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Matrix maps over planar near-rings

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Following a method by Meldrum and van der Walt, near-rings of matrix maps are defined for general near-rings, not necessarily with identity. The influence of one-sided identities is discussed. When the base near-ring is integral and planar, the near-ring of matrix maps is shown to be simple. Various types of primitivity of the near-ring of matrix maps are discussed when the base near-ring is planar but not integral. Finally, an open problem concerning bijective matrix maps is solved.

1. Introduction

For an additive group $(G, +)$, not necessarily abelian, the set $M(G)$ of all functions $f : G \to G$ under pointwise addition and function composition determines a structure $(M(G), +, \circ)$ that satisfies all the ring axioms, except perhaps that addition is commutative and that multiplication is left distributive over addition. Abstractly, an algebraic structure $(R, +, \cdot)$ is called a (right) *near-ring* if:

(1) $(R, +)$ is a (not necessarily abelian) group;

(2) $(R, \cdot)$ is a semigroup; and

(3) $(x + y)z = xz + yz$ for all $x, y, z \in R$.

Every near-ring can be embedded in an $M(G)$ for some suitable additive group G. For a comprehensive discussion on near-rings the reader is referred to [5, 11]. We will recall necessary notions along the way.

A natural equivalence relation exists in a near-ring R. Namely, for $a, b \in R$, $a \equiv_{\mathrm{m}} b$ if $xa = xb$ for all $x \in R$. In this case we say that a and b are *equal* *multipliers*. We say that R is *planar* if $|R/{\equiv_{\mathrm{m}}}| \geqslant 3$ and, for any $a, b, c \in R$ with $a \not\equiv_{\mathrm{m}} b$, there is a unique element $x \in R$ such that $xa = xb + c$. If a planar near-ring R is not a nearfield, then it has no (two-sided) identity. In this case it has many right identities. A planar near-ring R is *zero symmetric*, which means that $0x = x0 = 0$ for all $x \in R$. Given a planar near-ring R, the set of 0 *multipliers*, $\{r \in R \mid r \equiv_{\mathrm{m}} 0\}$, is of some importance. It is usually denoted by A.

Planarity has been proved to be a very good condition to pose on a near-ring. First of all, planar near-rings have rather simple ideal structures compared with general near-rings. Applications of planar near-rings to geometry, combinatorics, coding theory and cryptography have been developed (see [1] for more details).

In this paper we shall study the near-ring of 'matrices' over planar near-rings.

With square matrices having entries taken from a ring, one obtains a ring of matrices under the usual operations of matrix addition and multiplication. With square matrices having entries taken from a near-ring, however, under the same operations one obtains a near-ring of matrices only when the given near-ring is distributive, i.e. the near-ring satisfies both distributive laws. Moreover, the resulting near-ring of matrices is also distributive [3].

In [6], Meldrum and van der Walt used a strategy of considering matrices as mappings (rather than square arrays of elements from some near-ring) in order to define the notion of a matrix near-ring. Certain elementary maps were used to generate these matrix near-rings. These elementary maps imitate the well-known elementary matrices

$$rE_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & r & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

where r (from a ring R) occupies the (i, j)th entry of a square $n \times n$ array, and the other entries are 0. The idea in [6] was to consider the elementary matrices rE_{ij} as maps $f_{ij}^r : R^n \to R^n; f_{ij}^r v = \iota_i(r\pi_j v)$, where, in this case, R^n denotes the direct sum of n copies of the additive group of a near-ring R with identity and ι_i and π_j denote the usual ith coordinate injection function and the jth coordinate projection function, respectively. The $n \times n$ matrix near-ring over R, denoted $\mathcal{M}_n(R)$, is then defined to be the subnear-ring of the near-ring $M(R^n)$, generated by all the f_{ij}^r. A substantial amount of research has been done on the structure $\mathcal{M}_n(R)$ since its origin in 1986. See [9] for a general account on the development of matrix near-rings and related near-rings.

As we have in mind to study matrix near-rings over planar near-rings, we do not require that R has an identity in the following.

Definition 1.1. Let R be a right near-ring, not necessarily with identity. For a positive integer n, the *near-ring of $n \times n$ matrix maps over R*, denoted $\mathrm{Mat}_n(R)$, is defined to be the subnear-ring of $M(R^n)$ generated by the mappings $f_{ij}^r : R^n \to R^n$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, and $r \in R$, where each f_{ij}^r is defined as in our discussion above.

Note that if R happens to possess an identity element, then $\mathrm{Mat}_n(R) = \mathcal{M}_n(R)$, the $n \times n$ matrix near-ring over R, as defined in [6].

Remark 1.2. The matrix near-ring $\mathcal{M}_n(R)$ over a near-ring R without identity can also be defined. It may happen that for two different elements $r, s \in R$, the elementary matrix maps f_{ij}^r and f_{ij}^s are the same mapping on R^n, while the $n \times n$ elementary matrices having r and s, respectively, as the (i, j) entries and 0 elsewhere are different matrices. Therefore, special care should be taken to make an appropriate definition of $\mathcal{M}_n(R)$ in this case. Interested readers are referred to [6] for more detail on this issue.

We will need the following lemma.

Lemma 1.3. *If R is zero symmetric, so is* $\mathrm{Mat}_n(R)$.

Proof. This follows in exactly the same way as the proof where R is assumed to have an identity [6]. □

Now, if 1_l and $1_l'$ are left identities of R, then $f_{11}^{1_l} + f_{22}^{1_l'}$ is a (two-sided) identity of $\mathrm{Mat}_2(R)$. This follows immediately since we clearly have $(f_{11}^{1_l} + f_{22}^{1_l'})\langle x, y\rangle = \langle 1_l x, 1_l' y\rangle = \langle x, y\rangle$ for all $\langle x, y\rangle \in R^2$. The following theorem shows that the converse is also true.

Theorem 1.4. $\mathrm{Mat}_n(R)$ *has a two-sided identity element if and only if R has a left identity element.*

Proof. For simplicity we assume that $n = 2$. The general case follows in a similar way.

Suppose that $I \in \mathrm{Mat}_2(R)$ is an identity. Then $I = U + V$, where

$$U = \sum_i f_{11}^{r_i} A_i$$

for some $r_i \in R$ and $A_i \in \mathrm{Mat}_2(R)$, and

$$V = \sum_j f_{22}^{s_j} B_j$$

for some $s_j \in R$ and $B_j \in \mathrm{Mat}_2(R)$, and both sums are finite.

Each of the A_i and the B_j should be seen as an expression consisting of elementary matrix maps and opening and closing parentheses in appropriate positions. In [8], it was shown exactly how to determine those f_{ij}^r in these expressions that act 'first' on the components of vectors in R^2. For example, in $A = f_{11}^{r_1}(f_{11}^{r_2} + f_{12}^{r_3})$, the elementary maps $f_{11}^{r_2}$ and $f_{12}^{r_3}$ act first on x and y in $\langle x, y\rangle \in R^2$, and then the other elementary map $f_{11}^{r_1}$ comes into play: $A\langle x, y\rangle = \langle r_1(r_2 x + r_3 y), 0\rangle$. The positions of these elementary maps in the expression A that act first are denoted by the set $\mathcal{N}_A$. See [8] for a detailed discussion about this.

