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Unique continuation for the elastic transversely isotropic dynamical systems and its application

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ABSTRACT

In this paper we prove the unique continuation property of the solution for the elastic transversely isotropic dynamical systems with smooth coefficients satisfying some conditions and apply it to extending the Dirichlet to Neumann map. The proof is based on the localized Fourier–Gauss transformation and Carleman type estimate.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ and $C^\infty(\bar{\Omega})$ be the set of all smooth functions defined in $\bar{\Omega}$. Let the body $\mathcal{B}$ with reference configuration Ω be occupied by a transversely isotropic medium. More precisely, let the axis of rotational symmetry coincide with the x_3 axis, then the non-zero components of the elasticity tensor $C(x) = C_{ijk\ell}(x)$ are

$$C_{1111},\ C_{2222},\ C_{3333},\ C_{1122},\ C_{1133},\ C_{2233},\ C_{2323},\ C_{1313},\ C_{1212}$$

and they satisfy

$$C_{1111} = C_{2222}, \qquad C_{1133} = C_{2233}, \qquad C_{2323} = C_{1313}, \qquad C_{1212} = (C_{1111} - C_{1122})/2.$$

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For notational simplicity, we set

$$C_{1111} = A, \qquad C_{1122} = M, \qquad C_{1133} = F, \qquad C_{3333} = C, \qquad C_{2323} = L. \tag{1.1}$$

It should be noticed that the elasticity tensor $C(x)$ satisfies the full symmetry properties:

$$C_{ijkl} = C_{klij} = C_{jikl} \quad \text{for all } x \in \bar{\Omega}. \tag{1.2}$$

We assume that the elasticity tensor satisfies the strong convexity condition, i.e. there exists $\delta > 0$ such that for any real symmetric matrix E

$$C(x)E \cdot E \geqslant \delta |E|^2 \quad \text{for all } x \in \bar{\Omega}. \tag{1.3}$$

In other words, we assume that

$$C > \tilde{\delta}, \qquad L > \tilde{\delta}, \qquad (1/2)(A+M) > \tilde{\delta}, \qquad (1/2)(A-M) > \tilde{\delta}, \qquad (A+M)C - 2F^2 > \tilde{\delta} \tag{1.4}$$

in $x \in \bar{\Omega}$ for some $\tilde{\delta} > 0$. Now let $u(x,t)$ be the displacement vector, then the dynamic elastic equation is given by

$$\rho \partial_t^2 u - \mathcal{L}u = 0 \quad \text{in } \Omega \times (-T, T) \tag{1.5}$$

with

$$(\mathcal{L}u)_i = \sum_{j=1}^{3} \partial_j \sigma_{ij} \quad \text{in } \Omega \times (-T, T), \; 1 \leqslant i \leqslant 3, \tag{1.6}$$

where $\partial_j = \partial_{x_j}$ and the stress–strain relation is

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} A & M & F & 0 & 0 & 0 \\ M & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & (A-M)/2 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix},$$

where σ_{ij} and ε_{ij} denote the stress and strain tensors.

It should be noted that the strong convexity condition implies the strong ellipticity condition for the elasticity tensor, which ensures that the system of Eqs. (1.6) is strongly elliptic.

In this paper, we will study the weak unique continuation property of (1.5) by the method of the localized Fourier–Gauss transformation. The method was introduced by Lerner [8] for proving some uniqueness result for an ill-posed problem and it was also used by Robbiano [13] to prove some kind of unique continuation property. Henceforth we abbreviate this property by UCP. In this paper using the Calderón uniqueness theorem, we generalize the result in [13] to (1.5) with smooth coefficients satisfying some conditions. In addition, we apply the UCP to extend the Dirichlet to Neumann map given for large enough time interval to the infinite time interval. This is a generalization of the results [2,6] given for scalar equations and the result [1] given for the isotropic elastic system.

For the related results, the study of line unique continuation property was initiated by Cheng, Yamamoto and Zhou [3] for the wave equation and they showed it along each line in the hyperplane. They combined the localized Fourier–Gauss transformation abbreviated by LFGT to transform (1.5) to the Laplace equation with a small inhomogeneous term and the conditional stability estimate for the unique continuation of the solution of the Laplace equation along lines. Cheng, Lin and Nakamura [4]

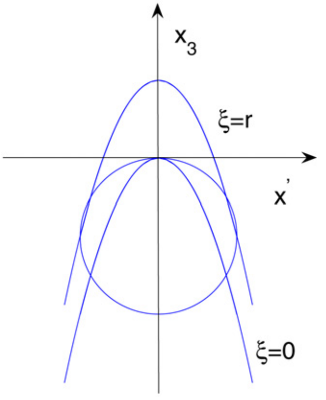

Fig. 1. Equation of the ball in the figure: $|x'|^2 + (x_3 + r)^2 = r^2$ and $\gamma = 2/r$.

extended the line unique continuation property to general hyperbolic systems with analytic coefficients.

For $x_0 \in \mathbb{R}^3$, $r > 0$, let $B(x_0, r) := \{x \in \mathbb{R}^3;\ |x - x_0| < r\}$. Let $y \in \partial B(x_0, r) := \{x \in \mathbb{R}^3;\ |x - x_0| = r\}$ and n_y be its unit outer normal with respect to $B(x_0, r)$. Also, let A_{y,n_0} be an affine transformation which transforms y to the origin and n_y to $n_0 := (0, 0, 1)^t$. Then, we define $U(\gamma, r_1, r_2) = \{x = (x', x_3) \in \mathbb{R}^3;\ |x'| < r_2,\ 0 \leqslant \xi := x_3 + \gamma |x'|^2 < r_1\}$ and $V(\gamma, r_1, r_2) = A^{-1}_{y,n_0} U(\gamma, r_1, r_2)$. Fig. 1 is about $U(2/r, r, r)$ with $\xi := x_3 + 2|x'|^2/r = 0, r$.

From now on, we assume that the elasticity tensor $C(x)$ satisfies the additional condition

$$F + L = 0$$

or

$$(A - L)(C - L) = (F + L)^2, \quad A > C.$$

Then, the main results of our paper are the following.

Theorem 1.1. *Assume $B(x_0, 3r) \subset \Omega$. Let y be the point at the boundary of $B(x_0, r)$ with its outer normal n_y not perpendicular to n_0. Suppose $u \in C^\infty(B(x_0, 3r) \times (-T, T))$ satisfying (1.5) in $B(x_0, 3r) \times (-T, T)$ and*

$$u(x, t) = 0 \quad \big((x, t) \in B(x_0, r) \times (-T, T)\big).$$

Then there exist a positive constant $r_0 < r$ depending on $\rho(x)$ and $\mathcal{L}$ in $B(x_0, 3r)$ and a positive constant $k < 1$ such that

$$u(x, t) = 0 \quad \text{for } (x, t) \in V(2/r_0, kr_0, kr_0) \times (-T_1, T_1), \tag{1.7}$$

where $T_1 = T - kr_0$. Moreover, the constants r_0 and k can be taken uniformly in Ω.

