ROOTLESS PAIRS OF EE_8-LATTICES

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ABSTRACT. We describe a classification of pairs M, N of lattices isometric to $EE_8 := \sqrt{2}E_8$ such that the lattice $M+N$ is integral and rootless and such that the dihedral group associated to them has order at most 12. It turns out that most of these pairs may be embedded in the Leech lattice. Complete proofs will appear in another article. This theory of integral lattices has connections to vertex operator algebra theory and moonshine.

1. INTRODUCTION

Associated to a sublattice S of an integral lattice L is an orthogonal transformation of order 2 on Euclidean space (defined to be -1 on S and 1 on the annihilator of S). This involution leaves L invariant if S satisfies the *RSSD condition* $(2L \leq S + ann(S))$ [9, 10]. An example of such an S is $\sqrt{2}U$, where U is an integral unimodular lattice, such as $EE_8 := \sqrt{2}E_8$. A pair of RSSD sublattices gives a dihedral group in the isometry group of L.

In this article, we describe our classification of pairs of EE_8-lattices which span an integral and rootless lattice and whose associated involutions (isometries of order 2) generate a dihedral group of order at most 12. There are exactly 11 such pairs. Most can be realized inside the Leech lattice, which we denote by Λ. We refer to an upcoming article [11] for proofs and details.

This work may be considered purely as a study of positive definite integral lattices. Our real motivation, however, is the evolving theory of vertex operator algebras (VOA) and their automorphism groups.

2. MOTIVATION FROM VOA THEORY

The primary connection between the Monster and vertex operator algebras was established in [3]. In [15], Miyamoto gave a theory for defining involutions on a VOA. Suppose that $(V, Y, \omega, \mathbf{1})$ is a VOA. A degree 2 element $e \in V$ is called a *rational conformal vector of central charge 1/2* if the components of $Y(e, z) = \sum_{n \in \mathbb{Z}} e_n z^{-n-1}$ (The e_i are in $End(V)$) together with Id_V span a Virasoro Lie algebra with central charge $\frac{1}{2}$ and the subVOA of V generated by w is a simple VOA, so is isomorphic to the simple Virasoro VOA $L(\frac{1}{2}, 0)$.

The simple Virasoro VOA $L(\frac{1}{2}, 0)$ has only three irreducible modules, $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{2})$, $L(\frac{1}{2}, \frac{1}{16})$, and the fusion rules give two formulas for an involution on V.

Since $L(\frac{1}{2}, 0)$ is rational, i.e., all $L(\frac{1}{2}, 0)$-modules are completely reducible, we can decompose $V := V_0 \oplus V_{\frac{1}{2}} \oplus V_{\frac{1}{16}}$ into the direct sum of V_c, where V_c denotes the sum of all $L(\frac{1}{2}, 0)$-submodules of V isomorphic to $L(\frac{1}{2}, c)$.

(1) Define a linear map $\tau_e : V \to V$ to be 1 on $V_0 \oplus V_{\frac{1}{2}}$ and -1 on $V_{\frac{1}{16}}$. Miyamoto[15] showed that τ_e is always an automorphism of V. This automorphism is called *a Miyamoto involution* if τ_e has order 2.

(2) If $\tau_e = Id_V$, i.e., $V_{\frac{1}{16}} = 0$, then the linear map σ_e defined by 1 on V_0 and -1 on $V_{\frac{1}{2}}$ is also an automorphism. It is called *a Miyamoto involution of σ-type*.

These involutions have been studied a lot. Miyamoto proved [16] the important result that there is a bijection between the conjugacy class of $2A$ involutions in the Monster simple group and conformal vectors of central charge $\frac{1}{2}$ in the moonshine vertex operator algebra $V^\natural$.

2.1. Lattice type vertex operator algebras.

Recall that a lattice VOA V_L associated with an even positive definite lattice L is given by $V_L = S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L]$, as a vector space, where $\mathfrak{h} := \mathbb{C} \otimes L$, $\hat{\mathfrak{h}}^- = \oplus_{n=1}^\infty t^{-n} \otimes \mathfrak{h}$, $S(\hat{\mathfrak{h}}^-)$ is the symmetric algebra of $\hat{\mathfrak{h}}^-$ and $\mathbb{C}[L] = \mathrm{span}_{\mathbb{C}}\{e^\alpha | \alpha \in L\}$ is the group algebra of L.