Using the fact that U is a first-row matrix, i.e. it satisfies $\iota_1\pi_1 U = U$, we have $U\langle a, b\rangle = \langle a, 0\rangle = U\langle a, 0\rangle$ for all $a, b \in R$. If we replace each occurrence in A_i of f_{k2}^r

positioned by $\mathcal{N}_{A_i}$ by $f_{k2}^{r:0}$ and denote the new expression by A_i', we would have

$$\left(\sum_i f_{11}^{r_i} A_i'\right)\langle a, 0\rangle = \left(\sum_i f_{11}^{r_i} A_i\right)\langle a, 0\rangle = \langle a, 0\rangle \quad \text{for all } a \in R.$$

But

$$\left(\sum_i f_{11}^{r_i} A_i'\right)\langle a, 0\rangle = \left\langle \sum_i r_i w_i, 0\right\rangle,$$

where each w_i is either $\zeta_{1,i} = s_i a + t_i 0$ for some $s_i, t_i \in R$, or a finite sum

$$\zeta_{2,i} = \sum_j x_{2,j}\zeta_{1,j},$$

or a finite sum

$$\zeta_{3,i} = \sum_k x_{3,k}\zeta_{2,k},$$

etc. Moreover, we observe that, for all $a \in R$, $s_i a + t_i 0 = s_i a + t_i 0a = (s_i + t_i 0)a$. Thus,

$$a = \sum_i r_i w_i = ea$$

for some $e \in R$ (independent of a), showing that e is a left identity of R. □

2. Near-rings of matrix maps over integral planar near-rings

Let R be a near-ring. An additive normal subgroup I of R is a *right ideal* if $IR \subseteq I$, and is a *left ideal* if $r(s+i) - rs \in I$ for all $r, s \in R$ and $i \in I$. We say that I is a (two-sided) *ideal* if I is both a right and a left ideal. The near-ring R is said to be *simple* if $\{0\}$ and R are the only ideals in R. Note that when R is zero symmetric and I a left ideal it holds that $RI \subseteq I$.

First, we consider zero-symmetric near-rings R such that R has a right identity 1_r, and for each $a \in R$ there exists an $\ell_a \in R$ such that $\ell_a a = a$. The main goal is to show that if such an R is simple, then $\mathrm{Mat}_n(R)$ is simple. This is known to be true in the case of near-rings with identity [6, proposition 4.9].

We start with a few lemmas. Let $\mathcal{A}$ be a two-sided ideal of $\mathrm{Mat}_n(R)$, and denote the subset $\{\pi_j(Av) \mid 1 \leqslant j \leqslant n,\ A \in \mathcal{A},\ v \in R^n\}$ of R by $\mathcal{A}_*$.

Lemma 2.1. *For $a \in R$, we have that $a \in \mathcal{A}_*$ if and only if $f_{11}^a \in \mathcal{A}$.*

Proof. Let $a \in \mathcal{A}_*$. Then $a = \pi_j(Av)$ for some $1 \leqslant j \leqslant n$, $A \in \mathcal{A}$ and $v \in R^n$. We may assume that $j = 1$ since $f_{1j}^{\ell_a} A \in \mathcal{A}$ by lemma 1.3. Let $v = \langle a_1, a_2, \ldots, a_r\rangle$, $a_1 = a$. Then $f_{11}^{\ell_{a_1}} Av = \langle a, 0, \ldots, 0\rangle$, and so

$$f_{11}^{\ell_{a_1}} A(f_{11}^{a_1} + f_{21}^{a_2} + \cdots + f_{n1}^{a_n})\langle 1_r, 0, \ldots, 0\rangle = \langle a, 0, \ldots, 0\rangle,$$

where

$$f_{11}^{\ell_{a_1}} A(f_{11}^{a_1} + f_{21}^{a_2} + \cdots + f_{n1}^{a_n}) = f_{11}^x \in \mathcal{A} \quad \text{for some } x \in R.$$

But $f_{11}^x\langle 1_r, 0, \ldots, 0\rangle = \langle a, 0, \ldots, 0\rangle$, which implies that $x = a$, and so $f_{11}^a \in \mathcal{A}$.

Conversely, if $f_{11}^a \in \mathcal{A}$, then $f_{11}^a\langle 1_r, 0, \ldots, 0\rangle = \langle a, 0, \ldots, 0\rangle$. Hence, $a \in \mathcal{A}_*$. □

LEMMA 2.2. *$\mathcal{A}_*$ is a two-sided ideal of R.*

Proof. If $a, b \in \mathcal{A}_*$, then $f_{11}^a, f_{11}^b \in \mathcal{A}$ by lemma 2.1. So $f_{11}^a - f_{11}^b = f_{11}^{a-b} \in \mathcal{A}$. This puts $a - b \in \mathcal{A}_*$. Now, for $r \in R$, $f_{11}^{ar} = f_{11}^a f_{11}^r \in \mathcal{A}$. Thus, $ar \in \mathcal{A}_*$. Also, for $r, s \in R$,

$$f_{11}^{r(a+s)-rs} = f_{11}^r(f_{11}^a + f_{11}^s) - f_{11}^r f_{11}^s \in \mathcal{A}.$$

This puts $r(a+s) - rs \in \mathcal{A}_*$. Finally, $f_{11}^{r+a-r} = f_{11}^r + f_{11}^a - f_{11}^r \in \mathcal{A}$, and so $r + a - r \in \mathcal{A}_*$. □

LEMMA 2.3. *Let I be a two-sided ideal of R. Then $I = (I^*)_*$, where I^* denotes the ideal $(I^n : R^n) = \{U \in \mathrm{Mat}_n(R) \mid U(R^n) \subseteq I^n\}$ of $\mathrm{Mat}_n(R)$.*

Proof. Let $a \in I$. Then $f_{11}^a \in I^*$ since $f_{11}^a\langle r_1, \ldots, r_n\rangle = \langle ar_1, 0, \ldots, 0\rangle$ for any $\langle r_1, \ldots, r_n\rangle \in R^n$, and $ar_1 \in I$. Thus, $a \in (I^*)_*$ by lemma 2.1.

Conversely, let $a \in (I^*)_*$. Then $f_{11}^a \in I^*$ by lemma 2.1. Since $f_{11}^a\langle 1_r, 0, \ldots, 0\rangle = \langle a, 0, \ldots, 0\rangle \in I^n$, we have $a \in I$, and the result follows. □

This brings us to one of the main results of this section.

THEOREM 2.4. *Let R be a zero-symmetric near-ring with a right identity 1_r, and for each $a \in R$ there exists an $\ell_a \in R$ such that $\ell_a a = a$. Then R is simple if and only if $\mathrm{Mat}_n(R)$ is simple.*

Proof. Assume that R is simple and let $\mathcal{A}$ be a non-zero ideal of $\mathrm{Mat}_n(R)$. Take a non-zero element $A \in \mathcal{A}$. Then for some $v \in R^n$, $Av = \langle a_1, a_2, \ldots, a_n\rangle$ with, say, $a_1 \neq 0$. Thus, $a_1 \in \mathcal{A}_*$. Since $\mathcal{A}_*$ is an ideal of R by lemma 2.2, we have $\mathcal{A}_* = R$. Hence, $f_{11}^r \in \mathcal{A}$ for all $r \in R$ by lemma 2.1, and so

$$f_{ij}^r = f_{i1}^{\ell_r} f_{11}^r f_{1j}^{1_r} = f_{i1}^{\ell_r}(f_{11}^r f_{1j}^{1_r} + 0) - f_{i1}^{\ell_r} \cdot 0 \in \mathcal{A} \quad \text{for all } r \in R \quad \text{and} \quad 1 \leqslant i, j \leqslant n.$$

Consequently, $\mathcal{A} = \mathrm{Mat}_n(R)$. Therefore, $\mathrm{Mat}_n(R)$ is simple.