Corollary 1.2 *(UCP). Let $B(x_0, r) \subset \Omega$ and given $T_2 > 0$. There exists a positive constant T_3 depending on $\mathcal{L}$ in Ω such that if $T > T_3$ and $u \in C^\infty((-T, T) \times \bar{\Omega})$ satisfies (1.5) in $(-T, T) \times \Omega$ and*

$$u(x, t) = 0 \quad \big((x, t) \in B(x_0, r) \times (-T, T)\big),$$

then

$$u(x, t) = 0 \quad \text{for } (x, t) \in \Omega \times (-T_2, T_2). \tag{1.8}$$

As an immediate byproduct of Corollary 1.2, we have the following corollary.

Corollary 1.3. *We can take $\tau_0 > 0$ (large enough) such that the following property is satisfied for the solutions of (1.5). For any $\tau > \tau_0$, let $u \in C^\infty(\bar{\Omega} \times [0, 2\tau])$ be a solution in $\Omega \times [0, 2\tau]$ of (1.5) such that $u = 0$ and $\nabla_x u = 0$ at $\partial\Omega \times [0, 2\tau]$. Then*

$$u(x, t) = 0 \quad (x \in \Omega),$$

if t is near τ.

The proof of Theorem 1.1 relies on the Carleman estimate of Calderón's uniqueness theorem which will be described in the next section. The rest of this paper is organized as follows. We review the Carleman estimate of Calderón's uniqueness theorem in Section 2. In Section 3, we diagonalize the associated elliptic system and check the conditions for applying Calderón's uniqueness theorem. The proofs of Theorem 1.1 and Corollary 1.2 are given in Section 4. In Section 5, we apply these results to extend the dynamical Dirichlet–Neumann map.

2. Carleman estimate of Calderón's uniqueness theorem for systems

To begin, we first review Carleman estimate of Calderón's uniqueness theorem from Zuily's book [15, Chapter 2]. The purpose for doing this is to make this paper as self-contained as possible. Let V be an open neighborhood of $x_0 \in \mathbb{R}^n$ ($n \in \mathbb{N}$). In this section we do not specify the dimension $n \in \mathbb{N}$. In the neighborhood of V we define a C^∞ hypersurface

$$S = \{x \in V\colon\ \psi(x) = \psi(x_0)\}. \tag{2.1}$$

Let

$$L(x, D) = P(x, D) + Q(x, D) \tag{2.2}$$

be a differential operator, where

$$P(x, D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha \quad \left(D = (D_1, \ldots, D_n),\ D_j = \sqrt{-1}\partial_{x_j}\right) \tag{2.3}$$

being an mth order differential operator with C^∞ coefficients and the lower order part $Q(x, D)$ has bounded coefficients. Denote $p(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ the full symbol of $P(x, D)$. As usual, the hypersurface S is assumed to be non-characteristic for L at x_0, i.e. $p(x_0, N_0) \neq 0$, where $N_0 = d\psi(x_0)$. Let u satisfy $Lu = 0$ near x_0 and $u = 0$ if $\psi(x) \leqslant \psi(x_0)$ near x_0. Before stating the main theorem of this section, we want to clearly describe the assumptions on the characteristic roots. For each x and ξ, we assume that

(C.1) there exist a conic neighborhood Γ_{N_0} of N_0 and m functions $\{\lambda_\ell(x, \xi, N)\}_{\ell=1}^m$ which are C^∞ in $(x, \xi, N) \in V \times (\mathbb{R}^n \setminus 0) \times \Gamma_{N_0}$ such that for every $\xi \nparallel N$, $p(x, \xi + \tau N)$ is written as

$$p(x, \xi + \tau N) = p(x, N) \prod_{\ell=1}^{m} \left(\tau - \lambda_\ell(x, \xi, N)\right)$$

in $V \times (\mathbb{R}^n \setminus 0) \times \Gamma_{N_0}$;

(C.2) for any ℓ, $1 \leqslant \ell \leqslant m$, if $\lambda_\ell(x, \xi, N)$ is real (or complex) at one point, then it remains real (or complex) at every point;

(C.3) the real roots are simple and the multiplicity of the complex roots is not more than two.

As in [15], assuming $x_0 = 0$ and using the Holmgren transform

$$x_i = x_i, \quad 1 \leqslant i \leqslant n-1, \qquad t = \langle x, N_0 \rangle + \delta |x|^2, \tag{2.4}$$

with a suitable constant $\delta > 0$, let $P(\tilde{x}, t, \tilde{\xi}, \tau)$ with $\tilde{x} = (x_1, \dots, x_{n-1})$ be the principal symbol of $P(\tilde{x}, t; D_{\tilde{x}}, D_t)$, where we abused the notation $P(\tilde{x}, t; D_{\tilde{x}}, D_t)$ to denote the operator (2.3) in terms of $(\tilde{x}, t)$. Then, there exist a function $c(\tilde{x}, t)$ and $\{\lambda_l(\tilde{x}, t, \tilde{\xi})\}_{l=1}^m$, such that

$$p(\tilde{x}, t, \tilde{\xi}, \tau) = c(\tilde{x}, t) \prod_{l=1}^{m} \bigl(\tau - \lambda_l(\tilde{x}, t, \tilde{\xi})\bigr)$$

in $\tilde{V} \times (\mathbb{R}^n \setminus 0)$, where $\tilde{V}$ is a small neighborhood of $(0, 0)$ and $c(\tilde{x}, t)$ is a C^∞ function with $c(0, 0) \neq 0$ and $\lambda_l(\tilde{x}, t, \tilde{\xi})$ is C^∞ in $\tilde{V} \times (\mathbb{R}^n \setminus 0)$ homogeneous of degree one in $\tilde{\xi}$, $1 \leqslant l \leqslant m$. That is, condition (C.1) is satisfied. Moreover, $\{\lambda_l(\tilde{x}, t, \tilde{\xi})\}_{l=1}^m$ satisfy conditions (C.2) and (C.3). Since the result is local near $(0, 0)$, it suffices to assume that the characteristic roots $\{\lambda_l\}_{l=1}^m = \{\lambda_l(0, 0, \xi)\}_{l=1}^m$ outside of a small neighborhood of $(0, 0)$. Furthermore, it is readily seen that transform $\tilde{u}$ of u by (2.4) satisfies

$$\operatorname{supp} \tilde{u} \subset \bigl\{(\tilde{x}, t) \in \mathbb{R}^n \colon\ t \geqslant \tilde{c}|x|^2\bigr\}$$

for some constant $\tilde{c}$. Then, we have the following Carleman estimate, which was given in [15].

Lemma 2.1. *There exist positive constants c, T_0, η_0 and r such that for $T \leqslant T_0$ and $\eta \geqslant \eta_0$ we have that*

$$\sum_{|\alpha| \leqslant m-1} \int_0^T e^{\eta(t-T)^2} \bigl\| D^\alpha w \bigr\|^2_{L^2(\mathbb{R}^{n-1})}\, dt \leqslant c\bigl(T^2 + \eta^{-1}\bigr) \int_0^T e^{\eta(t-T)^2} \| L w \|^2_{L^2(\mathbb{R}^{n-1})}\, dt \tag{2.5}$$

for any $w \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} w \subset \{(\tilde{x}, t)\colon\ 0 \leqslant t \leqslant T,\ |\tilde{x}| \leqslant r\}$, where $L = L(\tilde{x}, t, D_{\tilde{x}}, D_t)$ is the operator (2.2) in terms of $(\tilde{x}, t)$. Moreover, the constants c, T_0, η_0 and r only depend on the coefficients of $P(\tilde{x}, t, D_{\tilde{x}}, D_t)$.

