^3 that U contains $f(-1)e(-2)\mathbb{1}-f(-2)e(-1)\mathbb{1}$. Hence $f(-2)e(-1)\mathbb{1}\in U$ by (2.6), and so U is equal to $V(k,0)(0)$ by Theorem 2.1.

Now, $Y(u,z_1)Y(v,z_2)=Y(v,z_2)Y(u,z_1)$ for $u\in M_{\widehat{\mathfrak{h}}}(k,0)$ and $v\in N_0$. Since $h(-1)\mathbb{1}\in M_{\widehat{\mathfrak{h}}}(k,0)$ and $\omega, W^3\in N_0$, we conclude that N_0 is generated by ω and W^3. $\square$

Remark 3.2. Since $W^3_3W^3=36k^3(k-2)(k+2)(3k+4)\omega$, the vector W^3 in fact generates the vertex operator algebra N_0 by Theorem 3.1 if $k\geqslant 3$. In the case $k=2$, W^3 is contained in a unique maximal ideal of N_0 [3, Remark 2.3].

It is shown in [3, Lemma 2.6] that the Zhu algebra $A(\widetilde{\mathcal{W}})$ [22] is a commutative associative algebra. Thus the Zhu algebra $A(N_0)$ is commutative and every simple $A(N_0)$-module is one-dimensional.

4. The maximal ideal $\widetilde{\mathcal{I}}$ of N_0 and K_0

The vertex operator algebra $V(k,0)$ has a unique maximal ideal $\mathcal{J}$, which is generated by a weight $k+1$ vector $e(-1)^{k+1}\mathbb{1}$ [11]. The quotient algebra $L(k,0)=V(k,0)/\mathcal{J}$ is the simple vertex operator algebra associated to the affine Lie algebra $\widehat{sl}_2$ of type $A_1^{(1)}$ with level k. The Heisenberg vertex operator algebra $V_{\widehat{\mathfrak{h}}}(k,0)$ is again a simple subalgebra of $L(k,0)$ and $L(k,0)$ is a completely reducible $V_{\widehat{\mathfrak{h}}}(k,0)$-module. We have a decomposition

$$L(k,0)=\bigoplus_{\lambda\in 2\mathbb{Z}}M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes K_\lambda \tag{4.1}$$

as modules for $V_{\widehat{\mathfrak{h}}}(k,0)$, where

$$K_\lambda=\big\{v\in L(k,0)\;\big|\;h(m)v=\lambda\delta_{m,0}v\text{ for }m\geqslant 0\big\}.$$

Note that $M_{\widehat{\mathfrak{h}}}(k,0)=V_{\widehat{\mathfrak{h}}}(k,0)$ and K_0 is the commutant of $V_{\widehat{\mathfrak{h}}}(k,0)$ in $L(k,0)$.

Similarly, $\mathcal{J}$ is completely reducible as a $V_{\widehat{\mathfrak{h}}}(k,0)$-module. Hence by (3.2),

$$\mathcal{J}=\bigoplus_{\lambda}M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes(\mathcal{J}\cap N_\lambda).$$

In particular, $\widetilde{\mathcal{I}}=\mathcal{J}\cap N_0$ is an ideal of N_0 and $K_0\cong N_0/\widetilde{\mathcal{I}}$. It is proved in [3, Lemma 3.1] that $\widetilde{\mathcal{I}}$ is the unique maximal ideal of N_0. Thus K_0 is a simple vertex operator algebra.

For short we still use ω_{aff}, ω_γ, ω, W^3, W^4 and W^5 of $V(k,0)$ to denote their images in $L(k,0) = V(k,0)/\mathcal{J}$. Let $\mathcal{W}$ be the subalgebra of K_0 generated by ω, W^3, W^4 and W^5. Thus $\mathcal{W}$ is a homomorphic image of $\widetilde{\mathcal{W}}$. The following result is a direct consequence of Theorem 3.1

Theorem 4.1. *The simple vertex operator algebra K_0 is generated by ω and W^3. In particular, K_0 coincides with $\mathcal{W}$.*

We remark that this theorem in the case $k \leqslant 6$ has been obtained in [3] and was conjectured for general k. As mentioned in Remark 3.2, K_0 is generated by W^3 if $k \geqslant 3$, while $W^3 = 0$ in K_0 if $k = 2$.