Let V_L be a lattice VOA. It is common to study an involution on V which is a "lift" of the -1 map on L. The standard lift of -1, called θ, acts as -1 on $\hat{\mathfrak{h}}^-$ and takes e^α to $e^{-\alpha}$, for all α. The fixed point subVOA of θ is denoted V_L^+.

For VOAs of lattice type V_L or V_L^+, there are two known explicit general formulas for rational conformal vectors of central charge $1/2$.

First formula:
When α is a norm 4 vector of L,

$$(1) \qquad e = \frac{1}{16}(t^{-1} \otimes \alpha)^2 \pm \frac{1}{4}(e^\alpha + e^{-\alpha})$$

is a rational conformal vector of central charge $1/2$.

Second formula:
When E is a sublattice of L, $E \cong EE_8$,

$$(2) \qquad e = \frac{1}{8}q + \frac{1}{32} \sum_{\alpha \in E/\{\pm 1\}} \varphi(\alpha)(e^\alpha + e^{-\alpha})$$

is also a rational conformal vector of central charge $1/2$, (here q is the sum $\sum_i (t^{-1} \otimes u_i)^2$, where u_i is an orthonormal basis of $\mathbb{C} \otimes E$; φ is a homomorphism $E \to \{\pm 1\}$).

This second formula explains our interest in EE_8.

2.2. Why rootless?

If L is rootless, V_L^+ has zero degree 1 term and so (by general VOA theory) the degree 2 part is a commutative algebra (generally nonassociative). In this case, the automorphism group of V_L^+ is finite (not generally true for VOAs). Moreover, it is conjectured [14] that the two kinds of conformal vectors in (1) and (2) will exhaust all the rational conformal vectors of central charge $1/2$ in V_L^+ if L is rootless. This conjecture was proved when L is $\sqrt{2}$ times a root lattice or the Leech lattice [14, 13] but it is still open if L is a general rootless lattice. The results of this paper could help settle this conjecture.

2.3. **Why dihedral groups of order at most 12?** We may consider pairs of Miyamoto involutions and what they generate. For a general VOA, this is an arbitrary dihedral group. In the case of V_L^+ when L is rootless, the possibilities are limited to dihedral of order at most 12, by a recent theorem of Shinya Sakuma [17]. In fact, his hypotheses are more general: the VOA need not be lattice type, but it should be defined over $\mathbb{R}$ with a positive definite form, have no negative terms, the degree 0 component should be 1-dimensional, and the degree 1 component should be 0-dimensional. His proof is quite technical.

Even without the Sakuma theorem, one knows that the restriction to dihedral groups of order at most 12 is relevant to a study of sporadic groups. Consider the 2A class of involutions in the Monster [12] and classes within the Baby Monster, Fi_{24}, etc., discussed in [4].

3. THE CLASSIFICATION OF INTEGRAL, ROOTLESS EE_8-PAIRS

In this section, we shall discuss our classification of EE_8-pairs. The following is our main theorem.

Theorem 3.1. *Suppose that in Euclidean space, the integral lattice L is the sum $M + N$ of sublattices, where $M \cong N \cong EE_8$, and L is rootless. Let t_M be the involution defined by M and let t_N be the involution defined by N. Assume that the dihedral group $\langle t_M, t_N \rangle$ has order at most 12. Then the possibilities for L are listed in Table 1 and all these possibilities exist. The lattices in Table 1 are unique up to isometry of pairs M, N. Except for $DIH_4(15)$, all of them embed as sublattices of the Leech lattice.*