Conversely, suppose that $\mathrm{Mat}_n(R)$ is simple. Let I be a non-zero ideal of R and let a be a non-zero element of I. Then $f_{11}^a \neq 0$ since $f_{11}^a\langle 1_r, 0, \ldots, 0\rangle = \langle a, 0, \ldots, 0\rangle$. Thus, f_{11}^a is a non-zero element in I^*. As a non-zero ideal of $\mathrm{Mat}_n(R)$, $I^* = \mathrm{Mat}_n(R)$. Since $f_{11}^r \in I^*$ for all $r \in R$ by lemma 2.1, we conclude that $R \subseteq (I^*)_* = I$ by lemma 2.3. Therefore, $I = R$, and R is simple. □

If R is an integral planar near-ring (so that, for $a, b \in R$, $ab = 0$ if and only if $a = 0$ or $b = 0$), then it satisfies all the required conditions for this section. Therefore, we have the following corollary.

COROLLARY 2.5. *Let R be an integral planar near-ring. Then $\mathrm{Mat}_n(R)$ is simple.*

Note that when R is an integral planar near-ring, R is simple. Later we will show that if R is a finite simple planar near-ring, then $\mathrm{Mat}_n(R)$ is simple (see corollary 3.6).

Actually, corollary 2.5 is true for a much wider class of near-rings. We say that a near-ring R is *regular* if, for all $r \in R$, there exists $x \in R$ such that $rxr = r$. For example, an integral planar near-ring is regular [11, examples 9.154]. We introduce further terminology before we continue.

A group Γ is said to be an *R-group* if there is a function from $R \times \Gamma$ to Γ sending $(r, \gamma) \in R \times \Gamma$ to $r\gamma \in \Gamma$ such that, for all $\gamma \in \Gamma$ and $r, r' \in R$, $(r + r')\gamma = r\gamma + r'\gamma$ and $(rr')\gamma = r(r'\gamma)$. The additive group $(R, +)$ is naturally an R-group induced by the near-ring multiplication, and is usually denoted by ${}_RR$ when necessary. A subgroup Δ of an R-group Γ is said to be an *R-subgroup* of Γ if $R\Delta \subseteq \Delta$.

For any subsets S, T of Γ we set $(S : T) = \{r \in R \mid rT \subseteq S\}$. When S and/or T consists of just one element, we shall omit the brackets for sets. For example, $(0 : T)$, $(S : \gamma)$ or $(0 : \gamma)$ may be used. An R-group Γ is said to be *faithful* if $(0 : \Gamma) = \{0\}$. In this case R can be embedded into $M(\Gamma)$ (i.e. R can be viewed as a subnear-ring of $M(\Gamma)$).

Corollary 2.6. *Let R be a zero-symmetric regular near-ring with descending chain condition on the R-subgroups of ${}_RR$. Suppose that there is an $r \in R$ with $(0 : r) = \{0\}$. Then R is simple if and only if $\mathrm{Mat}_n(R)$ is simple.*

Proof. We notice that, for each $r \in R$, there is an $\ell_r \in R$ such that $\ell_r r = r$ since R is regular. Thus, we only have to show that R contains a right identity 1_r. Then the result would follow from theorem 2.4.

In case R is finite, R has a right identity by [11, remark 1.112]. The argument there could be carried over to when R is not finite but has the descending chain condition on R-subgroups of R [13, theorem 2.4]. □

As we have seen, the simplicity of a planar near-ring R carries over to $\mathrm{Mat}_n(R)$. It is not the case with planarity of R.

Proposition 2.7. *Let R be a planar near-ring. Then $\mathrm{Mat}_n(R)$ is not a planar near-ring if $n > 1$.*

Proof. Let 1_r be a right identity of R. From $f_{11}^{1_r} = f_{11}^{1_r} f_{11}^{1_r} \neq f_{11}^{1_r} f_{21}^{1_r} = 0$, we know that $f_{11}^{1_r} \not\equiv_{\mathrm{m}} f_{21}^{1_r}$. Now 0 and $f_{11}^{1_r} + f_{12}^{1_r}$ are two distinct solutions to the equation $X f_{11}^{1_r} = X f_{21}^{1_r}$ since $(f_{11}^{1_r} + f_{12}^{1_r}) f_{11}^{1_r} = f_{11}^{1_r} = (f_{11}^{1_r} + f_{12}^{1_r}) f_{21}^{1_r}$ and $0 f_{11}^{1_r} = 0 = 0 f_{21}^{1_r}$. □

But $\mathrm{Mat}_n(R)$ still has a right identity.

Proposition 2.8. *Let R be a planar near-ring and let $r_1, \ldots, r_n \in R$ be right identities. Then $f_{11}^{r_1} + f_{22}^{r_2} + \cdots + f_{nn}^{r_n}$ is a right identity in $\mathrm{Mat}_n(R)$.*

Proof. Again, we assume that $n = 2$ for simplicity, and note that the general case follows in a similar manner. We shall prove the result by induction on the *weight* of the elements of $\mathrm{Mat}_2(R)$. The weight of a matrix map A is basically the minimum number of elementary matrix maps needed to construct A. See [8] for a more detailed account of the notion of weight.

First of all, for all $r \in R$ and $\langle x, y \rangle \in R^2$, we have

$$f_{11}^{r}(f_{11}^{r_1} + f_{22}^{r_2})\langle x, y \rangle = f_{11}^{r}\langle r_1 x, r_2 y \rangle = \langle r r_1 x, 0 \rangle = \langle rx, 0 \rangle = f_{11}^{r}\langle x, y \rangle.$$

Hence,

$$f_{11}^{r}(f_{11}^{r_1} + f_{22}^{r_2}) = f_{11}^{r}.$$

Similarly, we have

$$f_{ij}^{r}(f_{11}^{r_1}+f_{22}^{r_2})=f_{ij}^{r} \quad \text{for all } 1\leqslant i,j\leqslant 2.$$

Now, if $U,V\in \mathrm{Mat}_2(R)$ are such that $U(f_{11}^{r_1}+f_{22}^{r_2})=U$ and $V(f_{11}^{r_1}+f_{22}^{r_2})=V$, then

$$(U+V)(f_{11}^{r_1}+f_{22}^{r_2})=U(f_{11}^{r_1}+f_{22}^{r_2})+V(f_{11}^{r_1}+f_{22}^{r_2})=U+V$$

and

$$(UV)(f_{11}^{r_1}+f_{22}^{r_2})=U(V(f_{11}^{r_1}+f_{22}^{r_2}))=UV.$$

Hence, $f_{11}^{r_1}+f_{22}^{r_2}$ is a right identity as claimed. □

3. Primitivity and ideals of near-rings of matrix maps over planar near-rings

In this section, we will study how the primitivity conditions on a planar near-ring affect that of the near-rings of matrix maps. We will see that the near-ring of matrix maps $\mathrm{Mat}_n(R)$ would be primitive when R is primitive and planar. Note that R has no identity element. This gives us the possibility of constructing various 1- and 2-primitive near-rings without identity. Hence, these will be primitive near-rings that are not isomorphic to the well-known primitive centralizer near-rings [11, theorem 4.52].

A brief review of some definitions seems appropriate.

Let R be a zero-symmetric near-ring and let $\mathit{\Gamma}$ be an R-group. A normal subgroup Δ of $\mathit{\Gamma}$ is called an *ideal* of $\mathit{\Gamma}$ if $r(\gamma+\delta)-r\gamma$ for all $\gamma\in\mathit{\Gamma}$, $\delta\in\Delta$, $r\in R$. We say that $\mathit{\Gamma}$ is *simple* if 0 and $\mathit{\Gamma}$ are the only ideals in $\mathit{\Gamma}$. This is not to be confused with $\mathit{\Gamma}$ being *R-simple*, which means that $\mathit{\Gamma}$ has no R-subgroups other than $\{0\}$ and $\mathit{\Gamma}$ itself.

