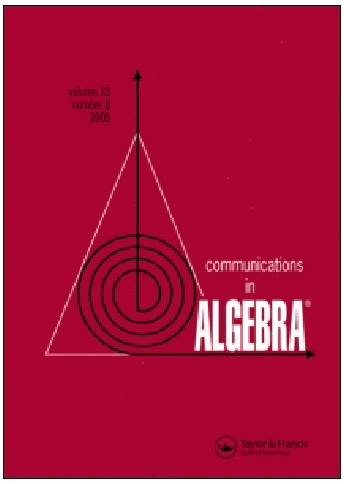

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<i>L</i>-Prime Rings Need Not Be Primary

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$\mathscr{L}$-PRIME RINGS NEED NOT BE PRIMARY

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We give an example of a commutative $\mathscr{L}$-prime ring with a non-nilpotent annihilator ideal, which solves a problem by Bergen, Montgomery, and Passman.

Key Words: Derivation; Lexicographic order; $\mathscr{L}$-prime ring; Primary ring; Prime ring.

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1. INTRODUCTION

Let K be a commutative ring, R a K-algebra, $\mathrm{Der}_K(R)$ the Lie algebra of K-derivations of R and $\mathscr{L}$ a K-submodule of $\mathrm{Der}_K(R)$. An ideal U of R is called *$\mathscr{L}$-stable* if $\sigma(U) \subseteq U$ for all $\sigma \in \mathscr{L}$. The ring R is called *$\mathscr{L}$-simple* if it has no non-trivial $\mathscr{L}$-stable ideals (in some articles, an additional restriction $R^2 \neq 0$ is required, see Fisher, 1975; Posner, 1960, for example). We say that R is *$\mathscr{L}$-prime* if, for $\mathscr{L}$-stable ideals U and V of R, $UV = 0$ implies $U = 0$ or $V = 0$ (see Bergen et al., 1987; Fisher, 1975).

It is known (see Fisher, 1975) that every $\mathscr{L}$-prime ring has characteristic zero or a prime number. The structures of $\mathscr{L}$-prime rings of characteristic 0 are quite different from those of prime characteristic. For example, let R be an $\mathscr{L}$-simple ring with $R^2 \neq 0$. If R is of characteristic zero, it is a prime ring (see Posner, 1960, Theorem 4). But if R is of prime characteristic, it need not be even semiprime. The standard counterexample is $R = K[x]/(x^p)$ with K a field of characteristic p, and $\mathscr{L}$ the space spanned by the derivation σ on R which is induced by the formal differentiation in $K[x]$.

An $\mathscr{L}$-prime ring can be prime under some additional assumptions, see Burkov (1980), Fisher (1975), and Jordan (1975). Some examples of $\mathscr{L}$-prime rings

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(even when $\mathscr{L}$ is generated by one derivation σ, that is, σ-prime rings) which are not prime have been known for a long time (see Fisher, 1975, p. 73; Jordan, 1975, Example 2.3, 1978, p. 448; Jordan and Jordan, 1978, p. 39). However, all these examples are of prime characteristic. Goodearl and Warfield (1982) posed the question whether there are σ-prime rings of characteristic zero which are not prime. An example answering this question was constructed by Bergen, Montgomery, and Passman (see Bergen et al., 1987, Proposition 1.3). This example is a Grassmann algebra generated by countably many elements $x_1, x_2, \dots$ over a field K and the derivation σ is the K-derivation with $\sigma(x_i) = x_{i+1}$ for $i \geq 1$.

Recall that a ring R is called *primary* if every annihilator ideal of R is nilpotent, that is, if I and J are ideals of R such that $IJ = 0$ or $JI = 0$, then either $I = 0$ or J is nilpotent. Certainly, prime rings are primary but not conversely. It was proved by Posner that an $\mathscr{L}$-simple ring R with $R^2 \neq 0$ is primary (see Posner, 1960, Theorem 3). As $\mathscr{L}$-prime rings need not be prime, it is natural to ask whether they are necessarily primary. Bergen, Montgomery, and Passman noticed that the $\mathscr{L}$-prime ring in their example is primary, and they mentioned that they did not know any $\mathscr{L}$-prime ring which is not primary, see Bergen et al. (1987, p. 172).

The goal of this article is to present such an example. We describe the idea of the construction briefly as follows. Denote by $\mathbb{Q}$ the field of rational numbers. First, we introduce a commutative $\mathbb{Q}$-algebra B as the factor algebra of the unital free commutative algebra $C = \mathbb{Q}\langle x_0, y_0, x_1, y_1, \dots \rangle$ by its ideal J generated by $\gamma_n = \sum_{i=0}^{n} x_i y_{n-i}$, $n = 0, 1, \dots$, and exhibit a basis of B over $\mathbb{Q}$. Then we choose a subspace $\mathscr{L}$ of $\mathrm{Der}_{\mathbb{Q}}(B)$ in a special way so that B is $\mathscr{L}$-prime. Since $(x_0 + J)(y_0 + J) = J$, the ideal generated by $y_0 + J$ annihilates the ideal generated by $x_0 + J$, but $y_0^n + J \neq J$ for all $n \geq 1$.

Note that the ideal J of C is designed so that

$$\sum_{i=0}^{\infty} x_i z^i \cdot \sum_{i=0}^{\infty} y_i z^i \equiv 0 \pmod{J}$$

in the ring $C[[z]]$ of formal power series over C. Thus, we hope that this algebra B can be used for some exotic examples of power series over commutative rings. A somehow similar construction has been used by Hamann and Swan (1986) to show that there exist a ring S and a nilpotent power series $f \in S[[z]]$ such that the nilpotency degrees of the coefficients of f are unbounded (see also Puczyłowski and Smoktunowicz, 1999).

