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Optimality Conditions for Vector Optimization Problems

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Abstract In this paper, some necessary and sufficient optimality conditions for the weakly efficient solutions of vector optimization problems (VOP) with finite equality and inequality constraints are shown by using two kinds of constraints qualifications in terms of the MP subdifferential due to Ye. A partial calmness and a penalized problem for the (VOP) are introduced and then the equivalence between the weakly efficient solution of the (VOP) and the local minimum solution of its penalized problem is proved under the assumption of partial calmness.

Keywords Vector optimization problem · Optimality condition · Partial calmness · Exact penalization · MP subdifferential

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1 Introduction

Let R^I and R^L be the I-dimensional and L-dimensional Euclidean spaces, respectively, where I and L are given positive integers. In this paper, all vectors are column vectors and the superscript T denotes the transpose. For any positive integers $a > b$, let $[a, b] = \{a, a+1, \ldots, b\}$. Denote by R^L_+ and $\mathrm{int}R^L_+$ the nonnegative orthant of R^L and the interior of R^L_+, respectively. Then, the order and weak order in R^L can be defined by respectively

$$x \le y \quad \Longleftrightarrow \quad y - x \in R^L_+, \quad x \not\le y \quad \Longleftrightarrow \quad y - x \notin R^L_+, \quad \forall x, y \in R^L,$$

and

$$y < x \quad \Longleftrightarrow \quad y - x \in -\mathrm{int}R^L_+, \quad y \not< x \quad \Longleftrightarrow \quad y - x \notin -\mathrm{int}R^L_+, \quad \forall x, y \in R^L.$$

Let X be a Banach space. In this paper, we consider the following vector optimization problem with finite equalities and inequalities constraints:

$$\begin{aligned} \text{(VOP)} \quad \min \quad & F(x), \\ \text{s.t.} \quad & u_i(x) \le 0, \quad i \in [1, m], \\ & v_j(x) = 0, \quad j \in [1, n], \end{aligned}$$

where $F = (f_1, \ldots, f_L)^T : X \to R^L$, $u_i : X \to R$ ($i \in [1, m]$) and $v_j : X \to R$ ($j \in [1, n]$) are functions and m, n are given nonnegative integers. In the case where $m = 0$ or $n = 0$, there are no explicit constraints of the above type. We can rewrite (VOP) as

$$\begin{aligned} \text{(VOP)} \quad \min \quad & F(x), \\ \text{s.t.} \quad & u(x) \le 0, \\ & v(x) = 0, \end{aligned}$$

where $u = (u_1, u_2, \ldots, u_m)^T : X \to R^m$ and $v = (v_1, v_2, \ldots, u_n)^T : X \to R^n$. We denote by

$$\begin{aligned} K &= \{x \in X : u_i(x) \le 0, v_j(x) = 0, i \in [1, m], j \in [1, n]\} \\ &= \{x \in X : u(x) \le 0, v(x) = 0\} \end{aligned}$$

the feasible set of (VOP) and, for any feasible solution $x^* \in K$, by $M(x^*) = \{i \in [1, m] : u_i(x^*) = 0\}$ the index set of the binding constraints. Recently, many authors have studied (VOP) under some different conditions. See e.g. [1–8]. It is well known that set-valued optimization is different from (VOP), even though one can claim that (VOP) is a special case of set-valued optimization.

A vector $x^* \in K$ is an efficient solution to (VOP) if and only if

$$F(x^*) \le F(y), \quad \forall y \in K,$$

or equivalently,

$$\forall y \in K, \quad \forall l \in [1, L] : f_l(y) - f_l(x^*) \geq 0.$$

A vector $x^* \in K$ is a weakly efficient solution to (VOP) if and only if

$$F(y) \nless F(x^*), \quad \forall y \in K,$$

or equivalently,

$$\forall y \in K, \quad \exists l_y \in [1, L] : f_{l_y}(y) - f_{l_y}(x^*) \geq 0.$$

Obviously, an efficient solution implies a weakly efficient solution.

If $F \equiv f$, then (VOP) reduces to the classical optimization problem (COP).

If $X = R^I$ and $F(x) = (c_1^T x, \dots, c_L^T x)^T$ for all $x \in R^I$, then (VOP) collapses to the multicriteria linear programming problem (MCLP, [9]) where $c_l \in R^I$ for each $l \in [1, L]$.

We denote by E_w and E the set of all weakly efficient solutions and the set of all efficient solutions to (VOP), by S_w and S the set of all weakly efficient solutions and the set of all efficient solutions to (MCLP), respectively. It is clear that $E \subseteq E_w$ and $S \subseteq S_w$. In this paper, we always assume that E_w, E, S_w and S are nonempty.

It is well known that both differentiability and Lipschitz continuity play an important role in establishing the optimality conditions for (COP). See [10–17]. Recently, Ye [18] studied (COP) with equality and inequality constraints on a Banach space where the objective and the binding constraints are either differentiable at the minimum solution or Lipschitz near the minimum solution, and derived necessary and sufficient optimality conditions and constraint qualifications in term of the Michel-Penot subdifferential. Ye, Zhu and Zhu [19] showed the equivalence between partial calmness and local exact penalization for (COP) with equality and inequality constraints.

The main goal of the present paper is to derive necessary and sufficient optimality conditions, partial calmness and the penalized problem for (VOP). The paper is organized as follows. By using the nondifferentiable Abadie constraints qualifications (ACQ) and generalized Zangwill constraints qualifications (ZCQ) in terms of the Michel-Penot subdifferential due to Ye [18], we derive necessary and sufficient optimality conditions for the weakly efficient solutions of (VOP) and (MCLP) with finite qualities and inequalities constraints in Sect. 2. In Sect. 3, we introduce notions of partial calmness and the penalized problem for (VOP) and then establish the relationship between the weakly efficient solution of (VOP) and the local minimum solution of the penalized problem of (VOP) under the assumption of the partial calmness.

2 Optimality Conditions for (VOP)

In this section, by using the nondifferentiable Abadie constraints qualifications (CQ) and the generalized Zangwill constraints qualifications (CQ) in terms of the Michel-Penot subdifferential [15, 16], we derive necessary and sufficient optimality conditions for the weakly efficient solutions of (VOP) and (MCLP) with finite equality and inequality constraints.

Let X, Y be Banach spaces, let X^* be the dual space of X. We recall first some definitions which are needed in our main results.

Definition 2.1 Let $h : X \to R$, and $\hat{x} \in X$.

(i) $h : X \to R$ is said to be quasiconvex at $\hat{x}$ if, for any $x \in X$,

$$h(x) \le h(\hat{x}), \quad 0 < t < 1 \quad \Longrightarrow \quad h((1-t)x + t\hat{x}) \le h(\hat{x}).$$

(ii) h is said to be Lipschitz near (or around) $\hat{x}$ if there exist constants $\delta, \theta_h > 0$ such that

$$|h(x) - h(y)| \le \theta_h \|x - y\|, \quad \forall x, y \in B(\hat{x}, \delta),$$

where θ_h is the Lipschitz constant and $B(\hat{x}, \delta)$ is the open ball with center $\hat{x}$ and radius δ.

It is clear that the convexity of h implies its quasiconvexity and that, if h is Lipschitz near $\hat{x}$, then it is Lipschitz continuous at $\hat{x}$.

Normal cones, contingent cones and cones of feasible directions are defined as follows.

Definition 2.2 Let Ω be a closed subset of X and $\hat{x} \in \Omega$. The normal cone $N_\Omega(\hat{x})$ of Ω at $\hat{x}$ is given by

$$N_\Omega(\hat{x}) = \{x^* \in X^* : \langle x^*, x - \hat{x} \rangle \le 0, \ \forall x \in \Omega\}.$$

Definition 2.3 Let $\Omega \subseteq X$ and $\hat{x} \in \mathrm{cl}\Omega$. The contingent cone of Ω at $\hat{x}$ is the closed cone defined by

$$T_\Omega(\hat{x}) = \{z \in X : \exists t_n \downarrow 0, z_n \to z, \text{ s.t. } \hat{x} + t_n z_n \in \Omega, \ \forall n\}.$$

Definition 2.4 Let $\Omega \subseteq X$ and $\hat{x} \in \mathrm{cl}\Omega$. The cone of feasible directions of Ω at $\hat{x}$ is given by

$$D_\Omega(\hat{x}) = \{z \in X : \exists \delta > 0, \text{ s.t. } \hat{x} + tz \in \Omega, \ \forall t \in (0, \delta)\}.$$

Based on the normal cone, the subdifferential and singular subdifferential of a proper lower semicontinuous and convex function are defined as follows.