Next, we study the ideal $\widetilde{\mathcal{I}}$ of N_0 in detail. For this purpose we recall that the Lie algebra sl_2 has an involution θ given by $h \mapsto -h$, $e \mapsto f$, $f \mapsto e$. The involution θ lifts to an automorphism of the vertex operator algebra $V(k,0)$ of order 2 naturally. We still denote it by θ. Then $\theta\omega = \omega$, $\theta W^3 = -W^3$, $\theta W^4 = W^4$ and $\theta W^5 = -W^5$.

It is proved in [3, Theorem 3.2] that $f(0)^{k+1}e(-1)^{k+1}\mathbb{1} \in \widetilde{\mathcal{I}}$. The following theorem is another conjecture in [3].

Theorem 4.2.

(1) *The unique maximal ideal $\widetilde{\mathcal{I}}$ of N_0 is generated by a weight $k+1$ vector $f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$.*
(2) *The automorphism θ acts as $(-1)^{k+1}$ on $f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$.*

Proof. We use the finite dimensional representation theory for the simple Lie algebra sl_2. In fact, $V(k,0)$ is an sl_2-module where $a \in sl_2$ acts as $a(0)$. Each weight subspace of the vertex operator algebra $V(k,0)$ is a finite dimensional sl_2-module and $V(k,0)$ is completely reducible as a module for sl_2.

We first show the assertion (1). Consider the sl_2-submodule X of $V(k,0)$ generated by $e(-1)^{k+1}\mathbb{1}$. We have $e(0)e(-1)^{k+1}\mathbb{1} = 0$ and $h(0)e(-1)^{k+1}\mathbb{1} = 2(k+1)e(-1)^{k+1}\mathbb{1}$, that is, $e(-1)^{k+1}\mathbb{1}$ is a highest weight vector with highest weight $2(k+1)$ for sl_2. Then X is an irreducible sl_2-module with basis $f(0)^i e(-1)^{k+1}\mathbb{1}$, $0 \leqslant i \leqslant 2(k+1)$ from the representation theory of sl_2. This implies that the ideal $\mathcal{J}$ of the vertex operator algebra $V(k,0)$ can be generated by any nonzero vector in X. In particular, $\mathcal{J}$ is generated by $f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$. Then $\mathcal{J}$ is spanned by $u_n f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$ for $u \in V(k,0)$ and $n \in \mathbb{Z}$ by [5, Corollary 4.2] or [18, Proposition 4.1]. Since $[h(0), Y(u,z)] = Y(h(0)u, z)$ and $h(0)f(0)^{k+1}e(-1)^{k+1}\mathbb{1} = 0$, it follows that

$$h(0)u_n f(0)^{k+1}e(-1)^{k+1}\mathbb{1} = \big(h(0)u\big)_n f(0)^{k+1}e(-1)^{k+1}\mathbb{1}.$$

Thus we see from (2.3) that $u_n f(0)^{k+1}e(-1)^{k+1}\mathbb{1} \in \mathcal{J} \cap V(k,0)(0)$ if and only if $u \in V(k,0)(0)$. Let $u = v \otimes w \in V(k,0)(0) = M_{\widehat{\mathfrak{h}}}(k,0) \otimes N_0$ with $v \in M_{\widehat{\mathfrak{h}}}(k,0)$ and $w \in N_0$. Then $Y(u,z) = Y(v,z) \otimes Y(w,z)$ acts on $M_{\widehat{\mathfrak{h}}}(k,0) \otimes N_0$. As a result we have that $\widetilde{\mathcal{I}}$ is spanned by $w_n f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$ for $w \in N_0$ and $n \in \mathbb{Z}$. That is, the ideal $\widetilde{\mathcal{I}}$ of the vertex operator algebra N_0 is generated by $f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$. Thus (1) holds.