TABLE 1. **NREE8SUMs: integral rootless lattices which are sums of EE_8s**

Name	$\langle t_M, t_N \rangle$	Isometry type of L (contains)	$\mathcal{D}(L)$	In Leech?
$DIH_4(12)$	Dih_4	$\geq DD_4^{\perp 3}$	$1^4 2^6 4^2$	Yes
$DIH_4(14)$	Dih_4	$\geq AA_1^{\perp 2} \perp DD_6^{\perp 2}$	$1^4 2^8 4^2$	Yes
$DIH_4(15)$	Dih_4	$\geq AA_1 \perp EE_7^{\perp 2}$	$1^2 2^{14}$	No
$DIH_4(16)$	Dih_4	$\cong EE_8 \perp EE_8$	2^{16}	Yes
$DIH_6(14)$	Dih_6	$\geq AA_2 \perp A_2 \otimes E_6$	$1^7 3^3 6^2$	Yes
$DIH_6(16)$	Dih_6	$\cong A_2 \otimes E_8$	$1^8 3^8$	Yes
$DIH_8(15)$	Dih_8	$\geq AA_1^{\perp 7} \perp EE_8$	$1^{10} 4^5$	Yes
$DIH_8(16, DD_4)$	Dih_8	$\geq DD_4^{\perp 2} \perp EE_8$	$1^8 2^4 4^4$	Yes
$DIH_8(16, 0)$	Dih_8	$\cong BW_{16}$	$1^8 2^8$	Yes
$DIH_{10}(16)$	Dih_{10}	$\geq A_4 \otimes A_4$	$1^{12} 5^4$	Yes
$DIH_{12}(16)$	Dih_{12}	$\geq AA_2 \perp AA_2$ $\perp A_2 \otimes E_6$	$1^{12} 6^4$	Yes

$X^{\perp n}$ denotes the orthogonal sum of n copies of the lattice X.

Notation 3.2. $A_1, \ldots, E_8$ *denote the root lattices of the corresponding type;*
  $AA_1, \ldots, EE_8$ *denote the lattices isometric to $\sqrt{2}$ times the lattices $A_1, \ldots, E_8$;*
  $\mathcal{D}(L) = L^*/L$ *is the discriminant group of L;*

$DIH_r(s)$ *means a pair of EE_8 lattices (M, N) such that t_M, t_N generates a dihedral group of order r and $\text{rank}(M + N) = s$;*

$DIH_8(16, P)$ *means $DIH_8(16)$ and that $P = ann_M(N) \cong ann_N(M)$.*

3.1. **Some highlights of the proof.** Next, we shall discuss the main steps for the classification. We go through cases $|t_M t_N| = 2, 3, 4, 5, 6$. Our respective analyses are called DIH_4*-theory, DIH_6-theory, DIH_8-theory, DIH_{10}-theory, DIH_{12}-theory.*

In $\langle t_M, t_N \rangle$, let $g := t_M t_N$. Then $\mathbb{Z}[\langle g \rangle]$ acts on L and it acts on $J := ann_L(Fix_L(g))$, where $Fix_L(g)$ denotes the set of all fixed points of g in L. The action is that of a ring of integers in a number field when $|g|$ is prime.

The main idea is to determine possibilities for $Fix_L(g)$, J, $ann_M(N)$, $ann_N(M)$ and related sublattices. Exhaustive case by case analysis gives a list of candidates. In all cases, the candidates are proved unique, given certain things we deduce about their sublattices.

First, we observe that $\langle t_M, t_N \rangle$ acts faithfully on L and leaves $Fix_L(g) = M \cap N$ invariant.

Case: $|g| = 2, 3, 5$.

In these cases, we determine all sublattices of E_8 which are direct summands and whose discriminant group is an elementary abelian p-group for $p = 2, 3, 5$. Exhaustive case by case analysis gives a list of candidates. It turns out $M \cap N$ is isometric to $\sqrt{2}$ times one of these lattices. In fact, $M \cap N \cong 0, AA_1, AA_1 \perp AA_1$, or DD_4 if $|g| = 2$; $M \cap N \cong 0$ or AA_2 if $|g| = 3$; and $M \cap N = 0$ if $|g| = 5$.

Given $M \cap N$, we then analyse J and its sublattices.

When $|g| = 2$, $\langle t_M, t_N \rangle$ is a four-group. Then we have $M \cap J = ann_M(N)$, $N \cap J = ann_N(M)$ and $M \cap N \perp ann_M(N) \perp ann_N(M)$ is an index 2 sublattice of L. In this case, the isometry type of L is uniquely determined by $M \cap N$.