3. Transversely isotropic dynamical systems

In this section, we will study the possibilities of (1.5) to have UCP. Having in our mind the solution u of (1.5) will be transformed by LFGT, we aim to apply (2.5) to $\rho\partial_s^2 + \mathcal{L}$ by diagonalizing its principal part. A direct way is to use the cofactor of the principal part of $\rho\partial_s^2 + \mathcal{L}$. The question is now whether the conditions (C.1), (C.2) and (C.3) for characteristic roots are satisfied. By assuming the elasticity tensor $C_{ijk\ell}(x) \in C^\infty(\bar{\Omega})$ and in view of the strong ellipticity (1.4), we only have to check the smoothness condition (C.1) of the characteristic roots and the multiplicity condition (C.3). It should be noted that when the characteristic roots are not smooth, Plis [11] constructed a fourth order elliptic differential operator in which the Cauchy problem is not unique. We will first discuss the multiplicity condition (C.3). It turns out that we need to exclude certain directions and put a condition (3.3) or (3.4) in order to guarantee (C.3). To begin with, we factor the determinant of the principal symbol of $\rho\partial_s^2 + \mathcal{L}$ by direct computations by a result in [9] (see Proposition 3.9, for a more general result see Lemma 2.7 in [12]), which also contain a detail discussion of the multiplicity of the eigenvalues for more general systems.

Proposition 3.1. *The determinant of the principal symbol of $\rho\partial_s^2 + \mathcal{L}$ can be factored as*

$$\det\left(\rho\delta_{ik}\eta_4^2 + \sum_{j,l=1}^{3} C_{ijkl}\eta_j\eta_l\right) = \left(\rho\eta_4^2 + \rho\lambda_0\right)\left(\rho\eta_4^2 + \rho\lambda_+\right)\left(\rho\eta_4^2 + \rho\lambda_-\right), \tag{3.1}$$

where

$$\rho\lambda_0(x,\eta) = (1/2)(A-M)\left(\eta_1^2+\eta_2^2\right) + L\eta_3^2,$$

$$\rho\lambda_\pm(x,\eta) = (1/2)(A+L)\left(\eta_1^2+\eta_2^2\right) + (1/2)(C+L)\eta_3^2 \pm (1/2)|D|,$$

$$D^2 = (A-L)^2\left(\eta_1^2+\eta_2^2\right)^2 + (C-L)^2\eta_3^4 - 2\left[(A-L)(C-L) - 2(F+L)^2\right]\left(\eta_1^2+\eta_2^2\right)\eta_3^2.$$

Lemma 3.1. *Let $\{\xi,\zeta\}$ be a pair of orthogonal vectors in $\mathbb{R}_x^3$. Consider the characteristic equation*

$$\det\left(\rho\delta_{ik}\eta_4^2 + \sum_{j,l=1}^{3} C_{ijk\ell}\eta_j\eta_l\right) = 0 \quad \text{in } \tau, \tag{3.2}$$

where $\eta = (\eta_1,\eta_2,\eta_3,\eta_4) = (\xi,\xi_4) + \tau(\zeta,0)$. Let $\ell := \xi\times\zeta$ and ϕ be the angle between ℓ and the x_3 axis. If

$$F + L = 0 \tag{3.3}$$

or

$$(A-L)(C-L) = (F+L)^2, \quad A > C, \tag{3.4}$$

then the characteristic roots of (3.2) *satisfy* (C.1) *and have at most double roots for $\phi \neq 0$ and π.*

Proof. Let

$$\mathbf{Q} = (Q_{ik}) = \rho\xi_4^2 I + \sum_{j,l=1}^{3} C_{ijkl}\xi_j\xi_l,$$

$$\mathbf{R} = (R_{ik}) = \sum_{j,l=1}^{3} C_{ijkl}\xi_j\zeta_l,$$

$$\mathbf{T} = (T_{ik}) = \sum_{j,l=1}^{3} C_{ijkl}\zeta_j\zeta_l,$$

where $i,k = 1,2,3$, and $\xi = (\xi_1,\xi_2,\xi_3)$, $\zeta = (\zeta_1,\zeta_2,\zeta_3)$, then the characteristic equation (3.2) is equivalent to

$$\det\left[\tau^2\mathbf{T} + \tau\left(\mathbf{R}+\mathbf{R}^t\right) + \mathbf{Q}\right] = 0. \tag{3.5}$$

From (1.3), we see that (3.5) contains only complex roots and they form conjugate pairs. Since the axis of rotational symmetry coincides with the x_3 axis, the elasticity tensor $C_{ijk\ell}$ is invariant under the orthogonal transform $\mathbf{O}$ rotating around the x_3 axis, i.e.

$$\mathbf{O}=\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, by the transform rule of tensor, the multiplicities are also invariant under the rotation of ξ and ζ in the plane spanned by ξ, ζ. Therefore, the multiplicities of the characteristic roots for (3.2) are invariant under the same transform $\mathbf{O}$ on ξ and ζ. Thus, it suffices to prove this proposition for

$$\xi=(\cos\phi, 0, -\sin\phi), \qquad \zeta=(0,1,0),$$

where $\ell=(\sin\phi, 0, \cos\phi)$ (see [14] or [10]).

First, we assume that (3.3) holds. Then from (3.1), we get

$$\begin{aligned} &\det\big[\tau^2\mathbf{T}+\tau\big(\mathbf{R}+\mathbf{R}^t\big)+\mathbf{Q}\big] \\ &\quad=\big[(1/2)(A-M)\tau^2+\rho\xi_4^2+(1/2)(A-M)\xi_1^2+L\xi_3^2\big] \\ &\qquad\times\big[A\tau^2+\rho\xi_4^2+A\xi_1^2+L\xi_3^2\big]\times\big[\tau^2L+\rho\xi_4^2+L\xi_1^2+C\xi_3^2\big] \\ &\quad=0. \end{aligned} \tag{3.6}$$

We take τ_1, τ_2, τ_3 to be three roots of (3.6) with positive imaginary part. Let τ_1, τ_2 and τ_3 satisfy

$$(1/2)(A-M)\tau_1^2+\rho\xi_4^2+(1/2)(A-M)\xi_1^2+L\xi_3^2=0, \tag{3.7}$$

$$A\tau_2^2+\rho\xi_4^2+A\xi_1^2+L\xi_3^2=0 \tag{3.8}$$

and

$$L\tau_3^2+\rho\xi_4^2+L\xi_1^2+C\xi_3^2=0. \tag{3.9}$$

From (3.7) and (3.8), the necessary condition for $\tau_1=\tau_2$ is

$$\frac{(1/2)(A-M)}{\rho\xi_4^2+(1/2)(A-M)\xi_1^2+L\xi_3^2}=\frac{A}{\rho\xi_4^2+A\xi_1^2+L\xi_3^2}. \tag{3.10}$$

We derive from (3.10) that

$$(1/2)(A+M)\big(\rho\xi_4^2+L\xi_3^2\big)=0$$

which implies from (1.4) that

$$\xi_3=0=\xi_4 \quad \text{and then} \quad \xi_1^2=1.$$

Therefore, we get the necessary conditions for triple roots $\tau_1=\tau_2=\tau_3$ and the triple root are i.