Definition 2.5 Let $h : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Denote $\operatorname{dom} h = \{x \in X : h(x) < +\infty\}$ and $\operatorname{epi} h = \{(x, t) \in X \times R : h(x) \le t\}$. The subdifferential and singular subdifferential of h at $\hat{x} \in \operatorname{dom} h$ are respectively the sets

$$\partial h(\hat{x}) = \{x^* \in X^* : (x^*, -1) \in N_{\mathrm{epi}h}(\hat{x}, h(\hat{x}))\}$$

and

$$\partial^\infty h(\hat{x}) = \{x^* \in X^* : (x^*, 0) \in N_{\mathrm{epi}h}(\hat{x}, h(\hat{x}))\}.$$

Clearly,

$$\partial h(\hat{x}) = \{x^* \in X^* : \langle x^*, x - \hat{x}\rangle \leq h(x) - h(\hat{x}),\ \forall x \in X\}$$

and

$$\partial^\infty h(\hat{x}) = N_{\mathrm{dom}\, h}(\hat{x}), \qquad \partial h(\hat{x}) = \partial^\infty h(\hat{x}) + \partial h(\hat{x}).$$

It is well known that (Theorem 1 of [20], Theorem 2.1 of [21]), if h is lower semicontinuous, then $\partial^\infty h(\hat{x}) = \{0\}$ is necessary and sufficient for h being Lipschitz near $\hat{x}$.

We now recall some definitions of the usual derivatives.

Definition 2.6 Let $h : X \to Y$, and $\hat{x} \in X$. The usual directional derivative of h at $\hat{x}$ in the direction $d \in X$ is given by

$$h'(\hat{x}, d) = \lim_{t \downarrow 0} \frac{h(\hat{x} + td) - h(\hat{x})}{t}$$

when this limit exists. We say that h is Gãteaux differentiable at $\hat{x}$ if there is $Dh(\hat{x}) \in L(X, Y)$ such that, for every $d \in X$, $h'(\hat{x}, d) = \langle Dh(\hat{x}), d\rangle$, where $L(X, Y)$ denotes the space of continuous linear operators from X to Y and $\langle \cdot, \cdot\rangle$ denotes the pairing. h is said to be Fréchet differentiable at $\hat{x}$ if $Dh(\hat{x}) \in L(X, Y)$ and the convergence in

$$h'(\hat{x}, d) = \lim_{t \downarrow 0} \frac{h(\hat{x} + td) - h(\hat{x})}{t} = \langle Dh(\hat{x}), d\rangle$$

is uniform with respect to d in bounded sets.

It is obvious that the Fréchet differentiability implies the Gãteaux differentiability. The following MP directional derivative and MP subdifferential were first investigated by Michel and Penot.

Definition 2.7 MP Subdifferential [15]. Let $h : X \to R$, and $\hat{x} \in X$. The MP directional derivative of h at $\hat{x}$ in the direction $d \in X$ is defined by

$$h^\diamond(\hat{x}, d) = \sup_{z \in X} \limsup_{t \downarrow 0} \frac{h(\hat{x} + t(d + z)) - h(\hat{x} + tz)}{t}$$

and the MP subdifferential of h at $\hat{x}$ is defined by

$$\partial^\diamond h(\hat{x}) = \{x^* \in X^* : \langle x^*, d\rangle \leq h^\diamond(\hat{x}, d),\ \forall d \in X\}.$$

Recall that the upper Dini directional derivative of $h : X \to R$ at $\hat{x} \in X$ in the direction $d \in X$ is defined by

$$h'_+(\hat{x}, d) = \limsup_{t \downarrow 0} \frac{h(\hat{x} + td) - h(\hat{x})}{t}.$$

Clearly, $h'_+(\hat{x}, d) \leq h^\diamond(\hat{x}, d)$.

From above definition, one has that $h^\diamond(\hat{x}, 0)$=0. It is clear that the MP subdifferential is a natural generalization of the Gãteaux derivative. When h is Gãteaux differentiable at $\hat{x}$, $h^\diamond(\hat{x}, d) = h'(\hat{x}, d)$ and $\partial^\diamond h(\hat{x}) = \{Dh(\hat{x})\}$; see [15]. If h is convex, then the MP subdifferential coincides with the subdifferential in the sense of convex analysis. Moreover, if h is linear, then from above definition, one has that $h^\diamond(\hat{x}, d) = h(d)$ for any $\hat{x}, d \in X$.

Based on the MP directional derivative, Ye [18] introduced the following concepts of MP regularity and the MP pseudoconvexity.

Definition 2.8 (See [18]) Let $h : X \to R$, and $\hat{x} \in X$.

(i) MP regularity: h is MP regular at $\hat{x} \in X$ if the usual directional derivative $h'(\hat{x}, d)$ exists and $h'(\hat{x}, d) = h^\diamond(\hat{x}, d)$ for all $d \in X$.
(ii) MP pseudoconvexity: $h : X \to R$ is said to be MP pseudoconvex at $\hat{x} \in X$ if, for any $x \in X$,

$$h^\diamond(\hat{x}, x - \hat{x}) \geq 0 \quad \Longrightarrow \quad h(x) \geq h(\hat{x}).$$

Remark 2.1 It is clear that, if $h : X \to R$ is linear, then it is MP pseudoconvex at any $\hat{x} \in X$. In fact, note that the linearity of h implies that $h^\diamond(\hat{x}, d) = h(d)$ for any $\hat{x}, d \in X$.

The following properties of the MP directional derivative and the MP subdifferential are useful.

Lemma 2.1 (*See* [15, 16, 22]) *Let $\hat{x} \in X$ and let $h, f : X \to R$ be either Gãteaux differentiable at $\hat{x}$ at $\hat{x}$ or Lipschitz near $\hat{x}$. Then, the following statements are true*:

(i) *The function $d \mapsto h^\diamond(\hat{x}, d)$ is finite, positively homogeneous, and subadditive on X.*
(ii) *For any scalar t, $\partial^\diamond(th)(\hat{x}) = t\partial^\diamond h(\hat{x})$; for every $d \in X$, $h^\diamond(\hat{x}, -d) = (-h)^\diamond(\hat{x}, d)$.*
(iii) *$\partial^\diamond(h + f)(\hat{x}) \subseteq \partial^\diamond h(\hat{x}) + \partial^\diamond f(\hat{x})$ and $(h + f)^\diamond(\hat{x}, d) \leq h^\diamond(\hat{x}, d) + f^\diamond(\hat{x}, d)$, for all $d \in X$. If in addition both h and f are MP regular at $\hat{x}$, then $\leq$ becomes $=$.*
(iv) *$\partial^\diamond h(\hat{x})$ is a nonempty, convex and weak*-compact subset of X^* and, for every $d \in X$, $h^\diamond(\hat{x}, d) = \max_{\xi^* \in \partial^\diamond h(\hat{x})} \langle \xi^*, d\rangle$.*
(v) *If $\hat{x}$ is a local minimum solution of h, then $0 \in \partial^\diamond h(\hat{x})$ and $h^\diamond(\hat{x}, d) \geq 0$ for all $d \in X$.*

Based on the MP directional derivative and the MP subdifferential, we next define the MP directional derivative and the MP subdifferential of vector-valued functions as follows.

Definition 2.9 Let $F = (f_1, \ldots, f_L)^T : X \to R^L$ and $\hat{x} \in X$. The MP directional derivative of F at $\hat{x}$ in the direction $d \in X$ is defined by

$$\begin{aligned} F^\diamond(\hat{x}, d) &= (f_1^\diamond(\hat{x}, d), \dots, f_L^\diamond(\hat{x}, d))^T \\ &= \left(\sup_{z\in X} \limsup_{t\downarrow 0} \frac{f_1(\hat{x}+t(d+z)) - f_1(\hat{x}+tz)}{t}, \dots,\right. \\ &\qquad \left. \sup_{z\in X} \limsup_{t\downarrow 0} \frac{f_L(\hat{x}+t(d+z)) - f_L(\hat{x}+tz)}{t}\right)^T \end{aligned}$$

and the MP subdifferential of F at $\hat{x}$ is given by

$$\begin{aligned} \partial^\diamond F(\hat{x}) &= \{x^* = (x_1^*, \dots, x_L^*)^T \in \underbrace{X^* \times \cdots \times X^*}_{L} : (\langle x_1^*, d\rangle, \dots, \langle x_L^*, d\rangle) \\ &\qquad \le F^\diamond(\hat{x}, d),\ \forall d \in X\} \\ &= \{x^* = (x_1^*, \dots, x_L^*)^T \in \underbrace{X^* \times \cdots \times X^*}_{L} : x_l^* \in \partial^\diamond f_l(\hat{x}),\ \forall l \in [1, L]\} \\ &= \partial^\diamond f_1(\hat{x}) \times \cdots \times \partial^\diamond f_L(\hat{x}). \end{aligned}$$

Similarly, we can define the MP pseudoconvexity of vector-valued functions.