When $|g| = 3$, we consider the $\mathbb{Z}[\langle g \rangle]$-submodule K generated by $M \cap J$. Then K is a sublattice of J and K is isomorphic as a lattice to $A_2 \otimes \frac{1}{\sqrt{2}}(M \cap J)$. The possibilities for $M \cap J$ in this case are $EE6$ or $EE8$. Again, the isometry type of L is uniquely determined by $M \cap N$.

When $|g| = 5$, $M \cap N = 0$. We show that for any norm 4 vector $\alpha \in L$, the $\mathbb{Z}[\langle g \rangle]$-submodule generated by α is isomorphic as a lattice to AA_4. In fact, we show that $L = M + N$ contains a sublattice U isometric to the orthogonal sum of 4 copies of AA_4 such that $M \cap U \cong N \cap U \cong AA_1^8$. The uniqueness of L is then shown by explicit glueings.

Case: $|g| = 4, 6$.

In each of these cases, let $h := g^2$. Then (M, Mh) and (N, Nh) are EE_8 pairs whose associated dihedral group has order 4 or 6. We then use the results for Dih_4 and Dih_6 to deduce the structures of L. It turns out that there is only one possible case for $|g| = 6$ but 3 different cases for $|g| = 4$.

A proof that the candidates are really rootless is made easier by a magic tool. Most candidates can be embedded in the Leech lattice by direct constructions (and use of a uniqueness result). Since the Leech lattice is rootless, so is our candidate L. See the last section for examples of embeddings.

4. Glauberman-Norton subgroups

In [4], Glauberman-Norton computed the centralizers in the Monster of a pair of 2A-involutions. We hope to study these subgroups by promoting pairs M, N of lattices from our list to involutions on the Moonshine VOA. An interesting point is that the Glauberman-Norton subgroups have "Levi factors" which look like index 2 subgroups of a direct product of a pair of Weyl groups modulo centers. Our theory may give a context for this by viewing such a subgroup as acting on the Moonshine VOA. At this time, we can see actions of direct products of Weyl groups on large subVOAs of the Moonshine VOA, in the following way. The Moonshine VOA is a twisted lattice type VOA based on the Leech lattice. The Leech lattice contains sublattices of types AA_1 or EE_8 which will give a concrete realization of each pair of $2A$-involutions in the Glauberman-Norton theory (by Formula (1) or Formula (2)). Since there are examples of tensor products of lattices contained in our $M+N$ when $|g| = 3$ or 5 (3.1), we have actions of a finite group with quotient a direct product of Weyl groups on a subVOA of lattice type.

5. Examples of some pairs M, N in the Leech lattice

In this section, we shall give a few examples of EE_8-pairs M, N in the Leech lattice. Because of space limitations, these are just for the DIH_6-theory. First, we recall some notation for describing the Leech lattice and its isometries [1, 7].

Notation 5.1. *Arrange the set $\Omega = \{1, \ldots, 24\}$ into a 4×6 array such that the six columns form a sextet. For each codeword in the Golay code $\mathcal{G}$, 0 and 1 are marked by a blanked and non-blanked space, respectively, at the corresponding positions in the array. For example, $(1^8 0^{16})$ is denoted by the array*

*	*				
*	*				
*	*				
*	*				

By the same notation, every vector in the Leech lattice Λ can be written in the form

$$X = \frac{1}{\sqrt{8}} \left[X_1 X_2 X_3 X_4 X_5 X_6 \right], \quad \textit{juxtaposition of column vectors.}$$

For example,

$$\frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

Definition 5.2. Let

$$A = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Define a linear map $\xi : \Lambda \to \Lambda$ by $X\xi = AXD$, where

$$X = \frac{1}{\sqrt{8}} [X_1 X_2 X_3 X_4 X_5 X_6]$$

is a vector in the Leech lattice Λ and D is the diagonal matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Recall that ξ defines an isometry of Λ (cf. [1, p. 288] and [7, p. 97]).