Secondly, we assume that (3.4) holds. Then

$$\begin{aligned}
&\det\left[\mathbf{Q}+\left(\mathbf{R}+\mathbf{R}^t\right)\tau+\mathbf{T}\tau^2\right]\\
&=\left[(1/2)(A-M)\tau^2+\rho\xi_4^2+(1/2)(A-M)\xi_1^2+L\xi_3^2\right]\\
&\quad\times\left[A\tau^2+\rho\xi_4^2+A\xi_1^2+C\xi_3^2\right]\times\left[L\tau^2+\rho\xi_4^2+L\xi_1^2+L\xi_3^2\right]\\
&=0. \qquad (3.11)
\end{aligned}$$

We take τ_1, τ_2, τ_3 to be three roots of (3.11) with positive imaginary part. Let τ_1, τ_2 and τ_3 satisfy

$$(1/2)(A-M)\tau_1^2+\rho\xi_4^2+(1/2)(A-M)\xi_1^2+L\xi_3^2=0, \qquad (3.12)$$

$$A\tau_2^2+\rho\xi_4^2+A\xi_1^2+C\xi_3^2=0 \qquad (3.13)$$

and

$$L\tau_3^2+\rho\xi_4^2+L\xi_1^2+L\xi_3^2=0. \qquad (3.14)$$

From (3.12) and (3.14) we have the necessary condition for $\tau_1=\tau_3$ is

$$A-M=2L. \qquad (3.15)$$

In this case, Eqs. (3.12) and (3.14) are the same. So, we consider $\tau_1=\tau_2$ under (3.15). Therefore, the necessary condition for $\tau_1=\tau_2=\tau_3$ is

$$\frac{L}{\rho\xi_4^2+L\xi_1^2+L\xi_3^2}=\frac{A}{\rho\xi_4^2+A\xi_1^2+C\xi_3^2}. \qquad (3.16)$$

We derive from (3.16) that

$$\rho(A-L)\xi_4^2+L(A-C)\xi_3^2=0. \qquad (3.17)$$

Plugging (3.15) into (3.17), it implies from (1.4) and (3.4) that

$$\xi_3=0=\xi_4 \quad \text{and then} \quad \xi_1^2=1.$$

On the other hand, the left-hand side of (3.2) is -1 times the determinant of the principal symbol of $\rho\partial_s^2+\mathcal{L}$ and it has the factorization (3.6) and (3.11). Hence, the smoothness of the characteristic roots can be easily verified. $\square$

Remark 3.2. Let $x_0\in\mathbb{R}^3$ and V be a neighborhood of x_0. Assume that $S=\{x\colon\ \psi(x)=\psi(x_0)\}$ is a C^∞ surface with $N_0=d\psi(x_0)$ satisfying

$$N_0 \text{ is not perpendicular to the } x_3 \text{ axis.}$$

Let the elasticity tensor $C_{ijk\ell}$ satisfy the additional condition (3.3) or (3.4) in V. With the help of Lemma 3.1, we will see in the next section that we can apply (2.5) for the system $\rho\partial_s^2+\mathcal{L}$ in some neighborhood V_0 of x_0.

4. Proofs of Theorem 1.1 and Corollary 1.2

We will use the method given in [3] to prove Theorem 1.1. We define LFGT $v_{a,\lambda}(x,s)$ of $u(x,t)$ by

$$v_{a,\lambda}(x,s) := \sqrt{\lambda/2\pi}\int_{-T}^{T} e^{-\lambda(is+a-t)^2/2}u(x,t)\,dt, \tag{4.1}$$

where $\lambda > 0$, $a, s \in \mathbb{R}$ and $i = \sqrt{-1}$. Associated with the operator $\rho\partial_t^2 - \mathcal{L}$, we define an elliptic operator $Q_{x,s} = Q(x,s,D_x,D_s)$ in $(x,s) \in \mathbb{R}_x^3 \times \mathbb{R}_s^1$ by $Q_{x,s} := \rho\partial_s^2 + \mathcal{L}$. Let $Q_{x,s}^{\mathrm{co}} = Q^{\mathrm{co}}(x,s,D_x,D_s)$ be the operator whose symbol is the cofactor matrix of the principal symbol of $Q_{x,s}$ and define $\tilde{Q}_{x,s}$ by

$$\tilde{Q}_{x,s} = \tilde{Q}(x,s,D_x,D_s) = Q_{x,s}^{\mathrm{co}} \circ Q_{x,s}.$$

Then, the principal symbol of $\tilde{Q}_{x,s}$ is

$$q_{x,s}(\eta_1,\eta_2,\eta_3,\eta_4) = -\det\left(\rho\delta_{ik}\eta_4^2 + \sum_{j,l=1}^{3} C_{ijkl}\eta_j\eta_l\right)I. \tag{4.2}$$

By Lemma 3.1, $q_{x,s}$ satisfies (C.1)–(C.3). We also define

$$\chi_{a,\lambda} := \tilde{Q}_{x,s}v_{a,\lambda}. \tag{4.3}$$

Let y be the point at the boundary of $B(x_0,r)$ and n_y be the unit normal of $\partial B(x_0,r)$ at y. By an affine transformation A_{y,n_0}, we can assume that $y = 0$ and $n_0 = (0,0,1)^t$. To be compared with Section 2, the C^∞ hypersurface $S = \{x\colon\ \psi(x) = \psi(x_0)\}$ in (2.1) is given by $S = \{(x_1,x_2,x_3,s)\colon\ \psi(x_1,x_2,x_3,s) = x_1^2 + x_2^2 + (x_3+r)^2 - r^2 = \psi(0,0,0,s) = 0,\ -r < x_1, x_2, (x_3 + r) < r,\ -T < s < T\}$. Here r is the radius of $B(x_0,r)$ and is independent of s. We now perform a change of coordinates near 0 by using the "Holmgren transform," i.e.,

$$s \to s, \quad x_j \to x_j \quad (j = 1,2), \quad \mu = x_3 + 2\big(s^2 + |x'|^2\big)/r,$$

where $x' = (x_1,x_2)$. For simplicity, we will use the same notations $\tilde{Q}_{x,s}$ and $v_{a,\lambda}$ even after applying the Holmgren transform to them. Then, in the region $V = \{(\mu,x',s);\ x_3 > -r,\ 0 < \mu < r\}$,

$$\operatorname{supp}(v_{a,\lambda}) \subset \bar{V} \cap \big\{(\mu,x',s);\ s^2 + |x'|^2 \leqslant r^2\big\} \tag{4.4}$$

and the new $q_{x,s}$ satisfies (C.1)–(C.3). For (4.4), the readers are referred to Fig. 2 in the proof of Corollary 1.2. Moreover, by the definition of LFGT, $v_{a,\lambda}(x,s)$ is smooth in $B(x_0,3r) \times \mathbb{R}$. Therefore, we are in a position to apply the following Carleman estimate which is deduced from (2.5).

Theorem 4.1. *There exist positive constants $r_0 < r$, η_0, and c depending on ρ and $\mathcal{L}$ in Ω such that for all $\eta \geqslant \eta_0$, we have that*

$$\sum_{|\nu|\leqslant 5}\int_0^{r_0} e^{\eta(\mu-r_0)^2}\big\|\partial^\nu v\big\|^2\,d\mu \leqslant c\big(\eta^{-1} + r_0^2\big)\int_0^{r_0} e^{\eta(\mu-r_0)^2}\|\tilde{Q}_{s,x}v\|^2\,d\mu \tag{4.5}$$

for all $v(x',\mu,s) \in C^\infty$ with $\operatorname{supp}(v) \subset \{(x',\mu,s)\colon\ s^2 + |x'|^2 \leqslant r_0^2,\ 0 \leqslant \mu \leqslant r_0\}$, where $\|\cdot\|^2 = (\cdot,\cdot)$ is the $L^2(\mathbb{R}^3)$ norm.