Definition 2.10 A function $F = (f_1, \dots, f_L)^T : X \to R^L$ is said to be MP pseudoconvex at $\hat{x} \in X$ if, for any $x \in X$,

$$F^\diamond(\hat{x}, x - \hat{x}) \not< 0 \quad \Longrightarrow \quad F(x) \not< F(\hat{x}).$$

Remark 2.2 If $F = (f_1, \dots, f_L)^T : X \to R^L$ reduces to $h : X \to R$, then Definition 2.9 collapses to Definition 2.7; also the MP pseudoconvexity of $F = (f_1, \dots, f_L)^T$ in Definition 2.10 becomes that in (ii) of Definition 2.8. Moreover, if f_l $(l \in [1, l])$ is linear, then

$$\begin{aligned} F^\diamond(\hat{x}, x - \hat{x}) &= (f_1^\diamond(\hat{x}, x - \hat{x}), \dots, f_L^\diamond(\hat{x}, x - \hat{x}))^T \\ &= (f_1(x - \hat{x}), \dots, f_L(x - \hat{x}))^T \\ &= (f_1(x) - f_1(\hat{x}), \dots, f_L(x) - f_L(\hat{x}))^T; \end{aligned}$$

thus, $F = (f_1, \dots, f_L)^T$ is MP pseudoconvex at any $\hat{x} \in X$.

The following properties of the MP directional derivative and the MP subdifferential of a vector-valued function generalize and extend those of a real-valued function in Lemma 2.1.

Proposition 2.1 *Let* $\hat{x} \in X$ *and let* $F = (f_1, \dots, f_L)^T, H = (h_1, \dots, h_L)^T : X \to R^L$. *Let* f_l, h_l $(l \in [1, L])$ *be either Gãteaux differentiable at* $\hat{x}$ *or Lipschitz near* $\hat{x}$. *Then, the following statements hold*:

(i) *The vector-valued function* $d \mapsto F^\diamond(\hat{x}, d)$ *is finite, positively homogeneous, and subadditive on* X.
(ii) *For any scalar* t, $\partial^\diamond(tF)(\hat{x}) = t\partial^\diamond F(\hat{x})$; *for every* $d \in X$, $F^\diamond(\hat{x}, -d) = (-F)^\diamond(\hat{x}, d)$.

(iii) $(F+H)^{\diamond}(\hat{x},d) \le F^{\diamond}(\hat{x},d) + H^{\diamond}(\hat{x},d)$, *for all* $d \in X$.
(iv) $\partial^{\diamond} F(\hat{x})$ *is a nonempty, convex and weak***-compact subset of* $\underbrace{X^* \times \cdots \times X^*}_{L}$; *for every* $d \in X$, $F^{\diamond}(\hat{x},d) = (\max_{\xi_1^* \in \partial^{\diamond} f_1(\hat{x})} \langle \xi_1^*, d\rangle, \dots, \max_{\xi_L^* \in \partial^{\diamond} f_L(\hat{x})} \langle \xi_L^*, d\rangle)^T$.
(v) *If* $\hat{x}$ *is a weakly efficient solution of* F *on* X, *i.e., if* $F(x) \nless F(\hat{x})$ *for all* $x \in X$, *then* $F^{\diamond}(\hat{x},d) \nless 0$ *for all* $d \in X$.

Proof (i) Since

$$\|F^{\diamond}(\hat{x},d)\| = \sqrt{(f_1^{\diamond}(\hat{x},d))^2 + \cdots + (f_L^{\diamond}(\hat{x},d))^2},$$

and $d \mapsto f_l^{\diamond}(\hat{x},d)$ ($l \in [1,L]$) is finite (from (i) of Lemma 2.1), so is $d \mapsto F^{\diamond}(\hat{x},d)$. For any $t>0$, $d, d_1, d_2 \in X$, it follows from the positive homogeneity and subadditivity of $f_l^{\diamond}(\hat{x},\cdot)([1,L])$ (see (i) of Lemma 2.1) that

$$\begin{aligned} F^{\diamond}(\hat{x},td) &= (f_1^{\diamond}(\hat{x},td),\dots,f_L^{\diamond}(\hat{x},td))^T \\ &= (tf_1^{\diamond}(\hat{x},d),\dots,tf_L^{\diamond}(\hat{x},d))^T \\ &= t(f_1^{\diamond}(\hat{x},d),\dots,f_L^{\diamond}(\hat{x},d))^T \\ &= tF^{\diamond}(\hat{x},d) \end{aligned}$$

and

$$\begin{aligned} F^{\diamond}(\hat{x},d_1+d_2) &= (f_1^{\diamond}(\hat{x},d_1+d_2),\dots,f_L^{\diamond}(\hat{x},d_1+d_2))^T \\ &\le (f_1^{\diamond}(\hat{x},d_1)+f_1^{\diamond}(\hat{x},d_2),\dots,f_L^{\diamond}(\hat{x},d_1)+f_L^{\diamond}(\hat{x},d_2))^T \\ &= (f_1^{\diamond}(\hat{x},d_1),\cdots,f_L^{\diamond}(\hat{x},d_1))^T + (f_1^{\diamond}(\hat{x},d_2),\dots,f_L^{\diamond}(\hat{x},d_2))^T \\ &= F^{\diamond}(\hat{x},d_1)+F^{\diamond}(\hat{x},d_2), \end{aligned}$$

which implies that $d \mapsto F^{\diamond}(\hat{x},d)$ is positively homogeneous and subadditive on X.

(ii) Conclusion (ii) of Lemma 2.1 implies that, for any scalar t,

$$\begin{aligned} \partial^{\diamond}(tF)(\hat{x}) &= \partial^{\diamond}(tf_1)(\hat{x}) \times \cdots \times \partial^{\diamond}(tf_L)(\hat{x}) \\ &= t(\partial^{\diamond} f_1(\hat{x}) \times \cdots \times \partial^{\diamond} f_L(\hat{x})) \\ &= t\partial^{\diamond} F(\hat{x}) \end{aligned}$$

and, for every $d \in X$,

$$\begin{aligned} F^{\diamond}(\hat{x},-d) &= (f_1^{\diamond}(\hat{x},-d),\dots,f_L^{\diamond}(\hat{x},-d))^T \\ &= ((-f_1)^{\diamond}(\hat{x},d),\dots,(-f_L)^{\diamond}(\hat{x},d))^T \\ &= (-F)^{\diamond}(\hat{x},d). \end{aligned}$$

(iii) For any $d \in X$,

$$(F+H)^{\diamond}(\hat{x},d) = ((f_1+h_1)^{\diamond}(\hat{x},d),\dots,(f_L+h_L)^{\diamond}(\hat{x},d))^T$$

$$\begin{aligned}&\leq (f_1^\diamond(\hat{x},d)+h_1^\diamond(\hat{x},d),\dots,f_L^\diamond(\hat{x},d)+h_L^\diamond(\hat{x},d))^T\\&=(f_1^\diamond(\hat{x},d),\dots,f_L^\diamond(\hat{x},d))^T+(h_1^\diamond(\hat{x},d),\dots,h_L^\diamond(\hat{x},d))^T\\&=F^\diamond(\hat{x},d)+H^\diamond(\hat{x},d);\end{aligned}$$

this follows from (iii) of Lemma 2.1.

(iv) Since for each $l \in [1,L]$, $\partial^\diamond f_l(\hat{x})$ is a nonempty, convex and weak*-compact subset of X^* and since for every $d \in X$, $f_l^\diamond(\hat{x},d)=\max_{\xi^*\in\partial^\diamond f_l(\hat{x})}\langle \xi^*,d\rangle$, it is easy to see that property (iv) holds.

