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Integral Equations and Operator Theory

Transitivity of Jordan Algebras of Linear Operators: On Two Questions by Grünenfelder, Omladič and Radjavi

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Abstract. Let $\mathcal{A}$ be a Jordan algebra of linear operators on a vector space over a field of characteristic different from 2. In this short note, we show that (1) if $\mathcal{A}$ is 2-transitive, then it is dense, and (2) if $\mathcal{A}$ is n-transitive, $n \geq 1$, then a nonzero Jordan ideal of $\mathcal{A}$ is also n-transitive. These answer two questions posed by Grünenfelder, Omladič and Radjavi.

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1. Introduction

Let $\mathcal{L} = \mathcal{L}(V)$ be the algebra of all linear operators on a vector space V over the field F. A subset $\mathcal{S} \subseteq \mathcal{L}$ is said to be *n-transitive* ($n \geq 1$) if for any linearly independent set $\{x_1, \ldots, x_n\}$ of V and any elements $y_1, \ldots, y_n \in V$, there is an $S \in \mathcal{S}$ such that $Sx_i = y_i$ for $i = 1, \ldots, n$. If $\mathcal{S}$ is transitive for all $n \geq 1$, it is called *dense*.

There are nice results on dense associative algebras of linear operators in $\mathcal{L}$. Burnside's theorem states that if V is finite-dimensional and F is algebraically closed, then the only transitive associative subalgebra of $\mathcal{L}$ is $\mathcal{L}$ itself. Jacobson [5] showed that if $\mathcal{S}$ is an associative subalgebra of $\mathcal{L}$ and $\mathcal{S}$ is 2-transitive, then $\mathcal{S}$ is dense.

In 1993 Grünenfelder, Omladič and Radjavi [3] studied transitive nonassociative algebras, and obtained the Jordan analogs of the Burnside's and Jacobson's theorems. By a Jordan algebra of linear operators over a field F of characteristic

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not 2 we mean a linear subspace $\mathcal{A}$ of $\mathcal{S}$ such that for all $A, B \in \mathcal{A}$, it holds that $A \circ B = AB + BA \in \mathcal{A}$. In [3], the following theorems were proved.

Theorem 1.1 ([3, Theorem 2.1]). *Let F be any formally real closed field. Then, the only transitive Jordan algebra of $\mathcal{S}_n(F)$, the symmetric $n \times n$ matrices over F, is $\mathcal{S}_n(F)$ itself.*

Theorem 1.2 ([3, Theorem 2.4]). *Let F be an algebraically closed field, and let $\mathcal{A}$ be a transitive Jordan algebra of $n \times n$ matrices over F. Then either $\mathcal{A} = \mathbb{M}_n(F)$ or there exists an invertible matrix T such that $T^{-1}\mathcal{A}T = \mathcal{S}_n(F)$.*

Theorem 1.3 ([3, Theorem 1.3]). *Let $\mathcal{A}$ be a Jordan algebra of finite rank operators on a vector space V If $\mathcal{A}$ is 2-transitive, then it is dense, i.e. n-transitive for all $n > 1$.*

Theorem 1.4 ([3, Theorem 3.2]). *Every Jordan ideal $\mathcal{J} \neq 0$ of an $(n+1)$-transitive Jordan algebra $\mathcal{A}$ of operators on a vector space V is n-transitive, $n > 1$.*

Two questions were raised in [3, p. 346] aiming at parts untouched by Theorems 1.3 and 1.4:

Question 1. Is there an n-transitive Jordan algebra $\mathcal{A}$ with a Jordan ideal $\mathcal{A} \neq 0$ which is not n-transitive?

Question 2. Is there an n-transitive Jordan algebra which is not $(n + 1)$-transitive for any $n \geq 2$?

The purpose of this paper is to complete the unfinished task left by Grünenfelder, Omladič and Radjavi. Namely, we are going to answer the two questions stated above. The answers to both questions are negative. We will first prove

Theorem 1.5. *Let $\mathcal{A}$ be a Jordan algebra of linear operators on a vector space V over the field of characteristic different from 2. If $\mathcal{A}$ is 2-transitive, then $\mathcal{A}$ is dense.*

Thus *Question 2* would be answered. It is interesting to note that there are 1-transitive Jordan algebras which are not dense. The Jordan algebra $\mathcal{S}_n(\mathbb{C})$ of symmetric $n \times n$ matrices, $n \geq 2$, over the complex number field $\mathbb{C}$ serves as a nice counterexample (see [3, Theorem 1.2 and Corollary 2.5] for details).