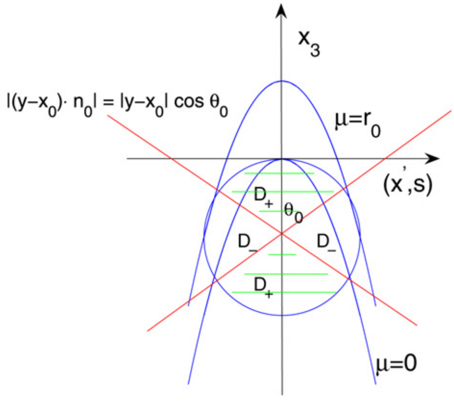

Fig. 2. Equation of the ball: $|x'|^2 + s^2 + (x_3 + r_0)^2 = r_0^2$ and $\mu = x_3 + 2(s^2 + |x'|^2)/r_0$.

It should be noted that the constants r_0, η_0, and c can be taken uniformly in $\bar{\Omega}$. As in the proof of [15], the constants r_0, η_0, and c are independent of the normal vector.

To prove the main theorem, we still need the following properties of LFGT given in [3].

Lemma 4.2. *Let $u \in C^6(B(x_0, 3r_0) \times [-T, T])$ and $s_0 \in (0, T)$ be fixed. If u satisfies $\rho u_{tt} - \mathcal{L}u = 0$ in $B(x_0, 3r_0) \times [-T, T]$, then for $s \in (-s_0, s_0)$ and $0 \leqslant \mu \leqslant r_0$, we have*

$$v_{a,\lambda}(x', \mu, 0) \to u(x', \mu, a) \quad \text{as } \lambda \to \infty,\ |a| < T, \tag{4.6}$$

$$\left|\left(\partial^\nu v_{a,\lambda}\right)(x', \mu, s)\right| \leqslant C_1 \lambda^{9/2} e^{\lambda s_0^2/2} \quad \left(|\nu| \leqslant 5\right), \tag{4.7}$$

$$\left|\chi_{a,\lambda}(x', \mu, s)\right| \leqslant C_2 \lambda^{11/2} e^{-\lambda[(T-|a|)^2 - s_0^2]/2}, \tag{4.8}$$

where $C_1 > 0$ depends on $\|u\|_{C^5(B(x_0,3r_0)\times[-T,T])}$ and $C_2 > 0$ depends on s_0, T, a and $\|u\|_{C^6(B(x_0,3r_0)\times[-T,T])}$.

Remark 4.3. In [3], (4.8) is shown for $\partial_s^2 + \Delta$. Here, we note that

$$\partial_s v_{a,\lambda}(x, s) = -i\sqrt{\lambda/2\pi} \int_{-T}^{T} e^{-\lambda(is+a-t)^2/2} \partial_t u(x, t)\, dt - i\sqrt{\lambda/2\pi}\, e^{-\lambda(is+a-T)^2/2} u(x, T)$$
$$+ i\sqrt{\lambda/2\pi}\, e^{-\lambda(is+a+T)^2/2} u(x, -T).$$

In this identity, only s-derivatives have been transformed into t-derivatives with end point values. So the proof for $\tilde{Q}_{s,x}$ is almost the same as that given in [3].

Now, let a C^∞ function $\theta(\mu) \in C_0^\infty(\mathbb{R})$ defined in $\mu \geqslant 0$ with $\theta(\mu) = 0$ for $\mu \geqslant r_0$ and $\theta(\mu) = 1$ for $\mu \leqslant 4r_0/5$. Denote $\eta = \lambda$ and $w_{a,\lambda} = \theta(\mu) v_{a,\lambda}$. Since $\tilde{Q}_{s,x} w_{a,\lambda} = \theta \tilde{Q}_{s,x} v_{a,\lambda} + [\tilde{Q}_{s,x}, \theta] v_{a,\lambda}$. We can apply (4.5) to $w_{a,\lambda}$ with (4.7) and (4.8) to get that

$$e^{(\lambda r_0^2/4)} \int_0^{r_0/2} \|v_{a,\lambda}\|^2\, d\mu \leqslant \int_0^{r_0} e^{\lambda(\mu - r_0)^2} \|w_{a,\lambda}\|^2\, d\mu$$
$$\leqslant c\left(\lambda^{-1} + r_0^2\right) \int_0^{r_0} e^{\lambda(\mu - r_0)^2} \|\theta \tilde{Q}_{s,x} v_{a,\lambda}\|^2\, d\mu$$

$$
\begin{aligned}
&+ c\big(\lambda^{-1} + r_0^2\big) \int_0^{r_0} e^{\lambda(\mu - r_0)^2} \big\| [\tilde{Q}_{s,x}, \theta] v_{a,\lambda} \big\|^2 \, d\mu \\
&\leqslant c\big(\lambda^{-1} + r_0^2\big)\lambda^{11} e^{\lambda r_0^2 - \lambda(T-|a|)^2 + \lambda s_0^2} + c\big(\lambda^{-1} + r_0^2\big)\lambda^9 e^{(\lambda r_0^2/25) + \lambda s_0^2}. \qquad (4.9)
\end{aligned}
$$

Multiplying $e^{-\lambda r_0^2/4}$ on both sides of (4.9), we have

$$
\int_0^{r_0/2} \|v_{a,\lambda}\|^2 \, d\mu \leqslant c\big(\lambda^{-1} + r_0^2\big)\lambda^{11} e^{(3\lambda r_0^2/4) - \lambda(T-|a|)^2 + \lambda s_0^2} + c\big(\lambda^{-1} + r_0^2\big)\lambda^9 e^{(-21\lambda r_0^2/100) + \lambda s_0^2}. \qquad (4.10)
$$

Let $|a| < T - \sqrt{7/8}\,r_0$ and $s_0 < r_0/10$, they imply the power exponents in the first and second terms on the right-hand side of (4.10) satisfy

$$
\begin{cases}
(3/4)\lambda r_0^2 - \lambda\big(T - |a|\big)^2 + \lambda s_0^2 < -(1/10)\lambda r_0^2, \\
-(21/100)\lambda r_0^2 + \lambda s_0^2 < -(1/5)\lambda r_0^2.
\end{cases} \qquad (4.11)
$$

By (4.9), (4.10) and (4.6), we have for $|a| < T - \sqrt{7/8}\,r_0$, $0 < \mu < r_0/2$ and $|x'| \leqslant r_0$ that

$$
v_{a,\lambda}(x', \mu, 0) \to 0 = u(x', \mu, a) \quad \text{as } \lambda \to \infty.
$$

This completes the proof of Theorem 1.1.

Proof of Corollary 1.2. First of all, we note that the constant r_0 in Theorem 1.1 can be taken uniformly with respect to $x \in \bar{\Omega}$. Hence, T_1 can be also taken uniformly with respect to $x \in \bar{\Omega}$. We use the following steps to continue u by zero from $B(x_0, r)$ onto the whole Ω.