(v) Let $\hat{x}$ be a weakly efficient solution of F on X, i.e., $F(x)\not< F(\hat{x})$ for all $x \in X$. Then,

$$F(x)-F(\hat{x})\in R^L\backslash(-\mathrm{int}R_+^L),\quad \forall x\in X,$$

which implies that, for any $d \in X$ and $t \in (0,1)$,

$$\begin{aligned}&\left(\frac{f_1(\hat{x}+td)-f_1(\hat{x})}{t},\dots,\frac{f_L(\hat{x}+td)-f_L(\hat{x})}{t}\right)^T\\&=\frac{F(\hat{x}+td)-F(\hat{x})}{t}\\&\in R^L\backslash(-\mathrm{int}R_+^L).\end{aligned}$$

We declare that

$$\left(\limsup_{t\downarrow 0}\frac{f_1(\hat{x}+td)-f_1(\hat{x})}{t},\dots,\limsup_{t\downarrow 0}\frac{f_L(\hat{x}+td)-f_L(\hat{x})}{t}\right)^T\in R^L\backslash(-\mathrm{int}R_+^L).$$

In fact, if

$$\left(\limsup_{t\downarrow 0}\frac{f_1(\hat{x}+td)-f_1(\hat{x})}{t},\dots,\limsup_{t\downarrow 0}\frac{f_L(\hat{x}+td)-f_L(\hat{x})}{t}\right)^T\in -\mathrm{int}R_+^L$$

for some $d \in X$, then

$$\limsup_{t\downarrow 0}\frac{f_l(\hat{x}+td)-f_l(\hat{x})}{t}<0,\quad \forall l\in[1,L];$$

hence, for each $l \in [1,L]$,

$$f_l(\hat{x}+td)-f_l(\hat{x})<0,$$

for $t>0$ small enough. Therefore,

$$F(\hat{x}+td)-F(\hat{x})=(f_1(\hat{x}+td)-f_1(\hat{x}),\dots,f_L(\hat{x}+td)-f_L(\hat{x}))^T<0,$$

for $t>0$ small enough, which contradicts the assumption that $\hat{x}$ is a weakly efficient solution of F on X. Since $0\in X$, by the MP directional derivative of F at $\hat{x}$, one has

$$F^\diamond(\hat{x},d)=\left(\sup_{z\in X}\limsup_{t\downarrow 0}\frac{f_1(\hat{x}+t(d+z))-f_1(\hat{x}+tz)}{t},\dots,\right.$$

$$\left.\sup_{z\in X}\limsup_{t\downarrow 0}\frac{f_L(\hat{x}+t(d+z))-f_L(\hat{x}+tz)}{t}\right)^T$$
$$\geq \left(\limsup_{t\downarrow 0}\frac{f_1(\hat{x}+td)-f_1(\hat{x})}{t},\ldots,\limsup_{t\downarrow 0}\frac{f_L(\hat{x}+td)-f_L(\hat{x})}{t}\right)^T$$

and thus

$$F^\diamond(\hat{x},d)\in R^L\backslash(-\mathrm{int}R^L_+),$$

i.e., $F^\diamond(\hat{x},d)\not< 0$, which yields the desired conclusion. □

We now establish some relationships between the MP pseudoconvexity of a vector-valued function and the MP pseudoconvexity of each of its components.

Proposition 2.2 *Let* $\hat{x}\in X$ *and* $F=(f_1,\ldots,f_L)^T: X\to R^L$. *Consider the following statements*:

(i) f_l *is MP pseudoconvex at* $\hat{x}$ *for each* $l\in[1,L]$.
(ii) $F=(f_1,\ldots,f_L)^T$ *is MP pseudoconvex at* $\hat{x}$.
(iii) *For any given* $x\in X$, *there exists* $l_x\in[1,L]$ *such that*

$$f^\diamond_{l_x}(\hat{x},x-\hat{x})\geq 0 \quad\Longrightarrow\quad f_{l_x}(x)\geq f_{l_x}(\hat{x}).$$

Then, (i)⇒(ii)⇒(iii).

Proof (i) ⇒(ii) Suppose that f_l is MP pseudoconvex at $\hat{x}$ for each $l\in[1,L]$. Let $x\in X$ such that $F^\diamond(\hat{x},x-\hat{x})\not< 0$. Then,

$$(f^\diamond_1(\hat{x},x-\hat{x}),\ldots,f^\diamond_L(\hat{x},x-\hat{x}))^T=F^\diamond(\hat{x},x-\hat{x})\not< 0;$$

so, there exists $l_x\in[1,L]$ such that $f^\diamond_{l_x}(\hat{x},x-\hat{x})\geq 0$. Since f_{l_x} is MP pseudoconvex at $\hat{x}$, it follows that $f_{l_x}(x)\geq f_{l_x}(\hat{x})$. Thus,

$$F(x)=(f_1(x),\ldots,f_{l_x}(x),\ldots,f_L(x))^T\not<(f_1(\hat{x}),\ldots,f_{l_x}(\hat{x}),\ldots,f_L(\hat{x}))^T=F(\hat{x}),$$

which implies that $F=(f_1,\ldots,f_L)^T$ is MP pseudoconvex at $\hat{x}$.

(ii)⇒(iii) Let $F=(f_1,\ldots,f_L)^T$ be MP pseudoconvex at $\hat{x}$. Suppose to the contrary that there is $x_0\in X$ and that, for each $l\in[1,L]$ with $f^\diamond_l(\hat{x},x_0-\hat{x})\geq 0$, one has $f_l(x_0)<f_l(\hat{x})$. Then it follows that

$$F^\diamond(\hat{x},x_0-\hat{x})^T=(f^\diamond_1(\hat{x},x_0-\hat{x}),\ldots,f^\diamond_L(\hat{x},x_0-\hat{x}))^T\not< 0$$

and

$$\begin{aligned}F(x_0)&=(f_1(x_0),\ldots,f_L(x_0))^T\\&<(f_1(\hat{x}),\ldots,f_L(\hat{x}))^T\\&=F(\hat{x}),\end{aligned}$$

which contradicts the fact that $F = (f_1, \dots, f_L)^T$ is MP pseudoconvex at $\hat{x}$. The proof is complete. □

Similar to the scalar case (see Theorem 2.1 in [18]), we obtain the following sufficient and necessary conditions for a weakly efficient solution of (VOP) under the MP pseudoconvexity.

Proposition 2.3 *Let Ω be a convex subset of X, $\hat{x} \in \Omega$, and let $F = (f_1, \dots, f_L)^T : X \to R^L$ be MP pseudoconvex at $\hat{x}$. Then, $\hat{x}$ is a weakly efficient solution of F on Ω, i.e., $F(x) \nless F(\hat{x})$, for all $x \in \Omega$, if and only if $F^\diamond(\hat{x}, x - \hat{x}) \nless 0$ for all $x \in \Omega$.*

Proof Suppose that $F^\diamond(\hat{x}, x - \hat{x}) \nless 0$ for all $x \in \Omega$. By the MP pseudoconvexity of F at $\hat{x}$, we have $F(x) \nless F(\hat{x})$, i.e., $\hat{x}$ is a weakly efficient solution of F on Ω.

Conversely, if $\hat{x}$ is a weakly efficient solution of F on Ω, i.e., $F(x) \nless F(\hat{x})$ for all $x \in \Omega$, then

$$F(x) - F(\hat{x}) \in R^L \backslash (-\mathrm{int} R^L_+).$$

The following proof is similar to that in (v) of Proposition 2.1 with replacing d by $x - \hat{x}$. We conclude that

$$F^\diamond(\hat{x}, x - \hat{x}) \in R^L \backslash (-\mathrm{int} R^L_+),$$

i.e., $F^\diamond(\hat{x}, x - \hat{x}) \nless 0$, which completes the proof. □

Similar to (COP) (see Lemma 3.1 in [18]), we have the following necessary conditions for a weakly efficient solution of (VOP) by considering the contingent cone and the cone of feasible directions.