As to the assertion (2), we prove a more general result here:

$$\theta\big(f(0)^i e(-1)^i \mathbb{1}\big) = (-1)^i f(0)^i e(-1)^i \mathbb{1} \tag{4.2}$$

for any positive integer i. Let U be the irreducible sl_2-submodule of $V(k,0)$ generated by the highest weight vector $e(-1)^i\mathbb{1}$ with highest weight $2i$ for sl_2. Then U has a basis $f(0)^j e(-1)^i \mathbb{1}$, $0 \leqslant j \leqslant 2i$. We express $f(0)^{2i}e(-1)^i\mathbb{1}$ as a linear combination of the vectors of the form (2.2). Let v be a vector of the form (2.2). Then $h(0)v = 2(q-r)v$ and

$$L_{\mathrm{aff}}(0)v = (i_1 + \cdots + i_p + j_1 + \cdots + j_q + m_1 + \cdots + m_r)v.$$

Since $h(0)f(0)^{2i}e(-1)^i\mathbb{1} = -2if(0)^{2i}e(-1)^i\mathbb{1}$ and $L_{\mathrm{aff}}(0)f(0)^{2i}e(-1)^i\mathbb{1} = if(0)^{2i}e(-1)^i\mathbb{1}$, we obtain that

$$f(0)^{2i}e(-1)^i\mathbb{1} = c_i f(-1)^i\mathbb{1} \tag{4.3}$$

for some constant c_i. In fact, this can be also seen in a different way. We consider the inner derivation $(\mathrm{ad}\, f(0))x = [f(0), x]$. Note that $f(0)\mathbb{1} = 0$, $(\mathrm{ad}\, f(0))e(-1) = -h(-1)$, $(\mathrm{ad}\, f(0))^2e(-1) = -2f(-1)$ and $(\mathrm{ad}\, f(0))^s e(-1) = 0$ for $s \geqslant 3$. Hence

$$f(0)^{2i}e(-1)^i\mathbb{1} = \big(\big(\mathrm{ad}\, f(0)\big)^{2i}e(-1)^i\big)\mathbb{1} = a_i\big(\big(\mathrm{ad}\, f(0)\big)^2e(-1)\big)^i\mathbb{1}$$

for some positive integer a_i. That is, $c_i = (-1)^i 2^i a_i$. More precisely, we have

$$a_i = \prod_{m=0}^{i-1}\binom{2i-2m}{2} = \frac{(2i)!}{2^i}.$$

Let j be an integer such that $0 \leqslant j \leqslant 2i$. Then

$$e(0)^j f(0)^{2i}e(-1)^i\mathbb{1} = c_i e(0)^j f(-1)^i\mathbb{1}$$

by (4.3). Moreover, one can obtain from the highest weight module structure for sl_2 that

$$e(0)^j f(0)^{2i}e(-1)^i\mathbb{1} = \frac{(2i)!j!}{(2i-j)!}f(0)^{2i-j}e(-1)^i\mathbb{1}.$$

In the case $j = i$, the above two equations imply that

$$f(0)^i e(-1)^i\mathbb{1} = \frac{c_i}{(2i)!}e(0)^i f(-1)^i\mathbb{1} = (-1)^i e(0)^i f(-1)^i\mathbb{1}.$$

Since $\theta(f(0)^i e(-1)^i\mathbb{1}) = e(0)^i f(-1)^i\mathbb{1}$, (4.2) holds. $\square$

Remark 4.3. The vector $f(0)^{k+1}e(-1)^{k+1}\mathbb{1}$ is a scalar multiple of W^3, W^4 or W^5 in the case $k = 2$, 3 or 4 [3, Section 5].

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