Step 1. Let

$$
D_+(x_0, r, \theta_0) := \big\{ y \in B(x_0, r);\ \big|(y - x_0)\cdot n_0\big| \geqslant |y - x_0| \cos\theta_0 \big\}
$$

with small $0 < \theta_0 < \pi/2$. In each boundary point $y \in \partial D_+(x_0, r, \theta_0)$, we can get a ball with radius r_0 inside $B(x_0, r)$ and y is in its boundary. By Theorem 1.1, we can continue u by zero onto a neighborhood $U(y)$ of each boundary point $y \in \partial D_+(x_0, r, \theta_0)$. We give Fig. 2 about $y = (0,0,0)$ and $n_y = (0,0,1)$. Hence, covering $\partial D_+(x_0, r, \theta_0)$ by finite numbers of such $U(y)$'s, we can continue u by zero onto $D_+(x_0, \tilde{r}, \theta_0)$ with $\tilde{r} > r$.

Step 2. Since the constant r_0 in Theorem 1.1 can be taken uniformly, we apply Theorem 1.1 to $y \in \partial D_+(x_0, r, \theta_0)$ to continue u by zero onto

$$
D_-(x_0, \hat{r}, \theta_0) := \big\{ y \in B(x_0, \hat{r});\ \big|(y - x_0)\cdot n_0\big| \leqslant |y - x_0| \cos\theta_0 \big\},
$$

where the small enough θ_0 is chosen dependently on r_0 and $r < \hat{r} < \tilde{r}$.

Step 3. Next take any $z \in \partial B(x_0, \hat{r})$ and repeat Steps 1 and 2. Then we can continue u by zero onto a more larger ball centered at x_0. It should be noted that the size of extending the radius of the ball in which u is zero can be kept uniform.

By repeating these steps, we can continue u by zero onto the whole Ω if T_3 is large enough. $\square$

5. Application to extending the Dirichlet–Neumann map

Let $u^f(t,x)$ be the solution of

$$\begin{cases} \partial_t^2 u^f(t,x) - \mathcal{M}u^f(t,x) = 0 & \big((t,x)\in Q^T\big), \\ u^f(t,x)\big|_{\Sigma^T} = f(t,x), \\ u^f(t,x)\big|_{t=0} = \partial_t u^f(t,x)\big|_{t=0} = 0, \end{cases} \tag{5.1}$$

where $Q^T := (0,T)\times\Omega$, $\Sigma^T := (0,T)\times\partial\Omega$, $f(t,x)\in H^1(\Sigma^T)$ and the αth component $(\mathcal{M}u)_\alpha(t,x)$ is given by

$$(\mathcal{M}u)_i(t,x) = \rho(x)^{-1}\sum_{j,l,k=1}^{3}\partial_j\big(C_{ijkl}(x)\partial_l u_k\big) \quad (1\leqslant i\leqslant 3),$$

here $C_{ijkl}(x)$ is the same elasticity tensor as above and $0<\rho(x)\in C^\infty(\bar{\Omega})$. We assume f to satisfy the compatibility condition: $f(0,x)=0$. It should be noted that $u^f\in C^1([0,T],L^2(\Omega))\cap C([0,T],H^1(\Omega))$ for $T>0$.

We define the response operator

$$R^T : H_0^1\big(\Sigma^T\big)\to L^2\big(\Sigma^T\big)$$

by

$$\big(R^T f\big)_i = \sum_{j,l,k=1}^{3}\nu_j C_{ijkl}(x)\partial_l u_k \quad (1\leqslant i\leqslant 3),$$

where $\nu := (\nu_1,\nu_2,\nu_3)$ is the outer normal vector of $\partial\Omega$, $u^f = (u_1^f,u_2^f,u_3^f)^t$ is the solution to (5.1) and $H_0^1(\Sigma^T) := \{f\in H^1(\Sigma^T),\ f(0,x)=0\}$. See Appendix A for a justification of the statement of this operator.

Theorem 5.1. *Let r be the radius of Ω. There exists a positive constant T_4 coming from Corollary* 1.3 *such that if we have $R^{\tilde{T}}f$ for some $\tilde{T}$, $\tilde{T}\geqslant T_4$, then we can determine $R^T f$ for every $T>0$.*

The proof follows that of Theorem 1.4 in [6]. To do it, we need the following key lemma whose proof is given by the standard argument using Corollary 1.3.

Lemma 5.2. *Let r be the radius of Ω. There exists a positive constant T_4 such that for any $T\geqslant T_4$ the set $\{(u^f(T),\partial_t u^f(T));\ f\in C_0^\infty([0,T]\times\partial\Omega)\}$ is dense in $H_0^1(\Omega)\times L^2(\Omega)$.*

Proof. Assume that the conclusion is false. Let $(\psi,\varphi)\in H^{-1}(\Omega)\times L^2(\Omega)$ be such that

$$\big\langle u^f(\cdot,T),\psi\big\rangle - \big\langle\partial_t u^f(\cdot,T),\varphi\big\rangle = 0$$

for all $f\in C_0^\infty(\Sigma^T)$, where the first and the second pairings are for the pairs in $H_0^1(\Omega)\times H^{-1}(\Omega)$ and $L^2(\Omega)\times L^2(\Omega)$ which use the natural extension of the pairing

$$\langle a,b\rangle = \int_\Omega ab\,dx \quad \big(a,b\in L^2(\Omega)\big).$$

Since $C_0^\infty(\Omega)$ is dense in both $L^2(\Omega)$ and $H_0^1(\Omega)$, we can assume $(\psi,\varphi)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$. To show that $\varphi=\psi=0$, we consider the solution $e\in C^1([0,T],H^{-1}(\Omega))$ of the following initial–boundary value problem:

$$\begin{cases} (\partial_t^2-\mathcal{M})e=0 \quad \text{in } Q^T,\\ e|_{\Sigma^T}=0,\\ e|_{t=T}=\varphi,\\ \partial_t e|_{t=T}=\psi. \end{cases} \tag{5.2}$$

Let $u^f(x,t)$ be the solution of the initial–boundary value problem (5.1). Taking the inner product of the equation in (5.1) with ρe and doing the same for the equation in (5.2) with ρu^f and integrating by parts, we have

$$\begin{aligned} 0&=\int\limits_{Q^T}\big[\big(\rho\big(\partial_t^2-\mathcal{M}\big)u^f\big)\cdot e-u^f\cdot\rho\big(\partial_t^2-\mathcal{M}\big)e\big]\,dx\,dt\\ &=\int\limits_{\Sigma^T} f\mathcal{N}e\,ds\,dt, \end{aligned} \tag{5.3}$$

where $(\mathcal{N}e)_i=\sum_{j,l,k=1}^3\nu_jC_{ijkl}(x)\partial_le_k$ for $e=(e_1,e_2,e_3)^t$. Since $f\in C_0^\infty(\Sigma^T)$ is arbitrary,

$$\mathcal{N}e|_{\Sigma^T}=0.$$

By Corollary 1.3, we obtain that

$$e|_{t=T/2}=\partial_te|_{t=T/2}=0. \tag{5.4}$$

Hence, $e(t,x)$ is the solution of the hyperbolic system (5.2) stated on $(T/2,T)\times\Omega$ with homogeneous initial conditions (5.4) on $T/2$. This implies that $e(t)=0$ for $t\in[T/2,T]$. Therefore,

$$\begin{aligned} &\varphi=e|_{t=T}=0,\\ &\psi=\partial_te|_{t=T}=0. \end{aligned}$$

This completes the proof. □

Proof of Theorem 5.1. Let T_4 be the value given in Lemma 5.2 and $T>T_4$. Suppose that we know R^T and set $\epsilon=(T-T_4)/2$. We are going to prove that $R^{T+\epsilon}$ is known. We take $f\in C_0^\infty([0,T+\epsilon]\times\partial\Omega)$. We write $f=g+h$ where $\operatorname{supp}h\subset[0,T_4]\times\partial\Omega$ and $\operatorname{supp}g\subset[\epsilon,T+\epsilon]\times\partial\Omega$. Since $u^f=u^h+u^g$, if we know $\mathcal{N}u^h|_{[0,T+\epsilon]\times\partial\Omega}$ and $\mathcal{N}u^g|_{[\epsilon,T+\epsilon]\times\partial\Omega}$ then we know $\mathcal{N}u^f|_{[0,T+\epsilon]\times\partial\Omega}$.