Next, $\mathit{\Gamma}$ is said to be *monogenic* if there is some $\gamma\in\mathit{\Gamma}$ such that $R\gamma=\mathit{\Gamma}$, and is said to be *strongly monogenic* if, for all $\gamma\in\mathit{\Gamma}$, either $R\gamma=\mathit{\Gamma}$ or $R\gamma=\{0\}$. When $\mathit{\Gamma}\neq\{0\}$ and is monogenic, it is of *type 0* if it is simple, of *type 1* if it is simple and strongly monogenic and of *type 2* if it is *R-simple*.

Let $i\in\{0,1,2\}$. The i-radical of R, denoted by $\mathcal{J}_i(R)$, is the intersection of all $(0:\mathit{\Gamma})$ of R-groups $\mathit{\Gamma}$ of type i. It is known that $\mathcal{J}_1(R)$ contains all nilpotent left ideals of R and $\mathcal{J}_2(R)$ contains all nilpotent R-subgroups of R [11, corollary 5.10]. To say that R is *i-primitive on the R-group $\mathit{\Gamma}$* means that $\mathit{\Gamma}$ is faithful and is of type i, and to say the R is *i-primitive* means that there exists some R-group $\mathit{\Gamma}$ such that R is i-primitive on $\mathit{\Gamma}$. Lastly, R is said to be *i-semisimple* if $\mathcal{J}_i(R)=\{0\}$ and *i-radical* if $\mathcal{J}_i(R)=R$.

We assume that R is a planar near-ring in the following discussions, and recall that A is the set of 0 multipliers.

Our first goal is to show that $\mathrm{Mat}_n(R)$ is 2-primitive if R is integral, and how it is related to the centralizer near-ring $\mathrm{M}_D(R^n)$, where D is the group of all $\mathrm{Mat}_n(R)$-automorphisms of $(R^n,+)$. Then we will show that the primitivity of $\mathrm{Mat}_n(R)$ follows from that of R. Finally, we discuss what happens when R is not primitive.

For $r\in R$, $r\not\equiv_{\mathrm{m}} 0$, define $\rho_r: R^n\to R^n$ by $\langle a_1,a_2,\ldots,a_n\rangle\mapsto\langle a_1r,a_2r,\ldots,a_nr\rangle$.

PROPOSITION 3.1. $\mathrm{Aut}_{\mathrm{Mat}_n(R)} R^n = \{\rho_r \mid r \in R, r \not\equiv_{\mathrm{m}} 0\}$.

Proof. Assume that $n = 2$ for simplicity.

First, let $r \in R$ with $r \not\equiv_{\mathrm{m}} 0$. For $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in R^2$, we have

$$\begin{aligned}\rho_r(\langle a_1, b_1\rangle + \langle a_2, b_2\rangle) &= \rho_r\langle a_1 + a_2, b_1 + b_2\rangle \\ &= \langle (a_1 + a_2)r, (b_1 + b_2)r\rangle \\ &= \rho_r(\langle a_1, b_1\rangle) + \rho_r(\langle a_2, b_2\rangle).\end{aligned}$$

If $U \in \mathrm{Mat}_2(R)$ and $U\langle a, b\rangle = \langle c, d\rangle$, then

$$\rho_r(U\langle a, b\rangle) = \langle cr, dr\rangle = U\langle ar, br\rangle = U(\rho_r(\langle a, b\rangle)).$$

Thus, ρ is a $\mathrm{Mat}_2(R)$-endomorphism of R^2.

Let $\rho_r(\langle a, b\rangle) = \langle ar, br\rangle = \langle 0, 0\rangle$. Thus, $ar = 0$ and $br = 0$. As $r \not\equiv_{\mathrm{m}} 0$, this is only possible when $a = b = 0$, since R is a planar near-ring and the right multiplication induced by r is an automorphism of $(R, +)$. This shows that ρ_r is injective.

Now, let $\langle c, d\rangle \in R^2$. To find $\langle x, y\rangle \in R^2$ such that $\rho_r(\langle x, y\rangle) = \langle c, d\rangle$ we need to solve the equations $xr = x0+c$ and $yr = y0+d$. Since $r \not\equiv_{\mathrm{m}} 0$ and R is a planar near-ring, these equations have (unique) solutions in R. Thus, ρ_r is surjective. Therefore, ρ_r is a $\mathrm{Mat}_2(R)$-automorphism of R^2.

Conversely, let $\varphi \in \mathrm{Aut}_{\mathrm{Mat}_n(R)} R^n$. Let $\varphi\langle 1_r, 0\rangle = \langle c, d\rangle$. Then

$$\langle c, d\rangle = \varphi\langle 1_r, 0\rangle = \varphi(f_{11}^{1_r}\langle 1_r, 0\rangle) = f_{11}^{1_r}\varphi\langle 1_r, 0\rangle = f_{11}^{1_r}\langle c, d\rangle = \langle c, 0\rangle.$$

Thus, $d = 0$. Since φ is a bijection, $\varphi\langle 1_r, 0\rangle \neq \langle 0, 0\rangle$, and so $c \neq 0$.

Now, for an arbitrary element $\langle x, y\rangle \in R^2$, set $U_{x,y} = f_{11}^x + f_{21}^y$. Then $U_{x,y}\langle 1_r, 0\rangle = \langle x, y\rangle$, and we have

$$\begin{aligned}\varphi\langle x, y\rangle &= \varphi(U_{x,y}\langle 1_r, 0\rangle) = U_{x,y}(\varphi\langle 1_r, 0\rangle) \\ &= U_{x,y}\langle c, d\rangle = (f_{11}^x + f_{21}^y)\langle c, d\rangle \\ &= \langle xc, yc\rangle = \rho_c\langle x, y\rangle.\end{aligned}$$

This shows that $\varphi = \rho_c$, and obviously, $c \not\equiv_{\mathrm{m}} 0$. □

LEMMA 3.2. *Let $x, y \in R$. Then $x \equiv_{\mathrm{m}} y$ if and only if $f_{ij}^x \equiv_{\mathrm{m}} f_{ij}^y$ for any i and j.*

Proof. As before, let $n = 2$ for simplicity. So we need to show that $Af_{ij}^x = Af_{ij}^y$ for all $A \in \mathrm{Mat}_2(R)$. We will proceed by induction on the weight of A.

Assume first that $x \equiv_{\mathrm{m}} y$ in R. Then $f_{kl}^r f_{ij}^x = f_{kl}^r f_{ij}^y$ for all k and l. Thus, $Uf_{ij}^x = Uf_{ij}^y$ for all $U \in \mathrm{Mat}_2(R)$ with weight 1.

If now $U, V \in \mathrm{Mat}_2(R)$ satisfy $Uf_{ij}^x = Uf_{ij}^y$ and $Vf_{ij}^x = Vf_{ij}^y$, then surely

$$(U + V)f_{ij}^x = (U + V)f_{ij}^y \quad \text{and} \quad (UV)f_{ij}^x = (UV)f_{ij}^y.$$

Thus, by induction, $f_{ij}^x \equiv_{\mathrm{m}} f_{ij}^y$.

Conversely, assume that $f_{ij}^x \equiv_{\mathrm{m}} f_{ij}^y$. Then for any $s \in R$, we have

$$sx = \pi_i f_{ij}^{sx}\langle 1_r, 1_r\rangle = \pi_i f_{ii}^s f_{ij}^x\langle 1_r, 1_r\rangle = \pi_i f_{ii}^s f_{ij}^y\langle 1_r, 1_r\rangle = \pi_i f_{ij}^{sy}\langle 1_r, 1_r\rangle = sy.$$

Since s is arbitrary, $x \equiv_{\mathrm{m}} y$ as required. □

We are now ready for the following theorem.