2. THE CONSTRUCTION

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ the field of rational numbers. Let X be the set $\{x_0, y_0, x_1, y_1, \dots\}$ of commuting indeterminates and $C = \mathbb{Q}\langle X \rangle$ the free commutative algebra in X over $\mathbb{Q}$. Let $d : C \to C$ be the derivation on C such that $d(x_i) = x_{i+1}$ and $d(y_i) = y_{i+1}$ for all $i \geq 0$. For $n \geq 0$, set $c_n = d^n(x_0 y_0) = \sum_{i=0}^{n} C_i^n x_i y_{n-i}$, where C_i^n is the binomial coefficient $n!/i!(n-i)!$ (Leibnitz Formula, cf. Beidar et al., 1996, Remark 1.1.1). Thus, $c_0 = x_0 y_0$, $c_1 = x_0 y_1 + x_1 y_0$, $c_2 = x_0 y_2 + 2x_1 y_1 + x_2 y_0$, and so on. Let I be the ideal of C generated by $\{c_n \mid n = 0, 1, \dots\}$. Let $A = C/I$ and let $\varphi : C \to A$ be the canonical epimorphism with kernel I. We denote $a_i = \varphi(x_i)$ and $b_i = \varphi(y_i)$ for each $i = 0, 1, \dots$.

Remark 2.1. From $c_n = \sum_{i=0}^{n} C_i^n x_i y_{n-i}$ it follows that

$$\sum_{i=0}^{n} C_i^n a_i b_{n-i} = 0. \tag{2.1}$$

Let $A_{\mathbb{Z}} = \mathbb{Z}\langle X\rangle/(I \cap \mathbb{Z}\langle X\rangle)$. Then $A_{\mathbb{Z}}$ is isomorphic to the subring $(\mathbb{Z}\langle X\rangle + I)/I$ of A and so we shall identify them. Both algebras A and $A_{\mathbb{Z}}$ are generated by $\{a_i, b_i \mid i \geq 0\}$. The difference is that A is an algebra over $\mathbb{Q}$ while $A_{\mathbb{Z}}$ is an algebra over $\mathbb{Z}$.

It is not very convenient to work with the coefficients C_i^n all the time. So we consider another algebra here. For $n \geq 0$, let $\gamma_n = \sum_{i=0}^{n} x_i y_{n-i}$. That is, $\gamma_0 = x_0 y_0$, $\gamma_1 = x_0 y_1 + x_1 y_0$, $\gamma_2 = x_0 y_2 + x_1 y_1 + x_2 y_0$, and so on. Let J be the ideal of C generated by $\{\gamma_n \mid n = 0, 1, \ldots\}$. Let $B = C/J$ and let $\psi : C \to B$ be the canonical epimorphism with kernel J. We denote $\alpha_i = \psi(x_i)$ and $\beta_i = \psi(y_i)$ for each $i \geq 0$.

Remark 2.2. From $\gamma_n = \sum_{i=0}^{n} x_i y_{n-i}$ it follows that

$$\sum_{i=0}^{n} \alpha_i \beta_{n-i} = 0. \tag{2.2}$$

Remark 2.3. There exists an isomorphism $\phi : A \to B$ such that $\phi(a_n) = n!\alpha_n$ and $\phi(b_n) = n!\beta_n$ for all $n \geq 0$.

Proof. Define two maps $\phi : A \to B$ and $\phi' : B \to A$ by setting $\phi(a_n) = n!\alpha_n$, $\phi(b_n) = n!\beta_n$, $\phi'(\alpha_n) = a_n/n!$, and $\phi'(\beta_n) = b_n/n!$ for all $n \geq 0$ and extending these rules additively and multiplicatively. Both ϕ and ϕ' are well defined, since

$$\phi\left(\sum_{i=0}^{n} C_i^n a_i b_{n-i}\right) = n! \sum_{i=0}^{n} \alpha_i \beta_{n-i} = 0,$$

and

$$\phi'\left(\sum_{i=0}^{n} \alpha_i \beta_{n-i}\right) = \frac{1}{n!} \sum_{i=0}^{n} C_i^n a_i b_{n-i} = 0.$$

It is clear that $\phi'\phi = 1_A$ and $\phi\phi' = 1_B$. Hence ϕ is an isomorphism. □

Lemma 2.4. $\alpha_0^{n+1}\beta_n = \alpha_n \beta_0^{n+1} = 0$ *for all* $n \geq 0$.

Proof. By symmetry, it suffices to prove only $\alpha_0^{n+1}\beta_n = 0$. We proceed by induction on n. For $n = 0$, we have $\alpha_0\beta_0 = 0$ by (2.2). Assume that $n > 0$ and $\alpha_0^{m+1}\beta_m = 0$ for $m < n$. Then, by (2.2) again, we have

$$\alpha_0 \beta_n = -\sum_{i=1}^{n} \alpha_i \beta_{n-i}.$$

It follows from the induction hypothesis that $\alpha_0^n\beta_j = 0$ for all $j < n$ and so

$$\alpha_0^{n+1}\beta_n = -\sum_{i=1}^{n}\alpha_i\alpha_0^n\beta_{n-i} = 0.$$

□

For a prime number p, let $T = \mathbb{Z}_p[t]/(t^p)$ and denote $\bar{t} = t + (t^p) \in T$. In the sequel, we shall make use of the algebra homomorphism $\theta : A_{\mathbb{Z}} \to T$ defined by extending additively and multiplicatively the rules $\theta(a_0) = \bar{t}$, $\theta(a_1) = 1$, $\theta(a_i) = 0$ for all $i > 1$, $\theta(b_0) = \bar{t}^{p-1}$, $\theta(b_1) = (p-1)\bar{t}^{p-2}$, $\theta(b_2) = (p-1)(p-2)\bar{t}^{p-3}, \ldots, \theta(b_{p-1}) = (p-1)!$ and $\theta(b_i) = 0$ for all $i > p-1$. One can verify that θ is well defined as follows.