Lemma 2.2 *Let Ω be a closed subset of X and $F = (f_1, \dots, f_L)^T : X \to R^L$. Let $\hat{x} \in \Omega$ be a weakly efficient solution of F on Ω, i.e., $F(x) \nless F(\hat{x})$ for all $x \in \Omega$. Then, the following statements are true*:

(i) *If f_l $(l \in [1, L])$ is either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, then*

$$F^\diamond(\hat{x}, d) \nless 0, \quad \forall d \in \mathrm{cl} D_\Omega(\hat{x}).$$

(ii) *If f_l $(l \in [1, L])$ is either Fréchet differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, then*

$$F^\diamond(\hat{x}, d) \nless 0, \quad \forall d \in T_\Omega(\hat{x}).$$

Proof (i) Suppose to the contrary that there exists $d \in D_\Omega(\hat{x})$ such that $F^\diamond(\hat{x}, d) < 0$. By the MP directional derivative of F at $\hat{x}$, one has

$$\left(\limsup_{t \downarrow 0} \frac{f_1(\hat{x} + td) - f_1(\hat{x})}{t}, \dots, \limsup_{t \downarrow 0} \frac{f_L(\hat{x} + td) - f_L(\hat{x})}{t} \right)^T$$
$$\leq \left(\sup_{z \in X} \limsup_{t \downarrow 0} \frac{f_1(\hat{x} + t(d + z)) - f_1(\hat{x} + tz)}{t}, \dots, \right.$$

$$\left.\sup_{z\in X}\limsup_{t\downarrow 0}\frac{f_L(\hat{x}+t(d+z))-f_L(\hat{x}+tz)}{t}\right)^T$$
$$= F^{\diamond}(\hat{x},d)$$
$$< 0.$$

It follows that

$$F(\hat{x}+td)-F_l(\hat{x})=(f_1(\hat{x}+td)-f_1(\hat{x}),\dots,f_L(\hat{x}+td)-f_L(\hat{x}))^T<0,$$

for $t>0$ small enough; however, this is a contradiction with the assumption that $\hat{x}$ is a weakly efficient solution of F on Ω. Thus,

$$F^{\diamond}(\hat{x},d)\not<0,\quad \forall d\in D_{\Omega}(\hat{x}),$$

or equivalently,

$$F^{\diamond}(\hat{x},d)\in R^L\backslash(-\mathrm{int}R^L_+),\quad \forall d\in D_{\Omega}(\hat{x}).$$

If f_l $(l\in[1,L])$ is either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, then from (i) of Proposition 2.1, $d\mapsto F^{\diamond}(\hat{x},d)$ is continuous. Again, since $R^L\backslash(-\mathrm{int}R^L_+)$ is closed, one has

$$F^{\diamond}(\hat{x},d)\not<0,\quad \forall d\in \mathrm{cl}D_{\Omega}(\hat{x}).$$

(ii) Assume that f_l $(l\in[1,L])$ is either Fréchet differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$. Let $d\in T_{\Omega}(\hat{x})$. Then, there exist $t_n\downarrow 0$ and $d_n\to d$ such that $\hat{x}+t_nd_n\in\Omega$ for each n. Since $\hat{x}$ is a weakly efficient solution of F on Ω and L is finite, there are $l_0\in[1,L]$ and infinitely many n such that $f_{l_0}(\hat{x}+t_nd_n)-f_{l_0}(\hat{x})\geq 0$. It follows that

$$(f_{l_0})'_+(\hat{x},d)\geq\liminf_{n\to\infty}\frac{f_{l_0}(\hat{x}+t_nd_n)-f_{l_0}(\hat{x})}{t_n}\geq 0,$$

where the first inequality holds by the assumption that f_{l_0} is either Fréchet differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$. Then, $f^{\diamond}_{l_0}(\hat{x},d)\geq 0$ and so

$$F^{\diamond}(\hat{x},d)=(f^{\diamond}_1(\hat{x},d),\dots,f^{\diamond}_{l_0}(\hat{x},d),\dots,f^{\diamond}_L(\hat{x},d))^T\not<0.$$

This completes the proof. □

The following lemma is well known in convex analysis (see e.g. [23]).

Lemma 2.3 *Let* $h(x)=\max_{1\leq l\leq L}h_l(x)$ *for all* $x\in X$, *where* $h_l:X\to R$ $(l\in[1,L])$. *Let* $\hat{x}\in X$, $h_l\ (\hat{x})\equiv\hat{z}$ *for each* $l\in[1,L]$. *If* $h_l(l\in[1,L])$ *is continuous and convex, then* $\partial h(\hat{x})=\mathrm{co}\{\bigcup_{l=1}^L\partial h_l(\hat{x})\}$.

The following nondifferentiable Abadie CQ and generalized Zangwill CQ were introduced and studied by Ye [18].

Nondifferentiable Abadie CQ (See [18]) Let $\hat{x} \in K = \{x \in X : u_i(x) \le 0, v_j(x) = 0, i \in [1,m], j \in [1,n]\}$. We say that the nondifferentiable Abadie CQ holds at $\hat{x}$, if u_i $(i \in [1,m])$ and $v_j (j \in [1,n])$ are either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, the convex cone generated by

$$A = \left\{ \bigcup_{i \in M(\hat{x})} \partial^\diamond u_i(\hat{x}) \right\} \cup \left\{ \bigcup_{j \in [1,n]} \partial^\diamond v_j(\hat{x}) \right\} \cup \left\{ \bigcup_{j \in [1,n]} [-\partial^\diamond v_j(\hat{x})] \right\} \tag{1}$$

is closed and

$$\left. \begin{array}{ll} u_i^\diamond(\hat{x}, d) \le 0, & \forall i \in M(\hat{x}) \\ v_j^\diamond(\hat{x}, d) = 0, & \forall j \in [1,n] \end{array} \right\} \quad \Longrightarrow \quad d \in T_K(\hat{x}).$$

Generalized Zangwill CQ (See [18]): Let $\hat{x} \in K = \{x \in X : u_i(x) \le 0, v_j(x) = 0, i \in [1,m], j \in [1,n]\}$. We say that the generalized Zangwill CQ holds at $\hat{x}$, if u_i $(i \in [1,m])$ and v_j $(j \in [1,n])$ are either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, the convex cone generated by the set A defined by (1) is closed and

$$\left. \begin{array}{ll} u_i^\diamond(\hat{x}, d) \le 0, & \forall i \in M(\hat{x}) \\ v_j^\diamond(\hat{x}, d) = 0, & \forall j \in [1,n] \end{array} \right\} \quad \Longrightarrow \quad d \in \mathrm{cl} D_K(\hat{x}).$$

Lemma 2.4 *Let $v_j : X \to R (j \in [1,n])$ be either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x} \in X$. Then, for any $d \in X$,*

$$\left. \begin{array}{r} v_j^\diamond(\hat{x}, d) \le 0 \\ (-v_j)^\diamond(\hat{x}, d) \le 0 \end{array} \right\} \quad \Longrightarrow \quad v_j^\diamond(\hat{x}, d) = 0, \quad \forall j \in [1,n].$$

Proof The conclusion follows immediately from (iii) of Lemma 2.1. □

We now derive the KKT condition for a weakly efficient solution of (VOP).

Theorem 2.1 KKT Condition for (VOP) in the Sense of Weakly Efficient Solution. *Let $\hat{x} \in E_w$. Suppose that one of the following conditions holds*:

(i) *the nondifferentiable Abadie CQ holds at $\hat{x}$ and f_l $(l \in [1,L])$ is either Fréchet differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$*;
(ii) *the generalized Zangwill CQ holds at $\hat{x}$ and f_l $(l \in [1,L])$ is either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$.*

Then, the KKT condition holds at $\hat{x}$; i.e., there exist $\alpha_i \ge 0$ $(i \in M(\hat{x}))$, $\beta_j \ge 0$ $(j \in [1,n])$, and $\gamma_j \ge 0 (j \in [1,n])$ such that

$$0 \in \mathrm{co}\left\{ \bigcup_{l \in [1,L]} \partial^\diamond f_l(\hat{x}) \right\} + \sum_{i \in M(\hat{x})} \alpha_i \partial^\diamond u_i(\hat{x}) + \sum_{j \in [1,n]} \beta_j \partial^\diamond v_j(\hat{x}) - \sum_{j \in [1,n]} \gamma_j \partial^\diamond v_j(\hat{x}).$$

Proof Following a similar idea to the proof of Theorem 3.1 in [18], we can prove Theorem 2.1. From assumptions, Lemma 2.2 implies that, in the case of condition (i),

$$F^\diamond(\hat{x}, d) \not< 0, \quad \forall d \in T_K(\hat{x}),$$

and in the case of condition (ii),

$$F^\diamond(\hat{x}, d) \not< 0, \quad \forall d \in \mathrm{cl} D_K(\hat{x}).$$