Partial answer to *Question 1* is already provided by Theorems 1.4 and 1.5. But we will prove the following result to have it covered completely.

Theorem 1.6. *Let $\mathcal{A}$ be an n-transitive Jordan algebra of linear operators on a vector space V over the field of characteristic different from 2, where $n \geq 1$. If $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{A}$, then $\mathcal{J}$ is also n-transitive.*

Related problems for Lie algebras can be found in [1] and [4]. Also, some very interesting results were obtained recently for more general situation of transitive spaces of operators, and the paper by Davidson, Marcoux and Radjavi [2] provides a comprehensive study of algebraic and topological transitivity for linear spaces of operators.

2. The results

Throughout this section F is a field of characteristic not 2, V a vector space over F, $\mathcal{L} = \mathcal{L}(V)$ the algebra of all linear operators on V, and $\mathcal{A}$ a Jordan algebra of operators from $\mathcal{L}$. Note that for $A, B \in \mathcal{A}$, we have $A^2 = \frac{1}{2}(A \circ A) \in \mathcal{A}$ and $ABA = \frac{1}{2}((A \circ B) \circ A - A^2 \circ B) \in \mathcal{A}$.

We start with the following auxiliary lemma.

Lemma 2.1. *Let $\mathcal{S}$ be a n-transitive linear subspace of $\mathcal{L}$, $n \geq 2$, and let $x_1, \ldots, x_{n+1} \in V$, be linearly independent. Then there exists an $S \in \mathcal{S}$ such that*

$$Sx_i = 0 \text{ for all } i = 3, \ldots, n+1, \text{ and } Sx_1 \notin \operatorname{span}\{Sx_2\} \tag{2.1}$$

Proof. Assume the contrary, i.e. $Tx_1 \in \operatorname{span}\{Tx_2\}$ for any $T \in \mathcal{S}$ with $Tx_3 = \cdots = Tx_n = 0$.

Suppose that there exists a $T_1 \in \mathcal{S}$ such that $T_1x_1 = T_1x_3 = \ldots = T_1x_{n+1} = 0$ and $T_1x_2 = u \neq 0$. Let $w \in V$ be a vector linearly independent with u. It exists since $\mathcal{S}$ is n-transitive with $n \geq 2$. By the n-transitivity of $\mathcal{S}$, there exists some $T_2 \in \mathcal{S}$ such that $T_2x_i = 0$ for $i \geq 3$ and $T_2x_1 = w$. By our assumption, $T_2x_1 \in \operatorname{span}\{T_2x_2\}$, and so $T_2x_2 = \alpha w$ for some nonzero $\alpha \in F$. We have $S = T_1 + T_2 \in \mathcal{S}$ satisfies (2.1), a contradiction. Therefore, for any $T \in \mathcal{S}$, if $Tx_1 = Tx_3 = \ldots = Tx_{n+1} = 0$, then readily $Tx_2 = 0$.

Let $S_1, S_2 \in \mathcal{S}$ with $S_1x_i = S_2x_i = 0$ for all $i \geq 3$ while $S_1x_1 \neq 0$ and $S_2x_1 \neq 0$. By our assumption, there are $\lambda_1, \lambda_2 \in F$ such that $S_1x_1 = \lambda_1S_1x_2$ and $S_2x_1 = \lambda_2S_2x_2$.

Now, it may be that S_1x_1 and S_2x_1 are linearly independent. In this situation, we set $S = S_1 + S_2$. Then

$$Sx_i = 0 \text{ for } i = 3, \ldots, n+1,\ Sx_1 = S_1x_1 + S_2x_1, \text{ and } Sx_2 = \lambda_1^{-1}S_1x_1 + \lambda_2^{-1}S_2x_1.$$

We see that if $\lambda_1 \neq \lambda_2$, then Sx_1 and Sx_2 are linearly independent, which cannot be. Hence we have

$$\text{if } S_1x_1 \text{ and } S_2x_1 \text{ are linearly independent, then } \lambda_1 = \lambda_2. \tag{2.2}$$