THEOREM 3.3. *Let R be a planar near-ring. Then R^n is a strongly monogenic* $\mathrm{Mat}_n(R)$*-group. If R is integral planar, then* $\mathrm{Mat}_n(R)$ *is 2-primitive.*

Proof. Since $\mathrm{Mat}_n(R)$ is a subnear-ring of $M_0(R^n)$, it acts on R^n faithfully.

Let $\langle a_1, a_2, \ldots, a_n\rangle$ and $\langle b_1, b_2, \ldots, b_n\rangle$ be two arbitrary elements of R^n. If $a_i \not\equiv_{\mathrm{m}} 0$ for some i, then there exist $x_1, \ldots, x_n \in R$ such that $x_j a_i = b_j$ for $j = 1, 2, \ldots, n$ (by the planarity of R). But then

$$(f_{1i}^{x_1} + f_{2i}^{x_2} + \cdots + + f_{ni}^{x_n})\langle a_1, a_2, \ldots, a_n\rangle = \langle x_1 a_i, x_2 a_i, \ldots, x_n a_i\rangle = \langle b_1, b_2, \ldots, b_n\rangle,$$

and so $\mathrm{Mat}_n(R)\langle a_1, a_2, \ldots, a_n\rangle = R^n$. On the other hand, if $a_i \equiv_{\mathrm{m}} 0$ for all i, then $f_{kl}^x\langle a_1, a_2, \ldots, a_n\rangle = \langle 0, 0, \ldots, 0\rangle$, i.e. $f_{kl}^x \in (0 : \langle a_1, \ldots, a_n\rangle)$, for all $x \in R$ and $k, l \in \{1, \ldots, n\}$. Therefore, $\mathrm{Mat}_n(R) \subseteq (0 : \langle a_1, \ldots, a_n\rangle) \subseteq \mathrm{Mat}_n(R)$, and we have

$$\mathrm{Mat}_n(R)\langle a_1, a_2, \ldots, a_n\rangle = \{0\}.$$

This shows that R^n is strongly monogenic.

Now, suppose that R is integral planar. Then $a \not\equiv_{\mathrm{m}} 0$ if and only if $a \neq 0$. Thus, from the above argument, we know that every non-zero element of R^n is a monogenic generator of R^n, and so R^n contains no non-trivial $\mathrm{Mat}_n(R)$-subgroup. This says that $\mathrm{Mat}_n(R)$ is 2-primitive. □

We have the following corollary as a direct consequence of theorem 3.3 and [5, theorem 3.35].

COROLLARY 3.4. *If R is a planar nearfield, then $D = \mathrm{Aut}_{\mathrm{Mat}_n(R)}(R^n)$ is fixed-point free, and either* $\mathrm{Mat}_n(R)$ *is a primitive ring on the faithful simple* $\mathrm{Mat}_n(R)$*-module R^n or* $\mathrm{Mat}_n(R)$ *is not a ring and is a dense subnear-ring of* $\mathrm{M}_D(R^n)$*. Here,* $\mathrm{M}_D(R^n)$ *denotes the centralizer near-ring*

$$\mathrm{M}_D(R^n) = \{f : R^n \to R^n \mid f \circ \delta = \delta \circ f \textit{ for all } \delta \in D\}.$$

We shall discuss further the primitivity of $\mathrm{Mat}_n(R)$ when R is not integral.

First of all, as a planar near-ring, if R is 0-primitive, then it is 1-primitive [12, theorem 2.5.2]. So we assume that R is 1-primitive; hence $\mathcal{J}_1(R) = \{0\}$. In this case, R is simple. Indeed, let U be a proper ideal of R. Then U is contained in A, the set of all zero multipliers, and so $U^2 = \{0\}$. This puts $U \subseteq \mathcal{J}_1(R) = \{0\}$, and so $U = \{0\}$. Therefore, R is a simple near-ring. It follows that R has no non-trivial left ideals, as the sum of all proper left ideals is a proper ideal in a planar near-ring [2]. This means that ${}_RR$ has no non-trivial R-ideals.

Next, we argue that $\mathrm{Mat}_n(R)$ is 1-primitive on R^n. Let S be a $\mathrm{Mat}_n(R)$-ideal of R^n with $S \neq R^n$. We want to show that $S = \{0\}$. For any $i \in \{1, 2, \ldots, n\}$, it is easy to see that the set $T_i = \{\pi_i(v) \mid v \in S\}$ is an R-ideal of ${}_RR$. Since R is simple, each T_i is either $\{0\}$ or R.

As $\mathrm{Mat}_n(R)$ is zero-symmetric, we have $\theta \cdot \langle 0, \ldots, 0\rangle = \langle 0, \ldots, 0\rangle$ for all $\theta \in \mathrm{Mat}_n(R)$. Therefore, $\mathrm{Mat}_n(R) \cdot S \subseteq S$. Since $\mathrm{Mat}_n(R)$ is strongly monogenic on R^n, we have $\mathrm{Mat}_n(R) \cdot S = \{0\}$. Now, for all $r \in R$, $\langle a_1, \ldots, a_n\rangle \in S$ and $i \in \{1, 2, \ldots, n\}$, it holds that $\iota_i(ra_i) = f_{ii}^r\langle a_1, \ldots, a_n\rangle = \langle 0, \ldots, 0\rangle$. Thus, $R \cdot T_i = \{0\}$ for all

$i \in \{1, 2, \ldots, n\}$. As an R-group, ${}_R R$ is strongly monogenic, and so T_i cannot be R. This puts $T_i = \{0\}$ for all $i \in \{1, 2, \ldots, n\}$. Thus, $S = \{0\}$ as desired.

Moreover, R^n is a faithful, strongly monogenic $\mathrm{Mat}_n(R)$-group. Thus, $\mathrm{Mat}_n(R)$ is 1-primitive. Hence, we have just shown the following theorem.

THEOREM 3.5. *Let R be a 1-primitive planar near-ring. Then $\mathrm{Mat}_n(R)$ is 1-primitive.*

COROLLARY 3.6. *Let R be a simple planar near-ring. Then $\mathrm{Mat}_n(R)$ is 1-primitive. Consequently, if R is finite, then $\mathrm{Mat}_n(R)$ is simple.*

Proof. As we have seen, R has no non-trivial left ideals since it is simple. By planarity, ${}_R R$ is a faithful, strongly monogenic R-group. The absence of non-trivial left ideals in R implies that ${}_R R$ is of type 1. Hence, R is a 1-primitive near-ring, and so $\mathrm{Mat}_n(R)$ is 1-primitive. In the case that R is finite, $\mathrm{Mat}_n(R)$ is simple by [11, theorem 4.46]. □

Suppose now that R is 2-primitive. This is equivalent to saying that the set of zero multipliers, $A = \{x \in R \mid x \equiv_{\mathrm{m}} 0\}$, contains no non-zero subgroup of R [12, theorem 2.5.4].