Since the nondifferentiable Abadie CQ (or the generalized Zangwill CQ) holds at $\hat{x}$, from Lemma 2.4 we obtain that $F^\diamond(\hat{x}, d) \not< 0$ for all d solving the following system:

$$\text{(S)} \qquad \begin{aligned} u_i^\diamond(\hat{x}, d) &\leq 0, \quad \forall i \in M(\hat{x}),\\ v_j^\diamond(\hat{x}, d) &\leq 0, \quad \forall j \in [1, n],\\ (-v_j)^\diamond(\hat{x}, d) &\leq 0, \quad \forall j \in [1, n]. \end{aligned}$$

Since u_i $(i \in M(\hat{x}))$ and v_j $(j \in [1, n])$ are either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, (ii) and (iv) of Lemma 2.1 imply that d solves the system (S) if and only if $\max_{a \in A}\langle a, d\rangle \leq 0$, where A is the set defined by (1). Denote by $\mathcal{A} = \mathrm{cone\,co} A$, $\mathcal{A}^0 = \{\xi^* \in X^* : \langle \xi^*, a\rangle \leq 0, \forall a \in \mathcal{A}\}$, and

$$\delta_{\mathcal{A}^0}(\xi^*) = \begin{cases} 0, & \xi^* \in \mathcal{A}^0,\\ +\infty, & \text{else}, \end{cases}$$

the convex cone generated by A, the polar cone of $\mathcal{A}$, and the indicator function of $\mathcal{A}^0$, respectively. It follows that

$$F^\diamond(\hat{x}, d) = (f_1^\diamond(\hat{x}, d), \dots, f_L^\diamond(\hat{x}, d))^T \not< 0, \quad \text{whenever } \max_{a \in \mathcal{A}}\langle a, d\rangle \leq 0. \tag{2}$$

Set $h(\cdot) = \max_{1 \leq l \leq L} h_l(\cdot)$ and $h_l(\cdot) = f_l^\diamond(\hat{x}, \cdot) (l \in [1, L])$. Then, the inequality (2) implies that

$$h(d) \geq 0 \quad \text{whenever } \max_{a \in \mathcal{A}}\langle a, d\rangle \leq 0. \tag{3}$$

Since $0 \in \mathcal{A}^0$ and $h_l(0) = f_l^\diamond(\hat{x}, 0) = 0$ for each $l \in [1, L]$, one has $h(0) = \max_{1 \leq l \leq L} f_l^\diamond(\hat{x}, 0) = 0$. Now Inequality (3) implies that the function $h(\cdot) + \delta_{\mathcal{A}^0}(\cdot)$ has its minimum at 0. From (i) of Lemma 2.1, $f_l^\diamond(\hat{x}, \cdot)$ $(l \in [1, L])$ is continuous and convex and, as a consequence, $h(\cdot)$ is continuous and convex. According to the sum rule (see e.g. [10]), one has

$$0 \in \partial h(0) + \partial \delta_{\mathcal{A}^0}(0). \tag{4}$$

It is easy to prove that $\partial h_l(0) = \partial^\diamond f_l(\hat{x})$ $(l \in [1, L])$. Since all the conditions in Lemma 2.3 hold, we obtain

$$\partial h(0) = \mathrm{co}\left\{\bigcup_{l=1}^{L} \partial h_l(0)\right\} = \mathrm{co}\left\{\bigcup_{l=1}^{L} \partial^\diamond f_l(\hat{x})\right\}. \tag{5}$$

Since $\partial\delta_{\mathcal{A}^0}(0)=\mathcal{A}^{00}=\mathcal{A}$, both (4) and (5) imply that

$$0\in \mathrm{co}\left\{\bigcup_{l=1}^{L}\partial^\diamond f_l(\hat{x})\right\}+\mathcal{A}.$$

From (ii) of Lemma 2.1, we have $\partial^\diamond(-v_j)(\hat{x})=-\partial^\diamond v_j(\hat{x})$. Therefore, there exist $t_\mu\ge 0$ ($\mu\in[1,k]$), with $\sum_{\mu=1}^{k}t_\mu=1$, $\xi_\mu^*\in\bigcup_{l=1}^{L}\partial^\diamond f_l(\hat{x})$ ($\mu\in[1,k]$), $\lambda_i^*\in\partial^\diamond u_i(\hat{x})$ ($i\in M(\hat{x})$), $\mu_j^*,\eta_j^*\in\partial^\diamond v_j(\hat{x})$ ($j\in[1,n]$), $\alpha_i\ge 0$ ($i\in M(\hat{x})$), $\beta_j\ge 0$ ($j\in[1,n]$) and $\gamma_j\ge 0$ ($j\in[1,n]$) such that

$$0=\sum_{\mu\in[1,k]}t_\mu\xi_\mu^*+\sum_{i\in M(\hat{x})}\alpha_i\lambda_i^*+\sum_{j\in[1,n]}\beta_j\mu_j^*-\sum_{j\in[1,n]}\gamma_j\eta_j^*;$$

thus, the proof is complete. □

If (VOP) reduces to (MCLP), then we obtain the following conclusion.

Theorem 2.2 KKT Condition for (MCLP) in the Sense of Weakly Efficient Solution. *Let $K\subseteq R^I$, $\hat{x}\in S_w$ and $F=(c_1^T,c_2^T,\dots,c_L^T)^T:R^I\to R^L$. Suppose that one of the following conditions holds*:

(i) *the nondifferentiable Abadie CQ holds at $\hat{x}$*;
(ii) *the generalized Zangwill CQ holds at $\hat{x}$.*

Then, the KKT condition holds at $\hat{x}$, i.e., there exist $\alpha_i\ge 0$ ($i\in M(\hat{x})$), $\beta_j\ge 0$ ($j\in[1,n]$) and $\gamma_j\ge 0$ ($j\in[1,n]$) such that

$$0\in \mathrm{co}\{c_1^T,c_2^T,\dots,c_L^T\}+\sum_{i\in M(\hat{x})}\alpha_i\partial^\diamond u_i(\hat{x})+\sum_{j\in[1,n]}\beta_j\partial^\diamond v_j(\hat{x})-\sum_{j\in[1,n]}\gamma_j\partial^\diamond v_j(\hat{x}).$$

Proof Since c_l^T is linear, differentiable and Lipschitz continuous, and since $\partial c_l^T(\hat{x})=\{c_l^T\}$ for each $l\in[1,L]$, Theorem 2.1 implies that the conclusion holds. □

The KKT sufficient condition for a weakly efficient solution of (VOP) can be constructed as follows.

Theorem 2.3 KKT Sufficient Condition for (VOP) in the Sense of Weakly Efficient Solution. *Let $\hat{x}\in K$. Suppose that f_l ($l\in[1,L]$), u_i ($i\in[1,m]$), and v_j ($j\in[1,n]$) are either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, and that there exist $\alpha_i\ge 0$ ($i\in M(\hat{x})$) and β_j ($j\in[1,n]$) such that*

$$0\in \mathrm{co}\left\{\bigcup_{l\in[1,L]}\partial^\diamond f_l(\hat{x})\right\}+\sum_{i\in M(\hat{x})}\alpha_i\partial^\diamond u_i(\hat{x})+\sum_{j\in[1,n]}\beta_j\partial^\diamond v_j(\hat{x}). \tag{6}$$

Let $[1,n]_+=\{j\in[1,n]:\beta_j>0\}$ and $[1,n]_-=\{j\in[1,n]:\beta_j<0\}$. Suppose that $F=(f_1,\dots,f_L)^T$ is MP pseudoconvex at $\hat{x}$, u_i ($i\in M(\hat{x})$), v_j ($j\in[1,n]_+$) and $-v_j$ ($j\in[1,n]_-$) are MP regular and quasiconvex at $\hat{x}$. Then, $\hat{x}\in E_w$.