Or, it may be that S_1x_1 and S_2x_1 are linearly dependent. We argue that $\lambda_1 = \lambda_2$ as well. Assume that $\lambda_1 \neq \lambda_2$. Note that we cannot have any $T \in \mathcal{S}$ and $\mu \in F$ with $Tx_i = 0$ for $i = 3, \ldots, n+1$, and $Tx_1 = \mu Tx_2 \neq 0$, such that Tx_1 and S_1x_1 are linearly independent (hence Tx_1 and S_2x_1 are linearly independent as well), otherwise a contradiction that $\lambda_1 = \mu = \lambda_2$ would arise by (2.2). Now, this means that for all $T \in \mathcal{S}$, $Tx_i = 0$ for all $i \geq 3$ implies $Tx_1 \in \operatorname{span}\{S_1x_1\}$. But this contradicts the fact that $\mathcal{S}$ is n-transitive. Therefore, we have

$$\text{if } S_1x_1 \text{ and } S_2x_1 \text{ are linearly dependent, then } \lambda_1 = \lambda_2. \tag{2.3}$$

From (2.2) and (2.3) we conclude that there is a $\lambda \in F$ such that if $S \in \mathcal{S}$ with $Sx_i = 0$ for $i \geq 3$ and $Sx_1 \neq 0$, then $Sx_1 = \lambda Sx_2$. By above if $Sx_1 = Sx_3 = \ldots = Sx_{n+1} = 0$, then $Sx_2 = 0$. But then for every $S \in \mathcal{S}$ with $Sx_i = 0$ for all $i \geq 3$, we have $S(x_1 - \lambda x_2) = 0$. Again, as $\mathcal{S}$ is n-transitive, this cannot happen. Therefore the lemma holds. $\square$

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. The theorem will be proved if we show that the n-transitivity of $\mathcal{A}$ with $n \geq 2$ implies that $\mathcal{A}$ is $(n+1)$-transitive.

Let $x_1, \ldots, x_{n+1} \in V$ be linearly independent, and $y_1, \ldots, y_{n+1} \in V$. Notice that if we can find $T_j \in \mathcal{A}$ $(j = 1, \ldots, n+1)$ such that $T_j x_i = 0$ for $i \neq j$ and $T_j x_j = y_j$, then $T = T_1 + \cdots + T_{n+1}$ will do the required job that $Tx_i = y_i$ for $i = 1, 2, \ldots, n+1$. And to achieve this goal, it suffices to show that for any $y \in V$, there is some $T \in \mathcal{A}$ such that $Tx_i = 0$ for $i \geq 2$ and $Tx_1 = y$.

Let $y \in V$. Certainly we may assume that y is nonzero, or the zero operator will do the job. Let $A \in \mathcal{A}$ be such that $Ax_1 = y$ and $Ax_i = 0$ for $i \geq 3$. If Ax_1 and Ax_2 are linearly independent, we may take $B \in \mathcal{A}$ such that $BAx_1 = x_1$ and $BAx_2 = 0$, and put $T = ABA \in \mathcal{A}$. Then $Tx_1 = ABAx_1 = Ax_1 = y$ and $Tx_i = ABAx_i = 0$ for all $i \geq 2$, and we are done. So we assume that $Ax_2 = \lambda Ax_1 = \lambda y$ for some $\lambda \in F$.

Assume that if $S \in \mathcal{A}$ with $Sx_2 \neq 0$ and $Sx_i = 0$ for $i = 3, \ldots, n+1$, then $Sx_1 \neq 0$. By Lemma 2.1 there is an S such that $Sx_i = 0$ for $i = 3, \ldots, n+1$, and $Sx_2 \notin \operatorname{span}\{Sx_1\}$. Thus Sx_1 and Sx_2 are linearly independent. Let $L \in \mathcal{A}$ be such that $LSx_1 = 0$ and $LSx_2 = x_2$. Then $SLS \in \mathcal{A}$ and $SLSx_1 = 0$, $SLSx_2 = Sx_2 \neq 0$, and $SLSx_i = 0$ for $i = 3, \ldots, n+1$, a contradiction. Therefore, there is an $S \in \mathcal{A}$ with $Sx_1 = Sx_i = 0$ for $i = 3, \ldots, n+1$ and $Sx_2 = v \neq 0$, and we fix it for what follows.

In the case that v and y are linearly independent, we set $C = A + S$. Then $Cx_1 = Ax_1 + Sx_1 = y$, $Cx_2 = Ax_2 + Sx_2 = \lambda y + v$, and $Ax_i = 0$ for $i \geq 3$. Since Cx_1 and Cx_2 are linearly independent, we can find $D \in \mathcal{A}$ with $DCx_1 = x_1$ and $DCx_2 = 0$. Then $T = CDC \in \mathcal{A}$ will do the job.