Let τ be the derivation of T such that $\tau(\bar{t}) = 1$. Then $\tau(\theta(a_i)) = \theta(a_{i+1})$ and $\tau(\theta(b_i)) = \theta(b_{i+1})$ for all $i \geq 0$. Then we have, by Leibnitz's Formula,

$$\begin{aligned}\theta\left(\sum_{i=0}^{n} C_i^n a_i b_{n-i}\right) &= \sum_{i=0}^{n} C_i^n \theta(a_i)\theta(b_{n-i}) \\ &= \sum_{i=0}^{n} C_i^n \tau^i(\theta(a_0))\tau^{n-i}(\theta(b_0)) \\ &= \tau^n\big(\theta(a_0)\theta(b_0)\big) \\ &= 0\end{aligned}$$

since

$$\theta(a_0)\theta(b_0) = \bar{t}\cdot\bar{t}^{p-1} = \bar{t}^p = 0.$$

Therefore, θ is well defined.

Lemma 2.5. *If p is a prime number, then $a_0^{p-1}b_{p-1}^m \neq 0$ and $a_{p-1}^m b_0^{p-1} \neq 0$ for all $m \geq 0$.*

Proof. By symmetry, it suffices to prove only $a_0^{p-1}b_{p-1}^m \neq 0$. Let $T = \mathbb{Z}_p[t]/(t^p)$ and let $\theta : A_{\mathbb{Z}} \to T$ be the homomorphism defined in the preceding paragraph. Since

$$\theta\big(a_0^{p-1}b_{p-1}^m\big) = \theta(a_0)^{p-1}\theta(b_{p-1})^m = [(p-1)!]^m\bar{t}^{p-1} \neq 0,$$

we have $a_0^{p-1}b_{p-1}^m \neq 0$. □

Corollary 2.6. *If p is a prime number, then $\alpha_0^{p-1}\beta_{p-1}^m \neq 0$ and $\alpha_{p-1}^m\beta_0^{p-1} \neq 0$ for all $m \geq 0$.*

Proof. Let $\phi : A \to B$ be the isomorphism defined in Remark 2.3 by $\phi(a_n) = n!\alpha_n$ and $\phi(b_n) = n!\beta_n$ for all $n \geq 0$. Then $\alpha_0^{p-1}\beta_{p-1}^m = [(p-1)!]^{-m}\phi\big(a_0^{p-1}b_{p-1}^m\big) \neq 0$. By symmetry, $\alpha_{p-1}^m\beta_0^{p-1} \neq 0$. □

Evidently, B is the $\mathbb{Q}$-linear span of the monomials $u = \alpha_{i_1} \cdots \alpha_{i_k} \beta_{j_1} \cdots \beta_{j_m}$, where $k \geq 0$, $m \geq 0$, $i_s \geq 0$, and $j_t \geq 0$. It is understood that $u = \beta_{j_1} \cdots \beta_{j_m}$ if $k = 0$, $u = \alpha_{i_1} \cdots \alpha_{i_k}$ if $m = 0$, and $u = 1$ if $k = m = 0$. We call $\alpha_{i_1} \cdots \alpha_{i_k}$ the *α-part* of u and k the *α-length* of u. Similarly, $\beta_{j_1} \cdots \beta_{j_m}$ is called the *β-part* of u and m the *β-length* of u. By the (total) length $|u|$ of u we mean the sum $k + m$. Following Hamann and Swan (1986), we call the sum of the indices of u the (total) *grade* of u and denote it by $\mathrm{gr}(u)$, that is,

$$\mathrm{gr}(u) = i_1 + \cdots + i_k + j_1 + \cdots + j_m.$$

Also, we call $i_1 + \cdots + i_k$ the *α-grade* and $j_1 + \cdots + j_m$ the *β-grade* of u.

From now on, we shall, unless otherwise stated, always write a monomial $u = \alpha_{i_1} \cdots \alpha_{i_k} \beta_{j_1} \cdots \beta_{j_m}$ in the way that $i_1 \leq \cdots \leq i_k$ and $j_1 \leq \cdots \leq j_m$. Let M be the set of monomials of the form $u = \alpha_{i_1} \cdots \alpha_{i_k} \beta_{j_1} \cdots \beta_{j_m}$ where $k \geq 0$, $m \geq 0$, $i_s \geq 0$ and $j_t \geq k$. We shall show that M is a basis of B over $\mathbb{Q}$.

For each $k \geq 0$, let Ω_k be the set of all monomials $\alpha_{i_1} \cdots \alpha_{i_k}$ in α_i of length k, and $\Omega = \bigcup_{k=0}^{\infty} \Omega_k$ the set of all monomials in α_i. First we proceed to show that B is the $\mathbb{Q}$-linear span of M, or equivalently, for any $u \in \Omega$, $u\beta_{j_1} \cdots \beta_{j_m}$ is a linear combination of elements in M.

Lemma 2.7. *For any $u \in \Omega$ and $j \geq 0$, $u\beta_j$ is a linear combination of elements of the form $u'\beta_l$ with $u' \in \Omega_{|u|}$, $l \geq |u|$ and $gr(u'\beta_l) = gr(u\beta_j)$.*

Proof. We shall proceed by induction on $|u|$. If $|u| = 0$, that is $u = 1$, then $u\beta_j = \beta_j$ and there is nothing to prove. If $|u| = 1$, that is $u = \alpha_i$ for some i, we see that $u\beta_j = \alpha_i\beta_j$ is already of the form as required for $j \geq 1$. For $j = 0$, $u\beta_j = \alpha_i\beta_0$, and the result holds since $\alpha_0\beta_0 = 0$ and for $i \geq 1$, by (2.2),

$$\alpha_i\beta_0 = -\sum_{l=1}^{i} \alpha_{i-l}\beta_l.$$

Assume now that $|u| = k > 1$ and the lemma is true for each $v \in \Omega$ with $|v| < k$.