Proof By (ii) of Lemma 2.1, one has $\partial^\diamond(-v_j)(\hat{x}) = -\partial^\diamond v_j(\hat{x})$ ($j \in [1,n]$). We remark that the inclusion (6) holds if and only if

$$\begin{aligned}
&t_\mu \geq 0 \quad \text{with} \sum_{\mu=1}^{k} t_\mu = 1,\ \forall \mu \in [1,k],\\
&\xi_\mu^* \in \bigcup_{l=1}^{L} \partial^\diamond f_l(\hat{x}), \quad \forall \mu \in [1,k],\\
&\lambda_i^* \in u_i(\hat{x}), \quad \forall i \in M(\hat{x}),\\
&\mu_j^* \in \partial^\diamond v_j(\hat{x}), \quad \forall j \in [1,n]_+,\\
&\eta_j^* \in \partial^\diamond(-v_j)(\hat{x}), \quad \forall j \in [1,n]_-,
\end{aligned}$$

such that

$$0 = \sum_{\mu \in [1,k]} t_\mu \xi_\mu^* + \sum_{i \in M(\hat{x})} \alpha_i \lambda_i^* + \sum_{j \in [1,n]_+} \beta_j \mu_j^* - \sum_{j \in [1,n]_-} \beta_j \eta_j^*. \tag{7}$$

As the proof of Theorem 3.2 in [18], one has that, for any $x \in K$,

$$\left\langle \sum_{i \in M(\hat{x})} \alpha_i \lambda_i^* + \sum_{j \in [1,n]_+} \beta_j \mu_j^* - \sum_{j \in [1,n]_-} \beta_j \eta_j^*, x - \hat{x} \right\rangle \leq 0. \tag{8}$$

It follows from (7) and (8) that

$$\sum_{\mu \in [1,k]} t_\mu \langle \xi_\mu^*, x - \hat{x} \rangle = \left\langle \sum_{\mu \in [1,k]} t_\mu \xi_\mu^*, x - \hat{x} \right\rangle \geq 0,$$

which implies that there are $\mu_x \in [1,k]$ and $l_x \in [1,L]$ with $\xi_{\mu_x}^* \in \partial^\diamond f_{l_x}(\hat{x})$ such that

$$\langle \xi_{\mu_x}^*, x - \hat{x} \rangle \geq 0.$$

By (iv) of Lemma 2.1, it follows that

$$f_{l_x}^\diamond(\hat{x}, x - \hat{x}) \geq \langle \xi_{\mu_x}^*, x - \hat{x} \rangle \geq 0$$

and so

$$F^\diamond(\hat{x}, x - \hat{x}) = (f_1^\diamond(\hat{x}, x - \hat{x}), \ldots, f_{l_x}^\diamond(\hat{x}, x - \hat{x}), \ldots, f_L^\diamond(\hat{x}, x - \hat{x}))^T \nless 0. \tag{9}$$

Since F is MP pseudoconvex at $\hat{x}$, it follows from (9) that $F(x) \nless F(\hat{x})$, i.e., $\hat{x} \in E_w$. This completes the proof. □

If $F \equiv f : X \to R$, then Theorem 2.3 reduces to Theorem 3.2 in [18].

Corollary 2.1 KKT Sufficient Condition for (COP) [18]. *Let $\hat{x} \in K$. Suppose that $F \equiv f : X \to R$, $u_i (i \in [1,m])$ and $v_j (j \in [1,n])$ are either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$ and there exist $\alpha_i \geq 0 (i \in M(\hat{x}))$ and $\beta_j (j \in [1,n])$ such that*

$$0 \in \partial^{\diamond} f(\hat{x}) + \sum_{i \in M(\hat{x})} \alpha_i \partial^{\diamond} u_i(\hat{x}) + \sum_{j \in [1,n]} \beta_j \partial^{\diamond} v_j(\hat{x}).$$

Let $[1,n]_+ = \{j \in [1,n] : \beta_j > 0\}$ and $[1,n]_- = \{j \in [1,n] : \beta_j < 0\}$. Suppose that f is MP pseudoconvex at $\hat{x}$, u_i $(i \in M(\hat{x}))$, v_j $(j \in [1,n]_+)$ and $-v_j$ $(j \in [1,n]_-)$ are MP regular and quasiconvex at $\hat{x}$. Then, $\hat{x}$ be a global minimum of f on K.

If (VOP) reduces to (MCLP), then we obtain the following conclusion.

Theorem 2.4 KKT Sufficient Condition for (MCLP) in the Sense of Weakly Efficient Solution. *Let $K \subseteq R^I$, $\hat{x} \in K$ and $F = (c_1^T, c_2^T, \ldots, c_L^T)^T : R^I \to R^L$. Suppose that u_i $(i \in [1,m])$ and v_j $(j \in [1,n])$ are either Gãteaux differentiable at $\hat{x}$ or Lipschitz near $\hat{x}$, and there exist $\alpha_i \geq 0$ $(i \in M(\hat{x}))$ and β_j $(j \in [1,n])$ such that*

$$0 \in \mathrm{co}\{c_1^T, c_2^T, \ldots, c_L^T\} + \sum_{i \in M(\hat{x})} \alpha_i \partial^{\diamond} u_i(\hat{x}) + \sum_{j \in [1,n]} \beta_j \partial^{\diamond} v_j(\hat{x}).$$

Let $[1,n]_+ = \{j \in [1,n] : \beta_j > 0\}$ and $[1,n]_- = \{j \in [1,n] : \beta_j < 0\}$. Suppose that u_i $(i \in M(\hat{x}))$, v_j $(j \in [1,n]_+)$ and $-v_j$ $(j \in [1,n]_-)$ are MP regular and quasiconvex at $\hat{x}$. Then, $\hat{x} \in S_w$.

Proof Since c_l^T is linear, differentiable and Lipschitz continuous, $\partial c_l^T(\hat{x}) = \{c_l^T\}$ for each $l = 1, 2, \ldots, L$; from Remark 2.2, $F = (c_1^T, c_2^T, \ldots, c_L^T)^T$ is MP pseudoconvex at $\hat{x}$. Thus Theorem 2.3 implies that the conclusion holds. □

3 Partial Calmness and Exact Penalization for (VOP)

Throughout this section, let $X = R^I$. The corresponding perturbed problem of (VOP) is given by

$$\begin{aligned} (\mathrm{VOP})_\epsilon \quad \min \quad & F(x), \\ \text{s.t.} \quad & u(x) \leq 0, \\ & v(x) = \epsilon, \end{aligned}$$

where $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T \in R^n$. We denote by $K_w^\epsilon = \{x \in R^I : u(x) \leq 0, v(x) = \epsilon\}$ the feasible set of $(\mathrm{VOP})_\epsilon$. If u_i $(i \in [1,m])$ is lower semicontinuous and v_j $(j \in [1,n])$ is continuous, then K_w^ϵ is closed. But K_w^ϵ is not convex in general.

In this section, we introduce the notions of partial calmness and the penalized problem for (VOP) with finite quality and inequality constraints; then, we establish the relation between the weakly efficient solution of (VOP) and the local minimum solution of the penalized problem of (VOP) under the assumption of partial calmness.

We now introduce the following notion of partial calmness for (VOP).

Partial Calmness for (VOP) Let $\hat{x} \in E_w$. (VOP) is said to be partially calm at $\hat{x}$ if there exist $\mu > 0, \delta > 0$ such that, for all $\epsilon \in B_{R^n}(0, \delta)$ and all $x \in B_{R^I}(\hat{x}, \delta) \cap K_w^\epsilon$, one has

$$h(x) - h(\hat{x}) + \mu \|v(x)\| \geq 0,$$

where $B_Z(z, t)$ is the open ball in Z with center z and radius t, $h(x) = \sum_{l \in [1,L]} f_l(x)$, $\forall x \in R^I$. The constants μ and δ are called the modulus and radius, respectively.
Remark 3.1 If $F \equiv f : R^I \to R$, then the partial calmness of (VOP) defined above reduces to the partial calmness of (COP) introduced by Ye and Zhu [24].

Define the penalized problem of (VOP) as follows:

$$\begin{aligned} (\text{VOP})(\mu) \quad \min \quad & h(x) + \mu \|v(x)\|, \\ \text{s.t.} \quad & u(x) \leq 0, \end{aligned}$$

where $\mu > 0$ is a constant, $h(x) = \sum_{l \in [1,L]} f_l(x)$, $\forall x \in R^I$.

It is well known that the notion of partial calmness is similar to, but different from, that of calmness introduced by Clarke [10] and Rockafellar [23]. In the definition of partial calmness, the restriction on the size of $\epsilon \in B_{R^n}(0, \delta)$ can be removed when function v is continuous.

Theorem 3.1 *Let the function v be continuous. If (VOP) is partially calm at $\hat{x}$ with modulus μ and radius δ, then there exists a $\hat{\delta} < \delta$ such that $\hat{x}$ is a $\hat{\delta}$-local minimum solution to (VOP)(μ), i.e.*,

$$h(x) - h(\hat{x}) + \mu \|v(x)\| \geq 0, \quad \forall x \in R^I \quad s.t. \quad u(x) \leq 0, \quad x \in B_{R^I}(\hat{x}, \hat{\delta}).$$

Proof We remark that $v(\hat{x}) = 0$, since $\hat{x} \in E_w$. From the continuity of v and the definition of partial calmness of (VOP), the conclusion is derived. This completes the proof. □

In [19], Ye, Zhu and Zhu proved that the calmness is equivalent to local exact penalization for (COP). Similarly, we establish now the relation between the weakly efficient solution of (VOP) and the local minimum solution of (VOP)(μ) under the assumption of the partial calmness of (VOP).