Finally, assume that $v = \mu y$ for some nonzero $\mu \in F$, and set $T = A - \frac{\lambda}{\mu} S$. Then $Tx_1 = Ax_1 - \frac{\lambda}{\mu} Sx_1 = y$, $Tx_2 = Ax_2 - \frac{\lambda}{\mu} Sx_2 = \lambda y - \frac{\lambda}{\mu} v = 0$, and $Tx_i = 0$ for $i \geq 3$. This completes the proof. $\square$

Remark 2.2. We note that the above proof is valid for any 2-transitive space $\mathcal{A}$ of linear operators on a vector space over any field satisfying $ABA \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.

As a corollary to Theorem 1.5 and Theorem 1.4 (or Corollary 3.3 of [3]), we have

Corollary 2.3. *If $\mathcal{A}$ is n-transitive Jordan algebra of linear operators on a vector space V with $n \geq 2$, and $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{A}$, then $\mathcal{J}$ is also n-transitive. Moreover, both $\mathcal{A}$ and $\mathcal{J}$ are dense.*

Thus, it remains to treat the 1-transitivity case in order to prove Theorem 1.6, which we shall present in the following. Note that if $A \in \mathcal{A}$ and $B \in \mathcal{J}$, then $ABA = \frac{1}{2}((A \circ B) \circ A - A^2 \circ B) \in \mathcal{J}$.

Theorem 2.4. *Let $\mathcal{A}$ be a 1-transitive Jordan algebra of linear operators on a vector space V, and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{A}$. Then $\mathcal{J}$ is also 1-transitive.*

Proof. Let $V_0 = \mathcal{J}V$. We first show that $V_0 = V$.

We claim that V_0 is a subspace of V. First of all, if $v_0 \in V_0$, then there is a $u_0 \in V$ and an $S \in \mathcal{J}$ such that $Su_0 = v_0$. Thus, for any $\lambda \in F$, $\lambda v_0 = \lambda S u_0 = S(\lambda u_0) \in V_0$. Now, take $v_1, v_2 \in V_0$. If either $v_1 = 0$ or $v_2 = 0$, then $v_1 + v_2 \in V_0$. So we assume that both v_1 and v_2 are nonzero. Let $u_1, u_2 \in V \setminus \{0\}$ and $T_1, T_2 \in \mathcal{J}$ be such that $T_1 u_1 = v_1$ and $T_2 u_2 = v_2$. In the case that $T_2 u_1 = v \neq 0$, we may pick $T \in \mathcal{A}$ such that $Tv = u_2$. Then $(T_1 + T_2 T T_2) \in \mathcal{J}$ and $(T_1 + T_2 T T_2) u_1 = v_1 + v_2$. On the other hand, in the case that $T_2 u_1 = 0$, we may pick $T \in \mathcal{A}$ such that $Tu_1 = u_2$. Then $(T_1 + T_2 \circ T) \in \mathcal{J}$ and $(T_1 + T_2 \circ T) u_1 = v_1 + v_2$. Hence we see that, in both cases, $v_1 + v_2 \in V_0$. Therefore V_0 is indeed a subspace of V.

Pick an arbitrary $w \in V$. Let $S \in \mathcal{J}$ be nonzero, and $u, v \in V$ such that $Su = v \neq 0$. Let $T \in \mathcal{A}$ be such that $Tv = w$. Then $(ST + TS)u = S(Tu) + w$. From $ST + TS \in \mathcal{J}$, it follows that $w = (ST + TS)u - S(Tu) \in V_0$. Therefore $V_0 = V$.

Now, let $u, v \in V$ with $u \neq 0$. We want to find some $R \in \mathcal{J}$ such that $Ru = v$. Let $S \in \mathcal{J}$ and $w \in V$ be such that $Sw = v$. If $Su = 0$, we can simply pick $T \in \mathcal{A}$ with $Tu = w$ and put $R = ST + TS \in \mathcal{J}$ to get $Ru = v$. If $Su = v' \neq 0$, we can pick $T \in \mathcal{A}$ such that $Tv' = w$, and put $R = STS \in \mathcal{J}$ to get $Ru = v$. Therefore, $\mathcal{J}$ is 1-transitive. □

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