We define a linear order on Ω_{k-1}. For words $z = \alpha_{p_1} \cdots \alpha_{p_{k-1}}$ and $w = \alpha_{q_1} \cdots \alpha_{q_{k-1}}$ in Ω_{k-1} with $p_1 \leq \cdots \leq p_{k-1}$ and $q_1 \leq \cdots \leq q_{k-1}$, we consider $z_l = \alpha_{p_1} \cdots \alpha_{p_l}$ and $w_l = \alpha_{q_1} \cdots \alpha_{q_l}$ for $l = 1, \ldots, k-1$. We define $z < w$ if $\mathrm{gr}(z) < \mathrm{gr}(w)$ or there exists an s such that $\mathrm{gr}(z_s) < \mathrm{gr}(w_s)$ and $\mathrm{gr}(z_l) = \mathrm{gr}(w_l)$ for all $l > s$. As one can see, the relation so defined is a linear order on Ω_{k-1} indeed.

For $u = \alpha_{i_1} \cdots \alpha_{i_k}$, we proceed by induction on the order of $u' = \alpha_{i_1} \cdots \alpha_{i_{k-1}} \in \Omega_{k-1}$. Note that the smallest element in Ω_{k-1} is α_0^{k-1} and consider the case $u' = \alpha_0^{k-1}$. If $j \geq k$, $u\beta_j = \alpha_0^{k-1}\alpha_{i_k}\beta_j$ is already of the form as required. If $j < k-1$, then $u\beta_j = \alpha_0^{k-1}\alpha_{i_k}\beta_j = (\alpha_0^{k-1}\beta_j)\alpha_{i_k} = 0$ by Lemma 2.4. As to the case $j = k-1$, we have by (2.2)

$$\alpha_{i_k}\beta_{k-1} = -\sum_{l=0, l\neq k-1}^{i_k+k-1} \alpha_{i_k+k-1-l}\beta_l,$$

and so, by the previous two cases,

$$\begin{aligned} u\beta_{k-1} &= \alpha_0^{k-1}\alpha_{i_k}\beta_{k-1} \\ &= -\sum_{l=0,l\neq k-1}^{i_k+k-1} \alpha_0^{k-1}\alpha_{i_k+k-1-l}\beta_l \\ &= -\sum_{l=k}^{i_k+k-1} \alpha_0^{k-1}\alpha_{i_k+k-1-l}\beta_l. \end{aligned}$$

Consider the case $u' > \alpha_1^{k-1}$. In this case, we have $i_{k-1} > 0$ and so $i_k > 1$. Assume that the lemma is true for each $v \in \Omega_{k-1}$ with $v < u'$.

If $j \geq k$, then $u\beta_j$ is already of the form as required. So assume that $j < k$. Suppose first that $j < k-1$. Set $w_1 = \alpha_{i_1}\cdots\alpha_{i_{j+1}}$ and $w_2 = \alpha_{i_{j+2}}\cdots\alpha_{i_k}$. Then

$$u\beta_j = \alpha_{i_1}\cdots\alpha_{i_k}\beta_j = w_1w_2\beta_j = (w_1\beta_j)w_2.$$

As $|w_1| = j+1 < k = |u|$, by the induction hypothesis, we have

$$w_1\beta_j = \lambda_1 v_1\beta_{l_1} + \cdots + \lambda_n v_n\beta_{l_n},$$

where $\lambda_s \in \mathbb{Q}$, $v_s \in \Omega_{j+1}$, $l_s \geq j+1$, and $\mathrm{gr}(v_s\beta_{l_s}) = \mathrm{gr}(w_1\beta_j)$ for each $s = 1, 2, \ldots, n$. Since $\mathrm{gr}(v_s) = \mathrm{gr}(v_s\beta_{l_s}) - l_s = \mathrm{gr}(w_1\beta_j) - l_s = \mathrm{gr}(w_1) + j - l_s < \mathrm{gr}(w_1)$, we see that

$$\mathrm{gr}(v_s\alpha_{i_{j+2}}\cdots\alpha_{i_{k-1}}) < \mathrm{gr}(w_1\alpha_{i_{j+2}}\cdots\alpha_{i_{k-1}}) = \mathrm{gr}(u')$$

and so

$$v_s\alpha_{i_{j+2}}\cdots\alpha_{i_{k-1}} < u'.$$

By the induction hypothesis, each $v_sw_2\beta_{l_s} = (v_s\alpha_{i_{j+2}}\cdots\alpha_{i_{k-1}})\alpha_{i_k}\beta_{l_s}$ is a linear combination of elements of the form $v\beta_l$ with $v \in \Omega_k$, $l \geq k$ and $\mathrm{gr}(v\beta_l) = \mathrm{gr}(v_sw_2\beta_{l_s}) = \mathrm{gr}(w_1w_2\beta_j) = \mathrm{gr}(u\beta_j)$ whence so is

$$u\beta_j = (w_1\beta_j)w_2 = \lambda_1 v_1w_2\beta_{l_1} + \cdots + \lambda_n v_nw_2\beta_{l_n}.$$

Suppose next that $j = k-1$. Using

$$\alpha_{i_k}\beta_{k-1} = -\sum_{l=0}^{k-2}\alpha_{i_k+k-1-l}\beta_l - \sum_{l=k}^{i_k+k-1}\alpha_{i_k+k-1-l}\beta_l,$$

we may write

$$u\beta_{k-1} = u'(\alpha_{i_k}\beta_{k-1}) = z_1 + z_2,$$

where

$$z_1 = -u'\sum_{l=0}^{k-2}\alpha_{i_k+k-1-l}\beta_l,$$

and

$$z_2 = -u' \sum_{l=k}^{i_k+k-1} \alpha_{i_k+k-1-l}\beta_l.$$

Note that each term $u'\alpha_{i_k+k-1-l}\beta_l$ in z_2 has β-grade at least k and so z_2 is a linear combination of elements of the form as required. On the other hand, each term $u'\alpha_{i_k+k-1-l}\beta_l$ in z_1 has β-grade less than $k-1$ and we are done by the previous case $j < k-1$. This completes the proof. $\square$