Notation 5.3. *Let $\mathcal{O}$ be an octad. Denote by $E(\mathcal{O})$ the sublattice of the Leech lattice Λ which is supported on $\mathcal{O}$. Then $E(\mathcal{O}) \cong EE_8$.*

Here are examples for the DIH_6 cases.

When $|g| = 3$, $M \cap N = 0$ or AA_2.

DIH$_6$(16)
Let

$$\mathcal{O}_1 = \begin{array}{|cc|c|c|} \hline * & * & & \\ * & * & & \\ * & * & & \\ * & * & & \\ \hline \end{array}$$

be an octad and $M := E(\mathcal{O}_1) \cong EE_8$. We choose a basis $\{\beta_1, \ldots, \beta_8\}$ of M, where

$$\beta_1 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \qquad \beta_2 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 0 & 4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array},$$

$$\beta_3 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \qquad \beta_4 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array},$$

$$\beta_5 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \qquad \beta_6 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array},$$

$$\beta_7 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline -4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \qquad \beta_8 = \frac{1}{\sqrt{8}} \begin{array}{|cc|cc|cc|} \hline 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

Let N be the lattice generated by the vectors

$$\alpha_1 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline 2 & \text{-2} & 2 & \text{-2} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ \hline \end{array}, \quad \alpha_2 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline 0 & 2 & 0 & 2 & 0 & 2 \\ \text{-2} & 0 & \text{-2} & 0 & 0 & \text{-2} \\ 0 & 0 & 0 & 0 & 2 & \text{-2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array},$$

$$\alpha_3 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & \text{-2} \\ 2 & \text{-2} & 2 & \text{-2} & 2 & 0 \\ 0 & 0 & 0 & 0 & \text{-2} & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ \hline \end{array}, \quad \alpha_4 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & \text{-2} & 2 \\ \text{-2} & 0 & \text{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{-2} \\ \hline \end{array},$$

$$\alpha_5 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & \text{-2} \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & \text{-2} & 2 & \text{-2} & 2 & 0 \\ 0 & 0 & 0 & 0 & \text{-2} & 0 \\ \hline \end{array}, \quad \alpha_6 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \text{-2} \\ 0 & 2 & 0 & 2 & \text{-2} & 2 \\ \text{-2} & 0 & \text{-2} & 0 & 0 & 0 \\ \hline \end{array},$$

$$\alpha_7 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline \text{-2} & \text{-2} & \text{-2} & \text{-2} & 0 & \text{-2} \\ 0 & 0 & 0 & 0 & \text{-2} & 0 \\ 0 & 0 & 0 & 0 & \text{-2} & 0 \\ 0 & 0 & 0 & 0 & \text{-2} & 0 \\ \hline \end{array}, \quad \alpha_8 = \frac{1}{\sqrt{8}}\begin{array}{|cc|cc|cc|} \hline 1 & 1 & 1 & 1 & \text{-3} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}.$$

By checking the inner products, it is easy to show that $N \cong EE_8$. Note that $\alpha_1, \ldots, \alpha_7$ are supported on octads and thus $N \le \Lambda$.

In this case, $M \cap N = 0$ and $\{\beta_1, \beta_2, \ldots, \beta_8, \alpha_1, \ldots, \alpha_8\}$ is a basis of $L = M + N$. The Gram matrix of $L = M + N$ is given by