Now, let $g^\epsilon(t,\cdot):=g(t+\epsilon,\cdot)$ and $g^0:=g$. By $g^\epsilon(t,\cdot)=g(t+\epsilon,\cdot)$, then $\mathcal{N}u^{g(t)}|_{[\epsilon,T+\epsilon]\times\partial\Omega}(t,\cdot)=\mathcal{N}u^{g(t+\epsilon)}|_{[0,T]\times\partial\Omega}(t,\cdot)$. As $\operatorname{supp}g^\epsilon(t,\cdot)\subset[0,T]\times\partial\Omega$, then $\mathcal{N}u^g|_{[\epsilon,T+\epsilon]\times\partial\Omega}$ is known since we know R^T. So, we have to continue $\mathcal{N}u^h$ from $[0,T]$ to $[0,T+\epsilon]$. To do it, we denote $t_0=T_4+\epsilon$. By Lemma 5.2, there exist $h_m\in C_0^\infty([0,T_4]\times\partial\Omega)$ such that

$$\lim_{m\to\infty}\big(u^{h_m}(T_4,\cdot),\partial_tu^{h_m}(T_4,\cdot)\big)=\big(u^h(t_0,\cdot),\partial_tu^h(t_0,\cdot)\big) \tag{5.5}$$

in $H_0^1(\Omega) \times L^2(\Omega)$. The functions $y^m(t,\cdot) := u^{h_m}(t,\cdot)$ with $t \in [T_4, T]$ are the solutions of the initial value problem

$$\begin{cases} \partial_t^2 y^m(t,x) - \mathcal{M} y^m(t,x) = 0 \quad \big((t,x) \in [T_4,T] \times \Omega\big), \\ y^m(t,x)\big|_{[T_4,T]\times\partial\Omega} = 0, \\ y^m(t,x)\big|_{t=T_4} = u^{h_m}(T_4,x), \\ \partial_t y^m(t,x)\big|_{t=T_4} = \partial_t u^{h_m}(T_4,x). \end{cases} \tag{5.6}$$

On the other hand, the function $y(t,\cdot) := u^h(t+\epsilon,\cdot)$ satisfies the same equation in (5.6) with different initial data

$$\begin{cases} \partial_t^2 y(t,x) - \mathcal{M} y(t,x) = 0 \quad \big((t,x) \in [T_4,T] \times \Omega\big), \\ y(t,x)|_{[T_4,T]\times\partial\Omega} = 0, \\ y(t,x)|_{t=T_4} = u^h(t_0,x), \\ \partial_t y(t,x)|_{t=T_4} = \partial_t u^h(t_0,x). \end{cases} \tag{5.7}$$

The continuous dependence of solutions on initial data, Lemma A.1 in Appendix A, and (5.5) imply that

$$\lim_{m\to\infty} \mathcal{N} y^m\big|_{[T_4,T]\times\partial\Omega} = \mathcal{N} y\big|_{[T_4,T]\times\partial\Omega} \tag{5.8}$$

in L^2 space. By (5.8), $\mathcal{N} u^h|_{[T_4+\epsilon,T+\epsilon]\times\partial\Omega}$ is determined. Combining the known information $\mathcal{N} u^h|_{[0,T]\times\partial\Omega}$, we can determine $\mathcal{N} u^h|_{[0,T+\epsilon]\times\partial\Omega}$. $\square$

Appendix A

The purpose of this appendix is to give the proof to following theorem which justifies the definition of the operator R^T and the well-posedness of the problem (5.6).

Lemma A.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is smooth. Denote $Q^T := (0,T) \times \Omega$ and $\Sigma^T := (0,T) \times \partial\Omega$. Assume that $F \in L^1((0,T); L^2(\Omega))$, $f \in H^1(\Sigma^T)$, $\psi_0 \in H^1(\Omega)$ and $\psi_1 \in L^2(\Omega)$ with the compatibility condition $f(0,\cdot) = \psi_0|_{\partial\Omega}$. Then, there exists a unique solution $u = u(t,x) \in C((0,T); H^1(\Omega)) \cap C^1((0,T); L^2(\Omega))$ to*

$$\begin{cases} \partial_t^2 u - \mathcal{M} u = F & \text{in } Q^T, \\ u|_{\Sigma^T} = f, \\ u|_{t=0} = \psi_0 & \text{in } \Omega, \\ \partial_t u|_{t=0} = \psi_1 & \text{in } \Omega. \end{cases} \tag{A.1}$$

Moreover, it satisfies

$$\begin{cases} \mathcal{N} u \in L^2\big(\Sigma^T\big), \\ \|\mathcal{N} u\|_{L^2(\Sigma^T)} \leqslant C(T)\left(\displaystyle\int_0^T \big\|F(t)\big\|_{L^2(\Omega)}\,dt + \|f\|_{H^1(\Sigma^T)} + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{L^2(\Omega)}\right), \end{cases} \tag{A.2}$$

where $\mathcal{N} u$ is given by (5.3) *replacing e by the solution u to* (A.1)*.*

Proof. (A.1) can be proven in the same way as Theorem 2.1 of [7]. This is because $\mathcal{M}$ is formally self-adjoint and it satisfies Gårding's inequality, and hence all the argument in Theorem 2.1 of [7] can be applied without any essential change to prove (A.1).