Corollary 2.8. *For any $u \in \Omega$, $u\beta_{j_1}\cdots\beta_{j_m}$ is a linear combination of elements of the form $u'\beta_{j'_1}\cdots\beta_{j'_m}$ where $u' \in \Omega_{|u|}$, $j'_s \geq |u|$ and $gr(u'\beta_{j'_1}\cdots\beta_{j'_m}) = gr(u\beta_{j_1}\cdots\beta_{j_m})$.*

Proof. We proceed by induction on m. If $m = 0$, there is nothing to prove. Assume that $m \geq 1$ and the corollary holds for $m-1$. Then we have

$$u\beta_{j_1}\cdots\beta_{j_{m-1}} = \sum_{s=1}^{n} \lambda_s u_s \beta_{j_{s,1}}\cdots\beta_{j_{s,m-1}},$$

where $\lambda_s \in \mathbb{Q}$, $u_s \in \Omega_{|u|}$, $j_{s,t} \geq |u|$ and $\operatorname{gr}(u_s\beta_{j_{s,1}}\cdots\beta_{j_{s,m-1}}) = \operatorname{gr}(u\beta_{j_1}\cdots\beta_{j_{m-1}})$. By Lemma 2.7, we may write each $u_s\beta_{j_m} = \sum_{t=1}^{n_s} \lambda_{s,t}u_{s,t}\beta_{l_{s,t}}$, where $\lambda_{s,t} \in \mathbb{Q}$, $u_{s,t} \in \Omega_{|u_s|} = \Omega_{|u|}$, $l_{s,t} \geq |u_s| = |u|$, and $\operatorname{gr}(u_{s,t}\beta_{l_{s,t}}) = \operatorname{gr}(u_s\beta_{j_m})$ for all $s = 1, 2, \ldots, n$. Thus

$$u\beta_{j_1}\cdots\beta_{j_m} = \left(\sum_{s=1}^{n} \lambda_s u_s \beta_{j_{s,1}}\cdots\beta_{j_{s,m-1}}\right)\beta_{j_m} = \sum_{s=1}^{n}\sum_{t=1}^{n_s} \lambda_s\lambda_{s,t}u_{s,t}\beta_{l_{s,t}}\beta_{j_{s,1}}\cdots\beta_{j_{s,m-1}}$$

is as required since

$$\operatorname{gr}(u_{s,t}\beta_{l_{s,t}}\beta_{j_{s,1}}\cdots\beta_{j_{s,m-1}}) = \operatorname{gr}(u_s\beta_{j_m}\beta_{j_{s,1}}\cdots\beta_{j_{s,m-1}}) = \operatorname{gr}(u\beta_{j_1}\cdots\beta_{j_{m-1}}\beta_{j_m})$$

for all $t = 1, 2, \ldots, n_s$ and $s = 1, 2, \ldots, n$. Thus the proof is complete. $\square$

As a consequence, we have the following lemma.

Lemma 2.9. *For any $m \geq 0$ and $n \geq 0$, both $\alpha_m^n\beta_0^m$ and $\alpha_0^n\beta_n^m$ are nonzero and $\mathbb{Q}$-dependent.*

Proof. By Corollary 2.8, the monomial $\alpha_m^n\beta_0^m$ can be written as a linear combination of monomials $\alpha_{i_1}\cdots\alpha_{i_n}\beta_{j_1}\cdots\beta_{j_m}$ with $i_s \geq 0$, $j_t \geq n$ for all s, t and $i_1 + \cdots + i_n + j_1 + \cdots + j_m = \operatorname{gr}(\alpha_m^n\beta_0^m) = mn$. It follows that $i_s = 0$ and $j_t = n$ for all $s = 1, \ldots, n$ and $t = 1, \ldots, m$. Hence we have $\alpha_m^n\beta_0^m = \lambda\alpha_0^n\beta_n^m$ for some $\lambda \in \mathbb{Q}$ and so $\alpha_m^n\beta_0^m$ and $\alpha_0^n\beta_n^m$ are $\mathbb{Q}$-dependent.

We claim that $\alpha_0^n\beta_n^m \neq 0$. Let p be a prime number such that $p > m$. It follows from Corollary 2.6 that $\alpha_{p-1}^n\beta_0^{p-1} \neq 0$. By the preceding paragraph, we get

$$\alpha_{p-1}^n\beta_0^{p-1} = \lambda'\alpha_0^n\beta_n^{p-1}$$

for some $\lambda' \in \mathbb{Q}$. Therefore $\alpha_0^n\beta_n^{p-1} \neq 0$ and so $\alpha_0^n\beta_n^m \neq 0$. By symmetry, we have also $\alpha_n^m\beta_0^n \neq 0$. $\square$

It follows immediately from the preceding lemma that neither α_0 nor β_0 is nilpotent.

Corollary 2.10. *$\alpha_0^m \neq 0$ and $\beta_0^m \neq 0$ for any $m \geq 0$.*

We shall make use of some derivations on B. For convenience, set $\alpha_i = \beta_i = 0$ for a negative integer i. For $i \geq 0$, let ∂_{α_i} and ∂_{β_i} be the derivations on B such that $\partial_{\alpha_i}(\alpha_j) = \alpha_{j-i}$, $\partial_{\alpha_i}(\beta_j) = 0$, $\partial_{\beta_i}(\beta_j) = \beta_{j-i}$, $\partial_{\beta_i}(\alpha_j) = 0$. First we need to check that ∂_{α_i} and ∂_{β_i} are well defined. Note that

$$\partial_{\alpha_i}\left(\sum_{j=0}^{n} \alpha_j\beta_{n-j}\right) = \sum_{j=i}^{n} \alpha_{j-i}\beta_{n-j} = \sum_{k=0}^{n-i} \alpha_k\beta_{n-i-k} = 0$$

which is the requirement for ∂_{α_i} to be well defined. Similarly, we can check that ∂_{β_i} is well defined.