$$\begin{bmatrix}
4 & \text{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \text{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{-2} & 4 & \text{-2} & 0 & 0 & 0 & \text{-2} & 0 & \text{-1} & 2 & \text{-1} & 0 & 0 & 0 & \text{-1} & 0 \\
0 & \text{-2} & 4 & \text{-2} & 0 & 0 & 0 & 0 & 0 & \text{-1} & 2 & \text{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & \text{-2} & 4 & \text{-2} & 0 & 0 & 0 & 0 & 0 & \text{-1} & 2 & \text{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & \text{-2} & 4 & \text{-2} & 0 & 0 & 0 & 0 & 0 & \text{-1} & 2 & \text{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & \text{-2} & 4 & 0 & 0 & 0 & 0 & 0 & 0 & \text{-1} & 2 & 0 & 0 \\
0 & \text{-2} & 0 & 0 & 0 & 0 & 4 & \text{-2} & 0 & \text{-1} & 0 & 0 & 0 & 0 & 2 & \text{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & \text{-2} & 4 & 0 & 0 & 0 & 0 & 0 & 0 & \text{-1} & 2 \\
2 & \text{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 4 & \text{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{-1} & 2 & \text{-1} & 0 & 0 & 0 & \text{-1} & 0 & \text{-2} & 4 & \text{-2} & 0 & 0 & 0 & \text{-2} & 0 \\
0 & \text{-1} & 2 & \text{-1} & 0 & 0 & 0 & 0 & 0 & \text{-2} & 4 & \text{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & \text{-1} & 2 & \text{-1} & 0 & 0 & 0 & 0 & 0 & \text{-2} & 4 & \text{-2} & 0 & 0 & 0 \\
0 & 0 & 0 & \text{-1} & 2 & \text{-1} & 0 & 0 & 0 & 0 & 0 & \text{-2} & 4 & \text{-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \text{-1} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \text{-2} & 4 & 0 & 0 \\
0 & \text{-1} & 0 & 0 & 0 & 0 & 2 & \text{-1} & 0 & \text{-2} & 0 & 0 & 0 & 0 & 4 & \text{-2} \\
0 & 0 & 0 & 0 & 0 & 0 & \text{-1} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \text{-2} & 4 \\
\end{bmatrix}$$

By looking at the Gram matrix, it is clear that $L = M + N \cong A_2 \otimes E_8$. The Smith invariant sequence is 1111111133333333.

DIH$_6$(12)

Let

$$\mathcal{O}_2 = \begin{array}{|c|c|c|c|c|} \hline & * & * & * & * & * \\ * & & & & & \\ * & & & & & \\ * & & & & & \\ \hline \end{array}$$

and $M := E(\mathcal{O}_2)$. Set $N := M\xi$, where ξ is the isometry defined in (5.2). Consider

$$\gamma_1 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \gamma_2 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\gamma_3 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & -4 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \gamma_4 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & 4 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\gamma_5 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \gamma_6 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\gamma_7 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \gamma_8 = \frac{1}{\sqrt{8}}\left[\begin{array}{cc|cc|cc} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Then $\{\gamma_1, \ldots, \gamma_8\}$ is a basis of M and $\{\gamma_1\xi, \ldots, \gamma_8\xi\}$ is a basis of N.

By the definition of ξ, it is easy to show that $\gamma_1\xi = -\gamma_1, \gamma_2\xi = -\gamma_2$. Moreover, for any $\alpha \in M = E(\mathcal{O}_2)$, $\alpha\xi$ is supported on $\mathcal{O}_2$ if and only if $\alpha \in \mathrm{span}_\mathbb{Z}\{\gamma_1, \gamma_2\}$. Hence, $F = M \cap N = \mathrm{span}_\mathbb{Z}\{\gamma_1, \gamma_2\} \cong AA_2$. Then $ann_M(F) \cong ann_N(F) \cong EE_6$ and $L = M + N$ is of rank 14.

Note that $\{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_8\}$ is a basis and $\{\gamma_1, \gamma_2, \gamma_3\xi, \ldots, \gamma_8\xi\}$ is a basis of N. Therefore,

$$\{\gamma_1, \gamma_2\} \cup \{\gamma_3, \ldots, \gamma_8\} \cup \{\gamma_3\xi, \ldots, \gamma_8\xi\}$$

is a basis of L and the Gram matrix of L is given by

$$\begin{bmatrix}
4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 4 & -2 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 2 & 0 \\
0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \\
0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 4 & -2 & 2 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 & 0 & 0 & -1 & -2 \\
0 & -2 & 0 & 1 & 0 & 0 & 2 & 0 & 4 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 2 & 4
\end{bmatrix}$$

whose Smith invariant sequence is 1111 11111 33366.

In this case, $ann_M(F) + ann_N(F) \cong A_2 \otimes E_6$ and thus L contains a sublattice isometric to $AA_2 \perp (A_2 \otimes E_6)$.