THEOREM 3.7. *Let R be a 2-primitive planar near-ring. Then, $\mathrm{Mat}_n(R)$ is a 2-primitive near-ring.*

Proof. Let $U \subseteq R^n$ be a proper $\mathrm{Mat}_n(R)$-subgroup of R^n and let $u = (u_1, \ldots, u_n) \in U$. Since R^n is a strongly monogenic $\mathrm{Mat}_n(R)$-group and U is a proper subgroup of R^n, we must have $u_i \in A$ for each $i \in \{1, \ldots, n\}$. Since $(U, +)$ is a subgroup of $(R^n, +)$ we must have $\langle u, + \rangle \subseteq (U, +)$, where $\langle u, + \rangle$ is the cyclic subgroup generated by u. Therefore, each coordinate of the vectors additively generated by u must be contained in A. In other words, for each $i \in \{1, \ldots, n\}$, the cyclic group $\langle u_i, + \rangle$ generated by the ith coordinate u_i of u must be contained in A. By the 2-primitivity of R, there is no non-zero subgroup contained in A. Thus, for each $i \in \{1, \ldots, n\}$, $u_i = 0$. Consequently, $U = \{0\}$. This shows that there are no proper $\mathrm{Mat}_n(R)$-subgroups in R^n. From the fact that R^n is a faithful, strongly monogenic $\mathrm{Mat}_n(R)$-group, we see that $\mathrm{Mat}_n(R)$ is 2-primitive. □

REMARK 3.8. When R is integral planar, R is 2-primitive, with ${}_R R$ being a faithful, simple, strongly monogenic R-group. Therefore, theorem 3.7 also infers that $\mathrm{Mat}_n(R)$ is 2-primitive (cf. theorem 3.3).

It may be of some interest to note a close connection between minimal left ideals of 2-primitive near-rings and planar near-rings. A *Ferrero pair* is a pair of groups (N, Φ) such that $\Phi \leqslant \mathrm{Aut}(N)$ is a fixed-point free automorphism group of N with more than one element, and each $\phi \in \Phi \setminus \{1\}$ has the property that $-1 + \phi$ is surjective. Note that the property being surjective is naturally fulfilled if N is finite, because ϕ is fixed-point free and so $-1 + \phi$ is always injective.

PROPOSITION 3.9 (Wendt [14, theorem 5.4]). *Let L be a minimal left ideal of a 2-primitive near-ring N. Let $\Phi = Aut_N L$. If (L, Φ) is a Ferrero pair, then L is a planar near-ring.*

Now, let R be an integral planar near-ring, let L be a minimal left ideal of $\mathrm{Mat}_n(R)$ and let $\Phi = \mathrm{Aut}_{\mathrm{Mat}_n(R)} L$. By the above proposition, if (L, Φ) is a Ferrero pair, then L, as a near-ring itself, is planar. We note that the assumption of (L, Φ) being a Ferrero pair is not a very strong one. By primitivity of $\mathrm{Mat}_n(R)$ we naturally have that Φ acts without fixed points on L (see [14, proposition 5.1 and lemma 5.2]). Thus, when L is finite, one only needs to be sure that Φ contains more than one element to have (L, Φ) a Ferrero pair and L a planar near-ring. We record this observation with the following theorem.

THEOREM 3.10. *Let R be an integral planar near-ring. Let L be a minimal left ideal of $\mathrm{Mat}_n(R)$ and $\Phi = \mathrm{Aut}_{\mathrm{Mat}_n(R)}(L)$. If (L, Φ) is a Ferrero pair, then L is a planar near-ring. When L is finite, and Φ contains more than one element, then (L, Φ) is a Ferrero pair.*

We next look at when R is 1-primitive but not 2-primitive. There is just one situation left for discussion, as the next general theorem shows.

PROPOSITION 3.11. *Suppose that R is a 1-primitive planar near-ring. Then R is either 2-primitive or 2-radical.*

Proof. Since R is 1-primitive, R is simple. Therefore, either $\mathcal{J}_2(R) = \{0\}$ or $\mathcal{J}_2(R) = R$. Now, every proper R-subgroup of ${}_RR$ is contained in A, and so is nilpotent and contained in $\mathcal{J}_2(R)$ by [11, corollary 5.45]. Suppose that R is not 2-radical. Then $\mathcal{J}_2(R) = \{0\}$, and so ${}_RR$ has no non-trivial proper R-subgroups. This means that ${}_RR$ is a faithful, strongly monogenic R-group of type 2, so that R is 2-primitive. □

As R is planar, $\mathrm{Mat}_n(R)$ is 1-primitive if R is 1-primitive, and is 2-primitive if R is 2-primitive according to theorems 3.5 and 3.7. For a finite planar near-ring R that is 1-primitive and 2-radical, we have the following theorem.

THEOREM 3.12. *Let R be a 1-primitive finite planar near-ring. If $\mathcal{J}_2(R) = R$, then $\mathcal{J}_2(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(R)$.*

Proof. By theorem 3.5, $\mathrm{Mat}_n(R)$ acts 1-primitively on R^n. Thus, by [11, theorem 4.46], $\mathrm{Mat}_n(R)$ is a simple near-ring. This shows that $\mathcal{J}_2(\mathrm{Mat}_n(R))$ is either $\{0\}$ or $\mathrm{Mat}_n(R)$. We have to show that $\mathcal{J}_2(\mathrm{Mat}_n(R)) = \{0\}$ is not the case.

Assume that $\mathcal{J}_2(\mathrm{Mat}_n(R)) = \{0\}$. Then $\mathrm{Mat}_n(R)$ is a direct sum of ideals, each of them being a 2-primitive near-ring (see [11, theorem 5.31]). The simplicity of $\mathrm{Mat}_n(R)$ now forces $\mathrm{Mat}_n(R)$ to be 2-primitive. Thus, there exists a $\mathrm{Mat}_n(R)$-group of type 2. Since $\mathrm{Mat}_n(R)$ is 1-primitive on R^n, it follows from [11, theorem 4.46] that R^n is itself an $\mathrm{Mat}_n(R)$-group of type 2.

Since R is planar, ${}_RR$ is a faithful R-group. Consequently, ${}_RR$ cannot be of type 2 under the assumption that $J_2(R) = R$. So, there exists a non-zero R-subgroup U in R. Also, U^n is a subgroup of R^n. As a consequence of planarity, $U \subseteq A$, and so $RU = \{0\}$. Therefore, for $i, j \in \{1, \dots, n\}$ and $r \in R$, we have $f_{ij}^r \in (0 : U^n)$. Consequently, $\mathrm{Mat}_n(R) \subseteq (0 : U^n) \subseteq \mathrm{Mat}_n(R)$, and so $\mathrm{Mat}_n(R)U^n = \{0\}$. This shows that U^n is a $\mathrm{Mat}_n(R)$-subgroup of R^n. Since $\mathrm{Mat}_n(R)$ is 2-primitive on R^n, we have $U^n = \{0\}$. It follows that $U = \{0\}$, and a contradiction is reached.

Therefore, we conclude that $\mathcal{J}_2(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(R)$, as desired. □

In general, R may not be primitive. Yet we have seen that $\mathrm{Mat}_n(R)$ is zero symmetric having R^n as a faithful, strongly monogenic $\mathrm{Mat}_n(R)$-group. We can still obtain some information about the ideal structure of $\mathrm{Mat}_n(R)$ if R is not simple.

THEOREM 3.13. *Let R be a planar near-ring and let*

$$I = \mathcal{J}_1(\mathrm{Mat}_n(R)).$$

Then $\mathrm{Mat}_n(R)/I$ is a 1-primitive near-ring and $I^2 = \{0\}$.

The theorem will follow from a more general result. Let N be a near-ring and Γ be an N-group. A result of [4, lemma 2.1] says that if Γ is a strongly monogenic N-group and N is zero symmetric, then Γ contains a greatest proper N-ideal. In this case, we denote by Δ this greatest proper N-ideal of Γ. Note that Γ/Δ is again an N-group by defining $n(g+\Delta) = ng + \Delta$ for all $n \in N$ and $g \in \Gamma$. Now, if N is strongly monogenic, then, for any $g \in \Gamma$, either $Ng = \Gamma$ or $Ng = \{0\}$. Thus, Γ/Δ is also strongly monogenic. As Δ is the greatest proper N-ideal of Γ, it makes Γ/Δ a simple N-group. Namely, Γ/Δ is an N-group of type 1.