We shall also use a sort of reversed lexicographical order on the α- and β-parts of the monomials in M. First, we set an order on α_i by $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$. Writing each monomial $\alpha_{i_1}\cdots\alpha_{i_k}$ with $\alpha_{i_1} \leq \cdots \leq \alpha_{i_k}$, we define $\alpha_{i_1}\cdots\alpha_{i_k} <_{\mathrm{r}} \alpha_{i'_1}\cdots\alpha_{i'_{k'}}$ if $\alpha_{i_k}\cdots\alpha_{i_1} < \alpha_{i'_{k'}}\cdots\alpha_{i'_1}$ lexicographically. Similarly, we set an order on β_i by $\beta_0 < \beta_1 < \beta_2 < \cdots$. And, writing each monomial $\beta_{j_1}\cdots\beta_{j_m}$ with $\beta_{j_1} \leq \cdots \leq \beta_{j_m}$, we define $\beta_{j_1}\cdots\beta_{j_m} <_{\mathrm{r}} \beta_{j'_1}\cdots\beta_{j'_{m'}}$ if $\beta_{j_m}\cdots\beta_{j_1} < \beta_{j'_{m'}}\cdots\beta_{j'_1}$ lexicographically.

Lemma 2.11. *M is a basis of B over $\mathbb{Q}$.*

Proof. Since B is the $\mathbb{Q}$-linear span of the monomials $\alpha_{i_1}\cdots\alpha_{i_k}\beta_{j_1}\cdots\beta_{j_m}$, it follows from Corollary 2.8 that B is spanned by M over $\mathbb{Q}$. Hence, we need only to show that M is linearly independent.

Suppose that some nontrivial linear combination w of monomials from M is equal to zero. For $\mu, v \in \mathbb{Q}$, let $f_{\mu,v} : B \to B$ be the homomorphism defined by $f_{\mu,v}(\alpha_i) = \mu\alpha_i$ and $f_{\mu,v}(\beta_j) = v\beta_j$ for all i and j. One can readily verify that the homomorphism $f_{\mu,v}$ is well defined indeed. Applying $f_{\mu,1}$ to w for $\mu = 1, 2, \ldots$, we see that, for each μ, the sum of all the monomials in w having α-length μ is equal to zero since the van der Monde determinant is nonzero. Hence we may assume that all the monomials in w have the same α-length. Similarly, we may also assume that all the monomials in w have the same β-length. Thus, assume that

$$w = \sum_{s=1}^{n} \lambda_s\alpha_{i_{s,1}}\ldots\alpha_{i_{s,k}}\beta_{j_{s,1}}\ldots\beta_{j_{s,m}} = 0$$

where all the n monomials $\alpha_{i_{s,1}}\ldots\alpha_{i_{s,k}}\beta_{j_{s,1}}\ldots\beta_{j_{s,m}} \in M$ are different and $\lambda_s \neq 0$ for all s.

Let $\alpha_{i_{t,1}}\ldots\alpha_{i_{t,k}}$ be the largest (with respect to the order $<_{\mathrm{r}}$) α-part among the monomials in w. Set $D_\alpha = \partial_{\alpha_{i_{t,1}}}\ldots\partial_{\alpha_{i_{t,k}}}$. Then $D_\alpha(\alpha_{i_{t,1}}\ldots\alpha_{i_{t,k}}) = \eta\alpha_0^k$ for some positive integer η, while $D_\alpha(\alpha_{i_{s,1}}\ldots\alpha_{i_{s,k}}) = 0$ for $\alpha_{i_{s,1}}\ldots\alpha_{i_{s,k}} <_{\mathrm{r}} \alpha_{i_{t,1}}\ldots\alpha_{i_{t,k}}$. Thus we have

$$D_\alpha(w) = \eta\alpha_0^k \sum_{s'=1}^{n'} \lambda'_{s'}\beta_{j'_{s',1}}\ldots\beta_{j'_{s',m}} = 0$$

where all the n' monomials $\beta_{j'_{s',1}}\ldots\beta_{j'_{s',m}}$ are different and $\lambda'_{s'} \neq 0$ for all s'.

Let $\beta_{j'_{t',1}} \dots \beta_{j'_{t',m}}$ be the largest (with respect to $<_{\mathrm{r}}$) β-part among the monomials in $D_\alpha(w)$. Set $D_\beta = \partial_{\beta_{j'_{t',1}-k}} \dots \partial_{\beta_{j'_{t',m}-k}}$. Then $D_\beta(\alpha_0^k \beta_{j'_{t',1}} \dots \beta_{j'_{t',m}}) = \zeta \alpha_0^k \beta_k^m$ for some positive integer ζ, while $D_\beta(\alpha_0^k \beta_{j'_{s',1}} \dots \beta_{j'_{s',m}}) = 0$ for $\beta_{j'_{s',1}} \dots \beta_{j'_{s',m}} <_{\mathrm{r}} \beta_{j'_{t',1}} \dots \beta_{j'_{t',m}}$ since $\alpha_0^k \beta_j = 0$ for $j < k$ by Lemma 2.4. Therefore, $D_\beta D_\alpha(w) = \lambda'_{t'} \eta \zeta \alpha_0^k \beta_k^m = 0$, contradicting Lemma 2.9. $\square$

Now each element v in B can be expressed as a linear combination of elements in M. We define the grade of v to be the largest grade of its nonzero summand.