(A.2) can be proven like Step 2 in the proof of Theorem 2.30 in [5]. However, we need to adapt the proof a little. We only give the part of the proof which differs from that given in [5]. Let ν be C^∞ vector field on $\bar{\Omega}$ such that $\nu(x)$ coincides with the outer unit normal vector field of $\partial\Omega$ if $x \in \partial\Omega$. Also let $\chi \in C^\infty(\mathbb{R})$ satisfy $\chi(t) = 1$ for $t \leqslant T$ and 0 for $t \geqslant 2T$. Continue F by zero onto the interval $[T, 2T)$. Write

$$\int_{Q^T} \rho\chi F \cdot (\nabla u\nu)\,dx\,dt = \int_{Q^T} \chi\rho\big(\partial_t^2 u - \nabla\cdot(C\nabla u)\big)\cdot(\nabla u\nu)\,dx\,dt = I_1 + I_2,$$

where $I_1 = \int_{Q^{2T}} \rho\chi\partial_t^2 u\cdot(\nabla u\nu)\,dx\,dt$ and $I_2 = -\int_{Q^{2T}} (\chi\nabla\cdot(C\nabla u))\cdot(\nabla u\nu)\,dx\,dt$. It should be noted that $\nabla u\nu$ stands for the matrix ∇u multiplied by the vector ν. Also, let

$$\begin{aligned} J_1 :=& \int_\Omega \rho\psi_1\cdot(\nabla\psi_0\nu)\,dx + \int_{Q^{2T}} \rho\partial_t\chi\,\partial_t u\cdot(\nabla u\nu)\,dx\,dt - (1/2)\int_{Q^{2T}} \rho\chi(\nabla\cdot\nu)|\partial_t u|^2\,dx\,dt \\ &- (1/2)\int_{Q^{2T}} \chi\nabla\rho\cdot|\partial_t u|^2\nu\,dx\,dt + (1/2)\int_{\Sigma^{2T}} \rho\chi|\partial_t u|^2\,ds\,dt. \end{aligned}$$

Then, by integration by parts, it is easy to see

$$\begin{aligned} I_1 &= \left[\int_\Omega \rho\chi\,\partial_t u\cdot(\nabla u\nu)\,dx\right]_0^{2T} - \int_{Q^{2T}} \rho\partial_t\chi\,\partial_t u\cdot(\nabla u\nu)\,dx\,dt - \int_{Q^{2T}} \rho\chi\,\partial_t u\cdot\big(\nabla(\partial_t u)\nu\big)\,dx\,dt \\ &= -\int_\Omega \rho\psi_1\cdot(\nabla\psi_0\nu)\,dx - \int_{Q^{2T}} \rho\partial_t\chi\,\partial_t u\cdot(\nabla u\nu)\,dx\,dt - (1/2)\int_{Q^{2T}} \rho\chi\big(\nabla|\partial_t u|^2\big)\cdot\nu\,dx\,dt \\ &= -\int_\Omega \rho\psi_1\cdot(\nabla\psi_0\nu)\,dx - \int_{Q^{2T}} \rho\partial_t\chi\,\partial_t u\cdot(\nabla u\nu)\,dx\,dt + (1/2)\int_{Q^{2T}} \rho\chi(\nabla\cdot\nu)|\partial_t u|^2\,dx\,dt \\ &\quad + (1/2)\int_{Q^{2T}} \chi\nabla\rho\cdot|\partial_t u|^2\nu\,dx\,dt - (1/2)\int_{\Sigma^{2T}} \rho\chi|\partial_t u|^2\,ds\,dt \\ &= -J_1, \end{aligned} \tag{A.3}$$

where the second identity use

$$\begin{aligned} \partial_t u\cdot\big(\nabla(\partial_t u)\nu\big) &= \sum_i \partial_t u_i \sum_j \partial_j(\partial_t u_i)\nu_j \\ &= (1/2)\sum_{i,j} \partial_j(\partial_t u_i)^2\nu_j = (1/2)\big(\nabla|\partial_t u|^2\big)\cdot\nu. \end{aligned}$$

On the other hand, the integration by parts yields

$$I_2 = \int_{Q^{2T}} \chi C\nabla u\cdot\nabla(\nabla u\nu)\,dx\,dt - \int_{\Sigma^{2T}} \chi\big((C\nabla u)\nu\big)\cdot(\nabla u\nu)\,ds\,dt.$$

By the direct computations, we get

$$\begin{aligned}
&C\nabla u\cdot\nabla(\nabla u\nu)\\
&\quad= C\nabla u\cdot\bigl((\nu\cdot\nabla)\nabla u\bigr)\\
&\quad= (1/2)(\nu\cdot\nabla)(C\nabla u\cdot\nabla u)-(1/2)\bigl((\nu\cdot\nabla)C\bigr)\nabla u\cdot\nabla u+C\nabla u\cdot(\nabla u\nabla\nu)\\
&\quad= (1/2)\operatorname{div}\bigl((C\nabla u\cdot\nabla u)\nu\bigr)-(1/2)\operatorname{div}\nu(C\nabla u\cdot\nabla u)-(1/2)\bigl((\nu\cdot\nabla)C\bigr)\nabla u\cdot\nabla u+C\nabla u\cdot(\nabla u\nabla\nu).
\end{aligned}$$

Therefore,

$$I_2=(1/2)\int\limits_{\Sigma^{2T}}\chi(C\nabla u\cdot\nabla u)\,ds\,dt-\int\limits_{\Sigma^{2T}}\chi\bigl((C\nabla u)\nu\bigr)\cdot(\nabla u\nu)\,ds\,dt-J_2 \tag{A.4}$$

with

$$\begin{aligned}
J_2:={}&(1/2)\int\limits_{Q^{2T}}\chi(\nabla\cdot\nu)(C\nabla u\cdot\nabla u)\,dx\,dt+(1/2)\int\limits_{Q^{2T}}\chi\bigl((\nu\cdot\nabla)C\nabla u\bigr)\cdot\nabla u\,dx\,dt\\
&-\int\limits_{Q^{2T}}\chi C\nabla u\cdot(\nabla u\nabla\nu)\,dx\,dt.
\end{aligned}$$

Now, we analyze

$$\begin{aligned}
K:={}&(1/2)\int\limits_{\Sigma^{2T}}\chi(C\nabla u\cdot\nabla u)\,ds\,dt-\int\limits_{\Sigma^{2T}}\chi\bigl((C\nabla u)\nu\bigr)\cdot(\nabla u\nu)\,ds\,dt\\
={}&\int\limits_{Q^{T}}\chi F\cdot(\nabla u\nu)\,dx\,dt+J_1+J_2
\end{aligned}\tag{A.5}$$

locally, that is we analyze K when we confine u to a neighborhood $\subset\bar{\Omega}$ around a point $\in\partial\Omega$ by using a partition of unity. In this neighborhood, we introduce boundary normal coordinates (y_1,y_2,y_3) such that $\partial\Omega$ and Ω are given locally by

$$\partial\Omega=\{y_3=0\},\qquad \Omega=\{y_3>0\}.$$

Define g and $\tilde{C}$ by

$$g=\bigl|\det(g_{ij})\bigr|,\quad g_{ij}=\sum_{k=1}^{3}\frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j}\quad(1\leqslant i,j\leqslant 2)$$

and

$$\tilde{C}=(\tilde{C}_{i\alpha k\beta}),\quad \tilde{C}_{i\alpha k\beta}=\sum_{j,l=1}^{3}C_{ijkl}\frac{\partial y_\alpha}{\partial x_j}\frac{\partial y_\beta}{\partial x_l},$$

respectively. Then K has the term

$$-(1/2)\int\limits_{\Sigma^{2T}}\chi\sum_{i,k=1}^{3}\tilde{C}_{i3k3}\partial_{y_3}u_i\partial_{y_3}u_k\sqrt{g}\,dy'\,dt, \tag{A.6}$$

where $y' = (y_1, y_2, y_3)$. The rest of the terms of K only contain at most $\partial_{y_3} u_i$ $(1 \leqslant i \leqslant 3)$. Since $(\tilde{C}_{i3k3})_{1 \leqslant i,k \leqslant 3}$ is positive definite, it is easy to prove the estimate in (A.2) from (A.3), (A.4), (A.5) and (A.6) by the argument of Step 2 in the proof of Theorem 2.1 in [5]. $\square$

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