PROPOSITION 3.14. *Let N be a zero-symmetric near-ring that has a faithful strongly monogenic N-group Γ, and let $I = \mathcal{J}_1(N)$. Then N/I is a 1-primitive near-ring and $I^2 = \{0\}$.*

Proof. Since Γ/Δ is an N-group of type 1, we have $I \subseteq (0 : \Gamma/\Delta)$. So Γ/Δ is an N/I-group of type 1 with $(n+\ I)(g+\ \Delta) = ng + \Delta$ for $n \in N$ and $g \in \Gamma$ [11, proposition 3.14].

Let $\bar{B} = \{n + I \in N/I \mid n\Gamma/\Delta = \{\Delta\}\}$ (the annihilator of Γ/Δ in N/I). Since $\bar{B}$ is an ideal in N/I, there is an ideal B of N with $I \subseteq B$ and $\bar{B} = B/I$. This means that $B\Gamma \subseteq \Delta \subseteq \{g \in \Gamma \mid Ng = \{0\}\}$. Consequently, $B^2\Gamma = \{0\}$. Since Γ is faithful, $B^2 = \{0\}$ and therefore $B \subseteq I$ by [11, theorem 5.37 and proposition 5.3]. This means that Γ/Δ is a faithful N/I-group of type 1. Hence, N/I is a 1-primitive near-ring. □

It is clear that theorem 3.13 follows directly from proposition 3.14. When R is finite, we can say more.

THEOREM 3.15. *Let R be a finite planar near-ring. Then $\mathcal{J}_1(\mathrm{Mat}_n(R))$ is the greatest proper ideal in $\mathrm{Mat}_n(R)$.*

Again, this theorem is a consequence of a more general result.

PROPOSITION 3.16. *Let N be a zero-symmetric near-ring with descending chain condition on the N-subgroups of N, and let $I = \mathcal{J}_1(N)$. Suppose that N has a faithful, strongly monogenic N-group Γ. Then*

(i) *$NI = \{0\}$ and I is a proper ideal,*

(ii) *if N has a multiplicative right identity, then I is the greatest proper ideal in N. Consequently, $NJ = \{0\}$ for all proper ideals J of N.*

Proof. From $I \subseteq (0 : \Gamma/\Delta)$, we have $I\Gamma \subseteq \Delta$, and so $NI\Gamma = \{0\}$. As Γ is strongly monogenic, we see that $N \neq I$. By the faithfulness of Γ we also have that $NI = \{0\}$.

Now, N/I satisfies the descending chain condition on N/I-subgroups of N/I by [11, theorem 2.35], and is 1-primitive by proposition 3.14. Thus, N/I is a simple near-ring by [11, theorem 4.46]. Consequently, I is a maximal ideal. Let J be an ideal of N. Then

$$\text{for all } n \in N,\ a \in I \text{ and } b \in J, \quad n(a+b) - na = n(a+b) \in J. \tag{3.1}$$

Therefore, if $J \not\subseteq I$, then $J + I = N$ by the maximality of I, and so $N^2 \subseteq J$ by (3.1).

Suppose that N has a right identity. Then $N = N^2 \subseteq J$. In this case, each proper ideal of N must be contained in I. This completes the proof. □

Proof of theorem 3.15. Since R has a right identity, $\mathrm{Mat}_n(R)$ has one as well, by proposition 2.8. The result follows from proposition 3.16. □

Next we shall describe the J_1-radical of $\mathrm{Mat}_n(R)$ for a finite planar near-ring R that is not 1-primitive. In this case, $J = \mathcal{J}_1(R) \neq \{0\}$ [12, theorem 2.5.3], and R is not a simple near-ring. As an ideal of $\mathrm{Mat}_n(R)$, $(J^n : R^n)$ is contained in the largest ideal $\mathcal{J}_1(\mathrm{Mat}_n(R))$. Whether the equality always holds is an open question. On the other hand, it is not hard to see that $\mathcal{J}_1(\mathrm{Mat}_n(R))$ is contained in $(A^n : R^n)$, which is just a subset of $\mathrm{Mat}_n(R)$.

Lemma 3.17. *Let R be a finite planar near-ring that is not 1-primitive. Let $N = \mathrm{Mat}_n(R)$, $I = \mathcal{J}_1(\mathrm{Mat}_n(R))$ and $J = \mathcal{J}_1(R)$. Then $(J^n : R^n) \subseteq I \subseteq (A^n : R^n)$. Consequently, if $A = J$, then $I = (A^n : R^n)$.*

Proof. Let $v \in R^n$. Then Iv is an N-subgroup of R^n. Since $NI = \{0\}$ by proposition 3.16(i), and R^n is a strongly monogenic N-group, we conclude that $Iv \subseteq A^n$. The last statement is clear. □

We close this section with a discussion of the case when R is neither 1-primitive nor 2-radical, and remark that we have no further information for $\mathcal{J}_2(\mathrm{Mat}_n(R))$ when R is 2-radical but not 1-primitive.

Theorem 3.18. *Let R be a planar near-ring with $\mathcal{J}_1(R) \neq \{0\}$ and $\mathcal{J}_2(R) \neq R$. Then $\mathcal{J}_1(\mathrm{Mat}_n(R)) = (\mathcal{J}_1(R)^n : R^n)$. Moreover, if R satisfies the descending chain condition on R-subgroups of R and $\mathcal{J}_2(\mathrm{Mat}_n(R)) \neq \mathrm{Mat}_n(R)$, then $\mathcal{J}_2(\mathrm{Mat}_n(R)) = \mathcal{J}_1(\mathrm{Mat}_n(R))$.*

Proof. Again, set $N = \mathrm{Mat}_n(R)$, $I_1 = \mathcal{J}_1(\mathrm{Mat}_n(R))$ and $I_2 = \mathcal{J}_2(\mathrm{Mat}_n(R))$. Also, let $J_1 = \mathcal{J}_1(R)$ and $J_2 = \mathcal{J}_2(R)$.

First of all, J_2 is a proper ideal of R by assumption. Therefore, $J_2 \subseteq A$, and so $J_2^2 = \{0\}$. This implies that $J_2 \subseteq J_1$ [11, corollary 5.10], and so $J_1 = J_2$.

Now, from lemma 3.17, we know that $(J_1^n : R^n) \subseteq I$. Let $v \in R^n$. Then, $U = I_1 v$ is an N-subgroup of R^n. Since R^n is a strongly monogenic N-group and $NI_1 = \{0\}$ by proposition 3.16, there is no vector $w \in I_1 v$ with $Nw = R^n$. Thus, U is a proper N-subgroup of R^n. Take an arbitrary $u = (u_1, \dots, u_n) \in U$. As we have seen in the proof of theorem 3.7, for each $i = 1, \dots, n$, the cyclic group $\langle u_i, + \rangle$ is contained in A, and so is a nilpotent R-subgroup of R. By [11, corollary 5.45], we have $\langle u_i, +\rangle \subseteq J_2$. From $J_1 = J_2$ we obtain that $u \in J_1^n$, and so $I_1 v = U \subseteq J_1^n$. Consequently, $IR^n \subseteq J_1^n$, and equivalently, $I_1 \subseteq (J_1^n : R^n)$.

Suppose $I_2 \neq N$ and R satisfies the descending chain condition on R-subgroups. Then $NI_2 = \{0\}$ by proposition 3.16, and so I_2 is nilpotent. This implies that $I_2 \subseteq I_1$; hence, $I_1 = I_2$. This completes the proof. □

4. Bijective matrix maps

In this section we solve a problem that was posed in [7]. The question is whether the inverse U^{-1} of a bijective matrix map $U : R^n \to R^n$, where R is a near-ring, is again a matrix map. We answer this in the affirmative in the case when R is finite, but in the infinite case the answer is in general negative, even if R is a nearfield.