We note that there exists a derivation δ on B with $\delta(\alpha_{i-1}) = i\alpha_i$ and $\delta(\beta_{i-1}) = i\beta_i$ for $i \geq 1$. To see this, we need only to show that δ is well defined:

$$\begin{aligned}
\delta\left(\sum_{i=0}^{n} \alpha_i \beta_{n-i}\right) &= \sum_{i=0}^{n} \big(\delta(\alpha_i)\beta_{n-i} + \alpha_i \delta(\beta_{n-i})\big) \\
&= \sum_{i=0}^{n} \big((i+1)\alpha_{i+1}\beta_{n-i} + (n-i+1)\alpha_i \beta_{n-i+1}\big) \\
&= \sum_{i=0}^{n} (i+1)\alpha_{i+1}\beta_{n-i} + \sum_{i=0}^{n} (n-i+1)\alpha_i \beta_{n-i+1} \\
&= \sum_{i=1}^{n+1} i\alpha_i \beta_{n-i+1} + \sum_{i=0}^{n} (n-i+1)\alpha_i \beta_{n-i+1} \\
&= \sum_{i=0}^{n+1} i\alpha_i \beta_{n-i+1} + \sum_{i=0}^{n+1} (n-i+1)\alpha_i \beta_{n-i+1} \\
&= \sum_{i=0}^{n+1} (n+1)\alpha_i \beta_{n-i+1} \\
&= (n+1) \sum_{i=0}^{n+1} \alpha_i \beta_{n+1-i} \\
&= 0.
\end{aligned}$$

Recall that there is an isomorphism $\phi : A \to B$ such that $\phi(a_n) = n!\alpha_n$ and $\phi(b_n) = n!\beta_n$ for all $n \geq 0$. Note that $\delta(\phi(a_n)) = \phi(a_{n+1})$ and $\delta(\phi(b_n)) = \phi(b_{n+1})$ for all $n \geq 0$. Hence we have the following lemma.

Lemma 2.12. $\delta(\phi(A_{\mathbb{Z}})) \subseteq \phi(A_{\mathbb{Z}})$.

We shall need two more derivations, ϵ_α and ϵ_β, of B which are defined by $\epsilon_\alpha(\alpha_i) = \alpha_i$, $\epsilon_\alpha(\beta_i) = 0$, $\epsilon_\beta(\alpha_i) = 0$, and $\epsilon_\beta(\beta_i) = \beta_i$ for all $i \geq 0$. Let $\mathscr{L}$ be the $\mathbb{Q}$-linear space spanned by $\{\delta, \epsilon_\alpha, \epsilon_\beta\} \cup \{\partial_{\alpha_i}, \partial_{\beta_i} \mid i \geq 1\}$.

Lemma 2.13. *For any nonzero $\mathscr{L}$-stable ideal U of B, there exist $s \geq 0$ and $t \geq 1$ such that $\alpha_0^s \beta_s^t \in U$.*

Proof. Let u be a nonzero element in U of the smallest grade. We claim that the α-part of each summand of u is of the form α_0^k. Assume the contrary. Let

$\alpha_0^{k_0}\alpha_{i_1}^{k_1}\cdots\alpha_{i_r}^{k_r}$ be the largest (with respect to $<_r$) α-part, where $r \geq 1$, $0 < i_1 < \cdots < i_r$ and $k_s \geq 1$ for all $s = 1, \ldots r$. Then set $D = \partial_{\alpha_{i_1}}^{k_1}\cdots\partial_{\alpha_{i_r}}^{k_r}$ and $D(u)$ would be a nonzero element in U of smaller grade than u, a contradiction.

Thus each monomial in u is of the form $\alpha_0^k\beta_{j_1}\cdots\beta_{j_m}$ with each $j_s \geq k$. We claim that in fact $j_s = k$ for each s. Assume the contrary. Let $\alpha_0^{k_0}\beta_{k_0}^{m_0}\beta_{j_1}^{m_1}\cdots\beta_{j_t}^{m_t}$ be a monomial in u such that $\beta_0^{m_0}\beta_{j_1-k_0}^{m_1}\cdots\beta_{j_t-k_0}^{m_t}$ is the largest (with respect to $<_r$), where $t \geq 1$, $k_0 < j_1 < \cdots < j_t$ and $m_s \geq 1$ for all $s = 1, \ldots t$. Then for $D = \partial_{\beta_{j_1-k_0}}^{m_1}\cdots\partial_{\beta_{j_t-k_0}}^{m_t}$, $D(u)$ would be a nonzero element in U of smaller grade than u, a contradiction.

Hence we have

$$u = \sum_{i=s}^{k}\sum_{j=0}^{m}\lambda_{i,j}\alpha_0^i\beta_i^j \in U,$$

where $\lambda_{i,j} \in \mathbb{Q}$ and $\lambda_{s,j} \neq 0$ for some j. Multiplying u by β_s we get

$$u\beta_s = \sum_{j=0}^{m}\lambda_{s,j}\alpha_0^s\beta_s^{j+1} \in U, \tag{2.3}$$

since $\alpha_0^i\beta_s = 0$ for $i > s$ by Lemma 2.4.

Now, we apply ϵ_β repeatedly to (2.3):

$$\begin{aligned}
\epsilon_\beta(u\beta_s) &= \sum_{j=0}^{m}\lambda_{s,j}\epsilon_\beta\left(\alpha_0^s\beta_s^{j+1}\right) = \sum_{j=0}^{m}\lambda_{s,j}(j+1)\alpha_0^s\beta_s^{j+1},\\
\epsilon_\beta^2(u\beta_s) &= \sum_{j=0}^{m}\lambda_{s,j}\epsilon_\beta^2\left(\alpha_0^s\beta_s^{j+1}\right) = \sum_{j=0}^{m}\lambda_{s,j}(j+1)^2\alpha_0^s\beta_s^{j+1},\\
&\ \ \vdots\\
\epsilon_\beta^m(u\beta_s) &= \sum_{j=0}^{m}\lambda_{s,j}\epsilon_\beta^m\left(\alpha_0^s\beta_s^{j+1}\right) = \sum_{j=0}^{m}\lambda_{s,j}(j+1)^m\alpha_0^s\beta_s^{j+1}.
\end{aligned}$$

Since U is $\mathcal{L}$-stable, $\epsilon_\beta^k(u\beta_s) \in U$ for $0 \leq k \leq m$. Using the van der Monde determinant and applying the argument as we did in the proof of Lemma 2.11, we conclude that these $m+1$ elements $\lambda_{s,0}\alpha_0^s\beta_s^1, \ldots, \lambda_{s,m}\alpha_0^s\beta_s^{m+1}$ are all in U. Hence $\alpha_0^s\beta_s^t \in U$ for some $t \geq 1$ with $\lambda_{s,t-1} \neq 0$ since $\mathbb{Q} \subseteq B$. □