Theorem 3.2 *Let $\hat{x} \in K$ and let f_l ($l \in [1, L]$) be continuous at $\hat{x}$. Then, the following conclusions hold*:

(i) *If $\hat{x} \in E_w$ and (VOP) is partially calm at $\hat{x}$, then there exists $\hat{\mu} > 0$ such that $\hat{x}$ is a local minimum solution of (VOP)(μ) for all $\mu \geq \hat{\mu}$.*
(ii) *If $\hat{x} \in E_w$ and $\hat{x}$ is a global minimum solution of h on K, then any local minimum solution x_μ of (VOP)(μ) with $\mu > \hat{\mu}$ with respect to the neighborhood of $\hat{x}$ in which $\hat{x}$ is a local minimum solution belong to E_w.*

Proof (i) Since f_l ($l \in [1, L]$) is continuous at $\hat{x}$, so is h. Suppose $\hat{x}$ is not a local minimum solution of (VOP)(μ) for any $\mu > 0$. Then, for each positive integer t, there exists a point $x_t \in B_{R^I}(\hat{x}, \frac{1}{t})$ and $u(x_t) \leq 0$ such that

$$h(x_t) + t\|v(x_t)\| < h(\hat{x}) + t\|v(\hat{x})\|.$$

Since $\hat{x} \in E_w$, $v(\hat{x}) = 0$, it follows that

$$h(x_t) + t\|v(x_t)\| < h(\hat{x}); \tag{10}$$

hence,

$$0 \leq \|v(x_t)\| < \frac{h(\hat{x}) - h(x_t)}{t} \leq \frac{1}{t},$$

for large enough t, since h is continuous at $\hat{x}$. Let $\epsilon(t) = v(x_t)$ for large enough t. Then, $x_t \in B_{R^I}(\hat{x}, \frac{1}{t}) \cap K_w^{\epsilon(t)}$. However, (10) contradicts the assumption that (VOP) is partially calm at $\hat{x}$. Thus, there exists $\hat{\mu} > 0$ such that $\hat{x}$ is a local minimum solution of (VOP)($\hat{\mu}$). It is clear that $\hat{x}$ is a local minimum solution of (VOP)(μ) whenever $\mu \geq \hat{\mu}$.

(ii) Let $\hat{x}$ is a global minimum solution of h on K. Let $\mu > \hat{\mu}$ and let x_μ be a local minimum solution of (VOP)(μ) in the neighborhood of $\hat{x}$ in which $\hat{x}$ is a local minimum solution. Then, $u(x_\mu) \leq 0$. Since $\hat{x} \in E_w$, $v(\hat{x}) = 0$ and

$$\begin{aligned} h(x_\mu) + \mu\|v(x_\mu)\| &= h(\hat{x}) + \mu\|v(\hat{x})\| \\ &= h(\hat{x}) \\ &\leq h(x_\mu) + \frac{1}{2}(\mu + \hat{\mu})\|v(x_\mu)\|. \end{aligned} \tag{11}$$

It follows that

$$(\mu - \hat{\mu})\|v(x_\mu)\| \leq 0,$$

which implies that $v(x_\mu) = 0$. Thus, $x_\mu \in K$. By (11), we have $h(x_\mu) = h(\hat{x})$. We conclude that $x_\mu \in E_w$; i.e., for any $x \in K$, there is $l_x \in [1, L]$ such that $F(x) \not< F(x_\mu)$. Suppose to the contrary that there exists $x_0 \in K$ and for any $l \in [1, L]$, we have $F(x_0) < F(x_\mu)$, i.e.,

$$(f_1(x_0), f_2(x_0), \ldots, f_L(x_0))^T < (f_1(x_\mu), f_2(x_\mu), \ldots, f_L(x_\mu))^T.$$

It follows that

$$h(x_0) = \sum_{l \in [1,L]} f_l(x_0) < \sum_{l \in [1,L]} f_l(x_\mu) = h(x_\mu) = h(\hat{x}),$$

which contradicts the assumption that $\hat{x}$ is a global minimum solution of h on K. This completes the proof. $\square$

References

1. Aghezzaf, B., Hachimi, M.: Generalized invexity and duality in multiobjective programming problems. J. Glob. Optim. **18**, 91–101 (2000)
2. Chen, G.Y., Huang, X.X., Yang, X.Q.: Vector Optimization: Set-valued and Variational Analysis. Lecture Notes in Economics and Mathematical Systems, vol. 541. Springer, Berlin (2005)
3. Chinchuluun, A., Pardalos, P.M.: A survey of recent developments in multiobjective optimization. Ann. Oper. Res. **154**, 29–50 (2007)
4. Deng, S.: Characterizations of the nonemptiness and compactness of solution sets in convex vector optimization. J. Optim. Theory Appl. **96**, 123–131 (1998)
5. Flores-Bazãn, F.: Ideal, weakly efficient solutions for vector optimization problems. Math. Program. Ser. A **93**, 453–475 (2002)
6. Liang, Z.A., Huang, H.X., Pardalos, P.M.: Efficiency conditions and duality for a class of multiobjective fractional programming problems. J. Glob. Optim. **27**, 447–471 (2003)
7. Luc, D.T.: Theory of Vector Optimization. Springer, Berlin (1989)
8. Yuan, D.H., Chinchuluun, A., Liu, X.L., Pardalos, P.M.: Optimality conditions and duality for multiobjective programming involving $(C; \alpha; \rho; d)$-type I functions. In: Konnov, I.V., Luc, D.T., Rubinov, A.M. (eds.) Generalized Convexity and Related Topics. Lecture Notes in Economics and Mathematical Systems, vol. 583, pp. 73–87. Springer, Berlin (2007)
9. Deng, S., Yang, X.Q.: Weak sharp minima in multicriteria linear programming. SIAM J. Optim. **15**, 456–460 (2004)
10. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley-Interscience, New York (1983)
11. Halkin, H.: Implicit functions and optimization problems without continuous differentiability of the data. SIAM J. Control **12**, 229–236 (1974)
12. Ioffe, A.D.: Necessary conditions in nonsmooth optimization. Math. Oper. Res. **9**, 159–189 (1984)
13. Mangasarian, O.L.: Nonlinear Programming. McGraw-Hill, New York (1969). Reprinted as Classics. Applied Mathematics, vol. 10. SIAM, Philadelphia (1994)
14. Mordukhovich, B.S.: On necessary conditions for an extremum in nonsmooth optimization. Sov. Math. Dokl. **283**, 215–220 (1985)
15. Michel, P., Penot, J.-P.: Calcus sous-différentiel pour des fonctions Lipschitziennes et non Lipschitziennes. C. R. Acad. Sci. Paris Ser. I Math. **12**, 269–272 (1984)
16. Michel, P., Penot, J.-P.: A generalized derivative for calm and stable functions. Diff. Integral Equ. **5**, 433–454 (1992)
17. Treiman, J.S.: Lagrange multipliers for nonconvex generalized gradients with equality, inequality, and set constraints. SIAM J. Control Optim. **37**, 1313–1329 (1999)
18. Ye, J.J.: Nondifferentiable multiplier rules for optimization and bilevel optimization problems. SIAM J. Optim. **15**, 252–274 (2004)
19. Ye, J.J., Zhu, D.L., Zhu, Q.J.: Exact penalization and necessary optimality conditions for generalized bilevel programming problems. SIAM J. Optim. **7**, 481–507 (1997)
20. Rockafellar, R.T.: Proximal subgradient, marginal functions, and augmented Lagrangians in nonsmooth optimization. Math. Oper. Res. **6**, 427–437 (1981)
21. Mordukhovich, B.S.: Approximation Methods in Problems of Optimization and Control. Nauka, Moscow (1988). English translation, Wiley/Interscience
22. Birge, J.R., Qi, L.: Semiregularity and generalized subdifferentials with applications to optimization. Math. Oper. Res. **18**, 982–1005 (1993)
23. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)
24. Ye, J.J., Zhu, D.L.: Optimality conditions for bilevel programming problems. Optimization **33**, 9–27 (1995)