Lemma 4.1. *Let R be a finite near-ring. Let $\theta : R \hookrightarrow M(G)$ be an embedding, where G is a finite additive group. If $r \in R$ is such that $\theta(r) : G \to G$ is bijective, then there is an $s \in R$ such that $\theta(s) = \theta(r)^{-1}$. As a consequence, R is a near-ring with identity.*

Proof. Denote by $\operatorname{Sym} G$ the symmetric group on G as a set. Since $\theta(r) \in \operatorname{Sym} G \subseteq M(G)$ and $\operatorname{Sym} G$ has finite order, we see that $\theta(r)^{-1} = \theta(r)^k = \theta(r^k)$ for some positive integer k. Now take $s = r^k$. Then $\theta(s) = \theta(r)^{-1}$. It follows that rs is the identity of R. □

Since the near-ring of matrix maps $\operatorname{Mat}_n(R)$, $n > 1$, is a subnear-ring of $M(R^n)$, we have

Corollary 4.2. *Let R be a finite near-ring. Let $U \in \operatorname{Mat}_n(R)$, $n > 1$. If $U : R^n \to R^n$ is a bijective map, then the inverse map $U^{-1} : R^n \to R^n$ also belongs to $\operatorname{Mat}_n(R)$. Consequently, $\operatorname{Mat}_n(R)$ has an identity and R has a left identity by theorem 1.4.*

Corollary 4.3. *Let R be a finite planar near-ring. Then $\operatorname{Mat}_n(R)$ contains no bijective maps.*

We conclude by giving an example that shows that corollary 4.2 is not necessarily true in the case when R is infinite. We adopt the notation $\partial p = \partial p(x)$ for the degree of a non-zero polynomial $p(x) \in \mathbb{Q}[x]$, and $\partial F = \partial F(x) = \partial p - \partial q$ denotes the degree of the (non-zero) rational form $F(x) = p(x)/q(x)$.

Example 4.4. Consider the right nearfield $(R, +, \circ)$, where $R = \mathbb{Q}(x)$ (the rational forms over $\mathbb{Q}$), $+$ is defined in the standard way and $\circ$ is defined by

$$\frac{p(x)}{q(x)} \circ \frac{s(x)}{t(x)} = \begin{cases} \dfrac{p(x + \partial s - \partial t)}{q(x + \partial s - \partial t)} \cdot \dfrac{s(x)}{t(x)} & \text{if } \dfrac{s(x)}{t(x)} \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\cdot$ denotes the standard multiplication in the field $(\mathbb{Q}(x), +, \cdot)$. See [11, example 8.29] for further details on this nearfield. Also, we simply write $f(x)$ for $f(x)/1$, etc.

Consider the matrix

$$U = f_{11}^{x} + f_{12}^{1} + f_{21}^{1} + f_{22}^{x}$$

in $\mathcal{M}_2(R)$. In order to show that $U : R^2 \to R^2$ is bijective, it suffices to show that, for every $\langle F, G\rangle \in R^2$, there exists a unique $\langle S, T\rangle \in R^2$ such that $U\langle S, T\rangle = \langle F, G\rangle$. This implies that the system

$$x \circ S + T = F, \qquad S + x \circ T = G$$

must have a unique solution for each pair $\langle F, G\rangle \in R^2$.

After a rather tedious, but relatively simple, computation, it is found that $\langle S, T\rangle$ is given as follows:

1. if $F \neq 0$, $G \neq 0$, $\partial F \geqslant \partial G$ and $F \neq x \circ G$, then
$$\langle S, T\rangle = \left\langle \frac{(x+\lambda_2)F - G}{(x+\lambda_1)(x+\lambda_2) - 1}, \frac{(x+\lambda_1)G - F}{(x+\lambda_1)(x+\lambda_2) - 1} \right\rangle;$$
2. if $F \neq 0$, $G \neq 0$, $\partial F \geqslant \partial G$ and $F = x \circ G$, then $\langle S,\ T\rangle = \langle G,\ 0\rangle$;
3. if $F \neq 0$, $G \neq 0$, $\partial F < \partial G$ and $G \neq x \circ F$, then
$$\langle S, T\rangle = \left\langle \frac{(x+\mu_2)F - G}{(x+\mu_1)(x+\mu_2) - 1}, \frac{(x+\mu_1)G - F}{(x+\mu_1)(x+\mu_2) - 1} \right\rangle;$$
4. if $F \neq 0$, $G \neq 0$, $\partial F < \partial G$, and $G = x \circ F$, then $\langle S, T\rangle = \langle 0, F\rangle$;
5. if $F = 0$ and $G \neq 0$, then
$$\langle S, T\rangle = \left\langle \frac{-G}{(x+\mu_2)(x+\mu_2-1) - 1}, \frac{(x+\mu_2-1)G}{(x+\mu_2)(x+\mu_2-1) - 1} \right\rangle;$$
6. if $F \neq 0$ and $G = 0$, then
$$\langle S, T\rangle = \left\langle \frac{(x+\lambda_1-1)F}{(x+\lambda_1)(x+\lambda_1-1) - 1}, \frac{-F}{(x+\lambda_1)(x+\lambda_1-1) - 1} \right\rangle;$$
7. if $F = 0$ and $G = 0$, then $\langle S, T\rangle = \langle 0,\ 0\rangle$,

where

$$\begin{aligned}
\lambda_1 &= \partial F - 1, \\
\lambda_2 &= \partial[(x + \partial F - 1) \cdot G - F] - 2, \\
\mu_1 &= \partial[(x + \partial G - 1) \cdot F - G] - 2, \\
\mu_2 &= \partial G - 1.
\end{aligned}$$

We proceed to show that the map U^{-1} is not a matrix map. Take $F = 1$ and $G_i = x^i$ for $i \leqslant -2$. Then, $\lambda_1 = -1$ and $\lambda_2 = -2$. Now, if U^{-1} is assumed to be a matrix map, then $f^1_{12}U^{-1}$ is a first-row matrix, and

$$f^1_{12}U^{-1}\langle F, G_i\rangle = \left\langle \frac{(x-1)x^i - 1}{(x-1)(x-2) - 1}, 0 \right\rangle.$$

But on the other hand, by [10, lemma 3], there exists a positive integer m such that, for all $i \leqslant -m$,

$$f_{12}^{1}U^{-1}\langle F,\ G_i\rangle = \langle P(x) + Q_i(x+i)x^i, 0\rangle,$$

where $P(x), Q_i(x) \in \mathbb{Q}(x)$ and the set $\{\partial Q_i\}_i$ is bounded from above. If we solve for $Q_i(x+i)$ from

$$\frac{(x-1)x^i - 1}{(x-1)(x-2)-1} = P(x) + Q_i(x+i)x^i,$$

we find that

$$Q_i(x+i) = \frac{x - 1 - x^{-i} - ((x-1)(x-2)-1)P(x)x^{-i}}{(x-1)(x-2)-1} \quad \text{for all } i \leqslant -m.$$

If $P(x) = 0$, then $\partial Q_i(x+i) = -i-2$, which could be made arbitrarily large, since $i \leqslant -m$ is arbitrary. If $P(x) \neq 0$, then $\partial Q_i(x+i) = -i + \max\{-2, \partial P\}$, which is again a number that could be made arbitrarily large. In both cases we obtain a contradiction to the fact that $\{\partial Q_i\}_i$ is bounded from above. We conclude that U^{-1} is not a matrix map.

In the above example we notice that $(R, +, \circ)$ is not a planar nearfield. Therefore, it would be interesting to know what happens in the case when R is an infinite planar near-ring.

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