Theorem 2.14. *B is an $\mathcal{L}$-prime ring which is not primary.*

Proof. Let U and V be nonzero $\mathcal{L}$-stable ideals of B. We proceed to show $UV \neq 0$. As Lemma 2.13 indicates, there exist $\alpha_0^{m_1}\beta_{m_1}^{n_1} \in U$ and $\alpha_0^{m_2}\beta_{m_2}^{n_2} \in V$ for some non-negative integers m_1, m_2 and positive integers n_1, n_2. We may assume that $m_1 = m_2 = p-1$ for some prime p and $n_1 = n_2 = n$ for some positive integer n. To see this, first we may write, by Lemma 2.9,

$$\alpha_0^{m_1}\beta_{m_1}^{n_1} = \mu_1\alpha_{n_1}^{m_1}\beta_0^{n_1} \in U \quad \text{and} \quad \alpha_0^{m_2}\beta_{m_2}^{n_2} = \mu_2\alpha_{n_2}^{m_2}\beta_0^{n_2} \in V \tag{2.4}$$

for some nonzero μ_1 and μ_2 in $\mathbb{Q}$. Take p a prime greater than both m_1 and m_2. Multiplying the two elements in (2.4) by $\alpha_{n_1}^{p-1-m_1}$ and $\alpha_{n_2}^{p-1-m_2}$, respectively, and using Lemma 2.9 again, we obtain

$$\mu_1\alpha_{n_1}^{p-1}\beta_0^{n_1} = \nu_1\alpha_0^{p-1}\beta_{p-1}^{n_1} \in U \quad \text{and} \quad \mu_2\alpha_{n_2}^{p-1}\beta_0^{n_2} = \nu_2\alpha_0^{p-1}\beta_{p-1}^{n_2} \in V \tag{2.5}$$

for some nonzero ν_1 and ν_2 in $\mathbb{Q}$. Take $n = \max\{n_1, n_2\}$ and we have

$$\nu_1\alpha_0^{p-1}\beta_{p-1}^{n} \in U \quad \text{and} \quad \nu_2\alpha_0^{p-1}\beta_{p-1}^{n} \in V,$$

by multiplying the two elements in (2.5) by $\beta_{p-1}^{n-n_1}$ and $\beta_{p-1}^{n-n_2}$, respectively, and so

$$w = \alpha_0^{p-1}\beta_{p-1}^{n} \in U \cap V.$$

Since $\delta^{p-1}(w) \in U \cap V$, our goal is to show that $\delta^{p-1}(w)^2 = \delta^{p-1}\big(\alpha_0^{p-1}\beta_{p-1}^{n}\big)^2 \neq 0$. Note that

$$\begin{aligned}[(p-1)!]^n\delta^{p-1}\big(\alpha_0^{p-1}\beta_{p-1}^{n}\big) &= \delta^{p-1}(\phi(a_0)^{p-1}\phi(b_{p-1})^n) \\ &= (p-1)!\phi(a_1)^{p-1}\phi(b_{p-1})^n + \sum_k \mu_k u_k\end{aligned}$$

where $\mu_k \in \mathbb{N}$ and each u_k is a monomial in $\phi(A_{\mathbb{Z}})$ involving either $\phi(a_i)$ with $i \geq 2$ or $\phi(b_j)$ with $j \geq p$. Thus we have

$$\phi^{-1}\big([(p-1)!]^n\delta^{p-1}\big(\alpha_0^{p-1}\beta_{p-1}^{n}\big)\big) = (p-1)!a_1^{p-1}b_{p-1}^{n} + \sum_k \mu_k u'_k,$$

where each $u'_k = \phi^{-1}(u_k)$ is a monomial in $A_{\mathbb{Z}}$ involving either a_i with $i \geq 2$ or b_j with $j \geq p$.

Let $T = \mathbb{Z}_p[t]/(t^p)$ and let $\theta : A_{\mathbb{Z}} \to T$ be the homomorphism such that $\theta(a_0) = \bar{t}$, $\theta(a_1) = 1$, $\theta(a_i) = 0$ for all $i \geq 2$, and $\theta(b_0) = \bar{t}^{p-1}$, $\theta(b_1) = (p-1)\bar{t}^{p-2}$, $\ldots$, $\theta(b_{p-1}) = (p-1)!$, and $\theta(b_i) = 0$ for all $i \geq p$. Then

$$\begin{aligned}&\theta\big(\phi^{-1}\big((p-1)!^n\delta^{p-1}\big(\alpha_0^{p-1}\beta_{p-1}^{n}\big)\big)\big) \\ &\quad = (p-1)!\theta(a_1)^{p-1}\theta(b_{p-1})^n + \sum_k \mu_k\theta(u'_k) = [(p-1)!]^{n+1}.\end{aligned}$$

Therefore,

$$\theta\big(\phi^{-1}\big(\delta^{p-1}\big(\alpha_0^{p-1}\beta_{p-1}^{n}\big)\big)\big) = (p-1)!$$

and so

$$\theta\big(\phi^{-1}\big(\delta^{p-1}\big(\alpha_0^{p-1}\beta_{p-1}^{n}\big)^2\big)\big) = [(p-1)!]^2$$

which is not zero in T. This shows that

$$\delta^{p-1}\left(\alpha_0^{p-1}\beta_{p-1}^n\right)^2 \neq 0.$$

Therefore B is an $\mathscr{L}$-prime ring.

Since $\alpha_0\beta_0 = 0$ while $\alpha_0 \neq 0$ and $\beta_0^m \neq 0$ for all $m \geq 1$ by Corollary 2.10, the commutative ring B is not a primary ring. The proof is now complete. $\square$

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