REFERENCES

[1] J. H. Conway and N. J. A. Sloane, "Sphere Packings, Lattices and Groups," 3rd Edition, With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 290, Springer-Verlag, New York, 1999. MR 1662447 (2000b:11077)

[2] C. Dong, H. Li, G. Mason and S. P. Norton, *Associative subalgebras of Griess algebra and related topics*, The Monster and Lie algebras (Columbus, OH, 1996), 27–42, Ohio State Univ. Math. Res. Inst. Publ., **7**, de Gruyter, Berlin, 1998. MR 1650629 (99k:17048)

[3] I. B. Frenkel, J. Lepowsky and A. Meurman, "Vertex Operator Algebras and the Monster," Pure and Applied Math., Vol. **134**, Academic Press, Inc., Boston, MA, 1988. MR 0996026 (90h:17026)

[4] G. Glauberman and S. P. Norton, *On McKay's connection between the affine E_8 diagram and the Monster*, Proceedings on Moonshine and related topics (Montreal, QC, 1999), 37–42, CRM Proc. Lecture Notes, **30**, Amer. Math. Soc., Providence, RI, 2001. MR 1877755 (2002k:20024)

[5] R. L. Griess, Jr., U. Meierfrankenfeld and Y. Segev, *A uniqueness proof for the Monster*, Ann. of Math.(2), **130** (1989), 567–602. MR 1025167 (90j:20030)

[6] R. L. Griess, Jr., *A vertex operator algebra related to E_8 with automorphism group $\mathrm{O}^+(10, 2)$*, in "The Monster and Lie algebras" (Columbus, OH, 1996), 43–58, Ohio State Univ. Math. Res. Inst. Publ., **7**, de Gruyter, Berlin, 1998. MR 1650633 (99i:17030)

[7] R. L. Griess, Jr., "Twelve Sporadic Groups," Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. MR 1707296 (2001c:20028)

[8] Robert L. Griess, Jr, *Pieces of eight: semiselfdual lattices and a new foundation for the theory of Conway and Mathieu groups*, Advances in Mathematics, **148** (1999), 75–104. MR 1736642 (2000m:20020)

[9] R. L. Griess, Jr., *Pieces of 2^d: existence and uniqueness for Barnes-Wall and Ypsilanti lattices*, Adv. Math., **196** (2005), 147–192. MR 2159298 (2006m:11096)

[10] R. L. Griess, Jr., *Corrections and additions to "Pieces of 2^d: existence and uniqueness for Barnes-Wall and Ypsilanti lattices"*, [Adv. Math., **196** (2005), 147–192], Advances in Mathematics- **211** (2007), 819–824. MR 2323546 (2008c:11095)

[11] R. L. Griess Jr., and C. H. Lam, *EE_8 lattices and dihedral groups*, preprint, about 85 pages. arXiv:0806.2753

[12] R. L. Griess, Jr., U. Meierfrankenfeld and Y. Segev, *A uniqueness proof for the Monster*, Ann. of Math. (2), **130** (1989), 567–602. MR 1025167 (90j:20030)

[13] C. H. Lam and H. Shimakura, *Ising vectors in the vertex operator algebra V_Λ^+ associated with the Leech lattice* Λ, Int. Math. Res. Not., (2007), Vol. **2007**: article ID rnm132, 21 pages. MR 2377016

[14] C. H. Lam, S. Sakuma and H. Yamauchi, *Ising vectors and automorphism groups of commutant subalgebras related to root systems*, Math. Z., **255** (2007), 597–626. MR 2270290 (2007h:17032)

[15] M. Miyamoto, *Griess algebras and conformal vectors in vertex operator algebras*, J. Algebra, **179** (1996), 523–548. MR 1367861 (96m:17052)

[16] M. Miyamoto, *A new construction of the moonshine vertex operator algebra over the real number field*, Ann. of Math, **159** (2004), 535–596. MR 2081435 (2005h:17052)

[17] S. Sakuma, *6-transposition property of τ-involutions of vertex operator algebras*, Int. Math. Res. Not. (2007), Vol. **2007** : article ID rnm030, 19 pages, doi:10.1093/imrn/rnm030. MR 2347298 (2008h:17033)

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